Mini-Tutorial:

Sensitivity Experiments

Thomas H Johnson, Institute for Defense Analyses

Greg Hutton, US Air Force, 96 Test Wing
Sensitivity Experiments Best Practices
Outline

1. Introduction to Binary Response Experiments
2. Binary Response Test Design Challenges
3. 1-D Sensitivity Test Designs
4. 2-D Sensitivity Test Designs
5. Case Study: Greg Hutto
Introduction to Binary Response Experiments
Types of Binary Response Experiments

**Pharmaceutical Industry**
- Lethal dose
- Effective dose

**Defense Industry**
- Lethality of munitions
- Survivability of systems
- Armor Characterization
Defense Industry Requirements

“Munition shall have a V50 less than 2,000 ft/ s”

“Armor shall have a v50 greater than 2,300 ft/ s”

Historically, an arithmetic mean estimator is used to calculated V50
## Regression Models

<table>
<thead>
<tr>
<th>Link Name (distribution)</th>
<th>Probability of perforation $\pi(v)$</th>
<th>Velocity where probability of perforation is $\pi$, $\hat{V}_\pi$</th>
<th>Estimated $V_{50}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logit (Logistic)</td>
<td>$\frac{e^{\hat{\beta}_0 + \hat{\beta}_1 v}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 v}}$</td>
<td>$\frac{\ln \left( \frac{\pi}{1 - \pi} \right) - \hat{\beta}_0}{\hat{\beta}_1}$</td>
<td>$-\frac{\hat{\beta}_0}{\hat{\beta}_1}$</td>
</tr>
<tr>
<td>Probit (Normal)</td>
<td>$\Phi \left( \hat{\beta}_0 + \hat{\beta}_1 v \right)$</td>
<td>$\frac{\Phi^{-1}(\pi) - \hat{\beta}_0}{\hat{\beta}_1}$</td>
<td>$-\frac{\hat{\beta}_0}{\hat{\beta}_1}$</td>
</tr>
</tbody>
</table>

![Graph showing probability of penetration vs. velocity with $V_{50}$ and $1\sigma_{true}$ and $2\sigma_{true}$ markers.]
<table>
<thead>
<tr>
<th>Estimators</th>
<th>Arithmetic Mean</th>
<th>Regression Modeling</th>
</tr>
</thead>
<tbody>
<tr>
<td>✅ Simple to calculate.</td>
<td>✗ Methodology and stopping criteria is tailored to this estimator.</td>
<td>✗Requires software to calculate.</td>
</tr>
<tr>
<td>✗ Unpredictable design and calculation.</td>
<td>✗ Does not always use all points in test.</td>
<td>✅ Does not require a customized test design for this analysis.</td>
</tr>
<tr>
<td>✗ Does not always use all points in test.</td>
<td>✗ Restricts analysis to $V_{50}$ only.</td>
<td>✅ Consistent framework.</td>
</tr>
<tr>
<td>✗ Restricts analysis to $V_{50}$ only.</td>
<td>✗ Does not yield reliable confidence intervals.</td>
<td>✅ Considers all test points in analysis.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>✅ Returns other useful estimators (such as $V_{90}$).</td>
</tr>
<tr>
<td></td>
<td></td>
<td>✅ Yields consistent confidence intervals.</td>
</tr>
</tbody>
</table>
Binary Response Test Design Challenges
Binary Response Designs Need Special Consideration

<table>
<thead>
<tr>
<th>Run #</th>
<th>Velocity</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>3000</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>3000</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>3000</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>3000</td>
<td>1</td>
</tr>
</tbody>
</table>

“Evidence of perfect fit” yields bad logistic model fit
Binary Response Designs Need Special Consideration

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<tr>
<th>Run #</th>
<th>Velocity</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1500</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1875</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2625</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>3000</td>
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<tr>
<td>7</td>
<td>3000</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>3000</td>
<td>1</td>
</tr>
</tbody>
</table>

A zone of mixed results provides a good rough estimate of the logistic model curve
Zone of Mixed Results

Example of Separation

Example of Quasi-Separation

Example of No Separation (Zone of Mixed Results Exists)
Test Designs to Achieve a Zone of Mixed Results

Sequential Methods with Initials Designs

Bayesian Methods
1-D Sensitivity Test Designs
# Up and Down

## Details of Implementation

### Rules
- If projectile does penetrate armor, decrease velocity.
- If projectile does not penetrate armor, increase velocity.

### Inputs
- Step size
- Velocity of projectile for trial number one

### Other details
- Fixed step size
- Step size calculated from anticipated standard deviation
- Initial shot typically taken at predicted $V_{50}$

## Background
- Most well-known sequential experimentation procedure, primarily due to its ease of implementation
- Developed by Dixon in 1948

## Advantages
- Useful for estimating $V_{50}$
- The rules are simple and practical to implement

## Disadvantages
- Not good for $V_{10}$
- Constant step size can lead to problems (especially for large steps)

## Example

![Outcome of One "Up and Down" Simulated Test](image)
Langlie Method

Details of Implementation

- START
- Choose initial inputs:
  1) Lower limit (LL)
  2) Upper limit (UL)
- Run Trial
- Record Velocity and Result
- Find next velocity (n+1):
  1) Start at the nth trial
  2) Find the previous nth trial so that there are an equal number of penetrations and no penetrations between nth and nth trial.
- nth trial DID NOT penetrate
- nth trial DID penetrate
- pth trial found
- pth trial NOT found
- Vp > Vn
- Vn+1 = Vn+(Vp-Vn)/2
- Vn+1 = Vp+(Vn-Vp)/2
- Vn+1 = LL+(Vn-LL)/2
- Vn+1 = UL-Vn/2

Background

- Numerous modified versions exist
- Developed in early 60s

Advantages

- Useful for estimating V50
- Has an adaptive step size

Example

Outcome of One Langlie Method Simulated Test

- No penetration
- Penetration
- True V50

Disadvantages

- Not designed for d-optimal curve fitting
- Not as easy to implement as up and down method
K-in-a-row

Details of Implementation

– If projectile does penetrate armor, decrease velocity.

– If projectile does not penetrate armor k times in a row, increase velocity.

– The step size is chosen based on the standard deviation of the predicted response curve.

– Targets Pth quantile of interest where

\[ P = 1 - \left( \frac{1}{2} \right)^{\frac{1}{k}} \]

– Typically, k=2 (P≈0.3) or k=3 (P≈0.2)

Background

– Similar to Up and Down Method

– Not typically used in armor testing

Advantages

– Useful for estimating percentiles away from the median

– Easy to implement (similar to Up and Down method)

Disadvantages

– Less accurate for estimating V50

– A constant step size is susceptible to problems
## Bias Coin Design

### Details of Implementation
- If projectile **does** penetrates armor, decrease velocity.
- If projectile **does not** penetrate armor, $\Gamma$ percent chance to increase velocity, $1 - \Gamma$ percent chance of staying at same velocity

$$\Gamma = \frac{P}{1 - P}$$

- Targets $P$th quantile of interest where (for $P \leq 0.5$)
- Random Bernoulli number generator used to determine if the velocity is increased or remains the same

### Background
- Competitor to the k-in-a-row method
- Not typically used in armor testing

### Advantages
- Useful for estimating percentiles away from the median

### Disadvantages
- A constant step size is susceptible to problems

### Example

Outcome of One "Biased Coin Design" Simulated Test

![Graph showing velocity over run number with different symbols and lines indicating no penetration, penetration, and true $V_{50}$]
# Robbins Monroe

## Details of Implementation
- Start the test at predicted V50.
- Determine the velocity of the next shot using

\[ x_{n+1} = x_n - c(y_n - P)/n \]

where \( c \) is an arbitrary constant, \( y_n \) is the outcome of the \( n \)th trial \((0,1)\), \( P \) is the desired percentile of interest and \( n \) is the number of trials. \( C \) is optimal when:

\[ c_{opt} = \left[F'(V_p)\right]^{-1} \]

where \( F \) is the response curve and \( V_p \) is the velocity at the \( p \)th percentile.
- Step size decreases as \( n \) increases.

## Background
- Developed in 1951
- Numerous variants of this method exist
- Used in armor testing by ARL

## Advantages
- Useful for estimating all quantiles
- A dynamic step size has advantages

## Disadvantages
- Justification for values of \( c \) may seem arbitrary, poor choices of \( c \) can lead to inaccurate results
- Poor guess of the velocity of the first shot can lead to slow convergence and/or convergence to an inaccurate result
Neyer’s Method

Details of Implementation

- **Phase 1:** Generate penetrations and non-penetrations. Bounds the problem. Determines if initial gate is too far left, right or narrow.

- **Phase 2:** Break separation. Provides unique MLE coefficient estimates and an indication that velocity is in the ballpark of V50.

- **Phase 3:** Refine model coefficients. Use D-optimality criterion to dictate ensuing shots.

Background

- Developed by Neyer in 1989
- First to propose a systemic method for generating a good initial design

Advantages

- Initial design is useful for quickly estimating model coefficients
- Robust to misspecification of input parameters

Disadvantages

- Requires coding and capability to do maximum likelihood estimation

Example

![Graph showing run number against velocity](chart.png)
3Pod

Details of Implementation

- **Phase 1:** Generate penetrations and non-penetrations. Similar to rules to Neyer's method. Uses slightly different logic and different step sizes.

- **Phase 2:** Break separation. Relies more heavily on conditional logic than Neyer's method.

- **Phase 3:** Refine model coefficients (and estimate of Vp). A portion of resources is devoted to D-optimal algorithm and the other portion is used for placing shots near Vp (velocity percentile value of interest) using Robbins Monroe Joseph method.

Background

- Developed by Wu in 2013

- Similar to Neyer's Method

Advantages

- Similar to Neyer's Method, good initial design

Disadvantages

- Requires maximum likelihood estimation

- More complex than Neyer's method

Initial Design

Example
Example of 3Pod Results

- Example of 30 Shots for 3-Phase Approach (3Pod)
Simulation Comparison
Simulation Factors and Responses

Response

1. V50 Error
2. V10 Error

Calculated as the difference between the “true” V50 (or V10) and the V50 (or V10) estimated with the simulated runs

Factors

1. Estimator (Probit-MLE, Arithmetic Mean)
2. Method (Up Down Method, 3Pod, Langlie, etc..).
3. Stopping criteria (“3&3”, break separation)
4. $\mu_{\text{guess}} \ (\mu_{\text{true}} - 2\sigma_{\text{true}}, \mu_{\text{true}}, \mu_{\text{true}} + 2\sigma_{\text{true}})$
5. $\sigma_{\text{guess}} \ (1/3\sigma_{\text{true}}, 1/2\sigma_{\text{true}}, 2\sigma_{\text{true}}, 3\sigma_{\text{true}})$
Runs for Stopping Criteria

Legend

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

Intercept = 10.5
Units in Runs
Recommend 3Pod or Neyer Method

Provides entire logistic model curve fit

Robust estimate for V50 and V10

D-optimal approach
2-D Sensitivity Test Designs
Sensitivity Test Designs with Two Factors

- Response is binary
- No interaction terms
- Two continuous factors
- Primary factor is velocity
Practical Multi-Factor Sequential Design

Practical multi-factor sequential designs:

1. Brute force use of single factor sequential designs in multi-dimensional design space
   - Intuitive design and easy to implement

<table>
<thead>
<tr>
<th>Armor Plate Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>40</td>
</tr>
</tbody>
</table>

   Each 3Pod uses velocity as factor

2. Propose a modified sequential design to search D-optimal points across multiple factors

3. Bayesian Sequential Design by Dror and Steinberg (2008)
   - Established, practical sequential design for multiple factors
   - Uses prior information about armor performance to search for D optimal points
Role of D-Optimality in Sequential Designs

1. 3Pod, Neyer, and DS focus on D-optimality
   - D-optimality is a widely accepted design criteria
   - D-optimality is a widely accepted design criteria
   - minimizes the confidence ellipsoid on coefficients

<table>
<thead>
<tr>
<th>Calculation of D-optimality</th>
</tr>
</thead>
<tbody>
<tr>
<td>The D-optimality designs criterion for fitting a logistic model maximizes the determinant of the information matrix among all competing designs ($\Omega$).</td>
</tr>
<tr>
<td>$Max_{\Omega}</td>
</tr>
<tr>
<td>The fisher information matrix is $I(\beta) =</td>
</tr>
<tr>
<td>$X$ is the $m \times p$ model matrix.</td>
</tr>
<tr>
<td>$\Sigma$ is the variance-covariance matrix for the $m \times 1$ vector of binomial variables, each being $\sum_j y_{ij}$, the sum of events at the $i^{th}$ design point.</td>
</tr>
<tr>
<td>$\Sigma$ is an $m \times m$ diagonal matrix with the $i^{th}$ diagonal element being $n_i P_i (1 - P_i)$.</td>
</tr>
</tbody>
</table>

2. Multi-factor sequential designs are compared in terms of D-efficiency
   - The D-efficiency of a candidate design is calculate as
     $$D\text{-efficiency} = \frac{|X'\Sigma X|_{\text{Candidate Design}}}{|X'\Sigma X|_{D\text{-optimal Design}}}$$
D-Optimal Design with 1 Factor

- The single factor logistic regression model, \( \ln \left( \frac{p}{1-p} \right) = \beta_0 + \beta_1 x_1 \), can be reparametrized in terms of location-scale parameters as 
  \[ \ln \left( \frac{p}{1-p} \right) = \frac{x_1 - \mu}{\sigma} \] 
  where \( \mu = -\frac{\beta_0}{\beta_1} \) and \( \sigma = \frac{1}{\beta_1} \)
  - \( \mu \) is \( V_{50} \) and \( \sigma \) is the amount of slope in the curve
  - Figure 1 illustrates various logistic model curve fits

- Abdelbasit and Plackett derived the determinant of the fisher information matrix: 
  \[ |I| = \frac{n^2 w_1 w_2}{\sigma^2} (x_1 - x_2)^2 \] 
  where \( w_i = p_i (1 - p_i) \) and \( x_i = \ln \left( \frac{p_i}{1-p_i} \right) \sigma + \mu \), for \( i = 1, 2 \).
  - Assumes a 2 point design where \( p_1 \) is symmetrical to \( p_2 \), and \( n \) is the number of runs at each point.

- Abdelbasit and Plackett showed the solution is the \( \delta \) that maximizes \( |I| \), where \( p_1 = \delta \) and \( p_2 = 1 - \delta \)

- The D-optimal solution (Figure 2) is \( p_1 = 0.176 \) and \( p_2 = 0.824 \)
  - Meaning that half of the shots are fired at \( V_{17.6} \) and the other half are fired at \( V_{82.4} \)

D-Optimal 1-Factor Design
Specifies Shots at \( V_{17.6} \) and \( V_{82.4} \)
D-Optimal Design with 2 Factors

- The dual factor logistic regression model can be expressed as
  \[ \ln \left( \frac{p}{1-p} \right) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \quad \text{or} \quad \ln \left( \frac{p}{1-p} \right) = u \]

  - obliquity angle
  - Impact velocity

- Sitter and Torsney (1995), and Jia and Meyers (2001) developed a 4 point D-optimal design
  - 2 points are placed at the lower obliquity angle setting \((\theta_L)\) and 2 points are placed at the upper setting \((\theta_U)\)
  - Results in a location-scale parametrization:
    \[
    \mu_L = -\frac{\beta_0}{\beta_2} - \frac{\beta_1 \theta_L}{\beta_2}, \quad \mu_U = -\frac{\beta_0}{\beta_2} - \frac{\beta_1 \theta_U}{\beta_2}, \quad \sigma = 1/\beta_2
    \]
  - 4 point D-optimal design:

<table>
<thead>
<tr>
<th>Location</th>
<th>Weight</th>
<th>Weight</th>
<th>Weight</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-u-\beta_0, 0))</td>
<td>(w)</td>
<td>(w)</td>
<td>(\frac{1}{2} - w)</td>
<td>(\frac{1}{2} - w)</td>
</tr>
</tbody>
</table>

- where \(u\) and \(w\) are numerically solved for using equations:

  \[
  u^2 (3 + 3e^u + 2u - 2ue^u) + \beta_0^2 (1 + e^u + 2u - 2ue^u) + \sqrt{u^4 + 14\beta_0^2 u^2 + \beta_0^4 (1 + e^u + u - ue^u)} = 0
  \]

  \[
  w = \left( -u^2 + 6u\beta_0 - \beta_0^2 + \sqrt{u^2 + 14\beta_0 u + \beta_0^2} \right) / 24\beta_0 u
  \]

Figure 3 – Example Model Fit

\[
\mu_L = 1392, \quad \mu_U = 1932, \quad \sigma = 120
\]

Figure 4 – Numerical Solution

\[\delta = 0.227\]

\[p_1 = 0.227, \quad p_2 = 0.773, \quad w = 0.225\]

D-Optimal 2-Factor Design

Specifies Shots at \(V_{22.7}\) and \(V_{77.3}\)
Proposed strategy to implement 3Pod in a two factor space:

1. Conduct initial design with velocity as the factor at zero degree obliquity angle.
2. Conduct an additional initial design with velocity as the factor at 45 degree obliquity angle.
3. Select next point by searching velocity settings that maximize the determinant of the Fisher information matrix.
   » Constrain search to velocities at 0 and 45 degree obliquity since we know that is where the 4 point locally d-optimal points is.
Theoretical Improvement

- We can calculate the improvement gained by expanding the search to additional factors, since we can analytically solve for the D-optimal design.

- Three 30 run designs considered:

<table>
<thead>
<tr>
<th></th>
<th>D-optimal Design</th>
<th>Design 1</th>
<th>Design 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obliquity Angle</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 deg</td>
<td>15 runs (7 runs @ V22.7, 8 runs @ V77.3)</td>
<td>15 runs (7 runs @ V17.6, 8 runs @ V82.4)</td>
<td>10 runs (5 runs @ V17.6, 5 runs @ V82.4)</td>
</tr>
<tr>
<td>45 deg</td>
<td>15 runs</td>
<td>15 runs (7 runs @ V22.7, 8 runs @ V77.3)</td>
<td>10 runs (5 runs @ V17.6, 5 runs @ V82.4)</td>
</tr>
</tbody>
</table>

|                  | | |
| $|X'\Sigma X|$: | 1.5E9 | 1.4E9 | 1.0E9 |
| D-efficiency:     | 1.0 | .896 | .600 |

- These designs are infeasible in practice because we don’t have prior knowledge of coefficients.
  - We must run simulations that include an initial design to determine practical improvement.
Simulation Setup

12 run factorial experiment

- **Response:** D-efficiency
- **Factors:**
  - **Methods**
    - 3Pod w/ 1-factor D-optimal search (3Pod-1D)
    - 3Pod w/ 2-factor D-optimal search (3Pod-2D)
    - Dror-Steinberg Method (D-S)
    - Langlie Method
  - **Sample Sizes**
    - 60, 120

**Method Input parameters**

- D-S requires prior uniform distributions on model coefficients
- 3Pod requires specification of $\sigma_G$ and $\mu_G$ at 0 and 45 degree obliquity angle
- To make a fair comparison, inputs for each method need to be equivalent

**Constant inputs into simulation**

- Assumed true logit model: $b_T = [b_{0T} \ b_{1T} \ b_{2T}] = [-11.6 \ -0.1 \ 0.0083]$
- Number of simulations per factorial trial: 1,000
Simulation Results

3Pod with 1 Factor D optimal Search
- Runs at 0 deg
- Runs at 45 deg
- True D-optimal point

3Pod with 2 Factor D optimal Search
- Runs at 0 deg
- Runs at 45 deg
- True D-optimal point

Dror-Steinberg
- Runs at 0 deg
- Runs at 45 deg
- True D-optimal point

Langlie Method
- Runs at 0 deg
- Runs at 45 deg
- True D-optimal point
Simulation Results

60 Runs

- 3Pod 1D (median=0.66)
- 3Pod 2D (median=0.66)
- D-S (median=0.70)
- D-S (median=0.70)
- Langlie (median=0.64)

120 Runs

- 3Pod 1D (median=0.76)
- 3Pod 2D (median=0.81)
- D-S (median=0.82)
- D-S (median=0.82)
- Langlie (median=0.64)
Recommendations

D-S and 3Pod2D perform best

Further investigation into the practicality, and robustness of D-S is needed
Case Study:
Greg Hutto
**Coded Coeffs (skinny)**

<table>
<thead>
<tr>
<th></th>
<th>Lower Range</th>
<th>Upper Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>B0</td>
<td>-2.70</td>
<td>-0.68</td>
</tr>
<tr>
<td>Bang</td>
<td>-4.5</td>
<td>-0.50</td>
</tr>
<tr>
<td>Bvel</td>
<td>1.67</td>
<td>16.67</td>
</tr>
</tbody>
</table>

**Coded Coeffs (fat)**

<table>
<thead>
<tr>
<th></th>
<th>Lower Range</th>
<th>Upper Range</th>
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</thead>
<tbody>
<tr>
<td>B0</td>
<td>-0.17</td>
<td>5.63</td>
</tr>
<tr>
<td>Bang</td>
<td>-4.5</td>
<td>-0.50</td>
</tr>
<tr>
<td>Bvel</td>
<td>3.34</td>
<td>33.34</td>
</tr>
</tbody>
</table>
Simulation Results

3Pod w/ 1-Factor D-optimal Search

3Pod w/ 2-Factor D-optimal Search

Dror-Steinberg Method

Langlie Method
One-Factor 3Pod D-optimal Search

- Second stage of 3-Pod searches the one-factor space for the candidate point that maximizes the determinant of the fisher information matrix
  - Uses the coefficients estimated from the initial design
  - After each additional run, coefficients are re-estimated and a new D-optimal location is found
Simulation Results

D-efficiency

Emperical Cumulative Distribution

- 60 Runs
  - 3Pod 1D (median=0.65)
  - 3Pod 2D (median=0.67)
  - D-S (median=0.87)
  - Langlie (median=0.65)

- 80 Runs
  - 3Pod 1D (median=0.7)
  - 3Pod 2D (median=0.74)
  - D-S (median=0.89)
  - Langlie (median=0.65)

- 100 Runs
  - 3Pod 1D (median=0.73)
  - 3Pod 2D (median=0.78)
  - D-S (median=0.9)
  - Langlie (median=0.65)

- 120 Runs
  - 3Pod 1D (median=0.75)
  - 3Pod 2D (median=0.81)
  - D-S (median=0.91)
  - Langlie (median=0.65)
Expanding 3Pod’s D-Optimal Search to Two Factors

• 3Pod places a constraint on the D-optimal search to prevent inaccurate coefficient estimates from generating an extreme D-optimal solution
  • ensures location parameter that is between the smallest x and largest x
  \[
  \hat{\mu}_k = \max \left( \min(x), \min(\mu_k, \max(x)) \right)
  \]
  • ensures scale parameter that is smaller than the range between the largest and smallest x
  \[
  \hat{\sigma}_k = \min(\sigma_k, \max(x) - \min(x))
  \]

• Following the intent of 3Pod’s one-factor D-optimal search, apply the following constraints on the coefficient estimates

First, calculate transform coefficients into location scale parameters:

\[
\mu_0 = -\frac{b_0}{b_2} - \frac{b_1 \theta_0}{b_2}, \quad \mu_{45} = -\frac{b_0}{b_2} - \frac{b_1 \theta_{45}}{b_2}
\]

Next, apply constraints on location parameters at low and high settings of obliquity angle:

\[
\hat{\mu}_0 = \max \left( \min(x_0), \min(\mu_0, \max(x_0)) \right), \quad \hat{\mu}_{45} = \max \left( \min(x_{45}), \min(\mu_{45}, \max(x_{45})) \right)
\]

Then, apply constraint on scale parameter:

\[
\hat{\sigma}_k = \min(\sigma_k, \max(x) - \min(x))
\]

Finally, transform location-scale parameters back into logistic model coefficients to be used in D-optimal search:

\[
\hat{b}_2 = 1/\hat{\sigma}_k, \quad \hat{b}_0 = \frac{-\hat{b}_2 \hat{\mu}_0 + \hat{b}_2 \hat{\mu}_{45} \theta_0 / \theta_{45}}{1 - \theta_0 / \theta_{45}}, \quad \hat{b}_1 = \frac{-\hat{b}_0 / \hat{b}_2 - \hat{\mu}_{45}}{\theta_{45} / \hat{b}_2}
\]
D-S Method Input Parameters

\[
\text{beta\_prior\_coded} = \begin{bmatrix}
-2 & 8 \\
-7 & 2 \\
1 & 40 \\
\end{bmatrix};
\]

\[
x2\text{lims} = [0 \ 4000];
\]
Binary Response Designs Need Special Consideration

<table>
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<tr>
<th>Run #</th>
<th>Velocity</th>
<th>Response</th>
</tr>
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<td>1</td>
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<td>0</td>
</tr>
<tr>
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<tr>
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<tr>
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<tr>
<td>8</td>
<td>4000</td>
<td>1</td>
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</tbody>
</table>

Firth bias adjustment can help, but can suffer from bias error
Primary limitation of D-optimal design is that it requires prior knowledge of the model fit
- If we knew the coefficients of the logistic model prior to testing, then we wouldn’t need to test
- For instance, the D-optimal solution for the one-variable logistic model dictates that shots should be placed at V17.6 and V82.4, but going into test we don’t know which velocities correspond to those solutions

There are two ways to mitigate this lack of prior knowledge
1. Bayesian Approach
   - Use prior knowledge of curve to estimate where D optimal locations are
2. Initial Design
   - Used to obtain a zone of mixed results gain quick estimate of model coefficients
   - First proposed by Neyer (1984), later refined by Wu in 3Pod (2014)
   - Generally requires 8-16 shots
1. Specify uniform prior distribution on logit model coefficients
2. Find median of each coefficient distribution, and record those coefficients
3. Build 8 run “starter” D-optimal design based on coefficients from Step 2
4. Loop over each run in “starter” design
   1. Build “candidate” design by combing a run from the starter design with the final design
   2. Search coefficient distributions for the coefficient set that is D-optimal for the candidate design
   3. Record D-optimality for that coefficient set
5. Record the “final” run from the “starter” design that maximized D-optimality
6. Update coefficient posterior distribution
7. Repeat steps 2-7 for as many runs as needed in final design