Can you play a fair game of craps with a loaded pair of dice?

and related problems about factorization in $\mathbb{R}^{\geq 0}[x]$

Ian Morrison

http://bit.ly/2hYjPHR

Department of Mathematics
Fordham University

morrison@fordham.edu

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Starting point: multiplication of polynomials as a mapping

Fix:

- a set or *sack* $\mathbf{S}$ of *ordered* factors $\mathbf{d}_i(x), i = 1, 2, \ldots m$, monic of degree $n_i - 1$.
- the product $\mathbf{T}_\mathbf{S}(x) = \prod_{i=1}^{m} \mathbf{d}_i(x))$ of degree $t - 1 = \sum_{i=1}^{m}(n_i - 1)$.

$$\mathbf{T}_\mathbf{S}(x) = \sum_{s=0}^{t-1} \left(\sum_{j_1+j_2+\cdots+j_m=s} \prod_{i=i}^{m} d_{i j_i} \right) x^s$$

- View this as map $\mu_\mathbf{S} : \mathbb{C}^{t-1} \to \mathbb{C}^{t-1}$: expect that μ is surjective and finite.
- Indeed $(\mathbf{d}_1, \mathbf{d}_2, \ldots \mathbf{d}_m) \mapsto \mathbf{T} \iff$ roots of $\mathbf{d}_i(x)$ partition those of $\mathbf{T}(x)$.
- When there are no repeated roots, $\deg(\mu) = \dfrac{(t-1)!}{\prod_{i=1}^{m}(n_i - 1)!}$

Example If $n_1 = n_2 = 6$, then $t = 11$ and $\deg(\mu) = \dfrac{10!}{5!\,5!} = 252$.

Next, consider restrictions to subrings of scalars

First, restriction $\mu : \mathbb{R}^{t-1} \to \mathbb{R}^{t-1}$ to real $\mathbf{d}_i(x)$.

- Now some fibers may be empty.
- If $\mathbf{T}(x)$ has r real roots and s complex conjugate pairs, then we cannot have more than r factors $\mathbf{d}_i(x)$ of odd degree.
- The partition of roots must respect complex conjugate pairs.
- Modulo this restriction, the picture is still quite uniform.

Consider $\mu : (\mathbb{R}^{\geq 0})^{t-1} \to (\mathbb{R}^{\geq 0})^{t-1}$ to real non-negative $\mathbf{d}_i(x)$

- Why? Question has a probabilistic interpretation.
- View $\mathbf{d}_i$ as independent n_i-sided dice with side probabilities d_{ij}.
  - Side numbers are shifted down 1 to simplify notation but n is the order of $\mathbf{d}$.
  - Probabilistic normalization $\sum_{j=1}^{n_i} d_{ij} = 1$ differs from monic normalization by a positive scaling: work with either at will, usually the latter.
  - Allow $d_{ij} = 0$ for some j so "spinner" is a better model than "die".
  - Confound $\mathbf{d}_i(x)$ with generating function for tautological random variable with value j on side j.
  - Then $\mathbf{T}_\mathbf{S}(x)$ is the generating function of the total random variable obtained by summing the random variables of the dice in $\mathbf{S}$.
- An example familiar in these terms is:

$$1+2x+3x^2+4x^3+5x^4+6x^5+5x^6+4x^7+3x^8+2x^9+x^{10} = (1+x+x^2+x^3+x^4+x^5)^2$$

The fair polynomials $\psi_n(x)$

$$\psi_n(x) := 1 + x + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$

- Then $\mathbf{d}_i$ is fair $\iff \mathbf{d}_i(x) = \psi_{n_i}(x)$.
- The roots of ψ_n are the non-trivial nth roots of unity.
- Only linear factor/real root of ψ_n is $(x+1)$/-1 for n even.
- If $\zeta = e^{\frac{2\pi i}{n}}$, we get irreducible ζ^k-factors
  $$(x - \zeta^k)(x - \zeta^{-k}) = x^2 - 2\cos(2\pi \frac{k}{n}) + 1,\ 0 < k < \frac{n}{2}$$
  which are non-negative $\iff k \geq \frac{n}{4}$.
- Note that all these factors are palindromic.

Goal: Convince you these questions are interesting by discussing two very special cases that have interesting answers

- Standard totals: the $\mathbf{T}$ obtained when all $\mathbf{d}_i$ are fair.
  - Example: What are the non-negative real solutions of
$$(d_0+d_1x+d_2x^2+d_3x^3+d_4x^4+d_5x^5)(d'_0+d'_1x+d'_2x^2+d'_3x^3+d'_4x^4+d'_5x^5) = (1+x+x^2+x^3+x^4+x^5)^2\ ?$$
  - Can you play a fair game of craps with a loaded pair of dice?
  - More generally, what sacks of unfair dice simulate a fair sack of the same orders?

Exercise Show that the probability of winning at craps is $\frac{244}{495}$ and find the expected number of rolls in a game.

- Fair totals: when $\mathbf{T}$ itself is fair.
  - What sacks $\mathbf{S}$ satisfy
$$T_{\mathbf{S}} := \prod_{\mathbf{d}\in\mathbf{S}} \mathbf{d}(x) = 1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}\ ?$$
  - What sacks of unfair dice simulate a fair dodecahedron?

Standard totals, joint with Dave Swinarski

For two cubical dice $\mathbf{d}$ and $\mathbf{d}'$, the standard problem in the probabilistic normalization is the algebraic system

$$
\begin{aligned}
d_0d'_0 &= \tfrac{1}{36} = d_5d'_5 \\
d_0d'_1 + d_1d'_0 &= \tfrac{2}{36} = d_4d'_5 + d_5d'_4 \\
d_0d'_2 + d_1d'_1 + d_2d'_0 &= \tfrac{3}{36} = d_3d'_5 + d_4d'_4 + d_5d'_3 \\
d_0d'_3 + d_1d'_2 + d_2d'_1 + d_3d'_0 &= \tfrac{4}{36} = d_2d'_5 + d_3d'_4 + d_4d'_3 + d_5d'_2 \\
d_0d'_4 + d_1d'_3 + d_2d'_2 + d_3d'_1 + d_4d'_0 &= \tfrac{5}{36} = d_1d'_5 + d_2d'_4 + d_3d'_3 + d_4d'_2 + d_5d'_1 \\
d_0d'_5 + d_1d'_4 + d_2d'_3 + d_3d'_2 + d_4d'_1 + d_5d'_0 &= \tfrac{6}{36} \\
\sum_{i=0}^{5} d_i &= 1 = \sum_{i=0}^{5} d'_i
\end{aligned}
$$

We took the obvious, but wrong, approach and solved this system.

Solutions in a moment but first . . .

A brief binary digression

- Exercise
  - Write down the system of equations for two coins and show it has no real unfair solution.
  - Show that the system of equations for any number of coins has no real unfair solution by finding all solutions.
- Both of these are obvious from the viewpoint of factorization maps.
  - We want coins that give the total $(1+x)^m$ of m fair coins.
  - The linear factor from each coin can only be $(1+x)$.

How we proceeded

- Using Magma, we first checked that the complex solutions were 0-dimensional of degree 252.
- This meant it was possible to use Gröbner techniques to perform an algebraic elimination and back-substitution procedure that showed that there were no unfair solutions.
- Our equations have rational coefficients and Magma can also produce the equations of irreducible components over $\mathbb{Q}$.
- These suggested that all solutions were defined over the field of 6th roots of unity and Magma was able to find all solutions there.
- There are 51 solutions (25 asymmetric pairs and the symmetric fair pair).
- The numerical solver Bertini, based on homotopy continuation methods, was able to find all 51 roots and give us a check.

Theorem You can't play a fair game of craps with a loaded pair of dice.

Here's half of the table of solutions in terms of $\zeta = \frac{1}{2}(1 + \sqrt{3}i)$ . . .

d_0	d_1	d_2	d_3	d_4	d_5	d'_0	d'_1	d'_2	d'_3	d'_4	d'_5
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{5}{18}$	$\frac{1}{6}$	$\frac{1}{18}$
$\frac{-1}{6}$	$\frac{-4\zeta+1}{6}$	$\frac{-4\zeta+7}{6}$	$\frac{4\zeta+3}{6}$	$\frac{4\zeta-3}{6}$	$\frac{-1}{6}$	$\frac{-1}{6}$	$\frac{4\zeta-3}{6}$	$\frac{4\zeta+3}{6}$	$\frac{-4\zeta+7}{6}$	$\frac{-4\zeta+1}{6}$	$\frac{-1}{6}$
$\frac{-\zeta-1}{9}$	$\frac{-5\zeta+4}{9}$	$\frac{-\zeta+5}{9}$	$\frac{\zeta+4}{9}$	$\frac{5\zeta-1}{9}$	$\frac{\zeta-2}{9}$	$\frac{\zeta-2}{12}$	$\frac{2\zeta-1}{4}$	$\frac{5\zeta+5}{12}$	$\frac{-5\zeta+10}{12}$	$\frac{-2\zeta+1}{4}$	$\frac{-\zeta-1}{12}$
$\frac{-\zeta}{3}$	$\frac{-\zeta+3}{3}$	$\frac{3\zeta-2}{3}$	$\frac{-3\zeta+1}{3}$	$\frac{\zeta+2}{3}$	$\frac{\zeta-1}{3}$	$\frac{\zeta-1}{12}$	$\frac{4\zeta-1}{12}$	$\frac{3\zeta+4}{12}$	$\frac{-3\zeta+7}{12}$	$\frac{-4\zeta+3}{12}$	$\frac{-\zeta}{12}$
$\frac{-\zeta-1}{12}$	$\frac{-2\zeta+1}{4}$	$\frac{-5\zeta+10}{12}$	$\frac{5\zeta+5}{12}$	$\frac{2\zeta-1}{4}$	$\frac{\zeta-2}{12}$	$\frac{\zeta-2}{9}$	$\frac{5\zeta-1}{9}$	$\frac{\zeta+4}{9}$	$\frac{-\zeta+5}{9}$	$\frac{-5\zeta+4}{9}$	$\frac{-\zeta-1}{9}$
$\frac{-\zeta}{6}$	$\frac{-\zeta+1}{3}$	$\frac{-\zeta+3}{6}$	$\frac{\zeta+2}{6}$	$\frac{\zeta}{3}$	$\frac{\zeta-1}{6}$	$\frac{\zeta-1}{6}$	$\frac{\zeta}{3}$	$\frac{\zeta+2}{6}$	$\frac{-\zeta+3}{6}$	$\frac{-\zeta+1}{3}$	$\frac{-\zeta}{6}$
$\frac{-2\zeta+1}{6}$	$\frac{1}{2}$	$\frac{2\zeta-1}{6}$	$\frac{-2\zeta+1}{6}$	$\frac{1}{2}$	$\frac{2\zeta-1}{6}$	$\frac{2\zeta-1}{18}$	$\frac{4\zeta+1}{18}$	$\frac{2\zeta+5}{18}$	$\frac{-2\zeta+7}{18}$	$\frac{-4\zeta+5}{18}$	$\frac{-2\zeta+1}{18}$
$\frac{-2\zeta+1}{9}$	$\frac{-\zeta+2}{9}$	$\frac{-2\zeta+4}{9}$	$\frac{2\zeta+2}{9}$	$\frac{\zeta+1}{9}$	$\frac{2\zeta-1}{9}$	$\frac{2\zeta-1}{12}$	$\frac{\zeta}{4}$	$\frac{\zeta+4}{12}$	$\frac{-\zeta+5}{12}$	$\frac{-\zeta+1}{4}$	$\frac{-2\zeta+1}{12}$
$\frac{-\zeta+1}{3}$	$\frac{\zeta}{3}$	$\frac{-\zeta+1}{3}$	$\frac{\zeta}{3}$	$\frac{-\zeta+1}{3}$	$\frac{\zeta}{3}$	$\frac{\zeta}{12}$	$\frac{2\zeta+1}{12}$	$\frac{\zeta+3}{12}$	$\frac{-\zeta+4}{12}$	$\frac{-2\zeta+3}{12}$	$\frac{-\zeta+1}{12}$
$\frac{-\zeta+2}{3}$	$\frac{\zeta-1}{1}$	$\frac{-5\zeta+4}{3}$	$\frac{5\zeta-1}{3}$	$\frac{-\zeta}{1}$	$\frac{\zeta+1}{3}$	$\frac{\zeta+1}{36}$	$\frac{2\zeta+5}{36}$	$\frac{\zeta+10}{36}$	$\frac{-\zeta+11}{36}$	$\frac{-2\zeta+7}{36}$	$\frac{-\zeta+2}{36}$
$\frac{-2\zeta+1}{12}$	$\frac{-\zeta+1}{4}$	$\frac{-\zeta+5}{12}$	$\frac{\zeta+4}{12}$	$\frac{\zeta}{4}$	$\frac{2\zeta-1}{12}$	$\frac{2\zeta-1}{9}$	$\frac{\zeta+1}{9}$	$\frac{2\zeta+2}{9}$	$\frac{-2\zeta+4}{9}$	$\frac{-\zeta+2}{9}$	$\frac{-2\zeta+1}{9}$
$\frac{-\zeta+1}{4}$	$\frac{1}{4}$	$\frac{\zeta}{4}$	$\frac{-\zeta+1}{4}$	$\frac{1}{4}$	$\frac{\zeta}{4}$	$\frac{\zeta}{9}$	$\frac{\zeta+1}{9}$	$\frac{\zeta+2}{9}$	$\frac{-\zeta+3}{9}$	$\frac{-\zeta+2}{9}$	$\frac{-\zeta+1}{9}$

Theorem You can't play a fair game of craps with a loaded pair of dice.

and here's the other half of the table of solutions:

d_0	d_1	d_2	d_3	d_4	d_5	d'_0	d'_1	d'_2	d'_3	d'_4	d'_5
$\frac{-\zeta+1}{6}$	$\frac{-\zeta+1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{\zeta}{6}$	$\frac{\zeta}{6}$	$\frac{\zeta}{6}$	$\frac{\zeta}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{-\zeta+1}{6}$	$\frac{-\zeta+1}{6}$
$\frac{-\zeta+2}{6}$	$\frac{0}{1}$	$\frac{\zeta+1}{6}$	$\frac{-\zeta+2}{6}$	$\frac{0}{1}$	$\frac{\zeta+1}{6}$	$\frac{\zeta+1}{18}$	$\frac{\zeta+1}{9}$	$\frac{\zeta+4}{18}$	$\frac{-\zeta+5}{18}$	$\frac{-\zeta+2}{9}$	$\frac{-\zeta+2}{18}$
$\frac{1}{3}$	$\frac{-\zeta}{3}$	$\frac{2\zeta}{3}$	$\frac{-2\zeta+2}{3}$	$\frac{\zeta-1}{3}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{\zeta+2}{12}$	$\frac{\zeta+2}{12}$	$\frac{-\zeta+3}{12}$	$\frac{-\zeta+3}{12}$	$\frac{1}{12}$
$\frac{\zeta+1}{3}$	$\frac{-\zeta}{1}$	$\frac{5\zeta-1}{3}$	$\frac{-5\zeta+4}{3}$	$\frac{\zeta-1}{1}$	$\frac{-\zeta+2}{3}$	$\frac{-\zeta+2}{36}$	$\frac{-2\zeta+7}{36}$	$\frac{-\zeta+11}{36}$	$\frac{\zeta+10}{36}$	$\frac{2\zeta+5}{36}$	$\frac{\zeta+1}{36}$
$\frac{-\zeta}{12}$	$\frac{-4\zeta+3}{12}$	$\frac{-3\zeta+7}{12}$	$\frac{3\zeta+4}{12}$	$\frac{4\zeta-1}{12}$	$\frac{\zeta-1}{12}$	$\frac{\zeta-1}{3}$	$\frac{\zeta+2}{3}$	$\frac{-3\zeta+1}{3}$	$\frac{3\zeta-2}{3}$	$\frac{-\zeta+3}{3}$	$\frac{-\zeta}{3}$
$\frac{-2\zeta+1}{18}$	$\frac{-4\zeta+5}{18}$	$\frac{-2\zeta+7}{18}$	$\frac{2\zeta+5}{18}$	$\frac{4\zeta+1}{18}$	$\frac{2\zeta-1}{18}$	$\frac{2\zeta-1}{6}$	$\frac{1}{2}$	$\frac{-2\zeta+1}{6}$	$\frac{2\zeta-1}{6}$	$\frac{1}{2}$	$\frac{-2\zeta+1}{6}$
$\frac{-\zeta+1}{6}$	$\frac{1}{3}$	$\frac{\zeta}{6}$	$\frac{-\zeta+1}{6}$	$\frac{1}{3}$	$\frac{\zeta}{6}$	$\frac{\zeta}{6}$	$\frac{1}{3}$	$\frac{-\zeta+1}{6}$	$\frac{\zeta}{6}$	$\frac{1}{3}$	$\frac{-\zeta+1}{6}$
$\frac{-\zeta+1}{9}$	$\frac{-\zeta+2}{9}$	$\frac{-\zeta+3}{9}$	$\frac{\zeta+2}{9}$	$\frac{\zeta+1}{9}$	$\frac{\zeta}{9}$	$\frac{\zeta}{4}$	$\frac{1}{4}$	$\frac{-\zeta+1}{4}$	$\frac{\zeta}{4}$	$\frac{1}{4}$	$\frac{-\zeta+1}{4}$
$\frac{-\zeta+2}{9}$	$\frac{\zeta+1}{9}$	$\frac{-\zeta+2}{9}$	$\frac{\zeta+1}{9}$	$\frac{-\zeta+2}{9}$	$\frac{\zeta+1}{9}$	$\frac{\zeta+1}{12}$	$\frac{1}{4}$	$\frac{-\zeta+2}{12}$	$\frac{\zeta+1}{12}$	$\frac{1}{4}$	$\frac{-\zeta+2}{12}$
$\frac{1}{3}$	$\frac{\zeta-1}{3}$	$\frac{-2\zeta+2}{3}$	$\frac{2\zeta}{3}$	$\frac{-\zeta}{3}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{-\zeta+3}{12}$	$\frac{-\zeta+3}{12}$	$\frac{\zeta+2}{12}$	$\frac{\zeta+2}{12}$	$\frac{1}{12}$
$\frac{-\zeta+1}{12}$	$\frac{-2\zeta+3}{12}$	$\frac{-\zeta+4}{12}$	$\frac{\zeta+3}{12}$	$\frac{2\zeta+1}{12}$	$\frac{\zeta}{12}$	$\frac{\zeta}{3}$	$\frac{-\zeta+1}{3}$	$\frac{\zeta}{3}$	$\frac{-\zeta+1}{3}$	$\frac{\zeta}{3}$	$\frac{-\zeta+1}{3}$
$\frac{-\zeta+2}{12}$	$\frac{1}{4}$	$\frac{\zeta+1}{12}$	$\frac{-\zeta+2}{12}$	$\frac{1}{4}$	$\frac{\zeta+1}{12}$	$\frac{\zeta+1}{9}$	$\frac{-\zeta+2}{9}$	$\frac{\zeta+1}{9}$	$\frac{-\zeta+2}{9}$	$\frac{\zeta+1}{9}$	$\frac{-\zeta+2}{9}$
$\frac{-\zeta+2}{18}$	$\frac{-\zeta+2}{9}$	$\frac{-\zeta+5}{18}$	$\frac{\zeta+4}{18}$	$\frac{\zeta+1}{9}$	$\frac{\zeta+1}{18}$	$\frac{\zeta+1}{6}$	$\frac{0}{1}$	$\frac{-\zeta+2}{6}$	$\frac{\zeta+1}{6}$	$\frac{0}{1}$	$\frac{-\zeta+2}{6}$

Small problem: $252 \neq 51$

- This actually provides a nice check.
- Roots of $\mathbf{T}$ all have multiplicity 2.
- When $\mathbf{d}$ and $\mathbf{d}'$ each have one copy, swapping those roots fixes both $\mathbf{d}$ and $\mathbf{d}'$.
- This gives ramification of orders 2^5, 2^3 and 2^1 when there are 5, 3 and 1 roots in common.
- Count each kind by choosing the common roots, then dividing the remaining pairs.

$$51 = \binom{5}{5}\binom{0}{0} + \binom{5}{3}\binom{2}{1} + \binom{5}{1}\binom{4}{2} = 1 + 20 + 30$$

$$252 = 2^5 \cdot 1 + 2^3 \cdot 20 + 2^1 \cdot 30$$

Of course, it's much easier to answer the craps question from the factorization point of view.

- For cubical dice:
  - Each die is quintic to must get one of copies of $(x+1)$ and two quadratic factors.
  - If the latter are different, we get fair dice. If the same, we get the other real solution.
- What about pairs of other sizes?
- It's also easy to check the existence of unfair sacks with standard totals.

Conjecture

There are unfair standard sacks of every pair of orders (n, n') with $2 \le n \le n'$ except for $(2, n')$, $(3, 3)$, $(3, 6)$, $(3, 9)$, $(4, 4)$, $(4, 8)$, $(5, 5)$, $(6, 6)$, $(7, 7)$, $(8, 8)$, $(9, 9)$, and $(11, 11)$.

Some small pairs with standard totals

- The first pair we found (numerically) has order 13 and swaps the ζ^4 and ζ^5 factors.

Face	0	1	2	3	4	5	6
$\mathbf{d}$	0.0992916	0.0210685	0.1381701	0.0410895	0.0693196	0.1241391	0.0138431
$\mathbf{d}'$	0.0595938	0.1065425	0.0732460	0.0499115	0.0997570	0.0877406	0.0464172

- The smallest unfair pairs with standard totals

Order	Swap	Face	0	1	2	3	4	5
10	ζ_3, ζ_4	$\mathbf{d}_{10}$	$\frac{5-\sqrt{5}}{20}$	0	$\frac{\sqrt{5}}{10}$	0	$\frac{5-\sqrt{5}}{20}$	
		$\mathbf{d}'_{10}$	$\frac{5+\sqrt{5}}{100}$	$\frac{5+\sqrt{5}}{50}$	$\frac{1}{10}$	$\frac{5-\sqrt{5}}{50}$	$\frac{15-\sqrt{5}}{100}$	
12	ζ_4, ζ_5	$\mathbf{d}_{12}$	$\frac{2-\sqrt{3}}{4}$	$\frac{2\sqrt{3}-3}{4}$	$\frac{2\sqrt{3}-3}{4}$	$\frac{2-\sqrt{3}}{4}$	$\frac{2\sqrt{3}-3}{4}$	$\frac{2-\sqrt{3}}{4}$
		$\mathbf{d}'_{12}$	$\frac{2+\sqrt{3}}{36}$	$\frac{1}{36}$	$\frac{4+\sqrt{3}}{36}$	$\frac{2-\sqrt{3}}{36}$	$\frac{5}{36}$	$\frac{4-\sqrt{3}}{36}$

Prior art

Theorem [Robertson-Shortt-Landry, Monthly 1988]
There are unfair standard sacks of every pair of equal orders n except for $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, $(6,6)$, $(7,7)$, $(8,8)$, $(9,9)$, $(11,11)$ and $(13,13)$.

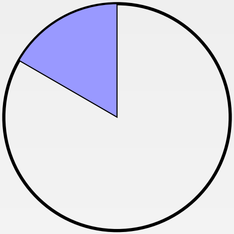
- Prove the identity $\dfrac{1-\alpha z+z^2}{1-\beta z+z^2} = 1 + (\beta - \alpha)\sum_{j=0}^{\infty} S_j(\beta) z^{j+1}$ where the S_j are the Chebyshev polynomials.
- Use this to estimate the coefficients of the $\mathbf{d}$ and $\mathbf{d}'$ obtained by swapping factors for the two largest powers of ζ.
- Deduce positivity of both for $n \geq 14$.
- Used a MACSYMA computation to “settle” smaller n.

Exercise If $\mathbf{d}$ and $\mathbf{d}'$ are rational and have standard totals, then both are fair.

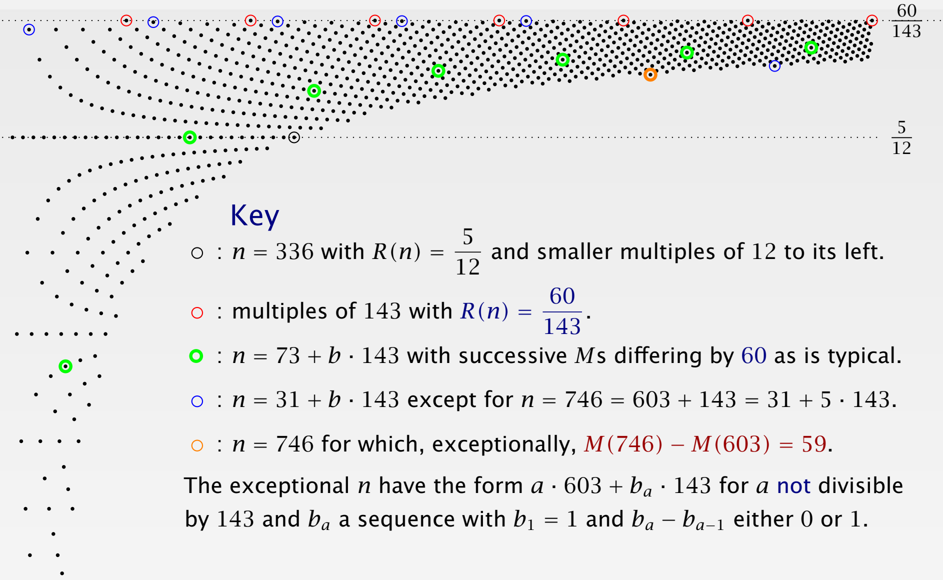
Asymptotics: when unfair sacks exist, how *many* are there?

- "Easiest" case is $(3, n)$.
- Which factors of ζ^k in $\psi_n(x)$ can you swap with the $1 + x + x^2$, keeping both non-negative?
- Need $k \geq \frac{n}{4}$ to keep the triangle non-negative.
- Experimentally the k that work form an interval $(\lfloor \frac{n}{4} \rfloor, M(n))$.

- Define $R(n) = \frac{M(n)}{n}$ so $\frac{1}{4} \leq R(n) < \frac{1}{2}$.
- Empirically $R(n) \cong \frac{5}{12}$.
- The ζ^k whose factors you can swap "fill" the blue sector from 90 to 150 degrees.



Scatter plot of $(n, R(n))$ for n up to 1001



Key

○ : $n = 336$ with $R(n) = \dfrac{5}{12}$ and smaller multiples of 12 to its left.

○ : multiples of 143 with $R(n) = \dfrac{60}{143}$.

○ : $n = 73 + b \cdot 143$ with successive Ms differing by 60 as is typical.

○ : $n = 31 + b \cdot 143$ except for $n = 746 = 603 + 143 = 31 + 5 \cdot 143$.

○ : $n = 746$ for which, exceptionally, $M(746) - M(603) = 59$.

The exceptional n have the form $a \cdot 603 + b_a \cdot 143$ for a not divisible by 143 and b_a a sequence with $b_1 = 1$ and $b_a - b_{a-1}$ either 0 or 1.

Weighted Dice

E 925 [1950, 416]. *Proposed by J. B. Kelly, University of Wisconsin*

Is it possible so to weight a pair of dice that the probability of occurrence of every sum from 2 to 12 shall be the same?

I. *Solution by Leo Moser and J. H. Wahab, University of North Carolina*. Let p_i and q_i denote the probabilities of an i appearing on the first and second die, respectively. Let s_j denote the probability of the occurrence of the sum j. Now assume $s_2 = s_{12}$. Then

$$p_1 q_1 = s_2 = s_{12} = p_6 q_6,$$

whence

$$(p_1 - p_6)(q_1 - q_6) \leqq 0,$$

so that

$$s_2 + s_{12} = p_1 q_1 + p_6 q_6 \leqq p_1 q_6 + p_6 q_1 \leqq s_7.$$

Thus no loading of the dice can yield equiprobable sums.

II. *Solution by J. V. Finch and P. R. Halmos, University of Chicago*. Define p_i and q_i as above. If $P(x) = \sum_{i=0}^{5} p_{i+1} x^i$ and $Q(x) = \sum_{i=0}^{5} q_{i+1} x^i$, then the requirement that all sums (between 2 and 12) are equally probable is equivalent to the identity $P(x)Q(x) = (1/11) \sum_{i=0}^{10} x^i$. The zeros of the polynomial on the right are all complex (they are, in fact, the complex 11th roots of unity) whereas either of the factors on the left (having real coefficients and odd degree) has at least one real root. It follows that no such factoring is possible and therefore that no loading of the dice can yield equiprobable sums.

Further work on the problem of dice with fair totals

- Several papers showing non-existence of fair sacks under various conditions on orders via inequalities.
- William Gasarch and Clyde Kruskal asked: *Do all fair sacks share some common structure?*
- They show necessity and sufficiency of one local and one global condition.

Gasarch-Kruskal Theorem, [Math. Mag., 1998]

A sack is fair if and only if:

- Each die in it is semifair.
- (Uniqueness of Totals) Each total is obtained from a unique effective roll.

Semifairness

- A die is semifair if
  - It is palindromic.
  - All coefficients are 0 or 1.
- The irreducible factors of $\psi_n(x)$ are palindromic so all dice in a fair sack are palindromic too.

Semifairness Lemma

If $\mathbf{d}(x) := \mathbf{d}'(x) \cdot \mathbf{d}''(x)$ is semifair and both $\mathbf{d}'$ and $\mathbf{d}''$ are palindromic, then both $\mathbf{d}'$ and $\mathbf{d}''$ are semifair.

- Induction of the order of a fair sack $\mathbf{S}$ then shows semifairness of its dice.

Proof of the Semifairness Lemma

- Assume that $n' \le n''$ and define $d'_j = 0$ for $n' \le j < n''$.
- First Claim: For $1 \le j \le n'' - 1$, either $d'_j = 0$ or $d''_j = 0$.

$$\sum_{j=1}^{n'-1} d'_j\, d''_j = \sum_{j=1}^{n'-1} d'_{n'-1-j}\, d''_j = d_{n'-1} - d'_{n'-1} d''_0 = d_{n'-1} - d'_0 d''_0 = d_{n'-1} - d_0 = 0\,.$$

- Second Claim: For $1 \le j \le n''-1$, one of d'_j and d''_j is 0 and the other 0 or 1.

$$d_j := \sum_{i=0}^{j} d'_i d''_{j-i} = d'_j + \sum_{i=1}^{j-1} d'_i d''_{j-i} + d''_j\,.$$

  - All terms are non-negative and, by induction on j, all red terms are 0 or 1.
  - If $d_j = 0$, all red terms and both d'_j and d''_j are 0.
  - If $d_j = 1$, either one red term is 1 and both d'_j and d''_j are 0, or,
  - All red terms are 0 and $d'_j + d''_j = 1$. One is 0 by the First Claim so the other must equal 1.

Uniqueness of totals and terms

- Each roll with total s of a sack $\mathbf{S}$ of semifair dice contributes 0 or 1 to the x^s term in $\mathbf{T}_\mathbf{S}(x)$. If it gives 1, we say the roll is effective.
- The coefficient of t^s in $\mathbf{T}_\mathbf{S}(x)$ is thus number of effective rolls with total s.
- There is 1 such roll for $s = 0$ and hence, by fairness of $\mathbf{T}_\mathbf{S}$, for all s: this is Uniqueness of totals

Uniqueness of terms

A fair sack can contain at most one die with non-zero x^s term and will contain such a die if and only if x^s does *not* arise as a product of terms of lower degree.

Corollary

No two dice in a fair sack can have the same order.

- This already subsumes all non-existence results prior to Gasarch-Kruskal.
- Of course, at most one die can have even order (odd degree) by reality.

Factorization sacks

- Factorization of t: $\mathbf{a} := (a_1, a_2, \ldots, a_\ell)$ with each $a_h \geq 2$ and $\prod_{h=1}^{\ell} a_h = t$.
- Fair sacks give factorizations: $a_h :=$ number of non-zero terms in $\mathbf{d}_h(x)$.
- The red equation holds since both sides count the number of effective rolls.
- Ordered factorizations give fair sacks: given $\mathbf{a}$, define $b_h := \prod_{h'<h} a_{h'}$ and

$$\mathbf{d}_h(x) := \psi_{a_h}(x^{b_h}).$$

- Check that $\prod_{h'<h} \psi_{a'_h}(x^{b'_h}) = \psi_{b_h}(x)$: the roots of both are exactly the non-trivial b_hth roots of unity.

Corollary

If t has m prime factors, then there are fair sacks of order ℓ with total t if and only if $\ell \leq m$.

Ordered factorizations a of 12 and their fair sacks $\mathbf{S_a}$

$a_1 \cdot a_2 \cdot \ldots \cdot a_\ell$	$\mathbf{d}_1(x) \cdot \mathbf{d}_2(x) \cdot \ldots \cdot \mathbf{d}_\ell(x))$
$2 \cdot 2 \cdot 3$	$(1+x)(1+x^2)(1+x^4+x^8)$
$2 \cdot 3 \cdot 2$	$(1+x)(1+x^2+x^4)(1+x^6)$
$2 \cdot 6$	$(1+x)(1+x^2+x^4+x^6+x^8+x^{10})$
$3 \cdot 2 \cdot 2$	$(1+x+x^2)(1+x^3)(1+x^6)$
$3 \cdot 4$	$(1+x+x^2)(1+x^3+x^6+x^9)$
$4 \cdot 3$	$(1+x+x^2+x^3)(1+x^4+x^8)$
$6 \cdot 2$	$(1+x+x^2+x^3+x^4+x^5)(1+x^6)$
12	$(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11})$

Partition-factorization sacks

- We can get further sacks from a factorization sack $\mathbf{S_a}$ by "subtotalling" dice as specified by a partition $\Pi := [\pi_i, \pi_2, \ldots, \pi_m]$ of $\{1, 2, \ldots, \ell\}$.
- For each part π_g of Π, define $\mathbf{d}_g(x) = \prod_{h \in \pi_g} \mathbf{d}_h(x)$ to get sack $\mathbf{S}_{\mathbf{a},\Pi}$.
- Subtotalling adjacent factors simply replaces them by their product.
- Π is interval free if no part contains any consecutive factors.
- For $t = 12$, there is one such $\Pi = [\{1, 3\}, \{2\}]$ which gives 3 new fair sacks.

Partition-factorization sacks with total 12 for $\Pi = [\{1, 3\}, \{2\}]$

$\mathbf{a}$	$\mathbf{S_a}$
$2 \cdot 2 \cdot 3$	$(1 + x + x^4 + x^5 + x^8 + x^9)(1 + x^2)$
$2 \cdot 3 \cdot 2$	$(1 + x + x^6 + x^7)(1 + x^2 + x^4)$
$3 \cdot 2 \cdot 2$	$(1 + x + x^2 + x^6 + x^7 + x^8)(1 + x^3)$

The Main Theorem

Theorem

Every fair sack $\mathbf{S}$ of size m and total t equals $\mathbf{S}_{\mathbf{a},\Pi}$ for Π an interval free partition with m parts of an ordered factorization $\mathbf{a}$ of t. Both $\mathbf{a}$ and Π are uniquely determined by $\mathbf{S}$.

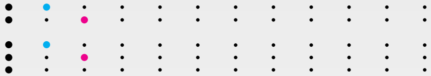
- I first proved uniqueness directly by induction on ℓ, the length of $\mathbf{a}$.
- Having a canonical description suggested that existence might follow by reading off $\mathbf{a}$ and Π from $\mathbf{S}$.
- The reconstruction is canonical so uniqueness is immediate from it.

The Main Theorem: Existence in a warmup case with $t = 12$

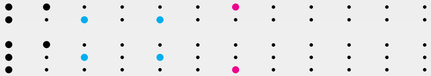
To warmup, let's do this "graphically" this for the two $\mathbf{a} = (2, 3, 2)$ sacks:

$$(1 + x + x^6 + x^7) \cdot (1 + x^2 + x^4) \text{ and } (1 + x) \cdot (1 + x^2 + x^4) \cdot (1 + x^6).$$

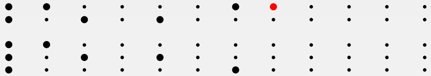
The cyan x term marks $\mathbf{d}_1$ as the top die. The magenta x^2 in $\mathbf{d}_2$ but not in $\mathbf{d}_1$ says $a_1 = 2$.



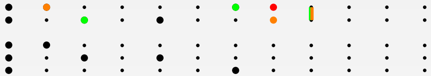
Repeat: $\mathbf{d}_2$ has an x^4 but no x^6 so $a_2 = 3$. The die with an x^6 has no x^{12} so $a_3 = 2$.



On the bottom, we have all totals. On top, no x^7. Why must this term lie in the top row as the red dot?

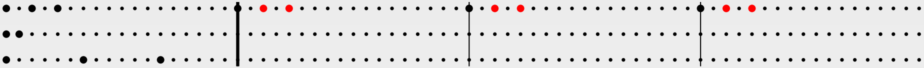


Because an x^7 in the second die would duplicate the total x^8 (sum green dots and orange dots)!

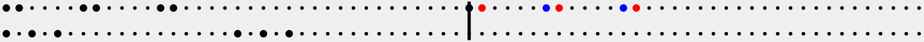


The Main Theorem: Existence in the general case

Existence follows in general by similar arguments but the bookkeeping is messier. We give the idea by diagramming two examples with $t = 72$.



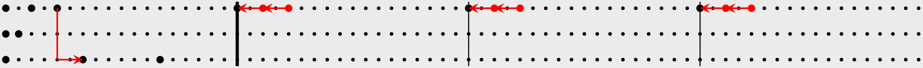
$$(\mathbf{a}', \Pi') = (2 \cdot 3 \cdot 3 \cdot 4, [\{2,4\}, \{1\}, \{3\}])$$



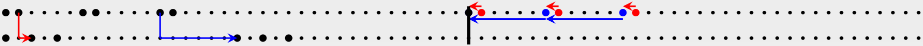
$$(\mathbf{a}', \Pi') = (2 \cdot 3 \cdot 3 \cdot 2 \cdot 2, [\{1,3,5\}, \{2,4\}])$$

- Latest b_ℓ (and multiples up to $a_\ell b_\ell$) are the thick bar (and thinner ones).
- Left of the b_ℓ-bar—i.e. for $(\mathbf{a}', \Pi')$—we inductively know all terms.
- Right of the b_ℓ-bar, only know the large black terms are present.
- Must show that all large colored terms on right are present and no others.

The Main Theorem: showing exactly the colored terms occur



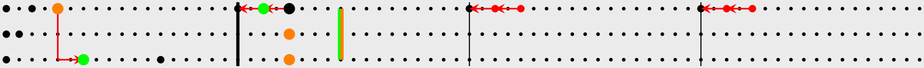
$(\mathbf{a}', \Pi') = (2 \cdot 3 \cdot 3 \cdot 4, [\{2,4\},\{1\},\{3\}])$, $h = 2, b_h = 2, b_{h+1} = 6$



$(\mathbf{a}', \Pi') = (2 \cdot 3 \cdot 3 \cdot 2 \cdot 2, [\{1,3,5\},\{2,4\}])$, $h = 1, b_h = 1, b_{h+1} = 2$, $h = 3, b_h = 6, b_{h+1} = 18$

- Work Inductively, left-to-right in chunks, one colored term at a time.
- The next colored degree must occur in some die: why must it be the top one?
- Each colored total $s = rb_\ell + s'$ gives an index h of a factor of the top die:
  - the h for which the s'-term comes from the factor $\psi_{a_h}(x^{b_h})$, or,
  - the largest h for which s is divisible by b_h.
- Two lines of the same color are drawn
  - left from s to the $s - b_h$ term, inductively known to be present.
  - from the $(b_{h+1} - b_h)$-term on top down and right to the b_{h+1}-term in a *lower* die.

The Main Theorem: showing exactly the colored terms occur



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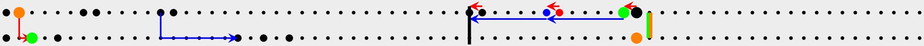

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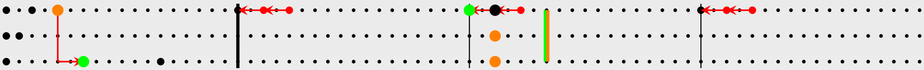
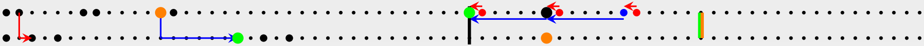
The Main Theorem: showing exactly the colored terms occur



$(\mathbf{a}', \Pi') = (2 \cdot 3 \cdot 3 \cdot 4, [\{2,4\}, \{1\}, \{3\}])$, $h = 2, b_h = 2, b_{h+1} = 6$



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Does every semifair die lie in some fair sack?

- No: simplest examples are $\mathbf{d}_s(x) := 1 + x^s + x^{2s-1} + x^{3s-1}$:
- The product giving the total $s - 1$ must use the constant term from $\mathbf{d}_s$ so can be used to duplicate the total $2s - 1$.
- Most semifair dice can be ruled out by:

Corollary

If a die lies in a sack with fair totals, then every non-zero degree is a multiple of the smallest one.

- The Corollary also yields a test for fairness of sacks of semifair dice $\mathbf{S}$ much faster than checking Uniqueness of totals.

Atomizations

- A polynomial in $\mathbb{R}^+[x]$ is atomic if it is not the product of two non-constant polynomials in $\mathbb{R}^+[x]$.
  - Every such polynomial has an atomization (can be written as a product of atoms) by induction on the degree. Hint: Factorization into primes!
  - Atomizations are not unique:
    $(1+x)(1+x^2)(1+x^4+x^8) = (1+x)(1+x^2+x^4)(1+x^6) = (1+x+x^2)(1+x^3)(1+x^6)$
- All atoms of any fair sack have the form $\psi_p(x^b)$ for p prime.
- If $\mathbf{d}$ is a semifair die in a sack with fair totals then all its atoms are semifair.
- Two questions:
  - Does every semifair die have a semifair atomization?
  - More greedily, is semifairness closed under atomization?

Elasticity

- Elasticity of $p(x)$ is the maximum of the ratios $\frac{n}{n'}$ for which p has atomizations with n and with n' factors.
- There are $p(x) \in \mathbb{R}^{\geq o}[x]$ with any rational elasticity ≥ 1.
- Dice in fair sacks have elasticity 1.
- Two questions:
  - What are the elasticities of semifair polynomials?
  - More greedily, are semifair polynomials inelastic?

Thank you for your attention, and,

for those who watched the clip from A Bronx Tale,

remember to “put ’em in the bathroom”.