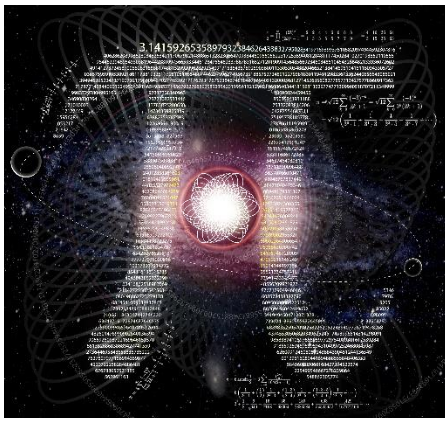
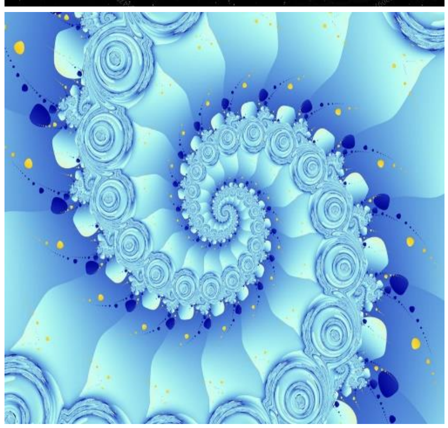
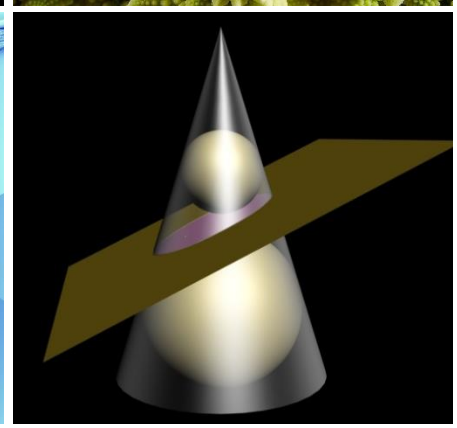
Washington State
Bridge to College Mathematics Course



Adapted from Math Ready
A Southern Regional Education Board Transition Course

Unit 1. Algebraic Expressions
Overview

Purpose

This unit was designed to solidify student conception of expressions while providing the students with an opportunity to have success early in the course. The recurring theme integrated in this unit focuses on engaging students using and expanding the concepts found within purposefully chosen activities. Through guided lessons, students will manipulate, create, and analyze algebraic expressions and look at the idea of whether different sets of numbers are closed under certain operations. The writing team selected content familiar to the students in this unit to build student confidence and acclimate students to the course's intended approach to instruction.

Essential Questions:

When is estimation appropriate?

How can you extend the properties of operations on numerical expressions to algebraic expressions?

How can you apply the properties of operations to generate equivalent expressions? How can you determine when two algebraic expressions are equivalent or not?

How can rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related?

How can you use the structure of an expression to identify ways to rewrite it?

COMMON CORE STATE STANDARDS

Quantities

Reason quantitatively and use units to solve problems.

- N-Q.1: Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

Seeing Structure in Expressions

Interpret the structure of expressions.

- A.SSE.1: Interpret expressions that represent a quantity in terms of its context.
- A.SSE.2: Use the structure of an expression to identify ways to rewrite it. Write expressions in equivalent forms to solve problems.
- A.SSE.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

Interpreting Functions

Analyze functions using different representations.

- F-IF.8: Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Prior Scaffolding Knowledge / Skills:

Expressions and Equations

Apply and extend previous understandings of arithmetic to algebraic expressions.

- 6.EE.1: Write and evaluate numerical expressions involving whole-number exponents.
- 6.EE.2: Write, read, and evaluate expressions in which letters stand for numbers.
- 6.EE.3: Apply the properties of operations to generate equivalent expressions.
- 6.EE.4: Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them).

Use properties of operations to generate equivalent expressions.

- 7.EE.1: Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.
- 7.EE.2: Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related.

The Number System

Apply and extend previous understandings of operations with fractions.

- 7.NS.1: Apply and extend previous understandings of addition and subtraction to add and sub- tract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.
- 7.NS.2: Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
- 7.NS.3: Solve real-world and mathematical problems involving the four operations with rational numbers.

Lesson Big Idea	Lesson Details	Content Standards	Standards for Mathematical Practice
Lesson 1: Numbers and Estimation	Students will be introduced to the course using an estimation activity that will be used to develop conception of numbers and reinforce numeral operation fluency. It is also an entry activity into the course showcasing the explicit incorporation of math practices including problem solving, reasoning and modeling using mathematics.	7.NS.1 7.NS.2 7.NS.3 N-Q.1	MP 2 MP 3 MP 4
Lesson 2: Interpreting Expressions	Students will begin this lesson by engaging in a “magic math” activity. This lesson will give students opportunities to explore and determine their understanding of expressions. They will be asked to consider, create and understand verbal representations of numbers and operations to symbolic representations using expressions. They will examine how symbolic manipulation of expressions affects values in real circumstances.	7.NS.3 A-SSE.1	MP 2 MP 6 MP 7 MP 8
Lesson 3: Reading and Evaluating	This lesson will give students an opportunity to fortify their understanding of interpreting and modifying expressions by analyzing symbolic manipulation of various expressions.	7.EE.2 A-SSE.1 A-SSE.2	MP 6 MP 7
Lesson 4: Constructing Equivalent Expressions	Students will begin this lesson by engaging in a real-life problem that encompasses some basic geometric concepts along with expression manipulation. This lesson will give students an– opportunity to fortify their understanding of writing expressions.	A-SSE.1 A-SSE.2 A-SSE.3	MP 2 MP 3 MP 4 MP 7
Lesson 5: Constructing Equivalent Expressions	Students will begin this lesson by engaging in a task on developing expressions for a particular geometric pattern. This lesson will strengthen the ability of students to compare expressions presented in different forms and determine equivalency.	A-SSE.3 F-IF.8	MP 1 MP 3 MP 7 MP 8
Lesson 6: Distributive Law	Students will begin this lesson with an engaging activity that will lead to an understanding of rewriting and interpreting expressions using the distributive property.	A-SSE.1 A-SSE.2	MP 3 MP 4
Lesson 7: Formative Assessment Lesson	This lesson is intended to help teachers assess how well students are able to translate between words, symbols, tables, and area representations of algebraic expressions. It is designed to identify and support students who have difficulty in these concepts. (Shell Center Formative Assessment Lesson: Interpreting Algebraic Expressions)	A-SSE.1 A-SSE.2	MP 2 MP 7

SREB Readiness Courses | *Transitioning to college and careers*

Math Ready: Unit 1

Algebraic Expressions
Lesson 1 of 7

Description:

Students will be introduced to the course using an estimation activity to develop conception of numbers and reinforce numeral operation fluency. It is also an entry activity into the course showcasing the explicit incorporation of math practices including problem solving, reasoning and modeling using mathematics. Students will practice estimating and determining a reasonable solution, prior to solving. They will breakdown a complicated task into smaller parts. The objective in this first lesson is for students to be efficient and reasonable estimators applying mental math strategies.

Common Core State Standard Addressed:

- 7.NS.1: Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.
- 7.NS.2: Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
- 7.NS.3: Solve real-world and mathematical problems involving the four operations with rational numbers.
- N-Q.1: Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

Mathematical Practice Standard(s) Emphasized:

- MP 2: Reason abstractly and quantitatively.
- MP 3: Construct viable arguments and critique the reasoning of others.
- MP 4: Model with mathematics.

Sequence of Instruction	Activities Checklist

Engage

Play initial clip of Bucky the Badger. This content is from Dan Meyer's Three Act Math, http://mrmeyer.com/threeacts/buckythebadger/. Following the initial video clip (Act 1), ask students to guess how many push-ups Bucky had to perform in the course of the game. If a student responds by saying 83, then explain again how the number of pushups is calculated. After asking several students for approximations, split students up into groups of three or so students to further explore this question.

Explore

MATH 3 PRACTICE

Ask the groups to construct viable arguments and critique the reasoning of others as they address the following questions:

INCLUDED IN THE STUDENT MANUAL

Task #1: Bucky the Badger

- Restate the Bucky the Badger problem in your own words.
- About how many total push-ups do you think Bucky did during the game?
- Write down a number that you know is too high.
- Write down a number that you know is too low.
- What further information would you need to know in order to determine the exact number of total push-ups Bucky did in the course of the game?
- If you're Bucky, would you rather your team score their field goals at the start of the game or the end?
- What are some numbers of pushups that Bucky will never do in any game?

The key here is that the total depends on the order in which the touchdowns and field goals were scored, not just how many touchdowns and field goals were scored.

Explanation

Play clip that explains how many push-ups in total Bucky did (whether it is Bucky or more than one person is still a mystery!)

Address any questions or issues that may have come up as you observed the groups discuss the questions above.

Teacher's Note – A blog discussing the Bucky the Badger problem and the incorporation of problem solving and communication can be found at http://blog.mrmeyer.com/2012/3acts-bucky-the-badger/. This may be used by the instructor to reflect on his/her own understandings and beliefs surrounding the Standards of Mathematical Practice.

Practice Together in Small Groups/Individually

MATH 2&3 PRACTICE

No calculator should be used for Tasks 1 and 2. It is important to stress that in Task #2, students are asked to find approximate values. If students find themselves wanting or needing to use a calculator, give them a hand in reasoning abstractly and quantitatively through useful approximation strategies that help find a good estimate while being easy to compute

INCLUDED IN THE STUDENT MANUAL

Task #2: Reasoning about Multiplication and Division and Place Value

Use the fact that 13×17=221 to find the following:

a. 13 • 1.7	e. 2210 ÷ 1
b. 130 • 17	f. 22100 ÷ 17
c. 13 • 1700	g. 221 ÷ 1.3
d. 1.3 • 1.7	

(http://illustrativemathematics.org/illustrations/272)

ommentary *for the Teacher:*

This task is NOT an example of a task asking students to compute using the standard algorithms for multiplication and division because most people know what those kinds of problems look like. Instead, this task shows what kinds of reasoning and estimation strategies students need to develop in order to support their algorithmic computations.

Possible Solutions:

All these solutions use the associative and commutative properties of multiplication (explicitly or implicitly).

a. 13•1.7=13(17•0.1)=(13•17)(0.1), so the product is one-tenth the product of 13 and 17. In other words:

13•1.7=22.1

b. Since one of the factors is ten times one of the factors in 13•17, the product will be ten times as large as well:

130•17=2210

c. 13•1700=13(17•100)=(13•17)(100), so

13•1700=22100

d. Since each of the factors is one tenth the corresponding factor in 13•17, the product will be one one-hundredth as large:

1.3•1.7=2.21

e. 2210÷13=? is equivalent to 13• ? =2210. Since the product is ten times as big and one of the factors is the same, the other factor must be ten times as big. So:

2210÷13=170

f. As in the previous problem, the product is 100 times as big, and since one factor is the same, the other factor must be 100 times as big:

22100÷17=1300

g. 221÷1.3= ? is equivalent to 1.3• ? =221. Since the product is the same size and one of the factors is one-tenth the size, the other factor must be ten times as big. So:

221÷1.3=170

NCLUDED IN THE STUDENT MANUAL

Task #3: Felicia's Drive

Felicia's Drive accessed on 8/5/2014, is licensed by Illustrative Mathematics under CC BY NC SA 4.0

As Felicia gets on the freeway to drive to her cousin's house, she notices that she is a little low on gas. There is a gas station at the exit she normally takes but she wonders if she will have to get gas before then. She normally sets her cruise control at the speed limit of 70mph and the freeway portion of the drive takes about an hour and 15 minutes. Her car gets about 30 miles per gallon on the freeway, and gas costs $3.50 per gallon.

a. Describe an estimate that Felicia might do in her head while driving to decide how many gallons of gas she needs to make it to the gas station at the other end.

b. Assuming she makes it, how much does Felicia spend per mile on the freeway?

(https://www.illustrativemathematics.org/content-standards/tasks/80)

Commentary *for the Teacher:*

This task provides students the opportunity to make use of units to find the gas need. It also requires them to make some sensible approximations (e.g., 2.92 gallons is not a good answer to part (a)) and to recognize that Felicia's situation requires her to round up. Various answers to (a) are possible, depending on how much students think is a safe amount for Felicia to have left in the tank when she arrives at the gas station. The key point is for them to explain their choices. This task provides an opportunity for students to practice reasoning abstractly and quantitatively, and constructing viable arguments and critiquing the reasoning of others.

Possible Solution:

a. To estimate the amount of gas she needs, Felicia calculates the distance traveled at 70 mph for 1.25 hours. She might calculate:

$$70 \cdot 1.25 = 70 + 0.25 \cdot 70 = 70 + 17.5 = 87.5 \text{ miles}$$

Since 1 gallon of gas will take her 30 miles, 3 gallons of gas will take her 90 miles, a little more than she needs. So she might figure that 3 gallons is enough.

Or, since she is driving, she might not feel like distracting herself by calculating 0.25x70 mentally, so she might replace 70 with 80, figuring that that will give her a larger distance than she needs. She calculates:

$$90 \cdot 1.25 = 80 + 14 \cdot 80 = 100$$

So at 30 miles per gallon, 313 gallons will get her further than she needs to go and should be enough to get her to the gas station.

b. Since Felicia pays $3.50 for one gallon of gas, and one gallon of gas takes her 30 miles, it costs her $3.50 to travel 30 miles. Therefore, $3.50/30 miles = $0.121, meaning it costs Felicia 12 cents to travel each mile on the freeway.

Evaluate Understanding

Ask some students to share their strategies for solving some of the questions above. Be sure to emphasize good (and bad) approximation strategies, paying attention to units when appropriate, and reviewing the properties of operations when working with numerical expressions. Do NOT mention anything about PEMDAS. Students should use any (correct) order of operations, and the order of those operations should be a result of useful strategies. For example:

$$1.3 \bullet 1.7 = (13)\left(\frac{1}{10}\right)(17)\left(\frac{1}{10}\right) = \frac{1}{100}(13 \bullet 17) = \left(\frac{1}{100}\right)(221) = 2.21$$

Here the strategy is using commutative and associative properties of multiplication rather than inventing a gimmicky trick with decimals that works in this one particular case. Reviewing and deepening the depth of understanding of these properties is crucial before moving on to working with algebraic expressions.

Closing Activity

Still working in groups, ask the students to model with mathematics the following situation: Let *x* denote the number of touchdowns Wisconsin scored in a game. Assuming the Wisconsin football team only scores touchdowns, write an algebraic expression to represent the total number of pushups Bucky must do in a game in which *x* touchdowns are scored.

Independent Practice

None available

Resources/Instructional Materials Needed:

Computer/Projector

Video Clip: Three Act Math: Bucky the Badger — http://mrmeyer.com/threeacts/buckythebadger/

Notes

Algebraic Expressions
Lesson 2 of 7

Description:

Students will begin this lesson by engaging in a "magic math" activity. This lesson will give students opportunities to explore and determine their understanding of expressions. They will be asked to consider, create and understand verbal representations of numbers and operations to symbolic representations using expressions. They will examine how symbolic manipulation of expressions affects values in real circumstances.

Common Core State Standards Addressed:

- 7.NS.3: Solve real-world and mathematical problems involving the four operations with rational numbers.
- A-SSE.1: Interpret expressions that represent a quantity in terms of its context.

Mathematical Practice Standard(s) Emphasized:

- MP 2: Reason abstractly and quantitatively.
- MP 3: Construct viable arguments and critique the reasoning of others.
- MP 6: Attend to precision.
- MP 7: Look for and make use of structure.
- MP 8: Look for and express regularity in repeated reasoning.

Sequence of Instruction	Activities Checklist

Engage

Magic Math: Number Guess Introduction

- Have each student choose a number between one and 20 and write it down at the top of the Magic Math template provided in the student manual.
- Students should complete the following steps on their paper under their original number and write the instructions given in the second column:
  - Double your original number.
  - Add 6.
  - Divide by 2.
  - Subtract the original number from the new number.
  - Fold the paper once so your work/answer cannot be seen.

Numbers and Operations

Magic Math

- Tell the students that you are going to come around and write your “guess” for their answer on the outside of the paper.
- Go around the room and write a 3 on the outside of their paper and turn your writing face down on the desk.
- Once you are finished writing your “guesses” on all of the papers ask them to look at your guess and if it matches their answer then have them raise their hand. (Hopefully all students will have their hand up at this point; however, if some do not just make note of it and address their calculation mistakes during the practice session).
- Make a “big deal” about how they must have all chosen the same original number.
- Ask two different students what numbers they chose. When you get two different original numbers, look puzzled, as if to say “How could that be?”
- Ask the class as a whole, “How is it that <John> started with <4> and <Jane> started with <11>? They both performed the same operations on the two different numbers, but ended up with the same answer.
- Tell the students that you are going to give some time to discuss it.

Explore

MATH 2&7 PRACTICE

Magic Math: Number Guess Exploration

- Have students pair up (one group of three if an odd number of students are present) with someone that chose a different original number.
- Have students discuss and write down on a sheet of paper their pair’s understanding of why this process always results in a “3.” Ask students to represent their thinking in as many ways as possible (e.g. verbal, symbolic, algebraic, pictorial). Have students look for and make use of structure as they create an expression representing all of the steps in the magic math number trick.
- Announce that if anyone did not get “3” as their answer (from a miscalculation), he/she should discuss the steps taken to arrive at the different solution with his/her partner. (Listen to the conversation surrounding the students’ process and be prepared to ask guiding questions as necessary to help students find their errors in the event they are unable to locate the miscalculation.)
- Give time for student pairs to both quantitatively and abstractly reason through the problem and provide sound justification for their decisions.
- Walk around the room observing the explanations/models. Pay attention to the different correct approaches. Make note of any incorrect

Explanation

MATH 6 PRACTICE

Magic Math: Number Guess Explanation

- Ask one to three groups to communicate methods and solutions precisely to others through a report of their processes. Try to select groups that have varying methods.
  - There is no need for the same exact process to be explained multiple times so choose pairs having some variations to share.
  - If you had one group that has an incorrect process you might sandwich them

between two correct groups. This way the students can solidify their thoughts with the first group. The second group will probably see their error and address it when presenting (but this provides a great opportunity for a group discussion about the reason for their miscalculation). Then the third group will provide reinforcement of the procedure.

- Leave students in their current groups and facilitate a whole group discussion about the process to include verbal, algebraic, and modeling representations.

- The following table shows an example of connections between various representations for each step of the Number Guess activity. You could also show additional columns with other representations or a number

Magic Math: Number Guess (Representation Connections)

When I asked you to choose a number between one and 20, I had no real idea what you would choose. And in math, if we know a number exists but we don't know what particular number it is, then we use a variable or a symbol to represent that number. So let's go through this problem with ★ and x representing the chosen number.

Pictorial	Verbal	Algebraic
★	Chosen Number	x
★ ★	Double it	$x \bullet 2 = 2x$
★ ● ● ● ★ ● ● ●	Add 6 to it	$2x + 6$
★ ● ● ● ~~★ ● ● ●~~	Divide by 2	$\frac{2x+6}{2} = x + 3$
~~★~~ ● ● ● ~~★~~	Subtract your original number	$x + 3 - x = 3$
● ● ●	Leaves 3	3

Practice Together in Small Groups

MATH 6&8 PRACTICE

Magic Math: Birthday Trick

- Have students work in pairs to complete the Math Magic: Birthday Trick from the Student Manual, taking turns with each student's own information.

INCLUDED IN THE STUDENT M

Do you believe that I can figure out your birthday by using simple math?

Get a calculator and ask your classmate to try the following. Your classmate must press equal (or enter) between every step.

a. Enter the month of his/her birth into the calculator. (Ex: enter 5 for May)
b. Multiply that number by 7.
c. Subtract 1 from that result.
d. Multiply that result by 13.
e. Add the day of birth. (Ex: For June 14th add 14)
f. Add 3.
g. Multiply by 11.
h. Subtract the month of birth.
i. Subtract the day of birth.
j. Divide by 10.
k. Add 11.
l. Divide by 100.

- Have the students look for and make use of repeated reasoning to model the process algebraically.
- Make sure that each of the members of the group can communicate the process that his/her pair used precisely.
- Have one student from each pair rotate to a different group.
- Have each student in the newly formed pairs explain to one another his/her model and the reasoning for each step.

Evaluate Understanding

Magic Math: Birthday Trick

- Monitor the different explanations in the groups and ask guiding questions aimed at correcting any misconceptions that may exist.

Closing Activity

Introduce Independent Practice

- In a whole-group discussion, introduce students to the independent practice where they are asked to create their own “magic trick.” The trick should include at least five steps and should be represented through both verbal and algebraic representations. This is to be competed without the use of technology.
- Allow time for students to ask clarifying questions and summarize the independent practice task.

Independent Practice

INCLUDED IN THE STUDENT MANUEL

Chocolate Math

1. Determine how many times per week you eat or want to eat chocolate. It must be a number between 1 and 10, including 1 and 10.
2. Multiply that number by 2.
3. Add 5 to the previous result.
4. Multiply that by 50.
5. Add the current year.
6. Subtract 250 if you've had a birthday this year. If you haven't had a birthday this year, subtract 251.
7. Subtract your birth year.
8. You'll end up with a 3 or 4 digit number.

The last two digits are your age (if you're under 10 years old there will be a zero before your age).

The remaining one or two digits will be the number of times per week you eat or want chocolate (the number you specified in the first step).

How does it work? Why does it work? What would an algebraic expression look like for each step?

- Ask the students to use quantitative and abstract reasoning to create his/her own "magic trick." This should be at least a five step math process and should be represented through both verbal and algebraic representations.

Notes

Algebraic Expressions
Lesson 3 of 7

Description:

This lesson will give students an opportunity to fortify their understanding of interpreting and modifying expressions by analyzing symbolic manipulation of various expressions.

Common Core State Standard Addressed:

- 7.EE.2: Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related.
- A.SSE.1: Interpret expressions that represent a quantity in terms of its context.
- A.SSE.2: Use the structure of an expression to identify ways to rewrite it.

Mathematical Practice Standard(s) Emphasized:

- MP 6: Attend to precision.
- MP 7: Look for and make use of structure.

Sequence of Instruction	Activities Checklist

Engage

Your Birthday!

Here's a fun trick to show a friend, a group, or an entire class of people. Tell the person (or class) to think of their birthday and you will guess it.

Step 1) Have them take the month number from their birthday: January = 1, Feb = 2, etc.
Step 2) Multiply that by 5.
Step 3) Then add 6.
Step 4) Then multiply that total by 4.
Step 5) Then add 9.
Step 6) Then multiply this total by 5 once again.
Step 7) Finally, have them add to that total the day in which they were born. If they were born on the 18th, they add 18, e.g.

Have them give you the total. In your head, subtract 165 from their total, and you will have the month and day they were born.

Ask students to construct a viable argument explaining why this birthday trick 'works.' Ask them to create an algebraic expression modeling the situations.

How It Works: Let M be the month number and D will be the day number. After the seven steps, the expression for their calculation is:

$$5\,(4\,(5M + 6\,) + 9\,) + D = 100M + D + 165$$

Thus, if you subtract 165, what remains will be the month in hundreds plus the day.

Explore

Put six different sheets of paper scattered around the room with **+, -, *, ÷, =, ( )**

Note: The left and right parentheses can be placed on the same sheet of paper.

Have the students quickly rotate throughout the room, writing key words that may be associated with the symbols provided on the different papers. This can be a timed event of four or five minutes. After the time is up, have different students present the symbols and discuss the words.

Ask students to provide scenarios and create verbal expressions where the various words might be used to denote a specific operation or structure. Have them generate the associated algebraic expression. Discuss any potential conflicts or misconceptions that students may have.

The lists can be edited following the discussion. Have the students record the lists as a future resource.

Group the students into pairs. Write the bolded statements on the board. Ask the groups to find an algebraic expression that represents each statement.

1. **Three times a number minus seven** --- 3x - 7
2. **A number minus seven, then multiplied by three** --- 3(x - 7)
3. **A shirt originally cost *c* dollars and is on sale for 60% of the original cost** --- $0.6c
4. **The total number of hours worked during *d* days for persons working seven hours each day** --- 7d
5. **Total amount of pay for working *h* hours at a wage of $7.25 per hour** --- 7.25h

Explanation

- After sharing the first two responses, ask students if the written statement and expression are equivalent. Have the students discuss how they determined their answer.
- Ideally some groups evaluated the expressions by substituting the same number into each expression. Others may perform symbolic manipulation to demonstrate that the expressions are not equivalent.
- If students cannot write the generalized algebraic expressions, the teacher may give specific examples to move student understanding to the general expressions.

For example, in number four, the teacher may want to use three days and then four days to move to the more general expression.

Practice Together in Small Groups/Individually

Print and cut out the following **I Have / Who Has** cards. The next two pages (of 16 cards) comprise one complete linked activity.

Algebraic Expressions

I Have / Who Has Practice Set (4 cards per set)

I have **7x** **Who has** **One half of the opposite of eight**	**I have** **9** **Who has** **The product of seven and a number**
I have **x - 5** **Who has** **The sum of three and six**	**I have** **-4** **Who has** **the difference of a number and five**
I have **7x** **Who has** **One half of the opposite of eight**	**I have** **9** **Who has** **The product of seven and a number**
I have **x - 5** **Who has** **The sum of three and six**	**I have** **-4** **Who has** **The difference of a number and five**

Algebraic Expressions

I Have / Who Has

I have $-4x$ **Who has** half of the difference of four times a number and eight	**I have** $2x-4$ **Who has** a third of the difference of eighteen and six times a number
I have $6-2x$ **Who has** three multiplied by the result of three subtracted from ten times a number	**I have** $30x-9$ **Who has** the difference of a number and seven
I have $x-7$ **Who has** six subtracted from the opposite of the sum of three times a number and five	**I have** $-3x-11$ **Who has** the difference of seven times a number and one
I have $7x-1$ **Who has** the difference of eight times a number and four times the same number	**I have** $4x$ **Who has** eight more than three times the sum of a number and one

Algebraic Expressions
I Have / Who Has

I have **3x+11** **Who has** **two times a number plus four**	**I have** **2x+4** **Who has** **add ten to three times a number subtracted from one**
I have **-3x+11** **Who has** **the difference of three times a number and eleven**	**I have** **3x-11** **Who has** **half the sum of double a number and fourteen**
I have **x+7** **Who has** **subtract ten from a number and multiply the result by three**	**I have** **3x-30** **Who has** **the difference of twice a number and six**
I have **2x-6** **Who has** **seven times a number subtracted from the same number**	**I have** **-6x** **Who has** **the difference of four times a number and eight times the same number**

Whole Class Game:

1. Distribute one card to each student. Then distribute the extras to strong students in the beginning and to random students as the class becomes more familiar with the deck.
2. As you distribute the cards, encourage students to begin thinking about what the question for their card might be so that they are prepared to answer. When all cards are distributed, select the student with the starter card to begin. Play continues until the game loops back to the original card. That student answers and then says "the end" to signal the end of the game.

Evaluate Understanding

INCLUDED IN THE STUDENT MANUAL

Task #4: Miles to Kilometers

The students in Mr. Sanchez's class are converting distances measured in miles to kilometers. To estimate the number of kilometers, Abby takes the number of miles, doubles it and then subtracts 20% of the result. Renato first divides the number of miles by 5 and then multiplies the result by 8.

a. Write an algebraic expression for each method.

b. Use your answer to part (a) to decide if the two methods give the same answer.

(http://illustrativemathematics.org/illustrations/433)

Commentary *for the Teacher:*

In this task students are asked to write two expressions from verbal descriptions and determine if they are equivalent. The expressions involve both percent and fractions. This task is most appropriate for a classroom discussion since the statement of the problem has some ambiguity.

Adapted from Algebra: Form and Function, McCallum et al., Wiley 2010

Possible Solution:

Writing and comparing expressions —

1. Abby's method starts by doubling *m,* giving 2*m*. She then takes 20% of the result, which we can write as 0.2(2*m*). Finally she subtracts this from 2*m*, giving:

   $$2m - (0.2)2m$$

   Renato's method starts by dividing *m* by 5 giving: $m \div 5 = \frac{m}{5}$ and then multiplies the result by 8 giving: $8\left(\frac{m}{5}\right)$

2. Abby's expression can be simplified as follows:

   $$2m - (0.2)2m = 2m - 0.4m = (2 - 0.4)m = 1.6m$$

   (the step where we rewrite 2m-0.4m as (2-0.4) uses the distributive property.)

   Renato's method gives: $8\left(\frac{m}{5}\right) = \frac{8m}{5} = \frac{8}{5}m = 1.6m$

So the two methods give the same answer and the expressions are equivalent.

Closing Activity

Assign Independent Practice and ask students how mathematical structure will be used in the assignment.

Independent Practice

INCLUDED IN THE STUDENT MANUEL

A Night Out

A new miniature golf and arcade opened up in town. For convenient ordering, a play package is available to purchase. It includes two rounds of golf and 20 arcade tokens, plus $\$3.00$ off the regular price. There is a group of six friends purchasing this package.

Let g represent the cost of a round of golf, and let t represent the cost of a token.

- Write two different expression that represent the total amount this group spent.
- Explain how each expression describes the situation in a different way.

Xander goes to the movies with his team. Each team member buys a ticket and two boxes of popcorn. There are 13 team members.

Let t represent the cost of a ticket and p represent the cost of a box of popcorn.

- Write two different expressions that represent the total amount the team spent.
- Explain how each expression describes the situation in a different way.

Commentary ***for the Teacher:***

Two equivalent expressions are as follows:

- $6(2g + 20t - 3)$: Each person will pay for two rounds of golf and 20 tokens and will be discounted three dollars. This expression is six times the quantity of each friends cost.
- $12g + 120t - 18$: The total cost is equal to 12 games of golf plus 120 tokens, minus 10 dollars off the entire bill.

- $13(t + 2b)$: 13 people each buy a ticket and two boxes of popcorn, so the cost is 13 times the quantity of a ticket and two boxes of popcorn.
- $13t + 26t$: There are 13 tickets and 26 boxes of popcorn. The total cost will be 13 times the cost of the tickets, plus 26 times the cost of the popcorn.

Resources/Instructional Materials Needed

Chart paper with operation labels: $+ \quad - \quad \times \quad \div \quad = \quad (\quad)$

I Have-Who Has practice cards (1 set of 4 per student, cut out)
I Have-Who Has cards (1 set per every 16 students, cut out)

Notes

Algebraic Expressions
Lesson 4 of 7

Description:

Students will begin this lesson by engaging in a real-life problem that encompasses some basic geometric concepts along with expression manipulation. This lesson will give students an opportunity to fortify their understanding of writing expressions.

Common Core State Standard Addressed:

- A-SSE.1: Interpret expressions that represent a quantity in terms of its context.
- A-SSE.2: Use the structure of an expression to identify ways to rewrite it.
- A-SSE.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

Mathematical Practice Standard(s) Emphasized:

- MP 2: Reason abstractly and quantitatively.
- MP 3: Construct viable arguments and critique the reasoning of others.
- MP 4: Model with mathematics.
- MP 7: Look for and make use of structure.

Engage

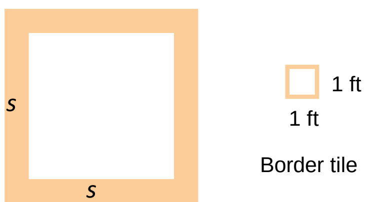
Begin by splitting the class into small groups (two to three students) and ask them to consider the following example:

INCLUDED IN THE STUDENT MANUAL

Task #5: Swimming Pool

Common Core Standards Writing Team. Progressions for The Common Core State Standards in Mathematics (Draft). Grades 6-8, Middle School, Expressions and Equations. Institute for Mathematics and Education, University of Arizona, April 22, 2011

You want to build a square swimming pool in your backyard. Let *s* denote the length of each side of the swimming pool (measured in feet). You plan to surround the pool by square border tiles, each of which is one foot by one foot (see figure).



A teacher asks her students to find an expression for the number of tiles needed to surround such a square pool, and sees the following responses from her students:

$4(s+1)$

s^2

$4s+4$

$2s+2(s+2)$

$4s$

Is each mathematical model correct or incorrect? How do you know?

Progressions for the Common Core State Standards in Mathematics (draft). Grade 6-8, Middle School, Equations and Expressions.

Explore

Ask the students to decide whether each answer is correct or incorrect. In addition, ask students to explain the method and logic (correct or incorrect) that each of these students used to determine their expression. What might each student have been thinking? Encourage students to use color to connect each expression to a physical representation. Teachers might want to make square tiles available for students to use to model each expression.

Explanation

MATH 7 PRACTICE

The response $4(s+1)$ is correct. This student may have thought that for a given side, you need *s* tiles plus tile additional one for a corner, then multiply by four since there are four sides.

The response s^2 is not correct. This student calculated the area of the pool, not the number of tiles needed to create a border.

The response $4s+4$ is correct. This student may have realized each of the four sides needs *s* tiles, then added the four tiles needed for each corner.

The response $2s+2(s+2)$ is correct. This student may have thought about using *s* tiles on two of the sides (say top and bottom edges). Then the remaining two sides would require $s+2$ tiles each.

The response $4s$ is not correct. This student forgot to take into account the corners.

- Here it is important that students see the structure of each expression, and they are able to connect the structure of the expression to an interpretation. Breaking an expression down into parts so each has meaning is the primary goal of this activity.
- You might ask your students how they could determine or show that the three correct expressions are equivalent while the two incorrect expressions are not equivalent to the correct answer.

Practice Together in Small Groups/Individually

Give each group a large piece of paper that they can write on and post on the wall. Have students work on the following examples and write their answers on the large piece of paper.

INCLUDED IN THE STUDENT MANUAL

MATH 2 PRACTICE

Task #6: Smartphones

Suppose *p* and *q* represent the price (in dollars) of a 64GB and a 32GB smartphone, respectively, where $p > q$. Interpret each of the expressions in terms of money and smartphones. Then, if possible, determine which of the expressions in each pair is larger.

- $p+q$ and $2q$
- $p+0.08p$ and $q+0.08q$
- $600-p$ and $600-q$

INCLUDED IN THE STUDENT MANUAL

Task #7: University Population

MATH 2 PRACTICE

Let *x* and *y* denote the number female and male students, respectively, at a university. where $x < y$. If possible, determine which of the expressions in each pair is larger? Interpret each of the expressions in terms of populations

- $x + y \text{ and } 2y$
- $\frac{x}{x+y} \text{ and } \frac{y}{x+y}$
- $\frac{x+y}{2} \text{ and } \frac{y+x}{2}$

Evaluate Understanding

After groups have finished with this activity and posted their answers around the room, call on various groups to share their answers and explanations. Be prepared to ask guiding questions with regard to interpreting the practical meaning of each of the expressions.

Independent Practice

INCLUDED IN THE STUDENT MANUAL

For each pair of expressions below, without substituting in specific values, determine which of the expressions in the given pairs is larger. Explain your reasoning in a sentence or two.

- $5 + t^2 \text{ and } 3 - t^2$
- $\frac{15}{x^2+6} \text{ and } \frac{15}{x^2-6}$
- $(s^2 + 2)(s^2 + 1) \text{ and } (s^2 + 4)(s^2 + 3)$
- $\frac{8}{k^2+2} \text{ and } k^2 + 2$

Resources/Instructional Materials Needed

Chart paper

Notes

Algebraic Expressions

Lesson 5 of 7

Description:

Students will begin this lesson by engaging in a task on developing expressions for a particular geometric pattern. This lesson will strengthen the ability of students to compare expressions presented in different forms and determine equivalency.

Common Core State Standard Addressed:

- A-SSE.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
- F-IF.8: Write a function defined by an expression in different but equivalent forms.

Mathematical Practice Standard(s) Emphasized:

- MP 1: Make sense of problems and persevere in solving them.
- MP 3: Construct viable arguments and critique the reasoning of others.
- MP 7: Look for and make use of structure.
- MP 8: Look for and express regularity in repeated reasoning.

Sequence of Instruction	Activities Checklist

Engage

Task #8: Sidewalk Patterns

Sidewalk Patterns that can be found at the Shell Center website http://map.mathshell.org/materials/tasks.php?taskid=254&subpage=apprentice and on the next page.

- Ask students to complete the grid on page one of the task, and have students share their results in pairs.
- Have students work in pairs to describe how they see the pattern growing. Ask students to think about the important pieces such as the white blocks, gray blocks or different parts within the pattern.
- Have volunteers share their ideas of how they see the pattern growing.

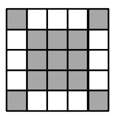
INCLUDED IN THE STUDENT MANUAL

Task #8: Sidewalk Patterns

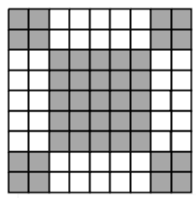
Sidewalk Patterns

In Prague some sidewalks are made of small square blocks of stone.

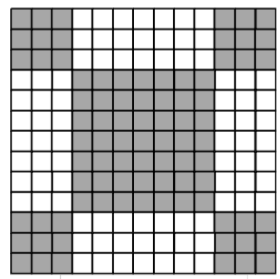
The blocks are in different shades to make patterns that are in various sizes.



Pattern #1

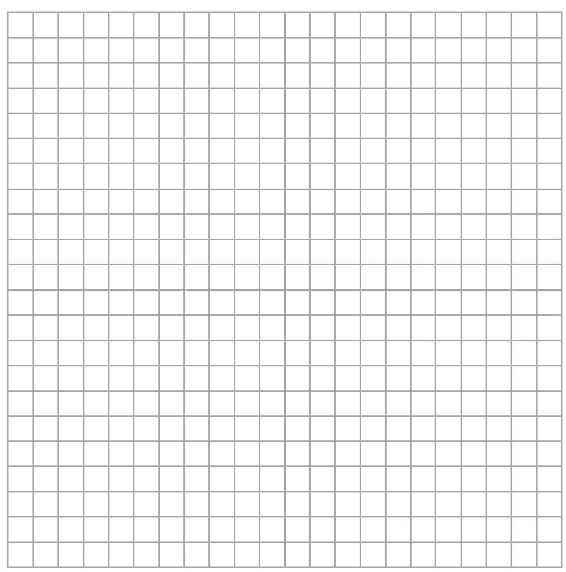


Pattern #2



Pattern #3

Draw the next pattern in this series.



Pattern #4

INCLUDED IN THE STUDENT MANUAL

1. Complete the table below

Pattern number, *n*	1	2	3	4
Number of white blocks	12	40		
Number of gray blocks	13			
Total number of blocks	25			

2. What do you notice about the number of white blocks and the number of gray blocks?

____

3. The total number of blocks can be found by squaring the number of blocks along one side of the pattern.

   a. Fill in the blank spaces in this list.

   $25 = 5^2$ $\quad 81 =$ ______ $\quad 169 =$ ______ $\quad 289 = 17^2$

   b. How many blocks will pattern #5 need? ______

   c. How many blocks will pattern #*n* need? ______

4. a. If you know the total number of blocks in a pattern you can work out the number of white blocks in it. Explain how you can do this.

____

____

   b. Pattern # 6 has a total of 625 blocks.
   How many white blocks are needed for pattern #6? ______
   Show how you figured this out.

Explore

Split students into small groups (two to three students)

- Ask them to make sense of the problem and complete page two of the task together. They should construct viable arguments for their choices and critique the reasoning of their partner.
- When the group completes the task, ask them to write their answer for part 3c on a large piece of sticky paper or record on the white board.
- After all groups have finished, ask each group to decide which answers are correct. Encourage students to connect the terms in the expression to the geometric pattern
- Ask a volunteer group to explain how they found their expression and justify by connecting the expression to the geometric pattern.
- Ask if another group solved the task differently, and how that was reflected in the structure of their expression, and connect back to the ideas discussed with the pool tiles from the previous lesson.
- If any answers were incorrect, ask how you could prove their expression is not the same as one of the correct answers.

Guiding questions:

- What does it mean for expressions to be equivalent?
- How can you prove that two expressions are not equivalent?

Explanation

At this point review the general concept of what it means for two expressions to be equivalent; namely two expressions are equivalent if they have the same value for every possible value(s) you substitute in for each of the variable(s). Stress that if you try one value and see the output is the same, that is not enough to claim the expressions are the same for ALL values you could substitute into the expression. For example:

- (x^2+y^2) and $(x+y)^2$ have the same value if you substitute in $x=0$ and $y=2$ into both expressions (you get 4). However you need to show this for all possible pairs of values you could substitute for x and y. Thus, since $x=1$ and $y=1$ give a value of 2 in the first expression and a value of 4 in the second, these two expressions are not equivalent.
- Showing expressions are not equivalent is easier than showing two expressions are equivalent since you presumably need to check all possible values you could substitute.
- You could ask students to think of other possible ways you might determine whether these two expressions are equivalent. Some students might explain you could use algebra to expand the second expression and get $x^2+2xy+y^2$ that is not the same as the first expression. You could use this as pre-assessment for distributing and collecting like terms covered in the next day's classes.

Practice Together in Small Groups/Individually

Divide the class up into small groups to complete Task #9: Expression Pairs: Equivalent or Not? Which pairs of algebraic expressions are equivalent and which are not equivalent. Ask them to specifically look for and make sense of the structure. If they believe the pair of expressions is not equivalent, ask them to provide values for the variable(s) that lead to different values when you evaluate.

If they believe they are equivalent, ask them to show or explain how they determined equivalence. For example, which properties of operations are being used (associative, commutative, and distributive)? You may need to review these properties with students.

Notice that some of the pairs highlight common student misconceptions.

INCLUDED IN THE STUDENT MANUAL

Task #9: Expression Pairs: Equivalent or Not?

a. $a + (3 - b)$ *and* $(a + 3) - b$
b. $2 + \frac{k}{5}$ *and* $10 + k$
c. $(a - b)^2$ *and* $a^2 - b^2$
d. $3(z + w)$ *and* $3z + 3w$
e. $-a + 2$ *and* $-(a + 2)$
f. $\frac{1}{(x+y)}$ *and* $\frac{1}{x} + \frac{1}{y}$
g. $x^2 + 4x^2$ *and* $5x^2$
h. $\sqrt{(x^2 + y^2}$ *and* $x + y$
i. $bc - cd$ *and* $c(b - d)$
j. $(2x)^2$ *and* $4x^2$
k. $2x + 4$ *and* $x + 2$

(More pairs could be added here if students need more practice.)

Evaluate Understanding

After the groups have had the opportunity to determine which pairs of algebraic expressions are equivalent and which are not equivalent, ask the groups to share their answers. If they all agree on the answer to the first set of expressions, move to the next pair. If they do not agree, ask two of the groups that disagree to come to the board and demonstrate how they determined that the expressions were equivalent or not equivalent.

It is important to emphasize that if you ever forget whether $\frac{1}{(x+y)} = \frac{1}{x} + \frac{1}{y}$, you can always check by substituting some values. If the results yield a false statement, clearly they are not equivalent. If the results are equivalent, then they still may not be equivalent for all values you substitute, so be careful.

It is also important to emphasize properties of operations with algebraic expressions that are exactly the same as the properties of operations on numerical expressions. We are not inventing new operations, rather extending previous understanding with numbers to algebraic expressions. Refrain from using gimmicks such as PEMDAS to tell students the order with which they MUST evaluate. When working with algebraic expressions, you still use distributive property just as with numbers.

To illustrate this point to teachers (you may not want to show this to students unless necessary), consider 7 – 2(3 – 8*x*). A student blindly recalling PEMDAS might simplify as follows $7\text{-}2(3\text{-}8x)=7\text{-}2(\text{-}5x)=7+10x$ since P comes first. Or a student may think $n–2+5=n–7$ since you do A before S. While strictly interpreting PEMDAS would lead one to (incorrectly) say 8(5+1) = 8(5) + 8(1). You first need to add 5 and 1.

Closing Activity

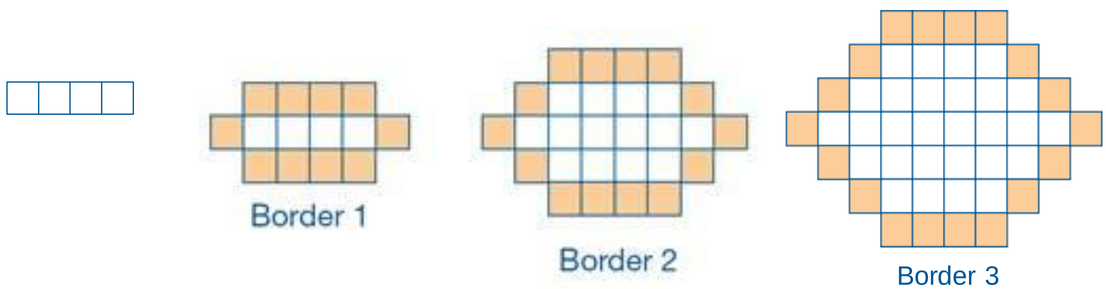
MATH PRACTICE 3,7 & 8

Kitchen Floor Tiles accessed on 8/5/2014, is licensed by Illustrative Mathematics under CC BY NC SA 4.0

INCLUDED IN THE STUDENT MANUAL

Task #10: Kitchen Floor Tiles

Fred has some colored kitchen floor tiles and wants to choose a pattern to make a border around white tiles. He generates patterns by starting with a row of four white tiles. He surrounds these four tiles with a border of colored tiles (Border 1). The design continues as shown below:



Fred writes the expression 4(*n*-1) + 10 for the number of tiles in each border, where *n* is the border number, $n \geq 1$.

- Explain why Fred's expression is correct.
- Emma wants to start with five tiles in a row. She reasons, "Fred started with four tiles and his expression was 4(*n*-1) + 10. So if I start with five tiles, the expression will be 5(*n*-1) + 10. Is Emma's statement correct? Explain your reasoning.
- If Emma starts with a row of *x* tiles, what should the expression be?

Adapted from: http://illustrativemathematics.org/illustrations/215

Commentary *for the Teacher:*

The purpose of this task is for students to practice reading, analyzing, and constructing algebraic expressions, attending to the relationship between the form of an expression and the context from which it arises. The context here is intentionally thin; the point is not to provide a practical application to kitchen floors but to give a framework that imbues the expressions with an external meaning.

Analyzing and generalizing geometric patterns such as the one in this task may be familiar to students from work in previous grades, so part (a) may be a review of that process. It requires students to make use of the structure in the expression, to notice and express the regularity in the repeated geometric construction and to explain and justify the reasoning of others. Part (b) requires a deeper analysis of the expression, identifying the referents for its various parts. Students may still need guidance in writing the formula for part (c) since it introduces a second variable.

Possible Solution

For Border 1, tiles are added above and below the original four tiles — a total of eight additional tiles — and a tile is added to each end of the row of original tiles — two additional tiles — for a total of 10 tiles.

For Border 2, we have four additional tiles needed to fill in the corners of the diagram (one tile for each corner gap), plus the original 10 tiles coming from the top and bottom rows of four tiles each and the two end tiles:

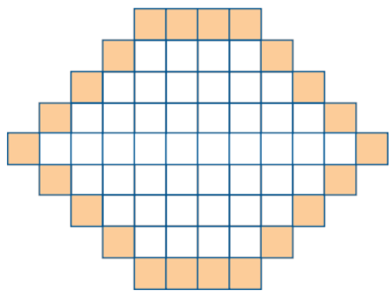
$4+10$ colored tiles in Border 2.

For Border 3, there are now two tiles in each of the four corners, plus the same 10 tiles from the top, bottom and ends, so there are:

$4(2)+10$ colored tiles in Border 3. For

Border 4, we have three tiles in each corner, for a total of:

$4(3)+10$ colored tiles in Border 4.



Border 4

The following table illustrates the pattern:

Border number	Number of tiles in the border
1	10
2	$4(1)+10$
3	$4(2)+10$
4	$4(3)+10$
b	$4(b-1)+10$

In Border b there are $b-1$ extra tiles needed at each of the four corners, so the number of border tiles needed is given by Fred's expression:

$$4(b-1)+10$$

- In part a, the number 10 comes from the top row of tiles, the bottom row of tiles, and the two tiles on the ends of the original four tiles. If Emma starts with five tiles, that number would change to 12 − 5 tiles above the originals — five tiles below the originals, and one tile on each end. Emma's formula is not correct. She has incorrectly assumed that the 4 in Fred's formula came from the number of tiles in the beginning row, when it actually comes from the number of corners in the diagram itself. Regardless of the number of tiles in the beginning row, there will always be "4" corners to be filled. If Emma wants a formula for the number of tiles in each border starting with five tiles in the original row, she could use:

$$t=4(b-1)+12$$

- In general, the number of tiles added at the top and the bottom in each border will always match the number in the original row (n) and there will always be one tile added to each end. If there are n tiles in the original row, the constant in the expression will be $2n+2$.

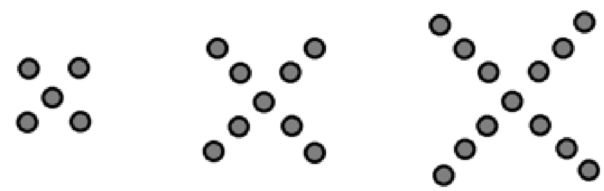
  The number of tiles needed for each corner will remain the same regardless of the number of tiles in the original row. If b is the number of the border, $4(b-1)$ corner tiles are needed. So if Emma starts with a row of n tiles, the number of tiles in the Border b is:

$$4(b-1)+(2n+2).$$

Independent Practice

INCLUDED IN STUDENT MANUAL

A. Luce is building a drone for her physics class. She has a total of 33 pieces available to build the propeller.



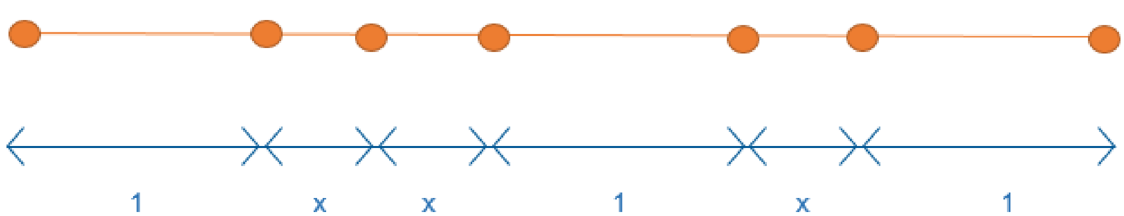
Propeller 1 Propeller 2 Propeller 3

1. Draw the next pattern in this series.
2. Create a table including the propeller number, p, and the number of pieces needed to build the propeller.
3. What is the expression describing the number of pieces in each propeller, where p is the propeller number?
4. What is the value of p when Luce has 33 pieces to build with? How do you know?

B. On the figure below, indicate intervals of length:

- $x+1$
- $3x+1$
- $3(x+1)$

What do your answers tell you about whether $3x+1$ and $3(x+1)$ are equivalent?



Notes

Algebraic Expressions
Lesson 6 of 7

Description:

Students will begin this lesson with an engaging activity that will lead to an understanding of rewriting and interpreting expressions using the distributive property.

Common Core State Standard Addressed:

- A-SSE.1: Interpret expressions that represent a quantity in terms of its context.
- A-SSE.2: Use the structure of an expression to identify ways to rewrite it.

Mathematical Practice Standard(s) Emphasized:

- MP 3: Construct viable arguments and critique the reasoning of others.
- MP 4: Model with mathematics.
-

Sequence of Instruction	Activities Checklist

Engage

Guess the Numbers on the Dice

Give a student a pair of dice. Tell the student you will deduce what the student rolled on the dice without seeing the dice.

Have the student roll the dice and ask them to:

- Multiply one of the numbers on the dice by 5.
- Add 8 to the product.
- Multiply the sum by 2.
- Add the number on the other die to the product.
- Give the teacher the result.

Subtract 16 from the student's final answer. The numbers on the dice are the two digits of the numeral of the resulting number.

Let the students discuss in groups and ask them to construct viable arguments on how this works. Have the students:

1. Reproduce the activity with their partner for another roll of the dice.
2. Verify that the process works.
3. Define x to be the number on one die and y the number on the other die.

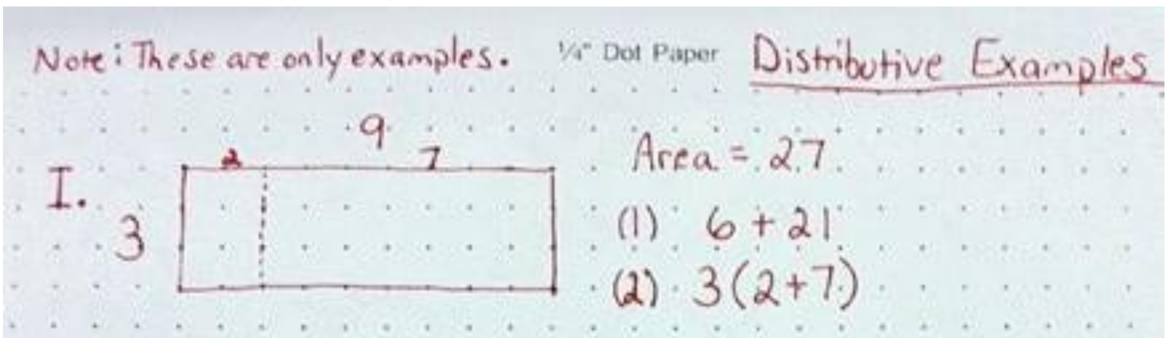
4. Write an equation to represent the calculations followed during the activity/
5. Rewrite the equation.
6. Analyze why the activity works.

(Note: The equation is (5x+8)*2+y that simplifies to 10x+y+16 and then subtracting 16 yields 10x+y=x(10)+y(1), and thus the ten's digit is the x and the one's digit is the y.)

Explore

Have students sketch a rectangle of any dimension on the dot paper provided, but make sure each vertex is on a dot (ex: 3 x 9).

- Have them label the length and width of the rectangle.
- Have them calculate the area of the rectangle (ex: 27).
- Tell them to draw a vertical line that cuts the rectangle into two pieces.



Example of Student Work

Now, the rectangle is two separate rectangles, say A and B.

- Have them label the dimensions of both rectangles A and B (ex: 3 x 2 and 3 x 7).

Now, write the area of the original rectangle in two ways: (1) as the sum of the areas of A and B (ex: 6+21), and (2) as the product of one length times width (ex: 3*(2+7)).

The focus here is to get them comfortable with expressing area in these two ways by modeling the mathematics.

Now, have the students draw another rectangle where the upper left and lower left vertices are the only vertices on dots.

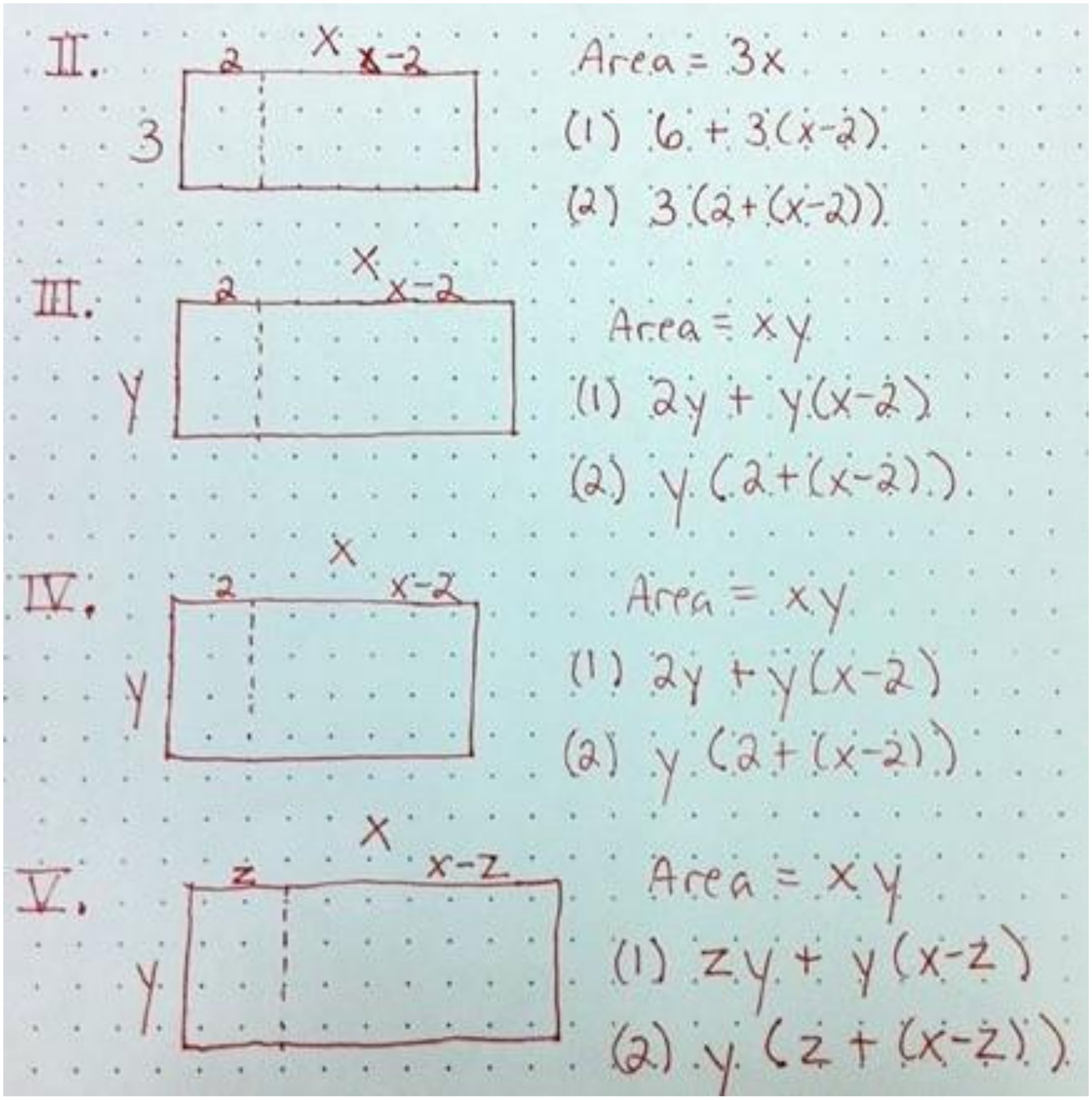
- Ask them how they should label the length and width (ex: "3" and "x").
- Calculate the area of the rectangle (ex: 3x).
- Now, draw a vertical line that cuts the rectangle into two pieces.

Now, the rectangle is two separate rectangles, say A and B.

- Have them label the dimensions of both rectangles A and B (ex: 3 by 2 and 3 by (x-2)).

Now, write the area of the original rectangle in two ways: (1) as the sum of the areas of A and B (ex: 6+3(x-2)), and (2) as the product of one length times width (ex: 3(2+(x-2)).

Continue with variations of rectangles (some are provided on the example sheet below getting students familiar with writing areas in two ways.



Example of Student Work

Explanation

MATH 4 PRACTICE

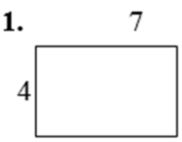
Have students work through the task. Provide support to students as needed.

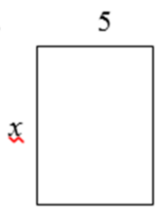
INCLUDED IN THE STUDENT MANUAL

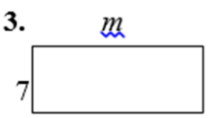
Task #11: Representing the Distributive Property with Area Models

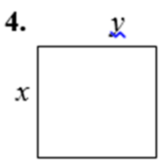
Name____

Write the expression that represents the area of each rectangle.

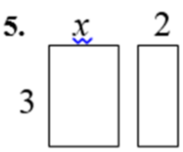
1.


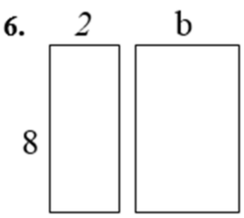
2.


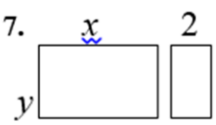
3.


4.


Find the area of each box in the pair.

5.


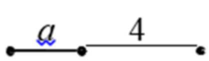
6.


7.


Write the expression that represents the total length of each segment.

8.


____

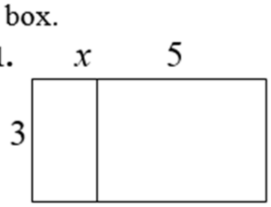
9.


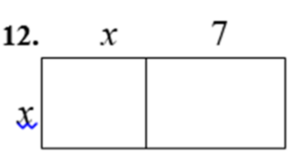
____

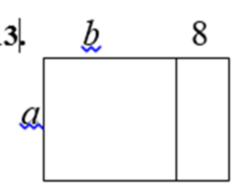
10.


____

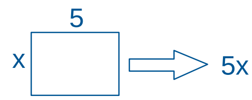
Write the area of each rectangle as the product of ***length*** × ***width*** and also as a sum of the areas of each box.

11.


12.


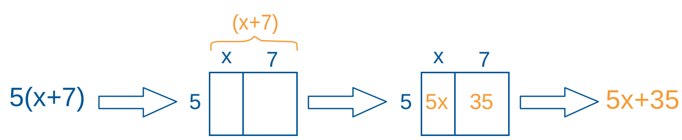
13.


The first section introduces students to the idea of writing the area of a rectangle as an expression of the length × width, even when one or more dimensions may be represented by a variable.



In the next section students learn to represent the length of a segment consisting of two parts as a *sum*.

The key section is next, having students represent the area of each rectangle *two ways* (modeling the mathematics) to distribute the common factor among all parts of the expression in parentheses.



INCLUDED IN THE STUDENT MANUAL

Task 12: Representing factoring a Common Factor with area models

NAME ________________________________

Fill in the missing information for each: dimensions, area as product, and area as sum

1. x 5
3 | |

2. x 4
| 5x | 20 |

3. 6
| $7x$ | 42 |

4. b
| $8b$ | 32 |

Area as Product: 3(x+5) Area as Product:__________ Area as Product: _________ Area as Product: ____

Area as Sum: 3x + 15 Area as Sum: ____________ Area as Sum: ____________ Area as Sum: _______

Fill in the missing dimensions from the expression given

5. $7x+35=7(\_\_\_\_\_\_)$

6. $2x+18=2(\_\_\_\_\_\_)$

7. $4x-20=\_\_(\_\_\_\_\_\_)$

8. $-7x+35=\_\_(\_\_\_\_\_)$

9. $3x-18=\_\_(\_\_\_\_\_)$

10. $-5x-20=\_\_(\_\_\_\_\_)$

Factoring is the process of writing a sum or difference as the product of factors.

Factor these:

11. $3x-12=$ _______________ **12.** $-9x-63=$ ________________

13. $6x-54=$ _______________ **14.** $3x+18=$ ___________

Give the students questions similar to the following.

a. Show why “2(x+y)” and “2x+2y” are the same.

b. Rewrite “3(x+z).”

c. Rewrite “a(p+q+r).”

d. Rewrite “5(2x+3y+z).”

Evaluate Understanding

Have students independently complete Task #13: Distributive Property.

INCLUDED IN THE STUDENT MANUAL

Task #13: Distributive Property

Are the expressions equivalent? Sketch and simplify to prove. If the two expressions are not equal write the correct equivalence.

1. 3(x+3) and 3x+6

2. 6(y+1) and 6y+6

3. x(x+4) and x^2+4

4. $y(x+2)$ and $xy+2y$

5. $x(x+y+2)$ and $x^2+xy+2x$

6. $2x(x+3)$ and $2x+6$

Distribute the following. Use a sketch or just distribute if you can.

1. $3(x+2)$
2. $4(y-1)$
3. $x(x+6)$
4. $x(y+4)$
5. $3x(x+y-1)$

Closing Activity

Write, 49, on a slip of paper.

- Fold the paper and give it to someone to hold for safekeeping.
- Ask a volunteer to toss a pair of dice (six sided) and write down the results of the following computations:
  - Multiply the two top numbers on the dice.
  - Multiply the two bottom numbers on the dice.
  - Multiply the top number on one die by the bottom number on the other die.
  - Multiply the other pair of top and bottom numbers.
  - Now, add up the four products and announce the sum.

- Then, ask the person with the folded slip of paper to unfold it and read your prediction. Tell the students that this trick always works because the sum of the two numbers on any pair of opposite faces is always seven.

Now, if we let *a* and *b* be the numbers that show after the dice are tossed, what are the products going to be from the steps given? (answer: ab, $(7\text{-}a)$, $(7\text{-}b)$, $a(7\text{-}b)$, and $b(7\text{-}a)$)

Now, have the students add the products and simplify to show that it must always equal 49.

Resources/Instructional Materials Needed

Dot Paper and Graph Paper

Teacher Note: The emphasis in this lesson is to get students comfortable with the area model for the distributive property, as it will be used in following unit(s).

Notes

Algebraic Expressions
Lesson 7 of 7

Description:

This lesson is intended to help you assess how well students are able to translate between words, symbols, tables and area representations of algebraic expressions. It is designed to identify and support students who have difficulty in these concepts. (Shell Center Formative Assessment Lesson: Interpreting Algebraic Expressions)

Common Core State Standard Addressed:

- A-SSE.1: Interpret expressions that represent a quantity in terms of its context.
- A-SSE.2: Use the structure of an expression to identify ways to rewrite it.

Mathematical Practice Standard(s) Emphasized:

- MP 2: Reason abstractly and quantitatively.
- MP 7: Look for and make use of structure.

The following *Formative Assessment Lesson* is a classroom-ready lesson included to help teachers assess and improve students' understanding of mathematical concepts and skills and their ability to use the "mathematical practices" described in the Common Core State Standards.

Research has shown that formative assessment, as embodied in the following lesson, is a powerful way to improve student learning and performance. This approach first allows students to demonstrate their prior understandings and abilities in employing the mathematical practices, and then resolve their own difficulties and misconceptions through structured discussion. This results in more secure long-term learning, reducing the need for re-teaching that otherwise takes so much classroom time.

Many often ask, why don't students just remember the procedures they have been taught and practiced? The fact is, that approach works fine in the short term but, as every teacher knows, if procedural knowledge is not underpinned by conceptual understanding, students will quickly forget "how to do it."

Read more about the *Formative Assessment Lesson* rationale, structure, and philosophy using the *Brief Guide for Teachers and Administrators* that can be found at http://map.mathshell.org/lessons.php?unit=9225&collection=8.

CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Interpreting Algebraic Expressions

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Interpreting Algebraic Expressions

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to translate between words, symbols, tables, and area representations of algebraic expressions. It will help you to identify and support students who have difficulty:

- Recognizing the order of algebraic operations.
- Recognizing equivalent expressions.
- Understanding the distributive laws of multiplication and division over addition (expansion of parentheses).

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-SSE: Interpret the structure of expressions.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their work and formulate questions for students to answer, to help them improve their solutions.
- During the lesson, students work in pairs or threes to translate between word, symbol, table of values, and area representations of expressions.
- In a whole-class discussion, students find different representations of expressions and explain their answers.
- Finally, students return to their original assessment task and try to improve their own responses.

MATERIALS REQUIRED

- Each student will need two copies of *Interpreting Expressions*, a mini-whiteboard, pen, and eraser.
- Each pair of students will need glue, a felt-tipped pen, a large sheet of poster paper, and cut-up copies of *Card Set A: Expressions*, *Card Set B: Words*, *Card Set C: Tables*, and *Card Set D: Areas*. Note that the blank cards are part of the activity.
- If you think you will need to continue with the activities into a second lesson, provide envelopes and paper clips for storing matched cards between lessons.

TIME NEEDED

10 minutes for the assessment task, a 90-minute lesson (or two 50-minute lessons), and 10 minutes in a follow-up lesson. All timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Interpreting Expressions* (10 minutes)

Have students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of *Interpreting Expressions*.

> *I want you to spend ten minutes working individually on this task.*
>
> *Don't worry too much if you can't understand or do everything. There will be a lesson [tomorrow] with a similar task that will help you improve.*

It is important that, as far as possible, students are allowed to answer the questions without assistance.

If students are struggling to get started, ask them questions that help them to understand what is required, but do not do the task for them.

Interpreting Expressions

1. Write algebraic expressions for each of the following:

   a. Multiply ***n*** by 5 then add 4. ....

   b. Add 4 to ***n*** then multiply by 5. ....

   c. Add 4 to ***n*** then divide by 5. ....

   d. Multiply ***n*** by ***n*** then multiply by 3. ....

   e. Multiply ***n*** by 3 then square the result. ....

2. The equations below were created by students who were asked to write equivalent expressions on either side of the equals sign.

   Imagine you are a teacher. Your job is to decide whether their work is right or wrong. If you see an equation that is false, then:

   a. Cross out the expression on the right and replace it with an expression that is equivalent to the one on the left.

   b. Explain what is wrong, using words or diagrams.

$$2(n + 3) = 2n + 3$$

$$\frac{10n - 5}{5} = 2n - 1$$

$$(5n)^2 = 5n^2$$

$$(n + 3)^2 = n^2 + 3^2 = n^2 + 9$$

Assessing students' responses

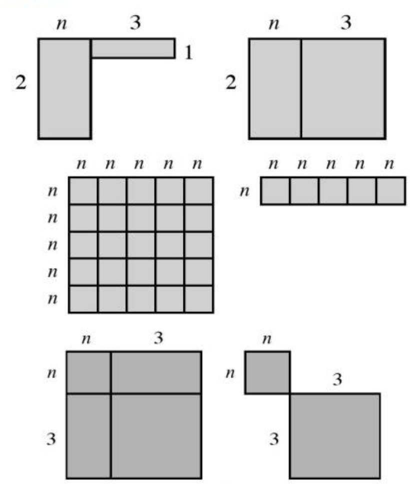
Collect students' responses to the task. Make some notes about what their work reveals about their current levels of understanding. The purpose of doing this is to forewarn you of the difficulties students will experience during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' papers. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given in the *Common issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions and highlight the questions relevant to each student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues:	Suggested questions and prompts:
Writes expressions left to right, showing little understanding of the order of operations implied by the symbolic representation For example: The student writes: Q1a. $n \times 5 + 4$ (not incorrect). Q1b. $4 + n \times 5$. Q1c. $4 + n \div 5$. Q1d. $n \times n \times 3$.	• Can you write answers to the following? $4 + 1 \times 5$ $4 + 2 \times 5$ $4 + 3 \times 5$ • Check your answers with your (scientific) calculator. How is your calculator working these out? • So what does $4 + n \times 5$ mean? Is this the same as Q1b?
Does not construct parentheses correctly or expands them incorrectly For example: The student writes: Q1b. $4 + n \times 5$ instead of $5(n + 4)$. Q1c. $4 + n \div 5$ instead of $\frac{4+n}{5}$. Or: The student counts: Q2. $2(n+3) = 2n+3$ as correct. Q2. $(5n)^2 = 5n^2$ as correct. Q2. $(n+3)^2 = n^2 + 3^2$ as correct.	• Which one of the following is the odd one out: *Think of a number, add 3, and then multiply your answer by 2.* *Think of a number, multiply it by 2, and then add 3.* *Think of a number, multiply it by 2, and then add 6.* Why?
Identifies errors but does not give explanations For example: The student corrects the first, third, and fourth statements, but no explanation or diagram is used to explain why they are incorrect (Q2).	• How would you write expressions for these areas?  • Can you do this in different ways?

SUGGESTED LESSON OUTLINE

Interactive whole-class introduction (10 minutes)

Give each student a mini-whiteboard, pen, and eraser and hold a short question and answer session. If students show any incorrect answers, write the correct answer on the board and discuss any problems.

On your mini-whiteboards, show me an algebraic expression that means:

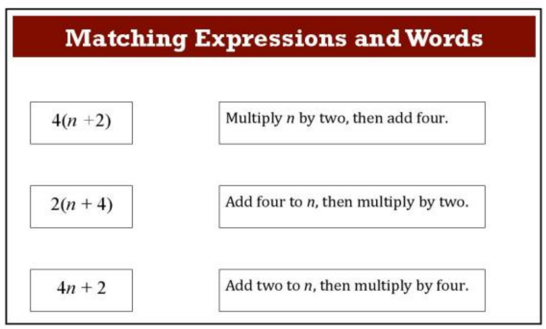
Multiply n by 4 and then add 3 to your answer.	$4n+3$
Add 3 to n and then multiply your answer by 4.	$4(3+n)$
Add 5 to n and then divide your answer by 3.	$\frac{n+5}{3}$
Multiply n by n and then multiply your answer by 5.	$5n^2$
Multiply n by 5 and then square your answer.	$(5n)^2$

Collaborative activity 1: matching expressions and words (20 minutes)

The first activity is designed to help students interpret symbols and realize that the way the symbols are written defines the order of operations.

Organize students into groups of two or three.

Display Slide P-1 of the projector resource:



Note that one of the algebraic expressions does not have a match in words. This is deliberate! It is to help you explain the task to students.

Model the activity briefly for students, using the examples on the projector resource:

I am going to give each group two sets of cards, one with expressions written in algebra and the other with words.

Take turns to choose an expression and find the words that match it. [4(n + 2) matches 'Add 2 to n then multiply by 4'; 2(n + 4) matches 'Add 4 to n then multiply by 2'.]

When you are working in groups, you should place these cards side by side on the table and explain how you know that they match.

If you cannot find a matching card, then you should write your own using the blank cards provided. [4n + 2 does not match any of the word cards shown on Slide P-1. The word card 'Multiply n by two, then add four' does not match any of the expressions.]

Give each small group of students a cut-up copy of *Card Set A: Expressions* and *Card Set B: Words*:

Card Set A: Expressions	
E1 $\dfrac{n+6}{2}$	E2 $3n^2$
E3 $2n+12$	E4 $2n+6$
E5 $2(n+3)$	E6 $\dfrac{n}{2}+6$
E7 $(3n)^2$	E8 $(n+6)^2$
E9 $n^2+12n+36$	E10 $3+\dfrac{n}{2}$
E11 n^2+6	E12 n^2+6^2
E13	E14

Card Set B: Words	
W1 Multiply ***n*** by two, then add six.	W2 Multiply ***n*** by three, then square the answer.
W3 Add six to ***n*** then multiply by two.	W4 Add six to ***n*** then divide by two.
W5 Add three to ***n*** then multiply by two.	W6 Add six to ***n*** then square the answer.
W7 Multiply ***n*** by two then add twelve.	W8 Divide ***n*** by two then add six.
W9 Square ***n***, then add six	W10 Square ***n***, then multiply by nine
W11	W12
W13	W14

Support students in making matches and explaining their decisions. As they do this, encourage them to speak the algebraic expressions out loud. Students may not be used to 'talking algebra' and may not know how to say what is written, or may do so in ways that create ambiguities.

For example, the following conversation between a teacher and student is fairly typical:

Teacher: *Tell me in words what this one says. [Teacher writes:* $3+\frac{n}{2}$*.]*

Student: *Three add n divided by two.*

Teacher: *How would you read this one then? [Teacher writes:* $\frac{(3+n)}{2}$*.]*

Student: *Three add n divided by two. Oh, but in the second one you are dividing it all by two.*

Teacher: *So can you try reading the first one again, so it sounds different from the second one?*

Student: *Three add … [pause] … n divided by two [said quickly]. Or n divided by two, then add three.*

Students will need to make word cards to match E10: $3+\frac{n}{2}$ and E12: n^2+6^2.

They will also need to make expression cards to match W3: *Add 6 to n, then multiply by 2* and W10: *Square n, then multiply by 9*.

Some students may notice that some expressions are equivalent, for example $2(n+3)$ and $2n+6$. You do not need to comment on this now as when *Card Set C: Tables* is given out, students will be able to notice this for themselves.

Collaborative activity 2: matching expressions, words, and tables (20 minutes)

Give each small group of students a cut-up copy of *Card Set C: Tables*:

Card Set C: Tables

T1

n	1	2	3	4
Ans	14	16	18	20

T2

n	1	2	3	4
Ans			81	144

T3

n	1	2	3	4
Ans		10	15	22

T4

n	1	2	3	4
Ans	3		27	48

T5

n	1	2	3	4
Ans			81	100

T6

n	1	2	3	4
Ans		10	12	14

T7

n	1	2	3	4
Ans		4		5

T8

n	1	2	3	4
Ans	6.5	7	7.5	8

Card Set C: Tables will make students substitute numbers into the expressions and will alert them to the fact that different expressions are equivalent.

Ask students to match these new cards to the two card sets they have been working on. Some tables have numbers missing and students will need to write these in.

Encourage students to use strategies for matching. There are shortcuts that will help to minimize the work. For example, some may notice that:

Since $2(n + 3)$ is an even number, we can just look at tables with even numbers in them.

Since $(3n)^2$ is a square number, we can look for tables with only square numbers in them.

Students will notice that there are fewer tables than expressions. This is because some tables match more than one expression. Allow students time to discover this for themselves. As they do so, encourage them to test that they match for all n. This is the beginning of a generalization.

Do $2(n + 3)$ and $2n + 6$ always give the same answer when $n = 1, 2, 3, 4, 5$?

What about when $n = 3246$, or when $n = -23$, or when $n = 0.245$?

Check on your calculator.

Can you explain how you can be sure?

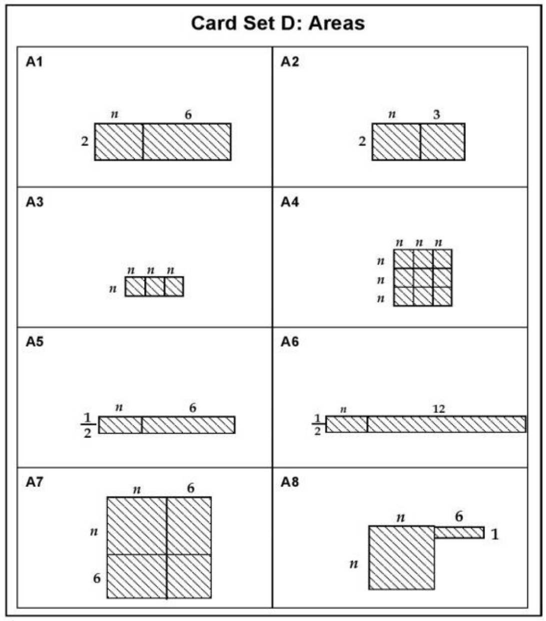
This last question is an important one and will be followed up in the next part of the lesson.

Extending the lesson over two days

It is important not to rush the learning. At about this point, some lessons run out of time. If this happens, ask pupils to stack their cards in order, so that matching cards are grouped together and fasten them with a paper clip. Ask students to write their names on an envelope and store the matched cards in it. These envelopes can then be reissued at the start of next lesson.

Collaborative activity 3: matching expressions, words, tables, and areas (20 minutes)

Give each small group of students a cut-up copy of the *Card Set D: Areas*, a large sheet of paper, a felt-tipped pen, and a glue stick.



The *Card Set D: Areas* will help students to understand *why* the different expressions match the same tables of numbers.

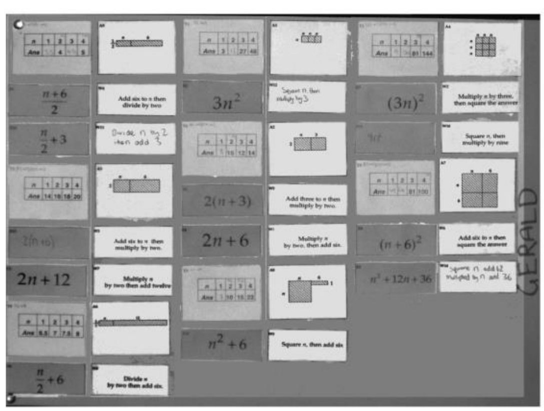
Each of these cards shows an area.

I want you to match these area cards to the cards already on the table.

When you reach agreement, paste down your final arrangement of cards onto the large sheet of paper, creating a poster.

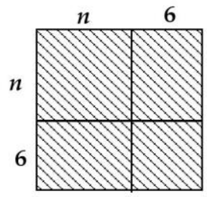
Next to each group of cards write down why the areas show that different expressions are equivalent.

The posters students produce will need to be displayed in the final whole-class discussion. They may look something like this:



As students match the cards, encourage them to explain and write down **why** particular pairs of cards go together.

Why does this area correspond to $n^2 + 12n + 36$?



Show me where n^2 is in this diagram. Where is 12n? Where is the 36 part of the diagram?

Now show me why it also shows $(n + 6)^2$.
Where is the $n + 6$?

Ask students to identify groups of expressions that are equivalent and explain their reasoning. For example, E1 is equivalent to E10, E8 is equivalent to E9, and E4 is equivalent to E5.

Whole-class discussion (20 minutes)

Hold a whole-class interactive discussion to review what has been learned over this lesson.

Ask each group of students to justify, using their poster, why two expressions are equivalent.

Then use mini-whiteboards and questioning to begin to generalize the learning:

Draw me an area that shows this expression: $3(x+4)$

Write me a different expression that gives the same area.

Draw me an area that shows this expression: $(4y)^2$

Write me a different expression that gives the same area.

Draw me an area that shows this expression: $(z+5)^2$

Write me a different expression that gives the same area.

Draw me an area that shows this expression: $\frac{w+6}{2}$

Write me a different expression that gives the same area.

Follow-up lesson: improving individual solutions to the assessment task (10 minutes)

Return students' work on the assessment task *Interpreting Expressions,* along with a fresh copy of the task sheet. If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Read through the solution you wrote [yesterday] and the questions (on the board/written on your script).

Answer the questions and then thinking about what you learned this lesson, write a new solution to see if you can improve your work.

Some teachers give this as a homework task.

SOLUTIONS

Assessment task: *Interpreting Expressions*

1a. $5n + 4$.

1b. $5(n + 4)$.

1c. $\frac{n+4}{5}$.

1d. $3n^2$.

1e. $(3n)^2$.

2. $2(n + 3) \neq 2n + 3$, $2(n + 3) = 2n + 6$.

$\frac{10n-5}{5} = 2n - 1$ is correct.

$(5n)^2 \neq 5n^2$, $(5n)^2 = 25n^2$.

$(n + 3)^2 \neq n^2 + 3^2$, $(n + 3)^2 = n^2 + 6n + 9$ *($n^2 + 3^2$ does however equal $n^2 + 9$).*

Lesson task

This table is for convenience only: it is helpful **not** to refer to cards by these letters in class, but rather to the content of the cards.

Expressions	Words	Tables	Areas
E1 E10	W4 *W13 (Blank) Divide n by 2 then add 3*	T7	A5
E2	*W11 (Blank) Square n then multiply by 3*	T4	A3
E3 *E13 (Blank) 2(n + 6)*	W3 W7	T1	A1
E4 E5	W1 W5	T6	A2
E6	W8	T8	A6
E7 *E14 (Blank)* $9n^2$	W2 W10	T2	A4
E8 E9	W6 *W14 (Blank) Square n, add 12 multiplied by n, add 36*	T5	A7
E11	W9	T3	A8
E12	*W12 (Blank) Square n then add 6 squared*		

Interpreting Expressions

1. Write algebraic expressions for each of the following:

   a. Multiply n by 5 then add 4. --

   b. Add 4 to n then multiply by 5. --

   c. Add 4 to n then divide by 5. --

   d. Multiply n by n then multiply by 3. --

   e. Multiply n by 3 then square the result. --

2. The equations below were created by students who were asked to write equivalent expressions on either side of the equals sign.

   Imagine you are a teacher. Your job is to decide whether their work is right or wrong. If you see an equation that is false, then:

   a. Cross out the expression on the right and replace it with an expression that is equivalent to the one on the left.

   b. Explain what is wrong, using words or diagrams.

$$2(n + 3) = 2n + 3$$

$$\frac{10n - 5}{5} = 2n - 1$$

$$(5n)^2 = 5n^2$$

$$(n + 3)^2 = n^2 + 3^2 = n^2 + 9$$

Card Set A: Expressions

E1 $\frac{n+6}{2}$	**E2** $3n^2$
E3 $2n+12$	**E4** $2n+6$
E5 $2(n+3)$	**E6** $\frac{n}{2}+6$
E7 $(3n)^2$	**E8** $(n+6)^2$
E9 $n^2+12n+36$	**E10** $3+\frac{n}{2}$
E11 n^2+6	**E12** n^2+6^2
E13	**E14**

Card Set B: Words

W1 Multiply n by two, then add six.	**W2** Multiply n by three, then square the answer.
W3 Add six to n then multiply by two.	**W4** Add six to n then divide by two.
W5 Add three to n then multiply by two.	**W6** Add six to n then square the answer.
W7 Multiply n by two then add twelve.	**W8** Divide n by two then add six.
W9 Square n, then add six	**W10** Square n, then multiply by nine
W11	**W12**
W13	**W14**

Card Set C: Tables

T1

n	1	2	3	4
Ans	14	16	18	20

T2

n	1	2	3	4
Ans			81	144

T3

n	1	2	3	4
Ans		10	15	22

T4

n	1	2	3	4
Ans	3		27	48

T5

n	1	2	3	4
Ans			81	100

T6

n	1	2	3	4
Ans		10	12	14

T7

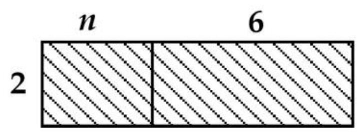
n	1	2	3	4
Ans		4		5

T8

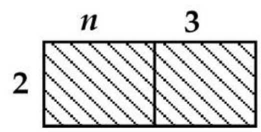
n	1	2	3	4
Ans	6.5	7	7.5	8

Card Set D: Areas

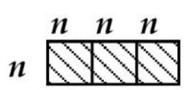
A1



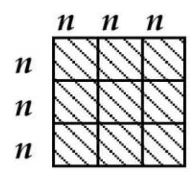
A2



A3

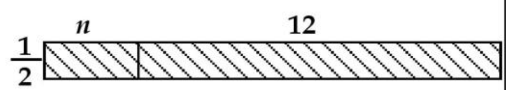


A4

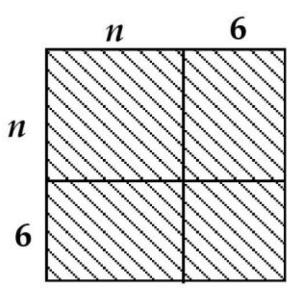


A5

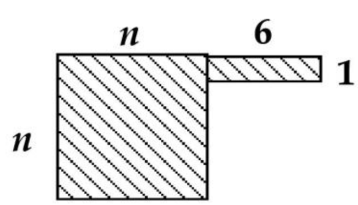
A6



A7



A8



Matching Expressions and Words

$4(n+2)$	Multiply *n* by two, then add four.
$2(n+4)$	Add four to *n*, then multiply by two.
$4n+2$	Add two to *n*, then multiply by four.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
University of Nottingham, England

Malcolm Swan, Clare Dawson, Sheila Evans,
Colin Foster and Marie Joubert
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by

Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Mat Crosier, Anne Floyde, Michael Galan, Nick Orchard, and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee