A Practical Look at Volatility in Financial Time Series

MATH 287C - Advanced Time Series Analysis
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June 4th, 2018
Outline

1. What is Volatility?

2. Normalizing and Variance Stabilizing (NoVaS) Transformation

3. Volatility Prediction

4. A Simple Volatility Trading Strategy
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What is Volatility?

- Volatility is a measure of price variability over some period of time.
- Typically described by the standard deviation $\sigma$ of the return series $\{X_t\}$.
- Volatility is peculiar in that we know it exists, but in some sense we can’t really measure it.
- Bachelier (1900) showed that $\{X_t\} \sim \text{iid. } N(0,1)$, but this is only good for a first order approximation.
Naive Measure - Realized Volatility
Stylized Facts

Further analysis of $\{X_t\}$ reveals other kinds of structure that cannot be explained by the gaussian assumption.

In particular, the return series displays the following distinctive behavior:

1. $\{X_t\}$ is heavy-tailed, much more so than the Gaussian white noise
2. Although $\{X_t\}$ is uncorrelated, the series $\{X_t^2\}$ is highly correlated
3. The changes in $\{X_t\}$ tend to be clustered, large changes tend to be followed by large changes and vice v
4. Effects are asymmetric, bad news results in larger downward price moves than positive news does to upward price moves
The Generalized ARCH (GARCH) model of Bollerslev (1986) and its variants are extremely popular (albeit imperfect) methods to model volatility.

GARCH(p,q) model can be expressed as:

\[ X_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \]

where

\[ \sigma^2_t = \alpha_0 + \sum_{i=1}^{p} \beta_i \sigma^2_{t-i} + \sum_{j=1}^{q} \alpha_j X^2_{t-j} \]

For the purposes of this talk, we’ll focus on GARCH(1,1) models where \( \sigma^2_t = \alpha_0 + \beta_1 \sigma^2_{t-1} + \alpha_1 X^2_{t-1} \)
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The NoVaS Transformation is defined as

\[ W_{t,a} = \frac{X_t}{\sqrt{\alpha s_{t-1}^2 + a_0 X_t^2 + \sum_{i=1}^{p} a_i X_{t-i}^2}} \]

for \( t = p + 1, p + 2, \ldots, n \)

It is a clever extension of the ARCH model where we include the value \( X_t \) in order to “studentize” the returns.

The order \( p \) and the vector of nonnegative parameters \( (\alpha, a_0, \ldots, a_p) \) are chosen by the practitioner with the twin goals of normalization and variance-stabilization.

Algorithm for Simple NoVaS:

- Let \( \alpha = 0 \) and \( a_i = \frac{1}{p+1} \) for all \( 0 \leq i \leq p \)
- Pick \( p \) such that \(|KURT_n(W_{t,p}^S)| \approx 3|\)
S&P500 Daily Returns Histogram
S&P500 Daily Returns Q-Q Plot
NoVaS Transformed S&P500 Daily Returns (p=16)
NoVaS Transformed S&P500 Histogram (p=16)
NoVaS Transformed S&P500 QQ-Plot (p=16)
BTC/USD Daily Returns (2010-2018)
BTC/USD Daily Returns Histogram (2010-2018)
BTC/USD Daily Returns QQ-Plot (2010-2018)
NoVaS Transformed BTC/USD Returns ($p=12$)
NoVaS Transformed BTC/USD Histogram (p=12)
NoVaS Transformed BTC/USD QQ-Plot (p=12)
5min Bar 10yr Treasury Futures (1983-2012)
5min Bar 10yr Treasury Futures (2010-2012)
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5min Bar 10yr Treasury Futures (2010-2012)
NoVaS 10yr Treasury Futures (1983-2012) (p=12)
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Volatility Prediction

- Forecasts of volatility are important when assessing and managing the risks of portfolios
- We focus on the problem of one-step ahead $X_{t+1}$ prediction based on the observed past
- For our purposes, volatility prediction = predicting $X_{t+1}^2$
- Even though $X_{t+1}^2$ is a noisy proxy for $\mathbb{E}(X_{t+1}^2 | \mathcal{F}_n)$, we’ll see that in under some conditions NoVaS allows us to predict the latter
- Assuming more realistically that financial returns are locally stationary, we use a rolling window size of 250 days to calculate our forecasts
Which Loss Function? $L_1$ or $L_2$?

- To assess the accuracy of forecasts, we need to decide on a loss function to use

- The MSE is most commonly used, however note that $\mathbb{E}(Y_{n+1}^2 - \hat{Y}_{n+1}^2)^2$ is essentially a fourth moment

- Thus the unconditional MSE is infinite if the returns process has infinite kurtosis!

- We find that this indeed the case and so focus on the Mean Absolute Deviation (MAD) loss function

- Under the objective of $L_1$-optimal prediction, the optimal predictor is $\text{Median}(X_{n+1}^2 | F_n)$
Empirical Kurtosis Plot S&P500

Empirical Kurtosis of S&P500

Kurtosis

0 10 20 30 40

0 2500 5000 7500 10000 12500 15000 17500
Empirical Kurtosis Plot BTC
Recall the GARCH(1,1) model can be expressed as:

\[ X_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \]

where \( \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \alpha_1 X_{t-1}^2 \)

To perform one-step ahead prediction with GARCH(1,1) we:

- Estimate the parameters \( \alpha_0, \beta_1 \) and \( \alpha_1 \) using MLE
- Let \( a = \frac{\alpha_0}{1-\beta_1} \) and \( a_i = \alpha_1 \beta_1^{i-1} \) for \( i = 1, 2, \ldots \)

Then the \( L_1 \)-optimal GARCH(1,1) predictor of \( X_{n+1}^2 \) is given by

\[
\text{Median}(X_{n+1}^2 | \mathcal{F}_n) = (a + \sum_{i=1}^{p} a_i X_{n+1-i}^2) \text{Median}(\varepsilon_{n+1}^2)
\]
Special Case: Uncorrelated NoVaS Series $W_{t,a}$
NoVaS Prediction in Special Case

From the ACF plot on the previous slide, we can conclude that the NoVaS transformed series \( \{W_{t,a}\} \) of the S&P500 returns appears to be uncorrelated.

As a result we can predict \( X_{n+1}^2 \) (under \( L_1 \) loss) by

\[
X_{n+1}^2 = \hat{\mu}_2 A_n^2
\]

where

\[
\hat{\mu}_2 = median\left\{ \frac{W_{t,a}^2}{1 - a_0 W_{t,a}^2} ; t = p + 1, p + 2, \ldots, n \right\}
\]

and

\[
A_n^2 = \alpha s_{t-1}^2 + \sum_{i=1}^{p} a_i X_{t+1-i}^2
\]
Bootstrap Prediction Intervals

In addition to calculating point estimates, we calculate prediction intervals.

Below is an outline of the procedure used for NoVaS (GARCH is almost identical):

1. Use Simple NoVaS to obtain transformation \( \{W_{t,a}\} \) and fitted parameter \( p \)
2. Calculate \( g(X_{n+1}) \) the point predictor of \( g(X_{n+1}) \)
3. Resample randomly (with replacement) the transformed variables \( \{W_{t,a}\} \) for \( t = p + 1, \ldots, n \) to create the pseudo-data \( W_{p+1}^*, \ldots, W_{n-1}^*, W_n^* \) and \( W_{n+1}^* \)
4. Let \( (X_1^*, \ldots, X_p^*)' = (X_{1+l}^*, \ldots, X_{p+l}^*)' \) where \( l \sim U\{0, n - p\} \)
Bootstrap Prediction Intervals

5. Generate the bootstrap pseudo-data $Y_t^*$ for $t = p + 1, \ldots, n$

6. Based on the bootstrap data $Y_1^*, \ldots, Y_n^*$ re-estimate the NoVaS transformation, then calculate the bootstrap predictor $g(Y_{n+1}^*)$

7. Calculate the bootstrap future value $Y_{n+1}^*$ and the bootstrap root: $g(Y_{n+1}^*) - g(Y_{n+1}^*)$

8. Repeat steps 3-7 B times - the B bootstrap root replicates are collected in the form of an empirical distribution whose $\alpha$-quantile is denoted $q(\alpha)$

9. The $(1 - \alpha)100\%$ equal-tailed prediction interval for $g(Y_{n+1})$ is given by

$$[\widehat{g(Y_{n+1})} + q(\alpha/2), \widehat{g(Y_{n+1})} + q(1 - \alpha/2)]$$
S&P500 Feb 2018 One Step Ahead Prediction

![Graph showing observed values, Simple NoVaS Values, and GARCH(1,1) Values for February 2018. The graph compares the observed values with predictions from different models, illustrating the performance of each in forecasting stock market volatility.](image-url)
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Estimating the volatility $\mathbb{E}(X_{n+1}^2|\mathcal{F}_n)$

In order to implement our volatility trading strategy we’d ideally like an estimate of $\mathbb{E}(X_{n+1}^2|\mathcal{F}_n)$.

Fortunately, it is straightforward to do so under the case were the NoVaS series $W_{t,a}$ is uncorrelated and independent.

$$\mathbb{E}(X_{n+1}^2|\mathcal{F}_n) = A_n^2 \mathbb{E}\left(\frac{W_{t,a}^2}{1 - a_0 W_{t,a}^2}\right)$$

a natural estimate is therefore

$$\frac{A_n^2}{n-p} \sum_{t=p+1}^{n} \left(\frac{W_{t,a}^2}{1 - a_0 W_{t,a}^2}\right)$$
Relevant Definitions:

- Implied Volatility (IV) - value calculated from an option price
- VIX - popular index which is a measure of the stock market’s expectation volatility implied by S&P500 index options
- \( VIX(t) = IV(t) \) current implied volatility
- VXX - ETN that (imperfectly) tracks VIX index
- RV\( (t+1) \) - is the GARCH or NoVaS predicted realized volatility for next period
- Expect RV\( (t+1) \) to be better predictor of VIX\( (t+1) \) than VIX\( (t) \)

Strategy:

If \( RV(t+1) - VIX(t) > 0 \) then BUY VXX. Vice Versa