

$$\sigma_b = \frac{M * c}{I}$$



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What is Finite Element Analysis?

Basic FEA Direct Formulation Theory
with Matrix Method, using a 1D
Element



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



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What is Finite Element Analysis?

- In a nut shell, Finite Element Analysis is the process of determining an approximate solution by discretizing or splitting the actual global structure into 'finite' elements, these elements and the model closely represent the actual structural stiffness, loading and constraints
- In simple terms it can be summarized as (kind of a spring equation):

$$\{F\} = [K] * \{x\}$$

Where:

- $\{F\}$ is the Applied Load Matrix (known)
- $\{K\}$ is the Global Stiffness Matrix (known)
- $\{x\}$ is the Displacements Matrix (unknown)



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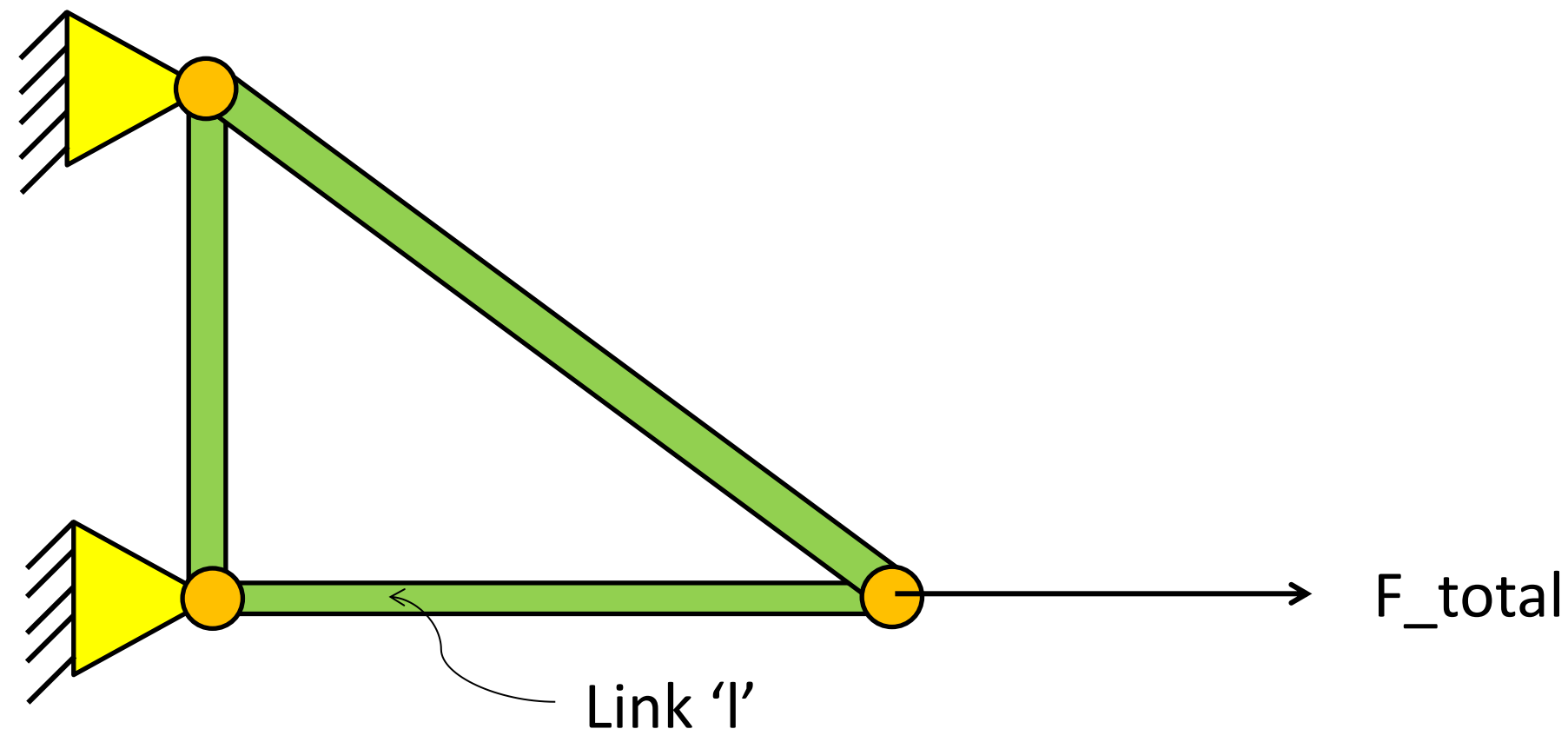


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- Let us consider a simple truss example made from three 1D elements (1D element examples are bars or beams or rods)
- Assume all the links are made from metal



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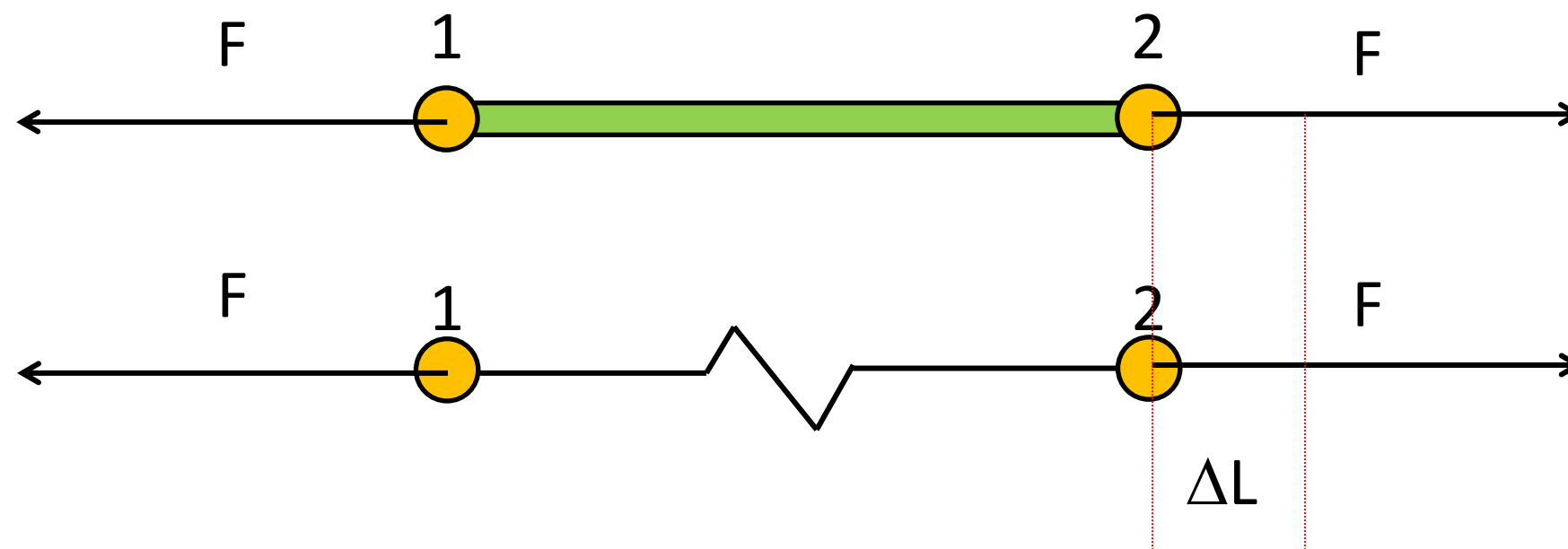


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- Now let us isolate the link 'l' shown in the previous slide as a balanced free body with its portion of the total load, say 'F'
- It has two nodes, 1 and 2
- It has material properties such as a Young's Modulus 'E', Poisson's ratio 'v' and has an initial length 'L', an area of cross section 'A'



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$

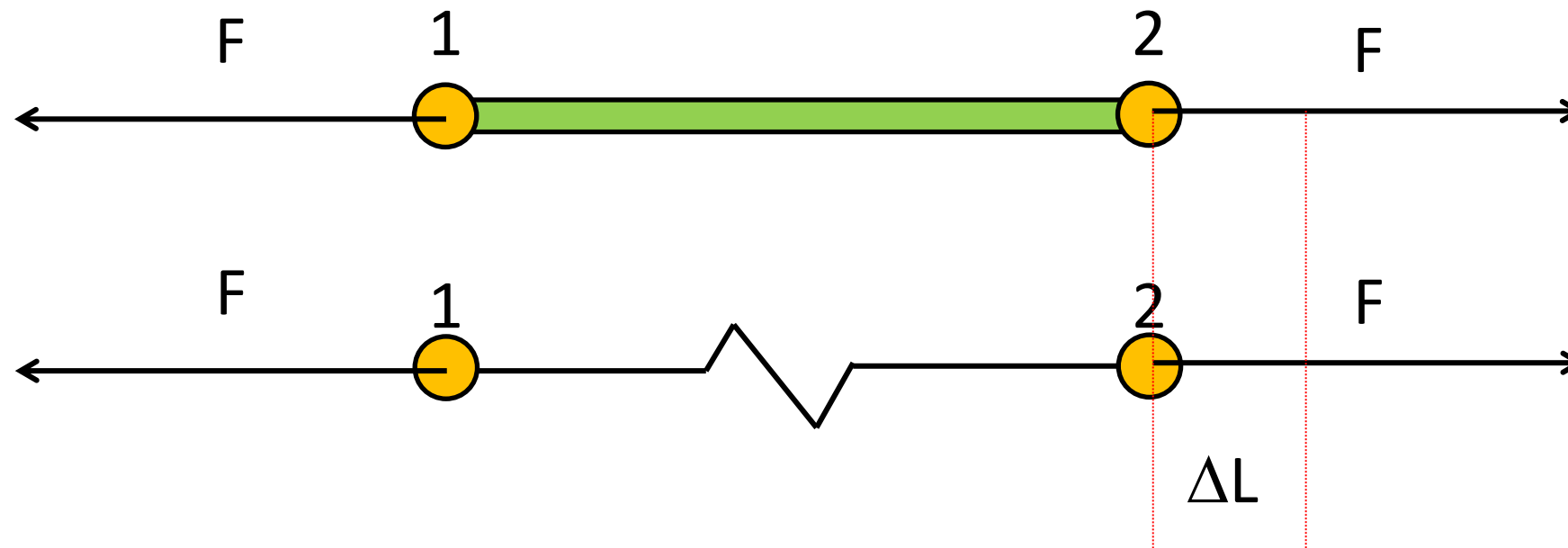


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- We can see the analogous system of a simple spring that stretches by ΔL
- Therefore Strain $\varepsilon = \Delta L/L$ and Stress $\sigma = F/A$
- Young's Modulus $E = \sigma/\varepsilon = F/A / \Delta L/L$
- Rearranging the terms in the equation for E, we get:

$$\left[\frac{EA}{L} \right] * \{\Delta L\} = \{F\}$$

Similar to a spring equation

$$\{F\} = [K]\{x\}$$



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



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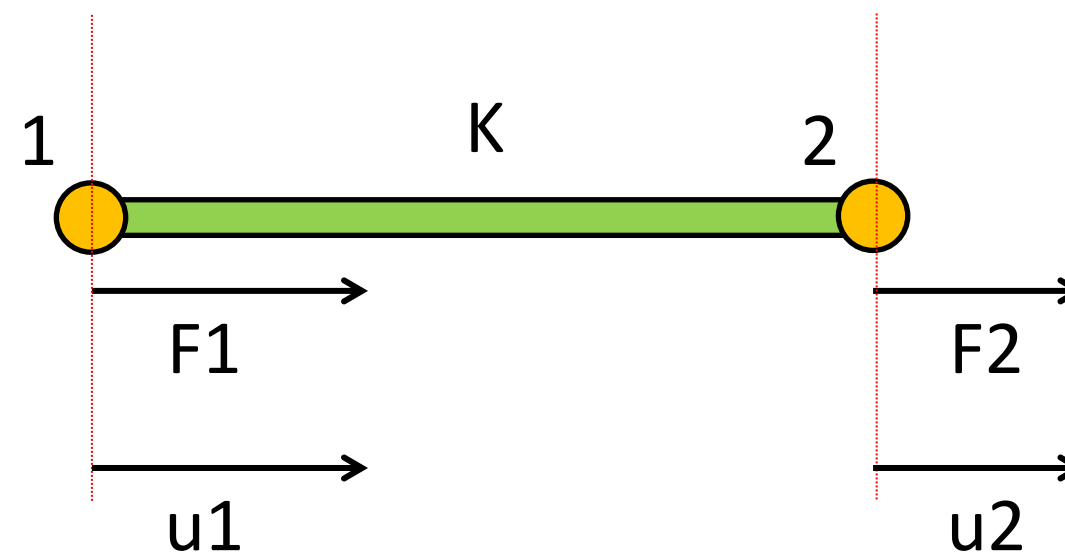
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- From the previous slide we have stiffness $K = \left[\frac{EA}{L} \right]$
- At the end, in the assembled matrix form, $[K]$ will be the global stiffness matrix. It will be assembled from the material properties and geometry of all the finite elements in the model
- So let us look at the matrix method

MATRIX METHOD



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$

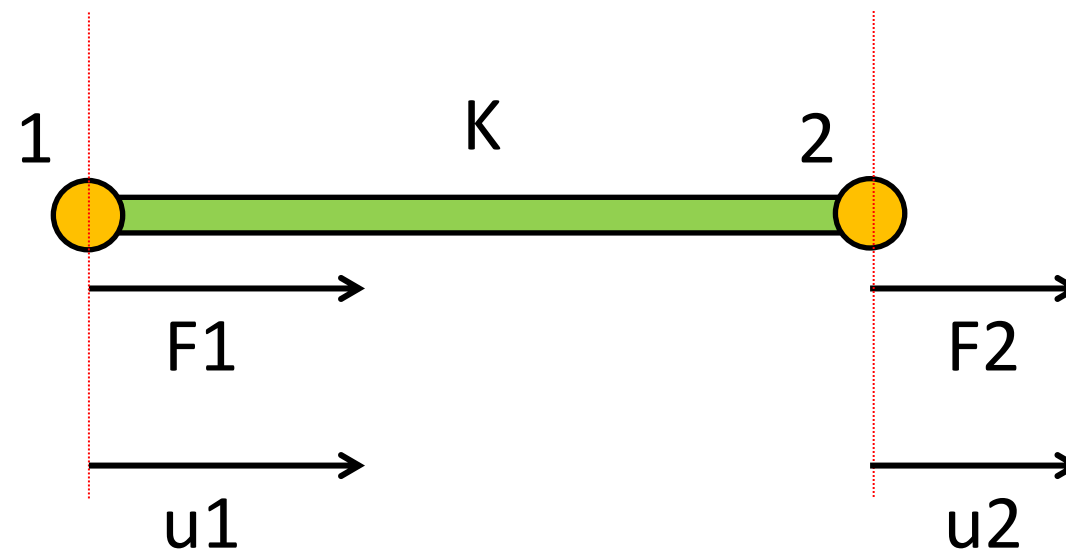


$$\sigma_b = \frac{M * c}{I}$$



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- We have the same element here, except we will assume a force and displacement at each node
- Assume the sign convention as shown above is positive. Don't worry at the end we will end up with the right signs, for now let us just go with this
- So we have our single element with stiffness K, end forces F1 and F2 and nodal displacements u1 and u2



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



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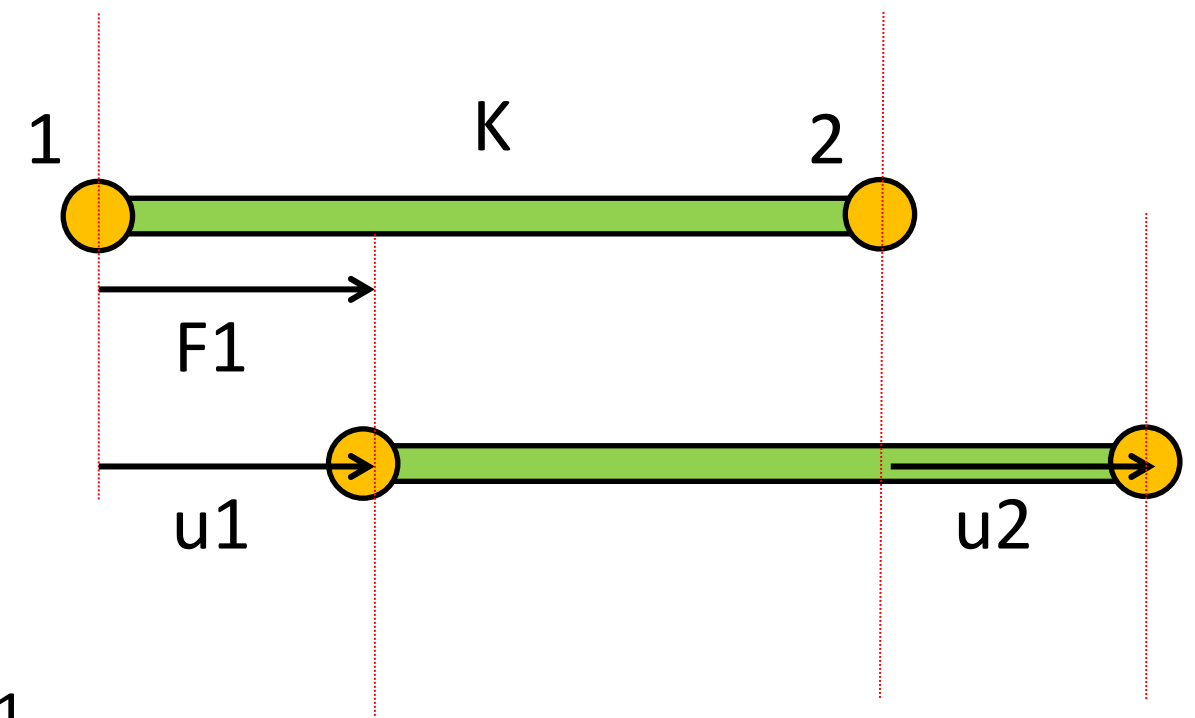
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Equation for node 1:

- $F1 = K * (u1 - u2) = AE/L * (u1 - u2)$
- Let us keep it consistent by using u1 first and then u2 for node 1
- This way, if u2 is zero then we get a positive $F1 = K * u1$, thus matching the sign convention we are using. So F1 is positive if (u1-u2) is positive, K is always positive
- Unknowns are F1, u1 and u2



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



$$\sigma_b = \frac{M * c}{I}$$



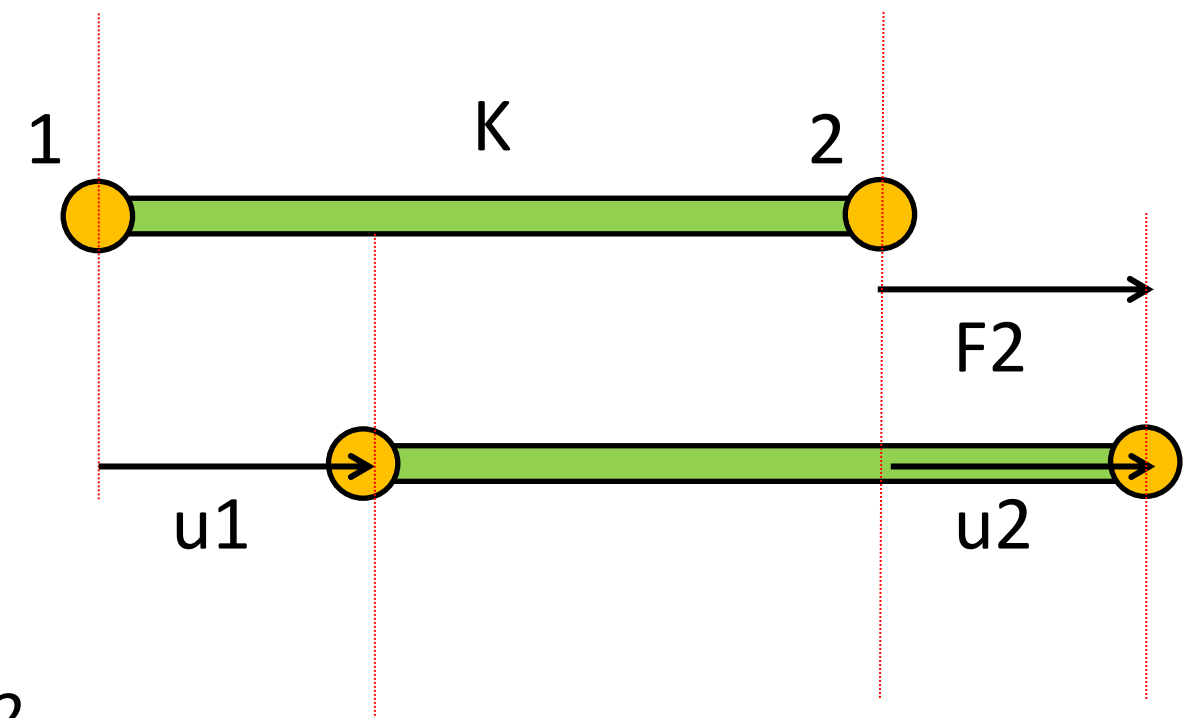
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Equation for node 2:

- $F2 = K * (u2 - u1) = AE/L * (u2 - u1)$
- Let us keep it consistent by using $u2$ first and then $u1$ for node 2
- This way, if $u1$ is zero then we get a positive $F2 = K * u2$, thus matching the sign convention we are using. So $F2$ is positive if $(u2 - u1)$ is positive, K is always positive
- Unknowns are $F2$, $u1$ and $u2$



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



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Load Vector, Element Stiffness Matrix and Displacement Vector:

- So we know that $F1 = AE/L * (u1 - u2)$
- And we also know that $F2 = AE/L * (u2 - u1) = AE/L * (-u1 + u2)$
- Thus, in matrix form, we now have the governing equations for the 1D bar element:

$$\begin{Bmatrix} F1 \\ F2 \end{Bmatrix} = \begin{bmatrix} AE/L & -AE/L \\ -AE/L & AE/L \end{bmatrix} \begin{Bmatrix} u1 \\ u2 \end{Bmatrix}$$

$\{F\}$

$[K]$

$\{x\}$

- Here, the element stiffness matrix is $[K]$ for the single element
- Similarly, the stiffness matrix, applied loads, unknown reactions and displacements for the entire system is “assembled” by the FEM solver, resulting in a single large stiffness matrix for all elements and a whole bunch of equations to solve
- Alright, let us now look at an example problem to make it easier to understand



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$

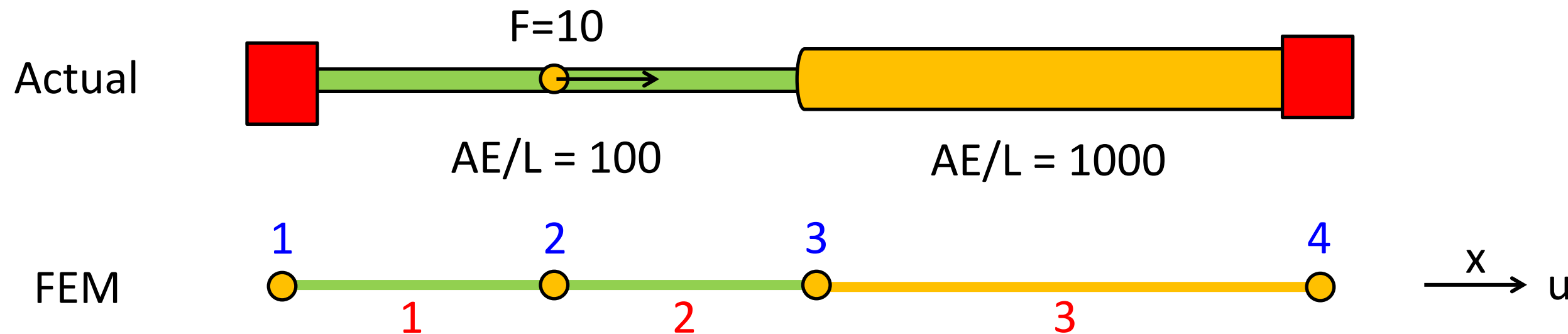


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Sample Problem of 1D Bar Elements:

- We have two bars with stiffness $AE/L = 100$ (**bar 1**) and $AE/L = 1000$ (**bar 2**)
- A load $F=10$ is applied at the mid point of the first bar
- So we split the structure into 3 elements (**1,2,3**) and 4 nodes (**1,2,3,4**)
- We use a node (**2**) at the middle of the first bar so we can apply the load there
- We have 4 degrees of freedom (DOF) in this model, one at each node – u_1, u_2, u_3, u_4



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$

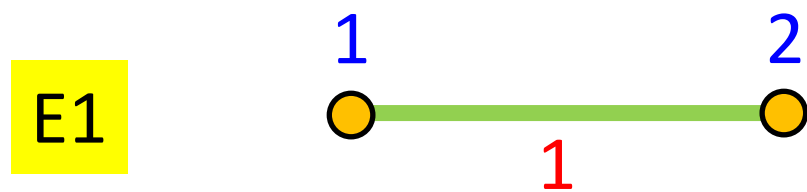
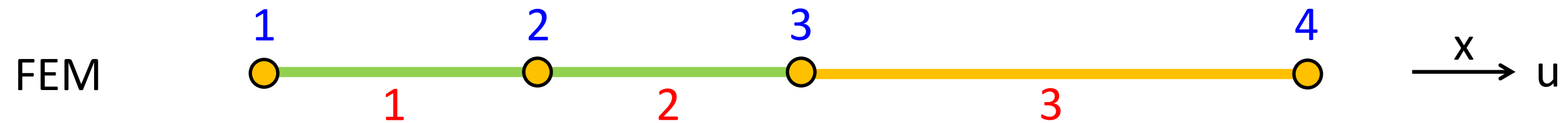


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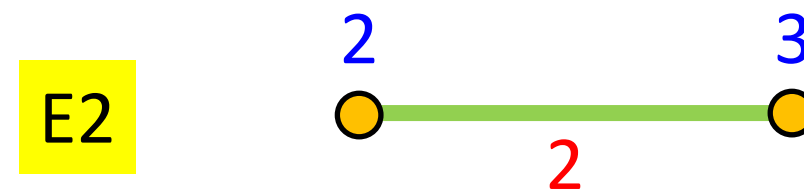
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For Element 1:

$$\begin{Bmatrix} F1 \\ F2 \end{Bmatrix} = \begin{bmatrix} AE/L & -AE/L \\ -AE/L & AE/L \end{bmatrix} \begin{Bmatrix} u1 \\ u2 \end{Bmatrix}$$

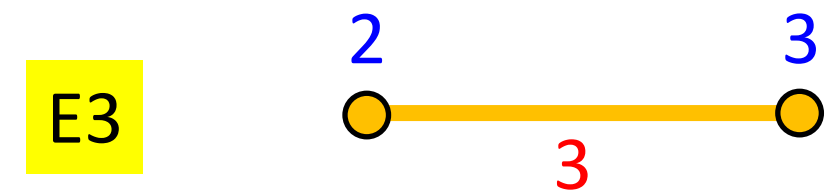
$$\begin{Bmatrix} F1 \\ F2 \end{Bmatrix} = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u1 \\ u2 \end{Bmatrix}$$



For Element 1:

$$\begin{Bmatrix} F2 \\ F3 \end{Bmatrix} = \begin{bmatrix} AE/L & -AE/L \\ -AE/L & AE/L \end{bmatrix} \begin{Bmatrix} u2 \\ u3 \end{Bmatrix}$$

$$\begin{Bmatrix} F2 \\ F3 \end{Bmatrix} = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u2 \\ u3 \end{Bmatrix}$$



For Element 1:

$$\begin{Bmatrix} F3 \\ F4 \end{Bmatrix} = \begin{bmatrix} AE/L & -AE/L \\ -AE/L & AE/L \end{bmatrix} \begin{Bmatrix} u3 \\ u4 \end{Bmatrix}$$

$$\begin{Bmatrix} F3 \\ F4 \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} u3 \\ u4 \end{Bmatrix}$$



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



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Global Element Stiffness Matrix Assembly

$$\begin{Bmatrix} F1 \\ F2 \\ F3 \\ F4 \end{Bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100+100 & -100 & 0 \\ 0 & -100 & 100+1000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} u1 \\ u2 \\ u3 \\ u4 \end{Bmatrix}$$

The matrix assembly is shown with three elements highlighted: Element 1 (green) covers the top-left 2x2 block, Element 2 (green) covers the middle 2x2 block, and Element 3 (yellow) covers the bottom-right 2x2 block. Red numbers 1, 2, and 3 are placed near the corresponding elements in the matrix.

Sample Problem of 1D Bar Elements:

- The elements are red numbers (1 2 3)
- We can see how at the common nodes, the stiffness terms are assembled by addition
- Next we apply the boundary conditions



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



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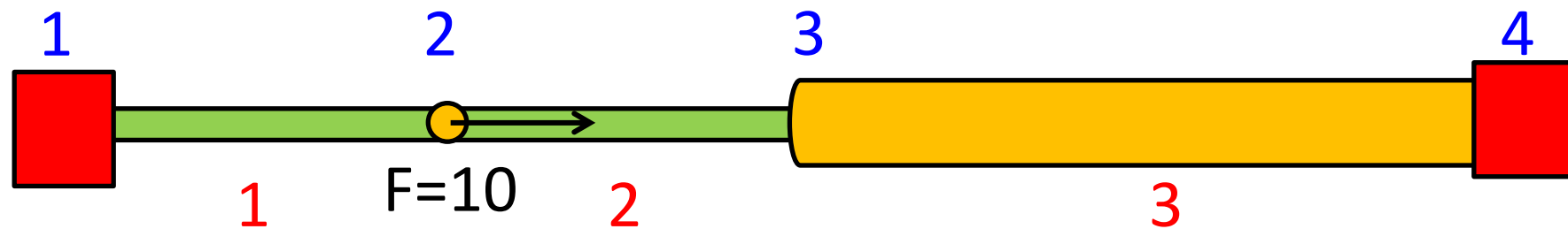
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Boundary Conditions and Loads

Actual



$$\begin{matrix} \text{Applied} = 10 \leftarrow \\ \text{Applied} = 0 \leftarrow \end{matrix} \begin{bmatrix} F1 \\ F2 \\ F3 \\ F4 \end{bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100+100 & -100 & 0 \\ 0 & -100 & 100+1000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{bmatrix} u1 \\ u2 \\ u3 \\ u4 \end{bmatrix} \begin{matrix} 0 \\ \\ \\ 0 \end{matrix}$$



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



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Two Equations and Two Unknown Displacements

$$\begin{bmatrix} F1 \\ 10 \\ 0 \\ F4 \end{bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100+100 & -100 & 0 \\ 0 & -100 & 100+1000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{bmatrix} 0 \\ u2 \\ u3 \\ 0 \end{bmatrix}$$

The matrix is partitioned into three colored regions: a green region (top-left 2x2), a yellow region (bottom-right 2x2), and a central green region (2x2) labeled with a red '2'. The top-left 2x2 region is labeled with a red '1', and the bottom-right 2x2 region is labeled with a red '3'. The central green region is labeled with a red '2'.

Solve using matrix method for u2 and u3:

- We will focus on first finding the displacements using matrix equations
- The FEM solver would use other techniques of Gaussian elimination or mathematical search techniques to solve all the equations in the solution algorithms



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Stiffness matrix reduction as u1 and u4 were 0

$$\begin{Bmatrix} 10 \\ 0 \end{Bmatrix} = \begin{bmatrix} 100+100 & -100 \\ -100 & 100+1000 \end{bmatrix} \begin{Bmatrix} u2 \\ u3 \end{Bmatrix}$$

Solve using matrix method for u2 and u3:

- So let us eliminate the first and last rows and columns in the stiffness matrix, then we have:
- $10 = 100(0) + 200(u2) - 100(u3) + 0(0)$
- $0 = 0(0) - 100(u2) + 1100(u3) - 1000(0)$
- Solving the equations we get $u2 = 5.2381E-2$ and $u3 = 4.7619E-3$



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



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Next solve for the unknown forces

$$\begin{bmatrix} F1 \\ 10 \\ 0 \\ F4 \end{bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100+100 & -100 & 0 \\ 0 & -100 & 100+1000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{bmatrix} 0 \\ u2 \\ u3 \\ 0 \end{bmatrix}$$

The matrix is partitioned into four 2x2 blocks with red numbers 1, 2, and 3 indicating sub-matrices:

- Block 1 (top-left): $\begin{bmatrix} 100 & -100 \\ -100 & 100+100 \end{bmatrix}$
- Block 2 (top-right): $\begin{bmatrix} 0 & 0 \\ -100 & 0 \end{bmatrix}$
- Block 3 (bottom-right): $\begin{bmatrix} 100+1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$

Solve using matrix method for F1 and F4:

- $F1 = 100(0) - 100(5.2381E-2) + 0(4.7619E-3) + 0(0)$
- $F4 = 0(0) + 0(5.2381E-2) - 1000(4.7619E-3) + 0(0)$
- Solving the equations we get F1 = -5.2381 and F4 = -4.7619
- Therefore F1+F4 = -10lb. Reaction forces balance the applied load of 10lb



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$



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Conclusions

So in summary:

- First we define all the FE entities in the FEA pre processor based on the actual structure
- In the FEA solver, the geometry and material properties are used to generate and assemble the global stiffness matrix
- Then the boundary conditions kick in to reduce the stiffness matrix
- In the FEA solver, typically the stiffness matrix is then inverted and numerical techniques are used to solve for the unknown displacements first
- Following that the reaction loads, strains, stresses and other derivative quantities are calculated based on what is requested in the output control



$$MS = \frac{1}{1.15 \sqrt{\left(\frac{P_t}{P_{ta}}\right)^2 + \left(\frac{P_s}{P_{sa}}\right)^2}} - 1$$

