Strong chromatic index of graphs with maximum degree four

Michael Santana

Joint Work with M. Huang and G. Yu

May 2017
Definition

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The **strong chromatic index** of $G$, denoted by $\chi'_s(G)$, is the minimum number of colors needed for a strong edge-coloring of $G$. 
For every graph $G$ with maximum degree $\Delta$, 

$$\Delta \leq \chi'(G) \leq \chi'_s(G)$$
## Bounds

### Proposition

For every graph $G$ with maximum degree $\Delta$,

\[
\Delta \leq \chi'(G) \leq \chi'_s(G) \leq 2\Delta(\Delta - 1) + 1.
\]

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- The lower bound is best possible due to $K_{1,\Delta}$.
- The order of magnitude of the upper bound is also best possible as

$$\chi'_s(K_{\Delta+1}) = \binom{\Delta+1}{2} \approx \frac{1}{2}\Delta^2.$$
Conjecture (Erdős-Nešetřil ‘85)

For any graph $G$ with maximum degree $\Delta$, 

$$\chi_s'(G) \leq \begin{cases} 
\frac{5}{4}\Delta^2, & \text{for even } \Delta \\
\frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{for odd } \Delta 
\end{cases}$$
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- If $G$ is $(2K_2)$-free, then $\chi'_s(G) = |E(G)|$. 

Theorem (Chung-Gyárfás-Trotter-Tuza '90)

The number of edges in a $(2K_2)$-free graph with max degree $\Delta$ is at most $\frac{5}{4} \Delta^2 - \frac{1}{2} \Delta + \frac{1}{4}$, for odd $\Delta$. Additionally, the blow-up of $C_5$ is the unique extremal graph.
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- For $\Delta = 4$, $\chi'_s(G) \leq 22$ (Cranston ‘06)
### Conjecture

**Conjecture (Erdős-Nešetřil ‘85)**

For any graph $G$ with maximum degree $\Delta$, $\chi'_s(G)$ satisfies:

$$\chi'_s(G) \leq \begin{cases} 
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- For $\Delta = 4$, $\chi'_s(G) \leq 21$ (Huang-S-Yu ‘17++)
Theorem (Huang-S-Yu ‘17++)

If $G$ is a multigraph with $\Delta(G) \leq 4$, then $\chi'_s(G) \leq 21$. 
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Among all counterexamples, choose $G$ so that $|V(G)| + |E(G)|$ is minimized.
Theorem (Huang-S-Yu ‘17++)
If $G$ is a multigraph with $\Delta(G) \leq 4$, then $\chi_s'(G) \leq 21$.

- Among all counterexamples, choose $G$ so that $|V(G)| + |E(G)|$ is minimized.
- So $\Delta(G) \leq 4$ and $\chi_s'(G) > 21$. 
Proof Sketch

Properties of a Minimal Counterexample $G$

- $G$ is 4-regular, simple, etc.

Partition the vertices of $G$ into three sets ($L$, $M$, and $R$), where $M$ is a cut-set. Show that $M$ contains some special vertices.

Case analysis and color.
Proof Sketch

Properties of a Minimal Counterexample $G$

- $G$ is 4-regular, simple, etc.
- $G$ has girth at least 6.
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Conjecture (Erdős-Nešetřil ‘85)
If $\Delta(G) \leq 4$, then $\chi'_s(G) \leq 20$. 

Conjecture (Faudree-Gyárfás-Schelp-Tuza ‘90)
Suppose $G$ is a bipartite graph with maximum degree $\Delta$. 

1. $\chi'_s(G) \leq \Delta^2$. 

2. If $\Delta \leq 3$ and $G$ has girth at least six, then $\chi'_s(G) \leq 7$. 

3. If $\Delta \leq 3$ and $G$ has 'large' girth, then $\chi'_s(G) \leq 5$. 

Theorem (Faudree et al. ‘90)
If $G$ is a planar graph with maximum degree $\Delta$, then 

$4\Delta - 4 \leq \chi'_s(G) \leq 4\Delta + 4$. 

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MIGHTY LVIII
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MIGHTY_LVIII@ gvsu.edu
Thanks for your attention!
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