Markov Decision Processes and Reinforcement Learning

An Introduction to Stochastic Planning

Path Planning Assumptions

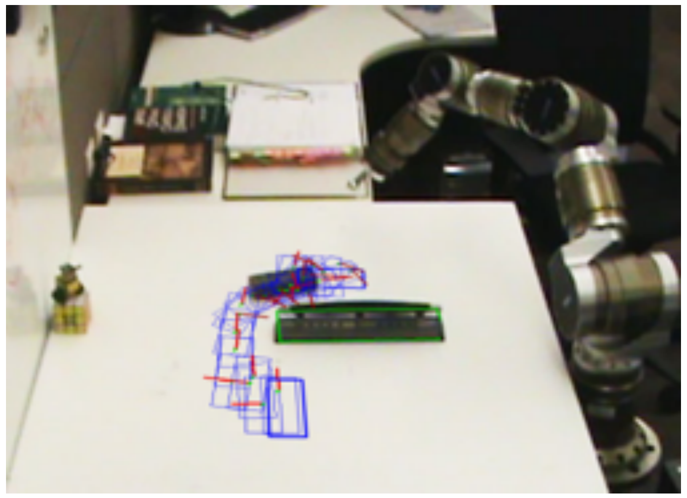
- Obstacles?

➡ Reliable Collision detection (assumes robust perception)

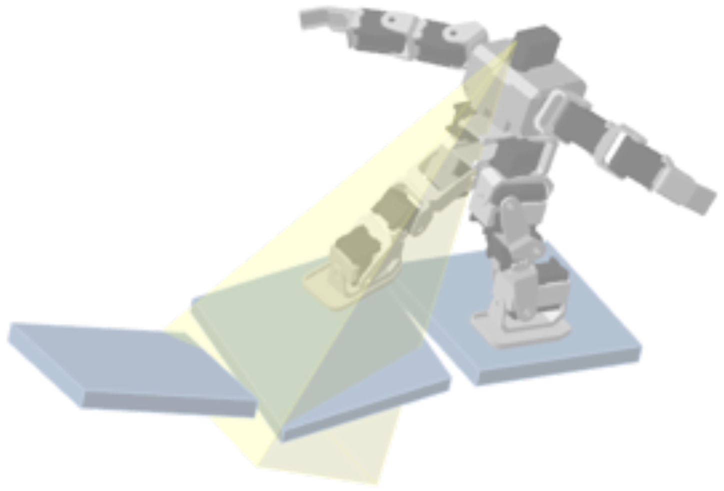
- Transitions

➡ Reliable mechanism for moving along path in graph (i.e., a controller)

move_block(x1,y1,x2,y2)



Two Sources of Error



- **State Estimation**
  - You don't know exactly where you are
  - Sensors have noise
  - No complete environment information
- **Action Execution**
  - Your actuators don't do what you tell them
  - Your system responds differently than you expect
  - Friction, gears, air resistance, etc.

Basic Idea: Your model of the world is incorrect!

Markov (Decision) Processes: A New Model for Planning

- Handles both forms of uncertainty in a *statistically principled way*
- Gives us back optimality!
- Of course, I'm talking about (PO)MDPs
  - All this flexibility comes at a cost, as we'll see...
  - Current research is largely about scalability

Handling Non-Deterministic Actions

- **Problem**: we don't know where our actions take us
- **Solution**: start thinking about *expected values*

➡ Weight each outcome by the *probability* of getting there

Formalizing the MDP Model

- **Step 1**: define the core problem representation
- Considerations?

1. should represent "rewards" somehow
2. should represent "state" somehow
3. should represent "actions" somehow

➡ next: what if actions aren't deterministic??

Formalizing the MDP Model

- **Step 2**: How to handle *stochastic* action effects ("transitions")?
  - replace transition rule with transition distribution

$$T(s, s') = P(s'|s) = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{bmatrix}$$

Formalizing the MDP Model

- Overall: $MDP = \{\mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{R}, \gamma\}$

$$\begin{aligned} \mathcal{S} &= \text{States} \\ \mathcal{A} &= \text{Actions} \\ \mathcal{T} &= \text{Transition Model} \\ \mathcal{R} &= \text{Rewards} \end{aligned}$$

Pacman states

➡ {all positions of pacman, ghosts, food, & pellets}

Pacman actions

➡ {N,S,E,W}

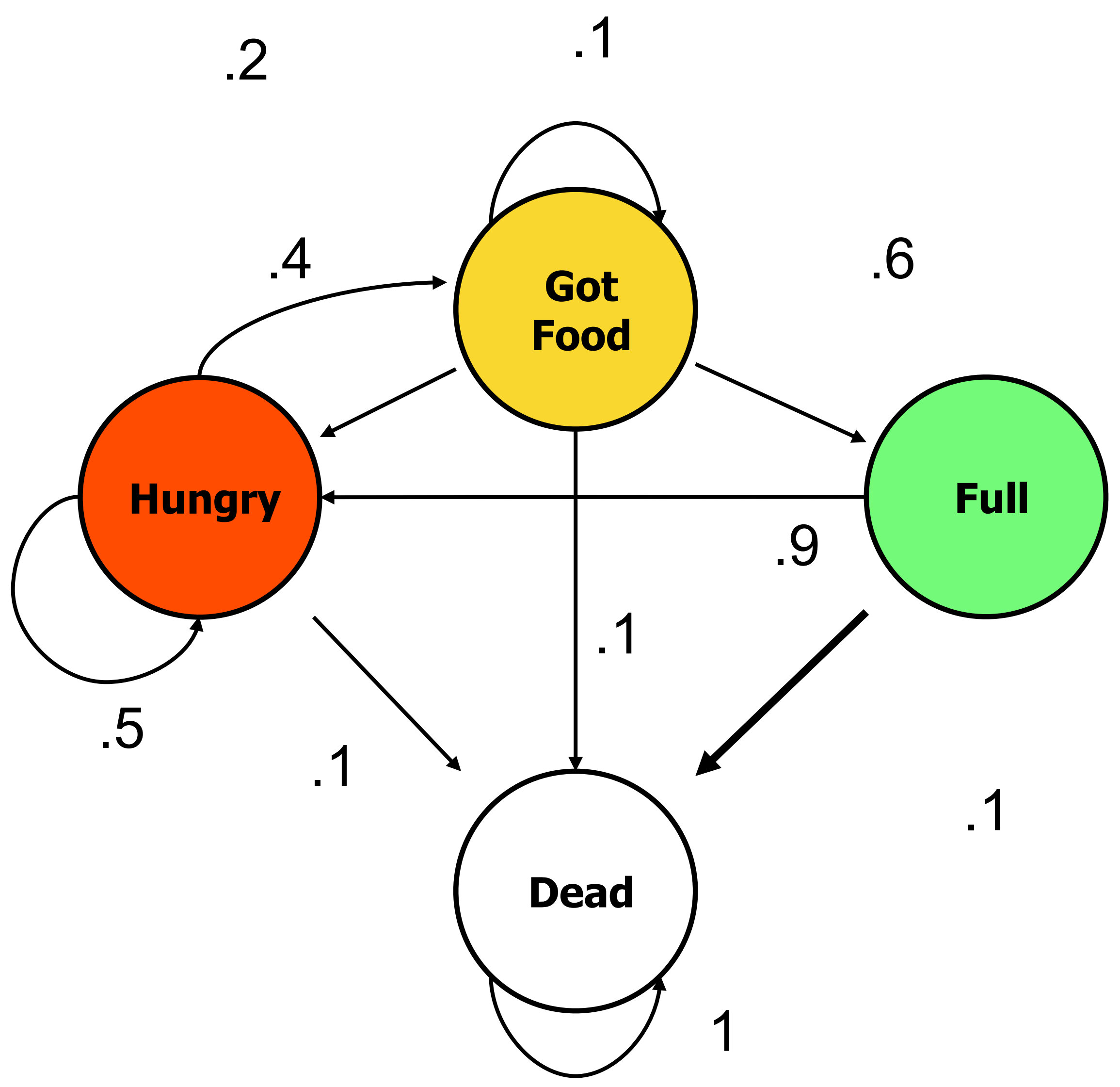
Pacman model

➡ {move directions, die from ghosts, eat food,...}

Pacman rewards

➡ -1 per step, +10 food, -500 die,+500 win,...

Markov Processes: Caveman's World



States: {H, G, F, D}

Actions: {}

(we'll get back to this)

Transition Model:

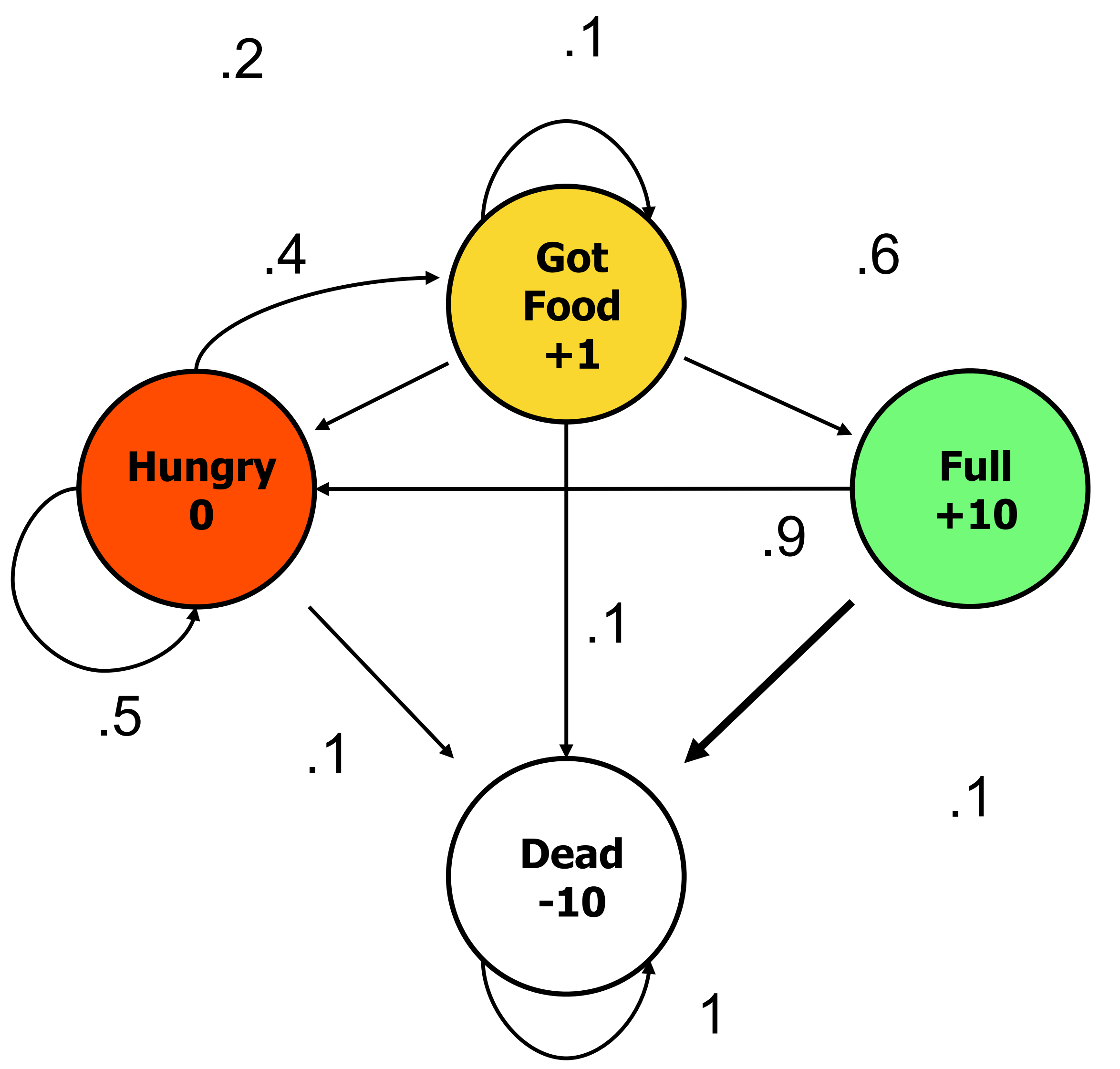
	H	G	F	D	
H	0.5	0.4	0.0	0.1	∑=1.0
G	0.2	0.1	0.6	0.1	∑=1.0
F	0.9	0.0	0.0	0.1	∑=1.0
D	0.0	0.0	0.0	1.0	∑=1.0

just a CPT ↗

Rewards:

H	G	F	D
0	1	10	-10

Markov Processes: Caveman's World



States: {H, G, F, D}

Actions: {}

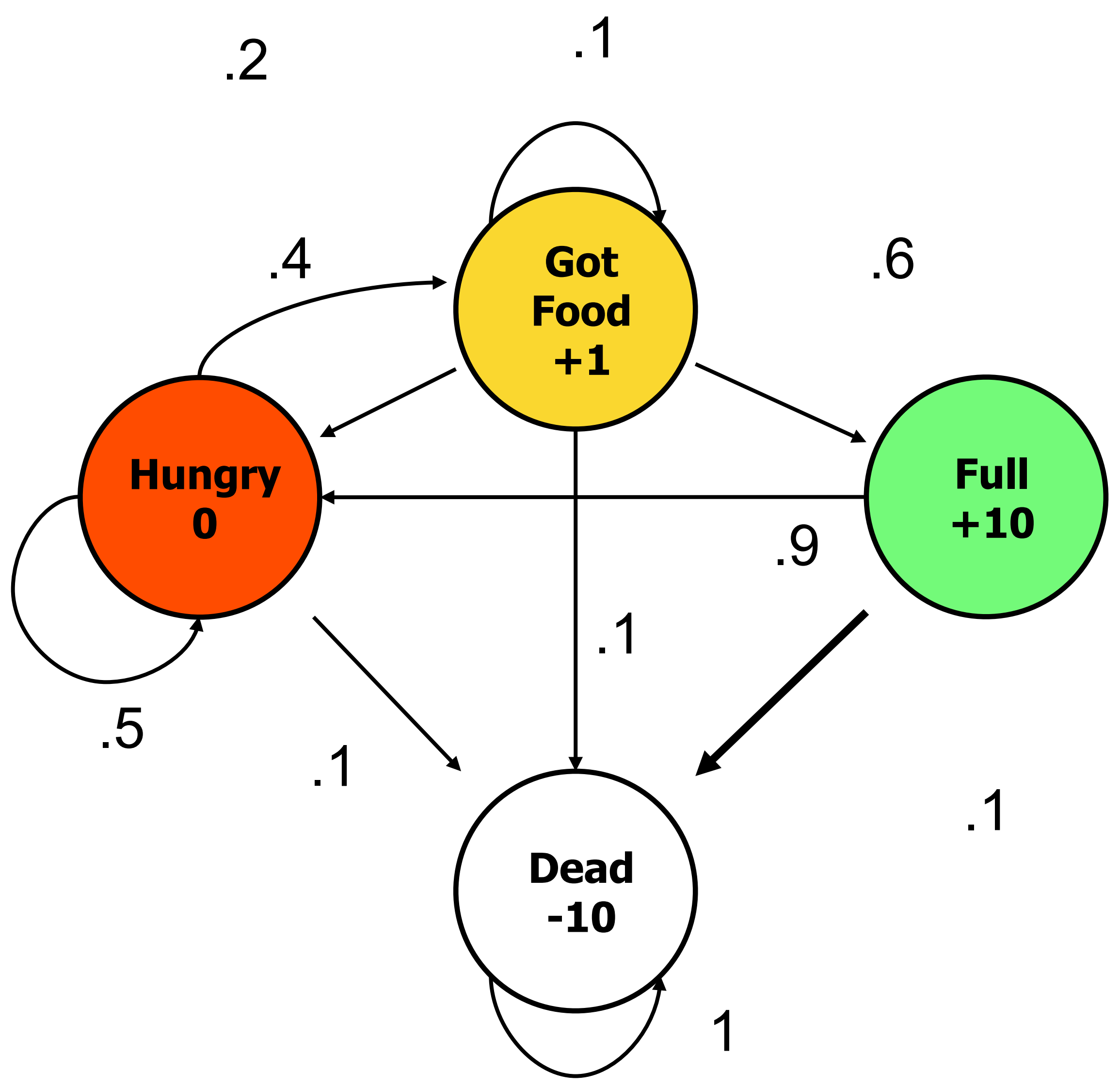
Transition Model:

	H	G	F	D
H	0.5	0.4	0.0	0.1
G	0.2	0.1	0.6	0.1
F	0.9	0.0	0.0	0.1
D	0.0	0.0	0.0	1.0

Rewards:

H	G	F	D
0	1	10	-10

Markov Processes: Caveman's World



Value

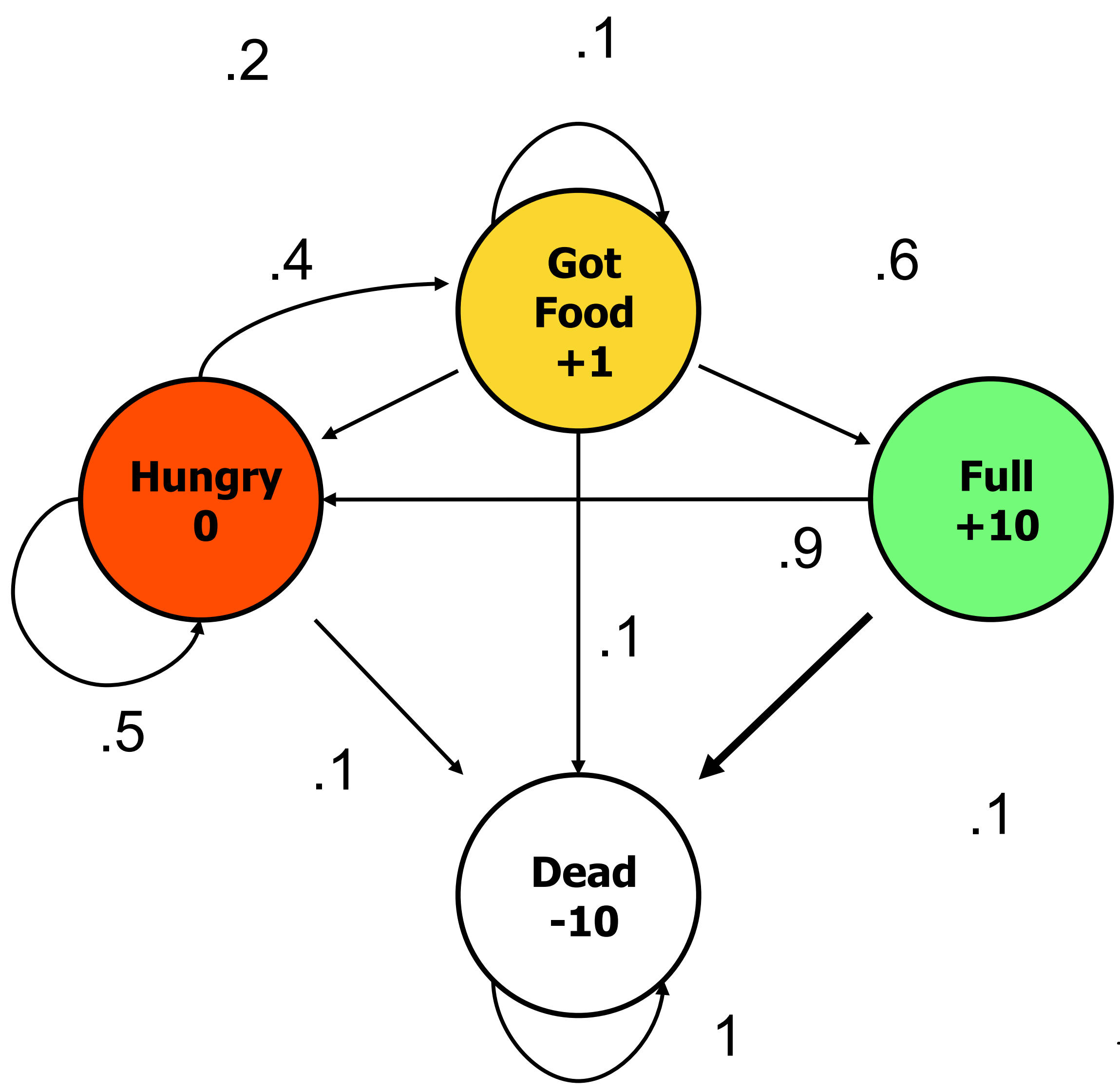
–How good is it to be in a state?

–Sum of DISCOUNTED expected rewards:

$$V(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

–Reward now is better than later. Why??

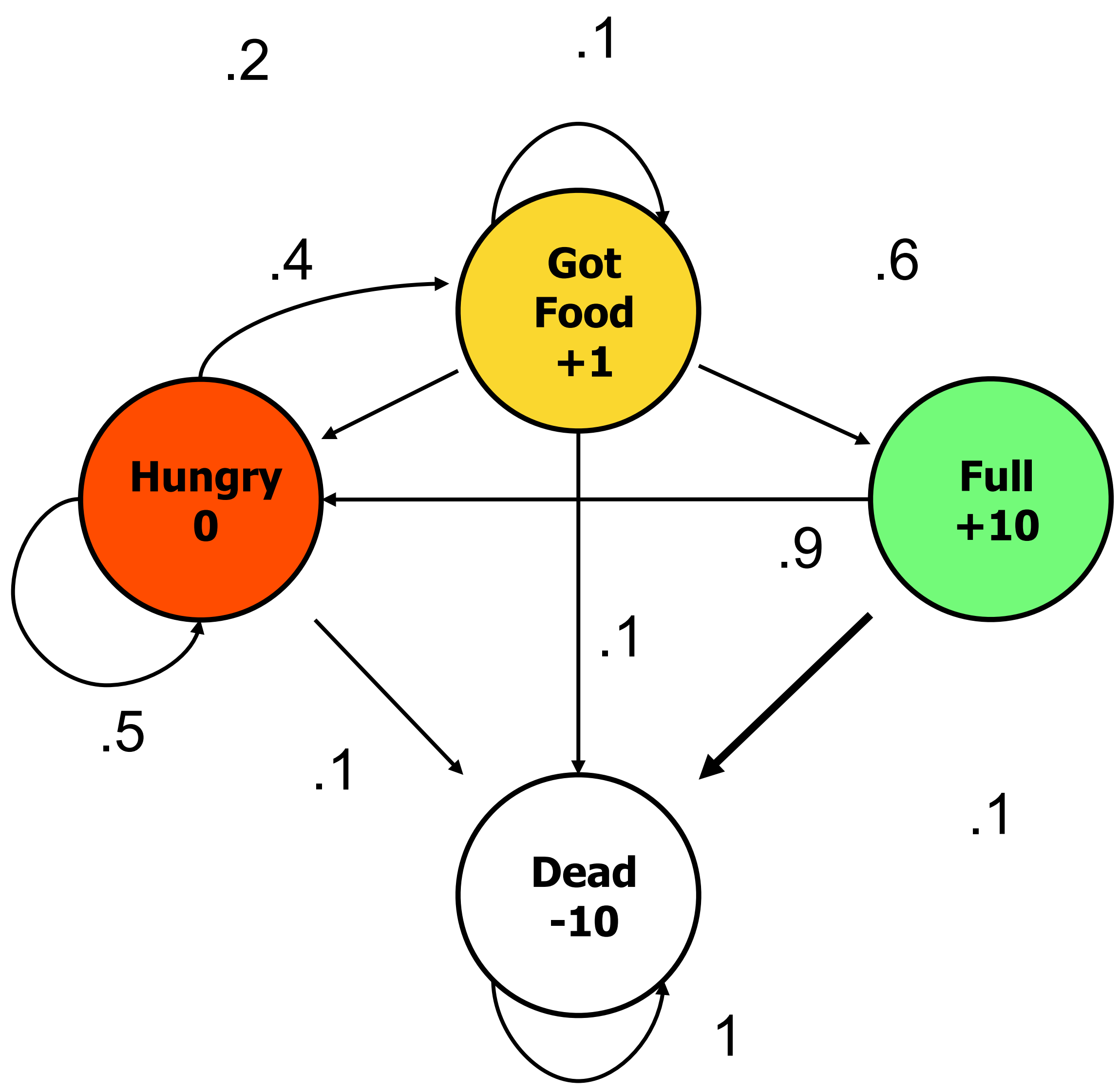
Value Iteration in Caveman's World



- Key idea: Bellman Recursion
  - Relates value in current state to expected value of next state

$$V(s) = R(s) + \gamma \sum_{s'} P(s'|s)V(s')$$

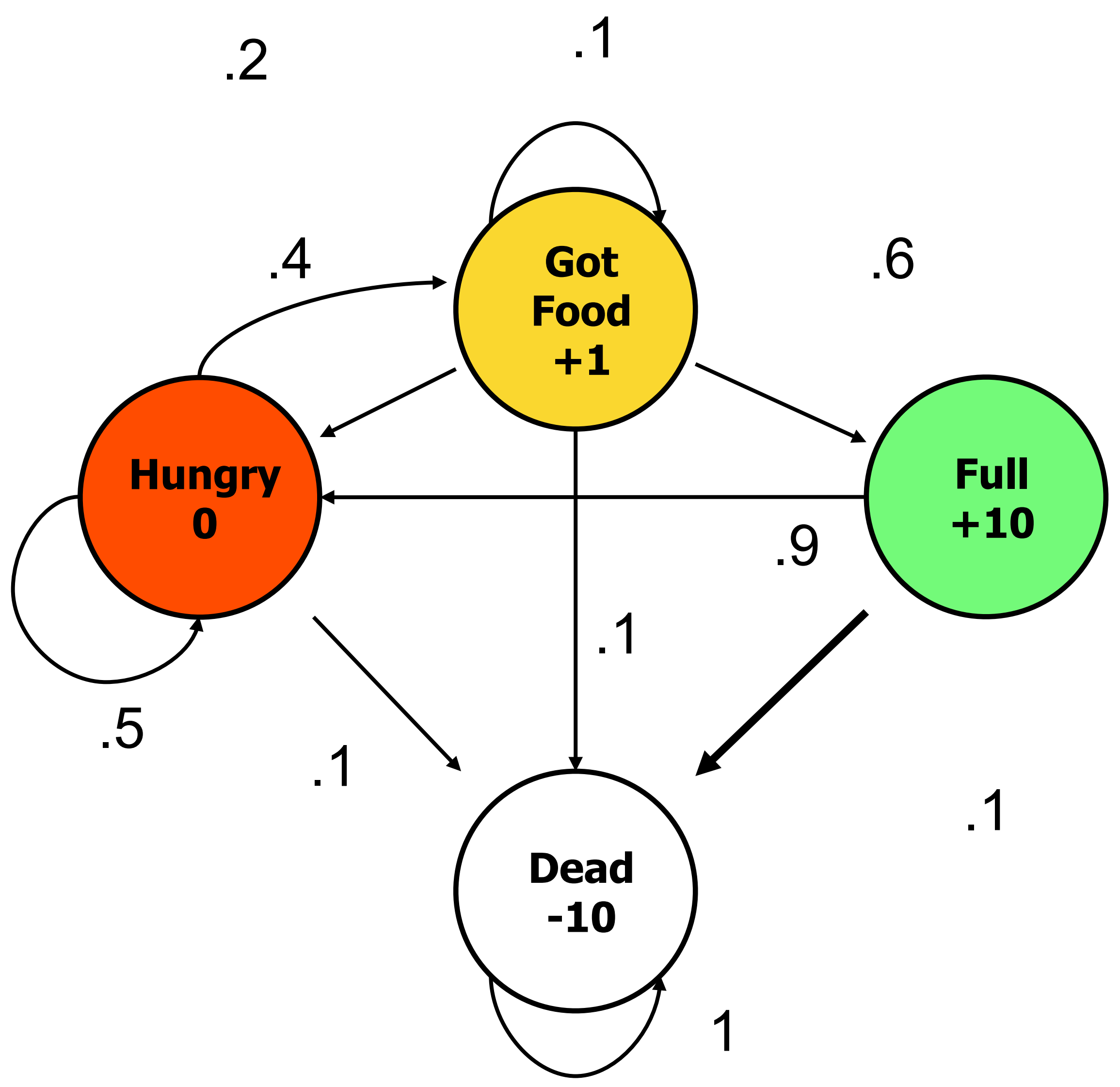
Value Iteration in Caveman's World



- Key idea: Bellman Recursion
    - Relates value in current state to expected value of next state

$$
\begin{aligned}
V(s=H) &= r + \gamma(P_{HH}(R_H) + P_{HG}(R_G) + P_{HF}(R_F) + P_{HD}(R_D)) \\
&= 0 + 0.9(0.5(0) + 0.4(1) + 0.1(10))
\end{aligned}
$$

Value Iteration in Caveman's World

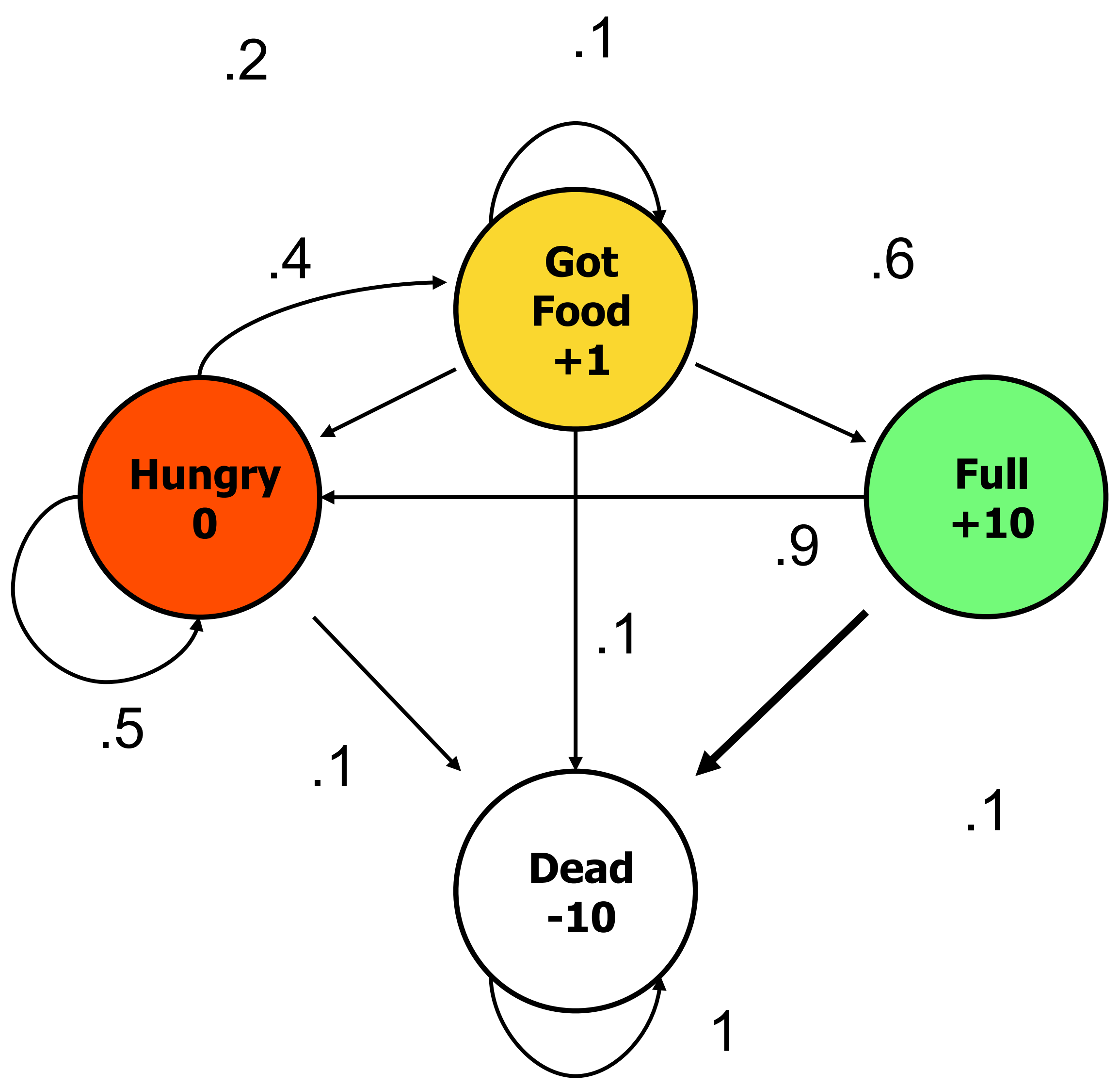


Value in k-steps

	H	G	F	D
1	0	1	10	-10
2	**-0.54**			

$$\begin{aligned} V(s=H) &= r + \gamma(P_{HH}(R_H) + P_{HG}(R_G) + P_{HF}(R_F) + P_{HD}(R_D)) \\ &= 0 + 0.9(0.5(0) + 0.4(1) + 0.1(10)) \end{aligned}$$

Value Iteration in Caveman's World



Value in k-steps

	H	G	F	D
1	0	1	10	-10
2	-.54	5.69	9.1	-19
3	.06	4.61	7.85	-27.1
4	-.75	3.23	7.61	-34.39
		...		
99	-39.08	-34.71	-30.66	-100.0
100	-39.09	-34.71	-30.66	-100.0

Value Iteration is Guaranteed to Converge

Summary

- Markov Processes represent uncertainty in state transitions
- It is possible to determine the overall value of a state
- What's next? Adding actions!

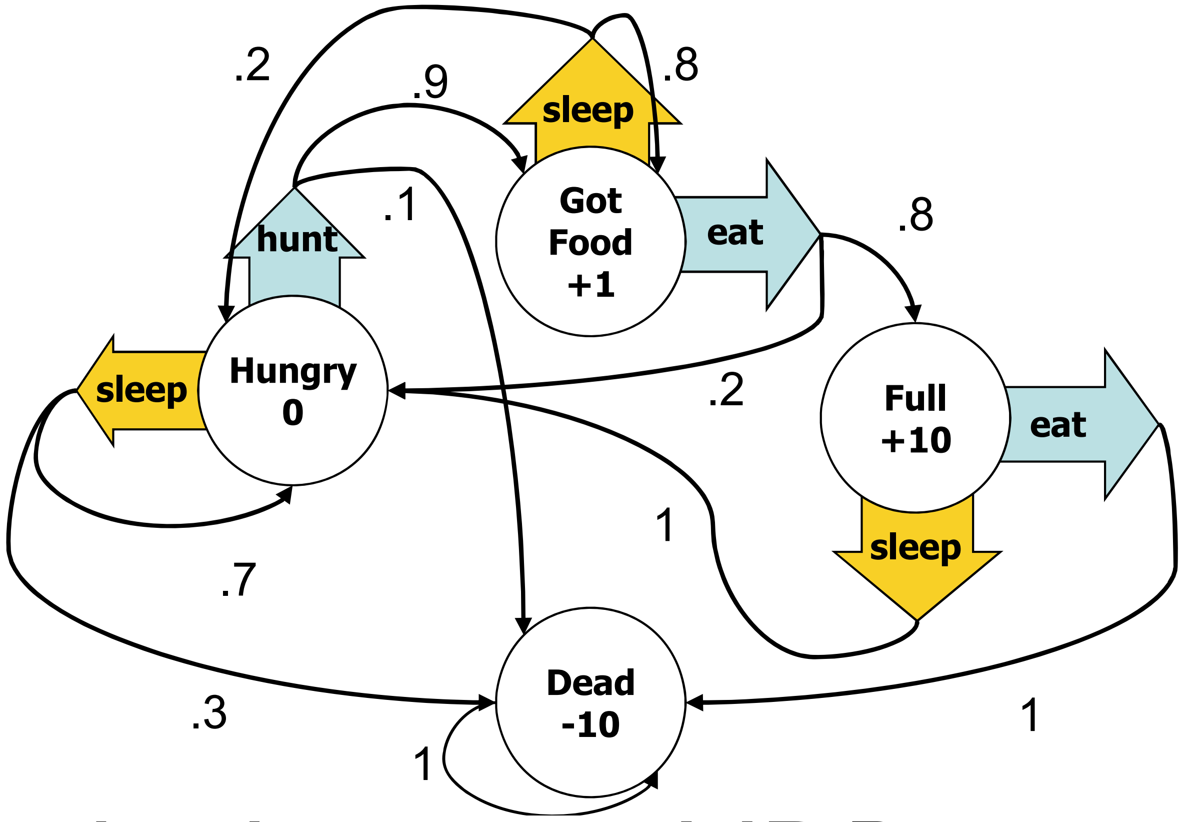
Actions: the value of free-will

- What'd we do so far?
  - Define values of *states*, and *transition probabilities* between them
- To add actions, what to we need to look at?

1. condition on actions: $P(s'|s) \to P(s'|s,a)$
2. values of actions: $V(s) = \max_a Q(s,a)$

- Turns out we need only (1), and (2) is RL

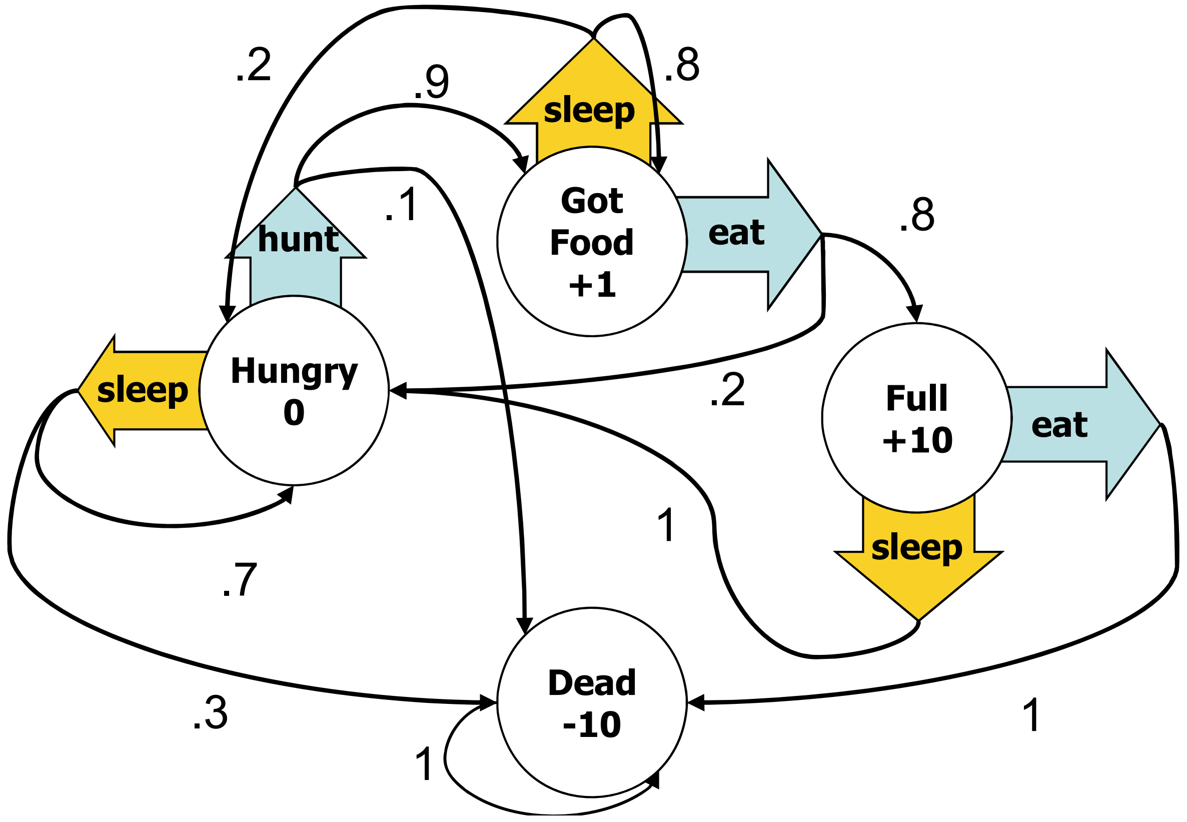
Actions: the value of free-will



- Adding actions back into an MDP:
- How? Make transitions conditional on action

$$T(s,a,s') = P(s'|s,a) = \begin{bmatrix} P^a_{11} & P^a_{12} & \dots & P^a_{1n} \\ P^a_{21} & P^a_{22} & \dots & P^a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ P^a_{n1} & P^a_{n2} & \dots & P^a_{nn} \end{bmatrix}$$

Actions: the value of free-will



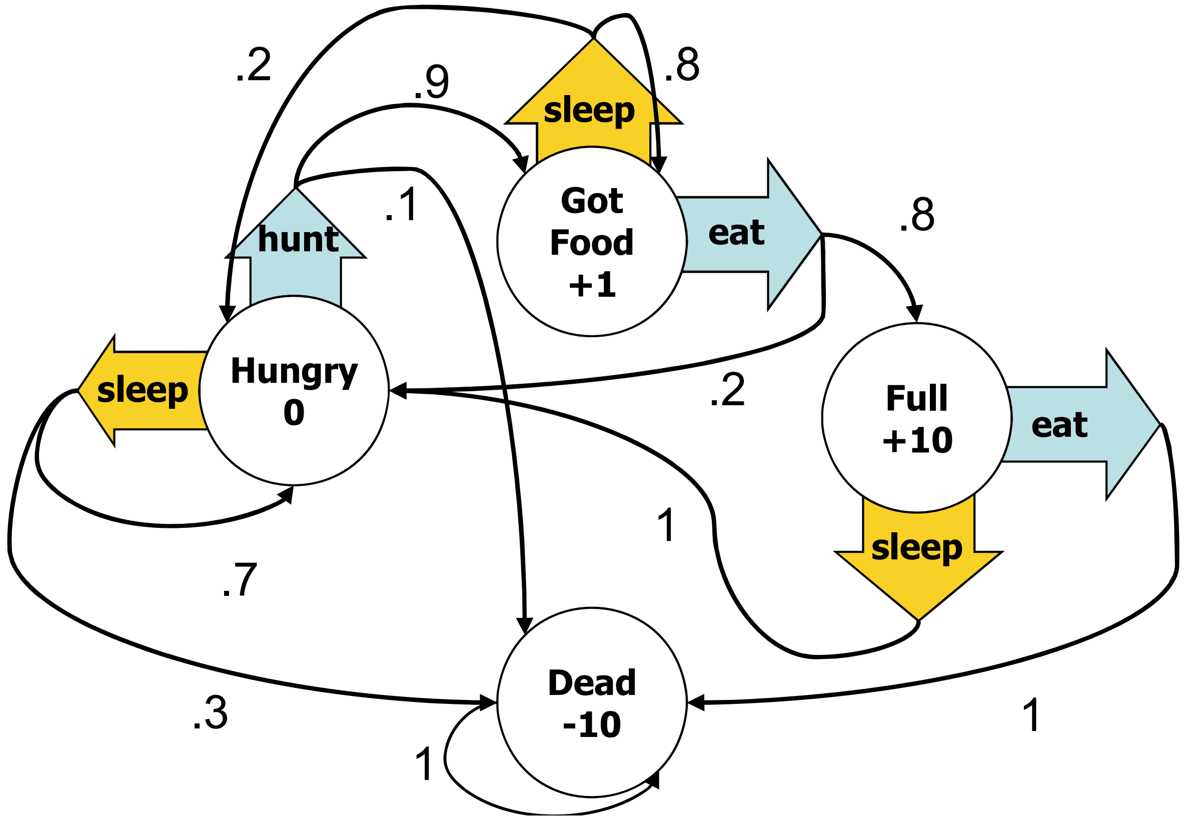
Value-Iteration needs one more thing:

$$\boxed{V(s) \leftarrow \max_a \left[R(s,a) + \gamma \sum_{s'} P(s'|s,a)V(s') \right]}$$

*Bellman Equation

added this max over actions

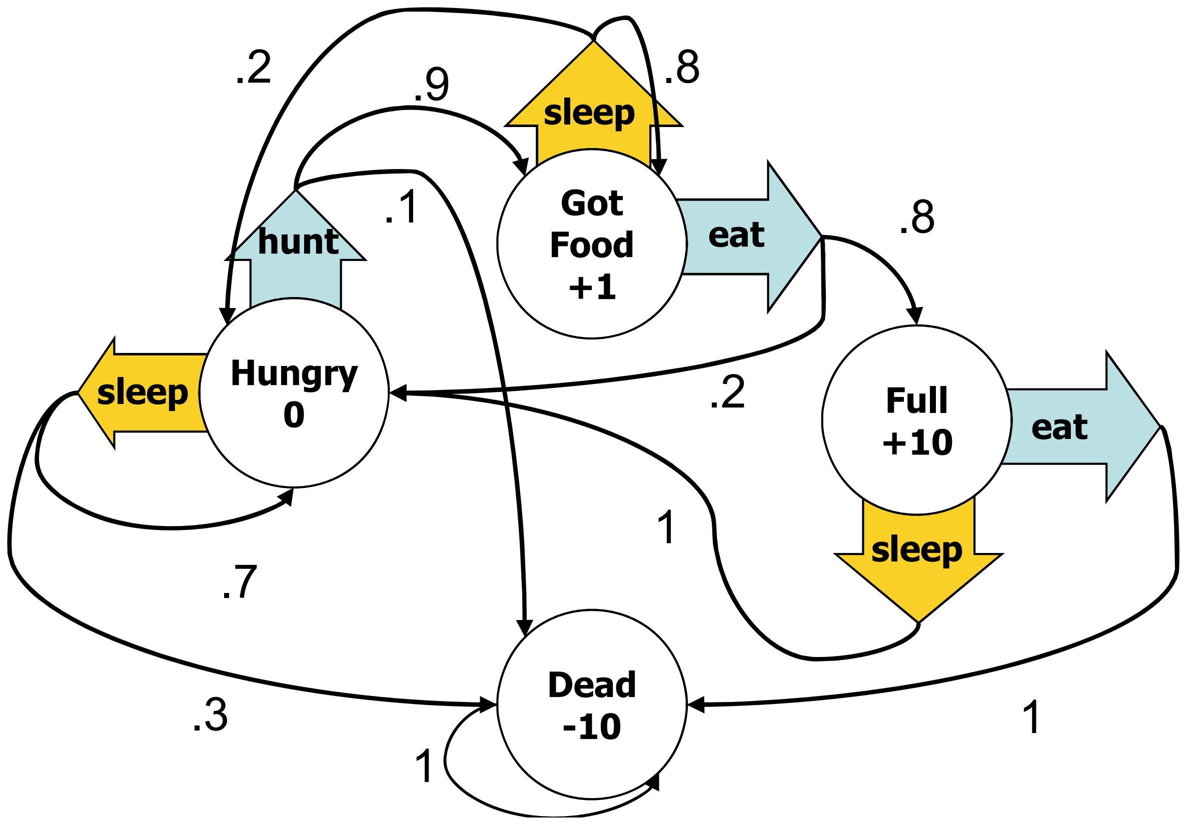
Actions: the value of free-will



“Free-Will” Values:

	H	G	F	D
1	0	1	10	-10
2	-.09	8.2		

Actions: the value of free-will



"Free-Will" Values:

	H	G	F	D
1	0	1	10	-10
2	-.09	8.2	10	-19
		...		
100	-7.16	2.27	3.56	-100

Value Iteration in Code

```
initialize $V(s)$ arbitrarily
loop until policy good enough
    loop for $s \in \mathcal{S}$
        loop for $a \in \mathcal{A}$
            $Q(s,a) := R(s,a) + \gamma \sum_{s' \in \mathcal{S}} T(s,a,s') V(s')$
        $V(s) := \max_a Q(s,a)$
    end loop
end loop
```

What's this "Q" function?

➡Topic for later, but short answer is to allow action selection without lookahead

MDP Planning: Core concepts

- Things to really understand about MDPs:
  - ➡what a value function is
  - ➡why we can converge to V* with these simple algorithms
  - ➡why V* is overkill sometimes
  - ➡why model is so important, and what to do without it
  - ➡why these algorithms can be (horribly) inefficient

Value Iteration: Big Questions

- Convergence?
- Efficiency?
- Assumptions?

Value Iteration Convergence

- **Proof Sketch:**

1. Defined in terms of max-norm between any two value functions (in particular V_i and V*)

2. Take advantage of basic property of max:

$$|max_a f(a) - max_a g(a)| \leq max_a |f(a) - g(a)|$$

3. Apply Bellman operator and rearrange

$$\begin{aligned}
|B(V_i) - B(V_j)|(s) &= \left| \left(R(s) + \gamma max_a \sum_{s'} P(s'|s,a) V_i(s') \right) - \left(R(s) + \gamma max_a \sum_{s'} P(s'|s,a) V_j(s') \right) \right| \\
&= \gamma (max_a E_{V_i}[s'] - max_a E_{V_j}[s'] \\
&\leq \gamma max_a (E_{V_i}[s'] - E_{V_j}[s']) \\
&\leq \gamma max_a (V_i(s') - V_j(s'))
\end{aligned}$$

tl;dr: max-norm (max difference w.r.t. V*) strictly contracts with each application of Bellman (with factor gamma)

But how important is convergence?

- Why does value matter again? To pick actions
  - ➡ IE, we're interested in π(s), not V(s)
- Can we optimize the policy directly?
  - ➡ Yes! This is "policy iteration"
  - ➡ We'll use the policy form of Bellman:

$$V_{t+1}^{\pi}(s) \longleftarrow R(s,\pi(s)) + \gamma \sum_{s'} P(s'|s,\pi(s)) V_t^{\pi}(s')$$

Policy Iteration

- Alternative approach:
  - Step 1: Policy evaluation: calculate value for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy using onestep look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges (it does)
- Facts about policy iteration:

➡ It's still optimal!

➡ Can converge faster under some conditions. Why??

Implementing Policy Iteration

- Simple change:

1. Evaluate policy somehow
   - ➡ option 1: solve as linear system
   - ➡ option 2: use Bellman for a while

$$\begin{aligned} V_0^\pi(s) &\longleftarrow 0 \\ V_{t+1}^\pi(s) &\longleftarrow R(s,\pi_t(s)) + \gamma \sum_{s'} P(s'|s,\pi_t(s)) V_t^\pi(s') \end{aligned}$$

2. Improve policy using 1-step lookahead

$$\pi_{k+1}^*(s) = \arg\max_a \left[R(s,a) + \gamma \sum_{s'} P(s'|s,a) V^{\pi_k}(s') \right]$$

Policy Iteration Convergence

Policy iteration convergence proof sketch:

(1) In every step the policy improves. Means that a given policy can be encountered at most once. This means that after we have iterated as many times as there are different policies (i.e., $|A|^{|S|}$), we must be done and hence have converged.

(2) By definition at convergence we have that $\pi_{k+1}(s) = \pi_k(s) \quad \forall s \in S$. This implies that $V^{\pi_k} = \max_a \left[R(s,a) + \gamma \sum_{s'} P(s'|s,a) V^{\pi_k}(s')\right]$ for all states. This satisfies the Bellman equation, which means V^{π_k} is equal to the optimal value function V^*.

Comparison to Value Iteration

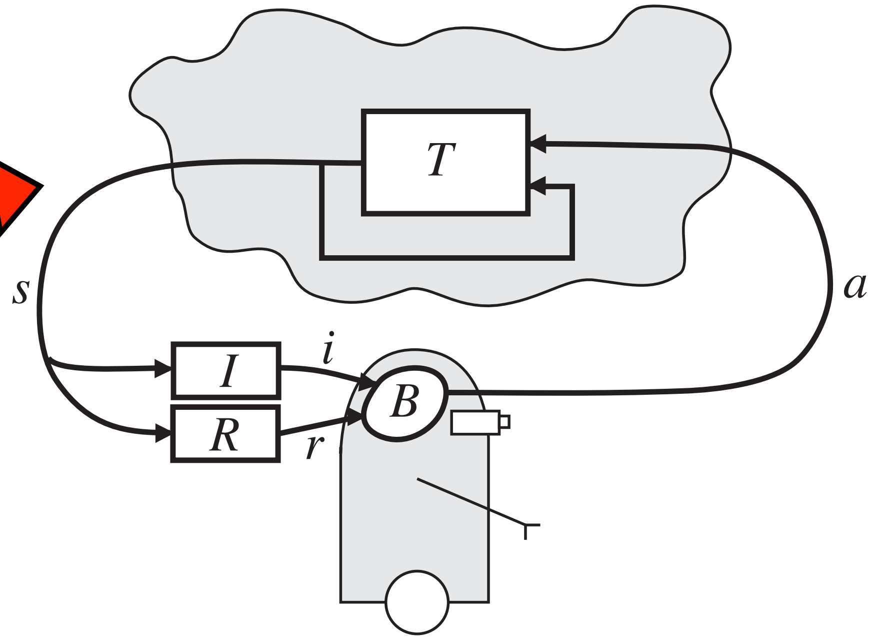
- What's the real difference vs. VI?
  - Just puts more effort into policy evaluations in between policy updates
  - Why might this be helpful??
    - ➡ Early convergence criterion (policy stops changing)
    - ➡ When we have lots of actions, so update is expensive

Reinforcement Learning

- Notice: all previous methods required the model
  - What if we don't have it? Can we learn from pure exploration??
- Yes! This is "reinforcement learning"
- Today we'll derive Q-learning, simplest model-free RL algorithm

Life of an RL Agent

Now "T" is *outside* the agent



T: Transition model (dynamics)

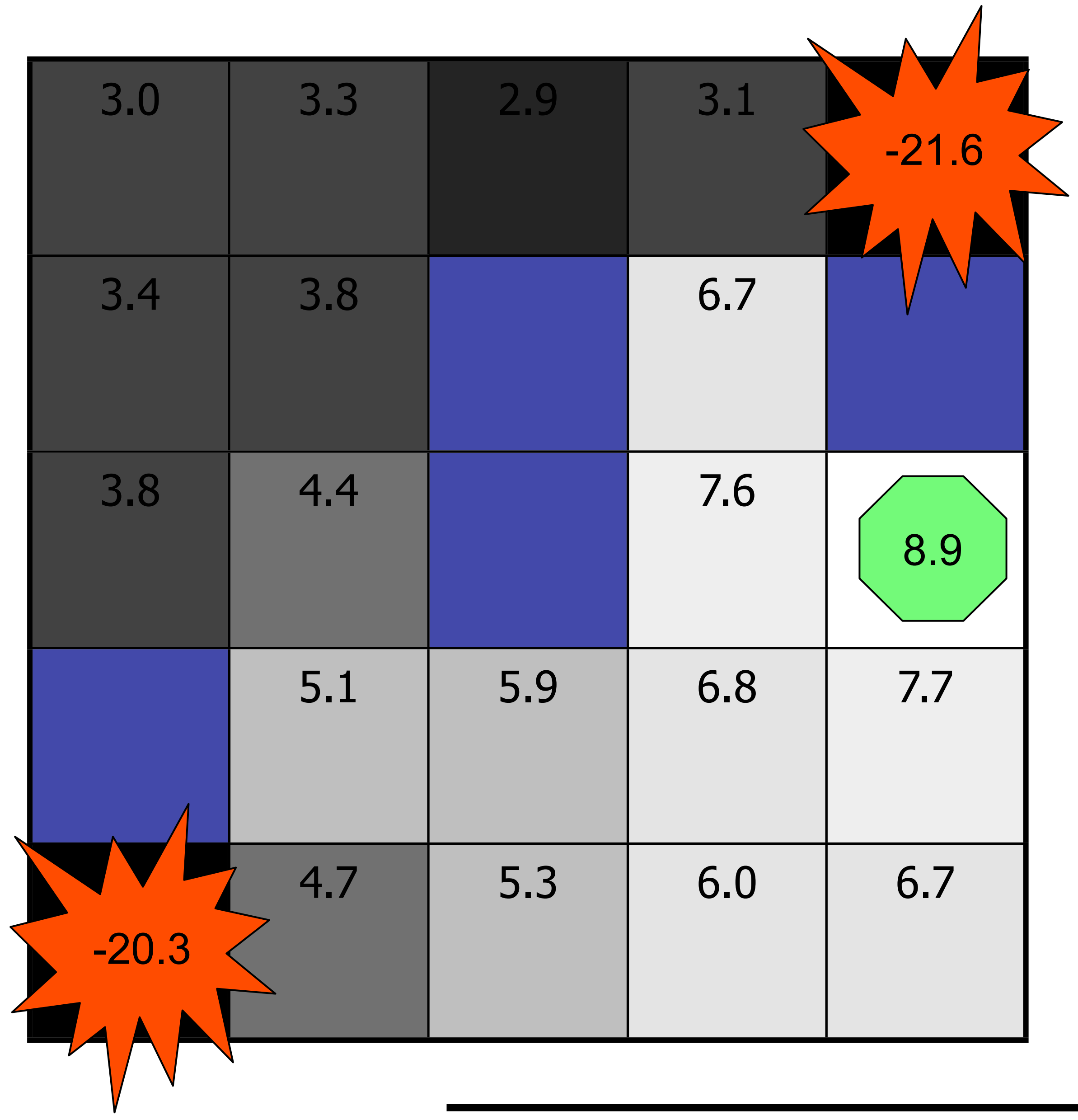
I: input (sensor reading)

R: reward (a real number)

B: behavior (an action)

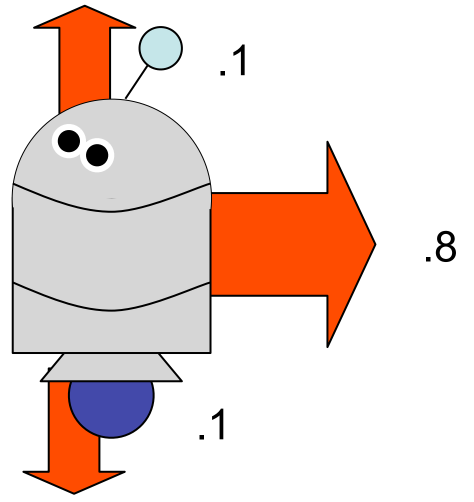
- Agent lives in loop:
  1. receive observation (eg camera image)
  2. select action
  3. receive reward

Review of MDPs



Optimal Value Function

Solved with value iteration



How do we use V(s) for planning?

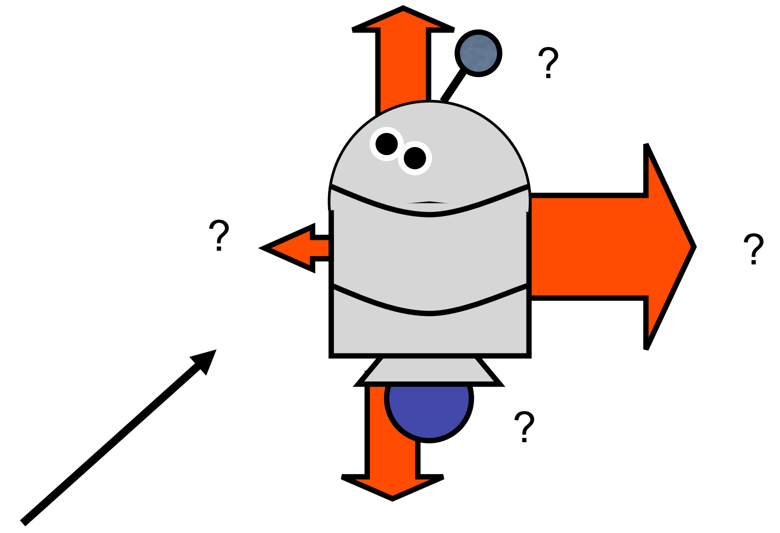
$$\pi^*(s) = \arg\max_a \left[R(s,a) + \gamma \sum_{s'} P(s'|s,a) V(s') \right]$$

(1-step look-ahead)

We assumed a model for P(s'|s,a)

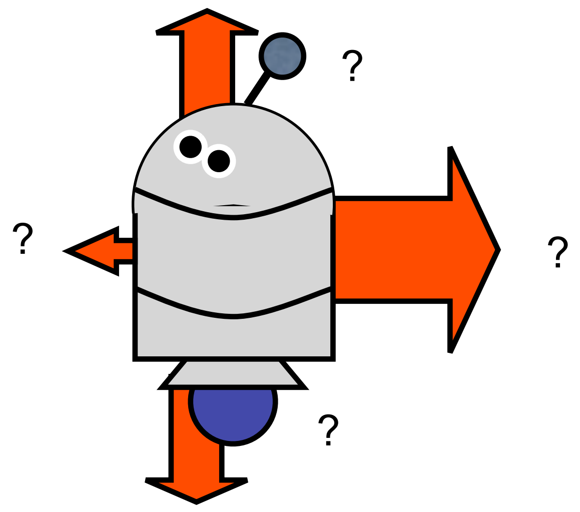
What do we do if such a model does not exist?

P(s'|s,a) unknown!



Model (without walls)

We assumed a model for P(j|i,a)



What do we do if such a model does not exist?

- **Learn one (e.g. Bayesian RL)**
- **"Model-Free" RL (e.g. Q-Learning)**

Want: $\pi^*(s)$ – Optimal policy in the state

Can't use: $V^*(s)$ – Value of a state

Learn instead: $Q^*(S_i,a)$ - Value of taking an action in a state

Q-function Definition

Key Relationship:

$$V(s) = \max_a Q(s,a)$$

Q(s,a) = Value of Taking action **a** in state **S**

How do we use Q(s,a) for planning? ➡ $\pi^*(s) = \arg\max_a Q(s,a)$

Q-function Definition

Definition of Q function:

$$\begin{aligned} Q(s,a) &= R(s,a) + \gamma \max_{a'} \mathbb{E}\left[Q(s',a')\right] \\ &= R(s,a) + \gamma \max_{a'} \underline{\sum_{s'} P(s'|s,a)}\, Q(s',a') \end{aligned}$$

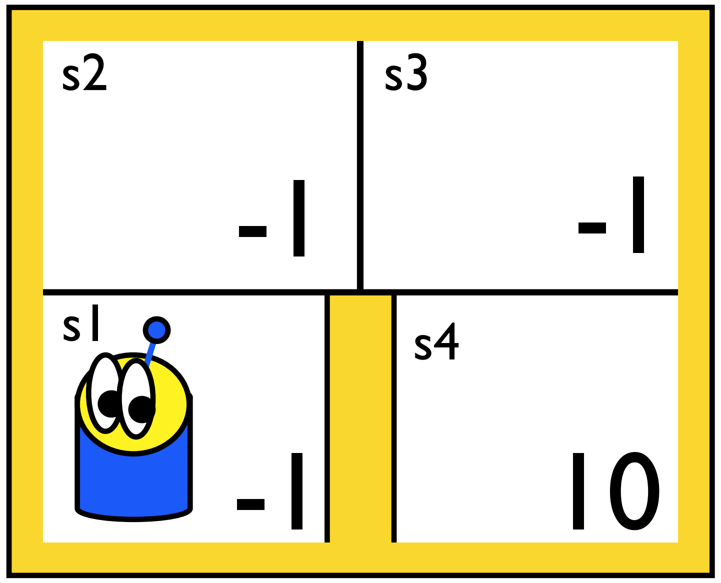
How to remove dependency on model?

From Q-function to Q-Learning

➡ Key question: How to remove dependency on model?

$$\begin{aligned}
Q(s,a) &= R(s,a) + \gamma \max_{a'} \sum_{s'} P(s'|s,a) Q(s',a') && \text{by definition} \\
&\approx R(s,a) + \gamma \max_{a'} Q(s',a'), \quad s' \sim P(s'|s,a) && \text{by sample approximation} \\
&\approx (1-\alpha) Q(s,a) + \alpha \left(R(s,a) + \gamma \max_{a'} Q(s',a') \right) && \text{smoothing} \\
&\approx Q(s,a) - \alpha Q(s,a) + \alpha R(s,a) + \alpha\gamma \max_{a'} Q(s',a') \\
&\approx Q(s,a) + \alpha \left(R(s,a) + \gamma \max_{a'} Q(s',a') - Q(s,a) \right) && \text{canonical form} \\
&\approx Q(s,a) + \alpha(\delta_{TD}) && \text{TD error}
\end{aligned}$$

Q-Learning Example

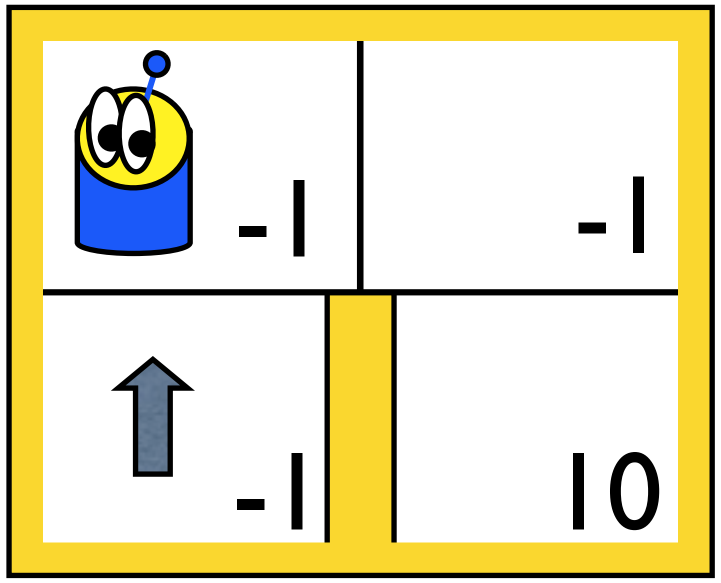


$\alpha = .7$

	⬆	⬇	⬅	➡
S_1	0	0	0	0
S_2	0	0	0	0
S_3	0	0	0	0
S_4	0	0	0	0

Q-Table

Q-Learning Example

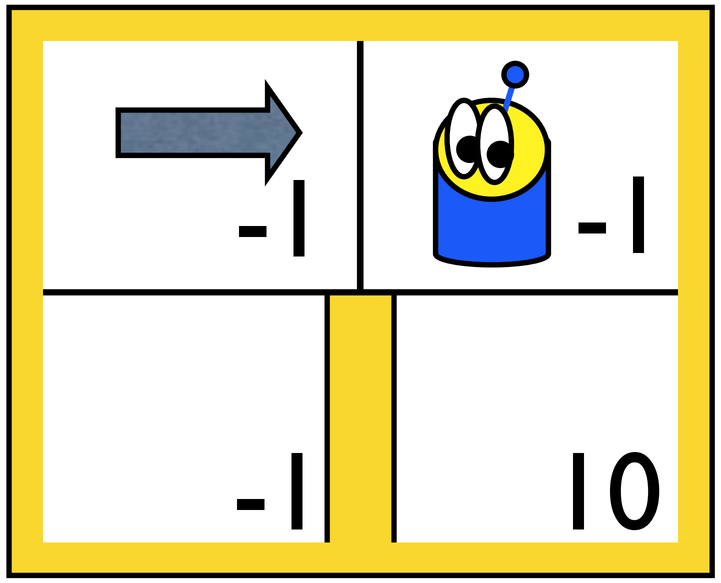


$$Q^{est}(S_1, \uparrow) =$$
$$.7(-1 + .9 \max(0, 0, 0, 0)) + .3 \times 0$$

	↑	↓	←	→
S_1	-.7	0	0	0
S_2	0	0	0	0
S_3	0	0	0	0
S_4	0	0	0	0

Q-Table

Q-Learning Example



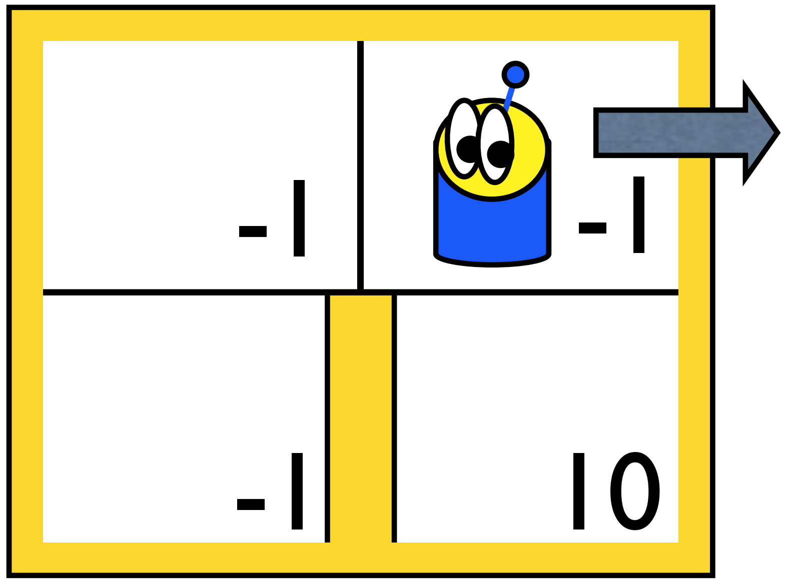
$Q^{est}(S_2,$ ⇨ $) =$

$.7(-1 + .9 \max(0, 0, 0, 0)) + .3 \times 0$

	⇧	⇩	⇦	⇨
S_1	-.7	0	0	0
S_2	0	0	0	-.7
S_3	0	0	0	0
S_4	0	0	0	0

Q-Table

Q-Learning Example

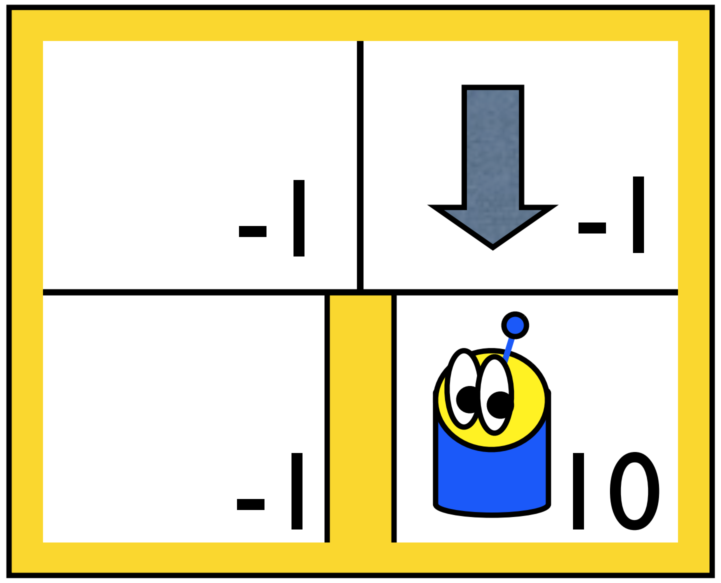


$Q^{est}(S_3,$ ➡ $) =$
$.7(-1 + .9 \max (0, 0, 0, 0)) + .3 \times 0$

	↑	↓	←	→
S_1	-.7	0	0	0
S_2	0	0	0	-.7
S_3	0	0	0	-.7
S_4	0	0	0	0

Q-Table

Q-Learning Example

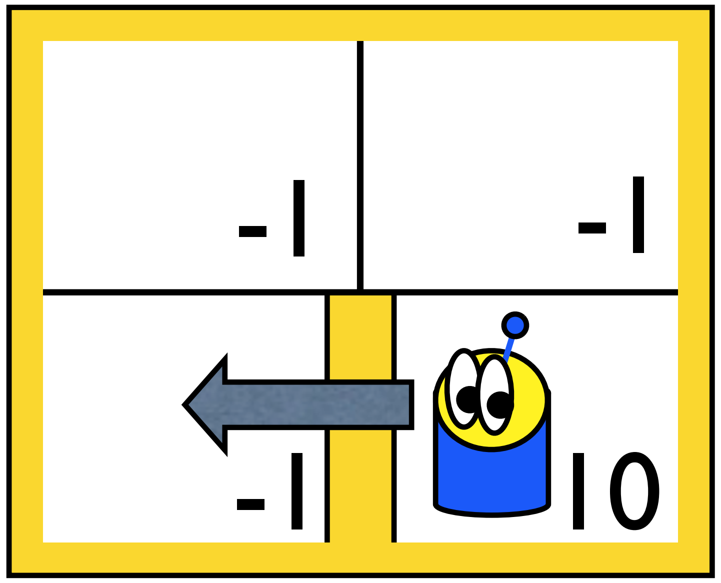


$$Q^{est}(S_3, \Downarrow) = .7(-1 + .9 \max(0, 0, 0, 0)) + .3 \times 0$$

	⇧	⇩	⇦	⇨
S_1	-.7	0	0	0
S_2	0	0	0	-.7
S_3	0	-.7	0	-.7
S_4	0	0	0	0

Q-Table

Q-Learning Example

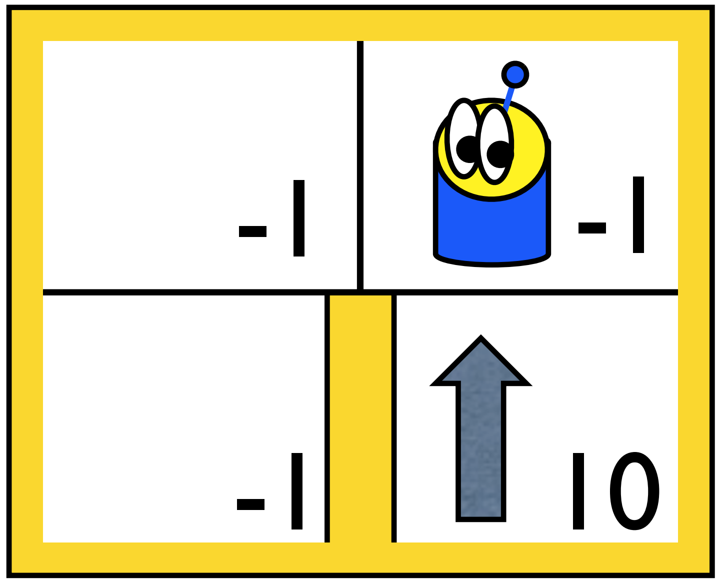


$Q^{est}(S_4, \leftarrow) =$
$.7(10 + .9 \max(0, 0, 0, 0)) + .3 \times 0$

	↑	↓	←	→
S_1	-.7	0	0	0
S_2	0	0	0	-.7
S_3	0	-.7	0	-.7
S_4	0	0	7	0

Q-Table

Q-Learning Example

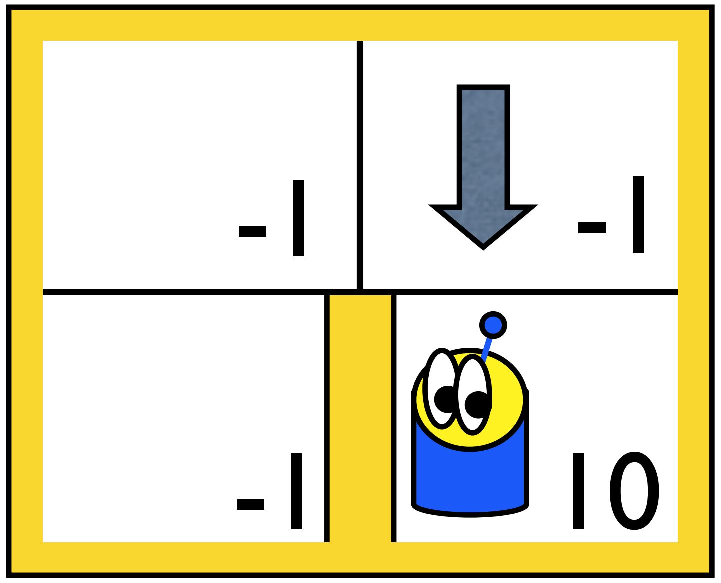


$$Q^{est}(S_4, \uparrow) =$$
$$.7(10 + .9 \max(0, -.7, 0, -.7)) + .3 \times 0$$

	↑	↓	←	→
S_1	-.7	0	0	0
S_2	0	0	0	-.7
S_3	0	-.7	0	-.7
S_4	7	0	7	0

Q-Table

Q-Learning Example



$Q^{est}(S_3, \downarrow) =$
$.7(-1 + .9 \max(7,0,7,0)) + .3 \times -.7$

	⇧	⇩	⇦	⇨
S_1	-.7	0	0	0
S_2	0	0	0	-.7
S_3	0	3.5	0	-.7
S_4	7	0	7	0

Q-Table

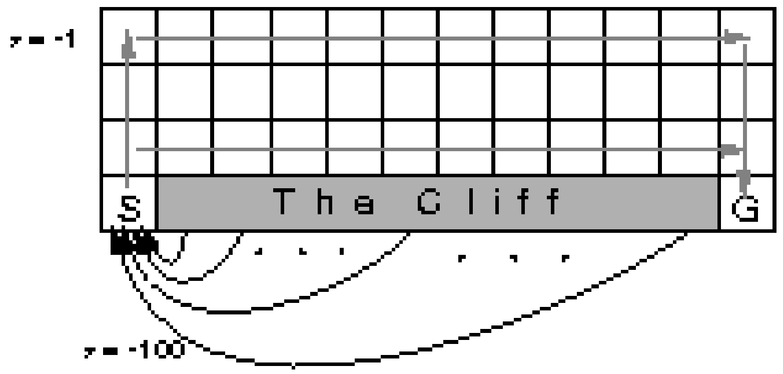
On-Policy Learning: SARSA

- Key idea: perform backups on action actually selected, rather than estimate of optimal action

$$Q(s,a) \leftarrow Q(s,a) + \alpha\left(R(s,a) + \gamma Q(s',a') - Q(s,a)\right)$$

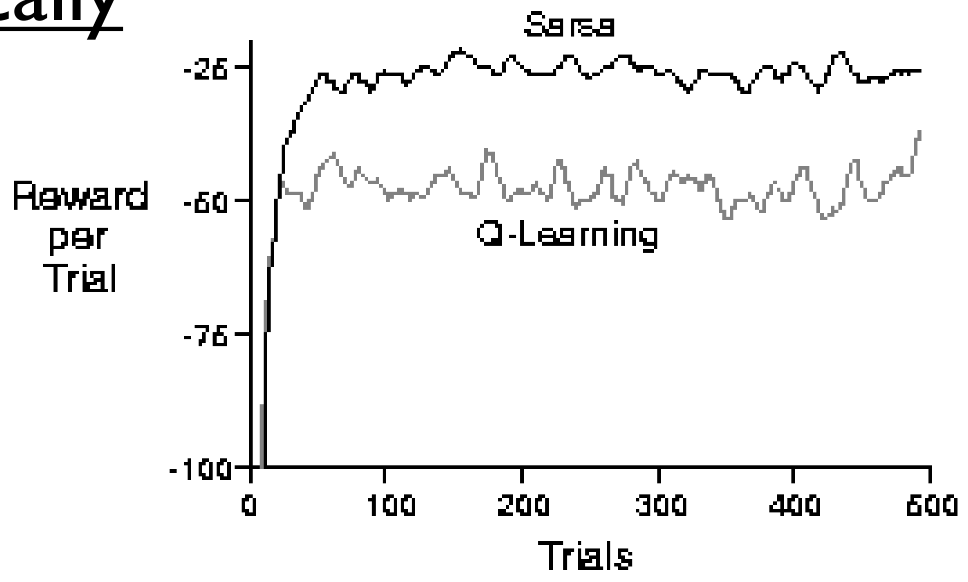
state + **a**ction + **r**eward + (next) **s**tate + (next) **a**ction = **SARSA**

- Otherwise same as Q-learning, but "on-policy"
- Less greedy, so addresses problem of locally high-reward/risk states (e.g. cliff task)



safe path

optimal path



TD and eligibility traces

- Problem: Q-values spread slowly
- Solution: Propagate over history
- Mechanism: exponential decay w.p. λ

TD error for last action

$$\delta_{td} = R(s,a) + \gamma \max_{a'} Q(s',a') - Q(s,a)$$

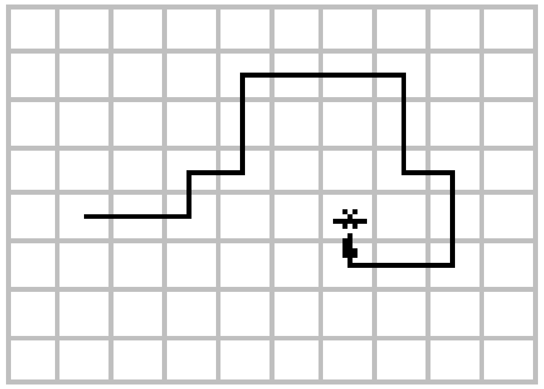
TD error at time T-t

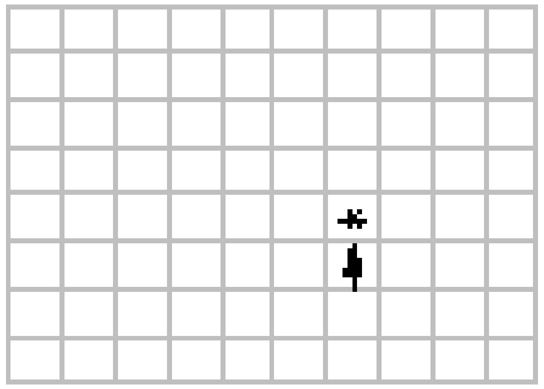
$$e(s,a) = \gamma^t \lambda^t \delta_{td}$$

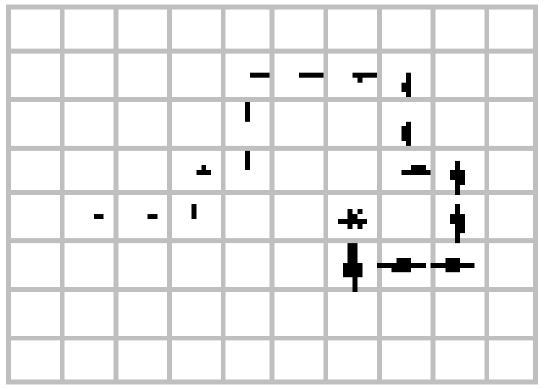
SARSA(λ)

```
Initialize $Q(s,a)$ arbitrarily and $e(s,a)=0$, for all $s,a$
Repeat (for each episode):
   Initialize s, a
   Repeat (for each step of episode):
      Take action a, observe r, $s'$
      Choose $a'$ from $s'$ using policy derived from Q
         (e.g., ε-greedy)
      $\delta \leftarrow r + \gamma Q(s',a') - Q(s,a)$
      $e(s,a) \leftarrow e(s,a) + 1$
      For all s,a:
         $Q(s,a) \leftarrow Q(s,a) + \alpha\delta e(s,a)$
         $e(s,a) \leftarrow \gamma\lambda e(s,a)$
      $s \leftarrow s'$; $a \leftarrow a'$
   until s is terminal
```

This is the TD-error


Path Taken


action values increased by 1-step SARSA


action values increased by SARSA(λ), λ=0.9

Take-home

- Use Q-learning/SARSA when:
  - state space is tiny
  - interested in full policy
  - don’t have access to model
- Use eligibility traces when:
  - always.

Monte-Carlo Reinforcement Learning

- Recall:
  - ➡V.I., P.I., Q-Learning, & SARSA are all direct implementations of bellman recursion, via *dynamic programming*
- MCRL is direct implementation of reward expectation, via sampling

$$V_\pi(s) = \mathbb{E}_\pi\left[\sum_{t=0}^{T} \gamma^t r_t\right]$$

- Returns are simply averaged together
- Variance of the error decreases as 1/n

Monte-Carlo Reinforcement Learning

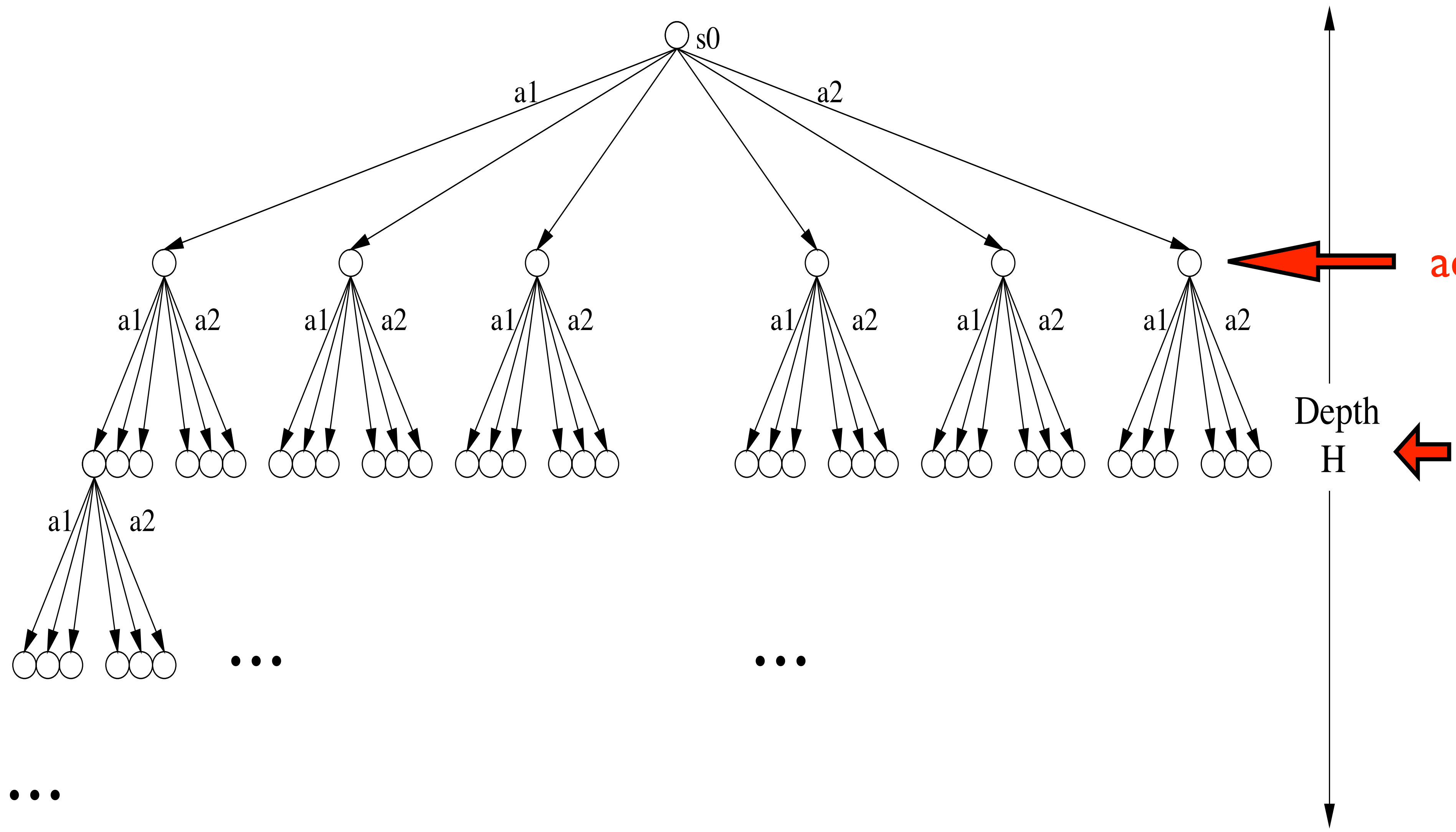
Unpacking the bellman recursion:

$$
\begin{aligned}
V_\pi(s) &= \mathbb{E}_\pi\left[\sum_{t=0}^{\infty}\gamma^t r_t | s_0 = s\right] \\
&= \mathbb{E}_\pi\left[r_0 + \sum_{t=0}^{\infty}\gamma^{t+1} r_{t+1} | s_0 = s\right] \\
&= \mathbb{E}_\pi\left[r_0 + \gamma r_1 + \sum_{t=0}^{\infty}\gamma^{t+2} r_{t+2} | s_0 = s\right] \\
&= R(s) + \gamma \sum_a P_\pi(a|s) \sum_{s'} P(s'|s,a)\left[R(s') + \mathbb{E}_\pi\left[\sum_{t=0}^{\infty}\gamma^{t+1} r_{t+1} | s_0 = s'\right]\right] \\
&= R(s) + \gamma \sum_a P_\pi(a|s) \sum_{s'} P(s'|s,a) V(s') \quad \Leftarrow \text{back to bellman}
\end{aligned}
$$

The point: you can approximate bellman using <u>finite sums</u>

Monte-Carlo Reinforcement Learning

How to visualize:



action selection according to some exploration policy π

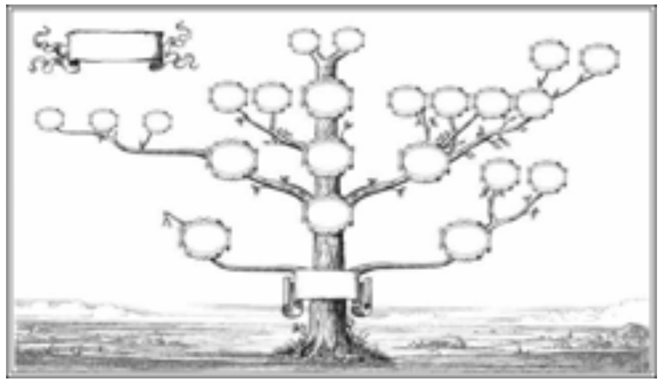
transitions sampled from model or environment

Monte-Carlo Reinforcement Learning

- Key properties:
  - runtime independent of |S| (!)
  - can learn from actual and simulated experience
  - can target parts of the state space we care about (!)
- Problems:
  - Slow
  - When to stop?
    - Variance falls as 1/n, can we do better?

Sparse-Sampling (MCTS)

- ONLINE MCRL algorithm with provable loss bounds
  - Kearns, Mansour, Ng (ML 2002)
- Key idea: rewards in future matter less than rewards now
- Outputs:
  - ε-optimal policy



$$\boxed{|V^{\mathcal{A}}(s) - V^*(s)| \leq \varepsilon}$$

Sparse-Sampling (MCTS)

Running Time: $O((kC)^H)$

- Hairy Math:

$$H = \lceil \log_\gamma(\lambda/V_{\max}) \rceil$$ Planning horizon

$$C = \frac{{V_{\max}}^2}{\lambda^2}\left(2H\log\frac{kH{V_{\max}}^2}{\lambda^2} + \log\frac{R_{\max}}{\lambda}\right)$$ Number of rollouts

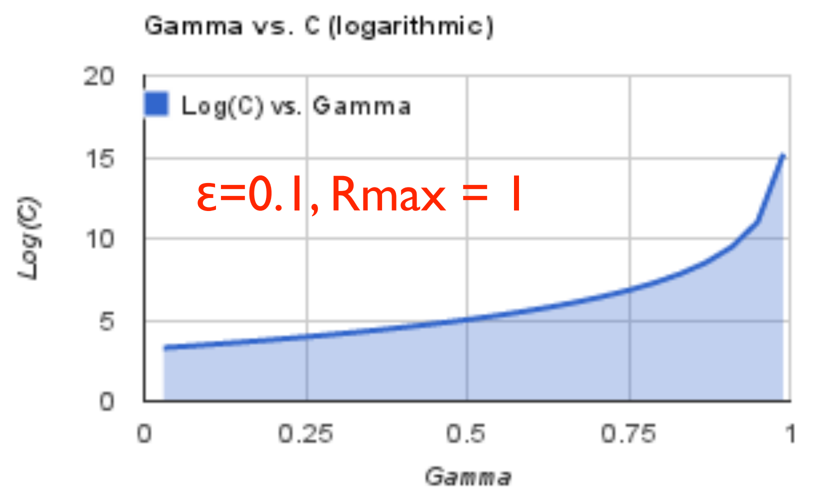
$$\lambda = (\epsilon(1-\gamma)^2)/4,\ V_{\max} = R_{\max}/(1-\gamma)$$ Useful Constants

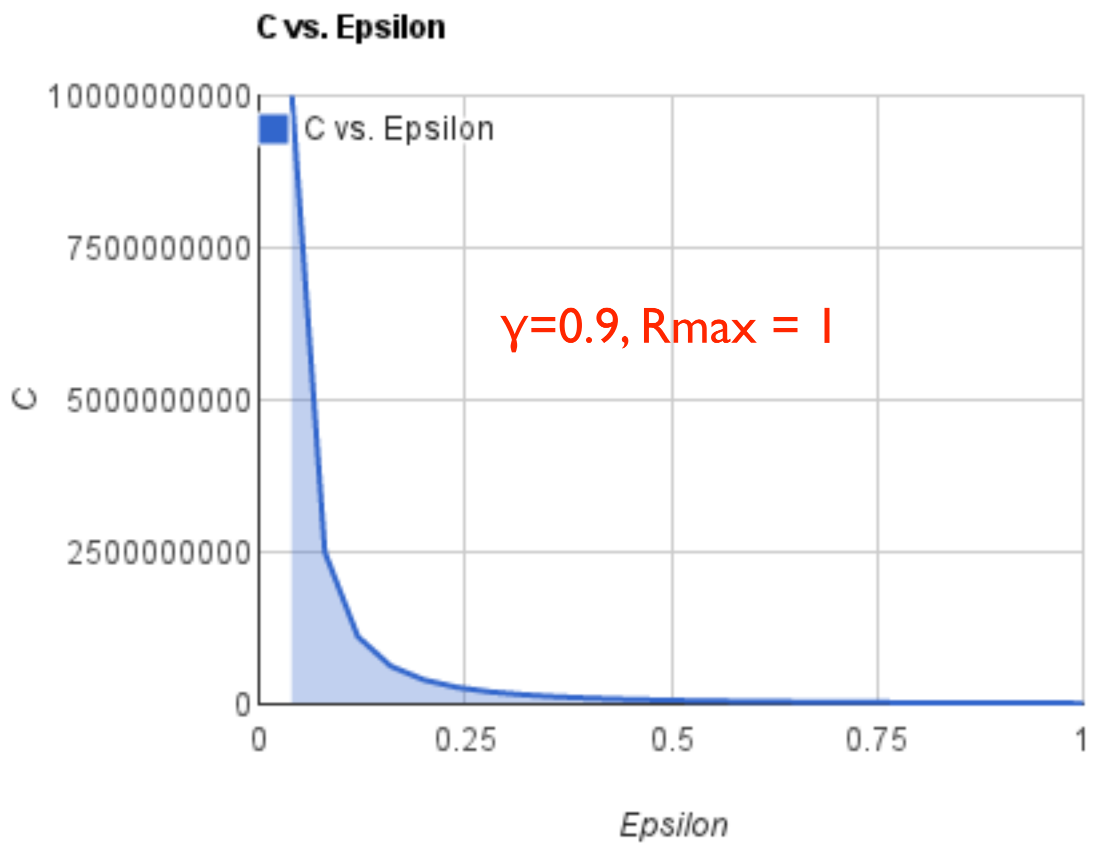
- Running time depends only on $R_{\max}$, ε, and γ!

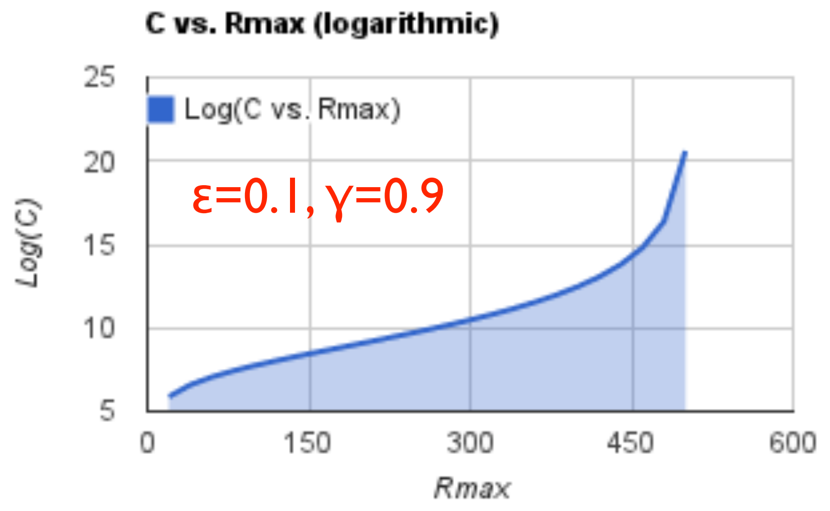
➡ The point: can do MCRL with provable guarantees. But how useful??

Sparse-Sampling (MCTS)

- Problem: C can be HUGE







Take-home

- Use MCRL/MCTS when:
  - state space is huge
  - interested in subset of S (online planning)
  - planning horizon is small
  - can efficiently sample from model
- Related work:
  - UCT (Kocsis et al 2006) ⇦ Reigning GO champion!
  - FSSS (Walsh et al 2010)