A selective version of Lin’s Theorem

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Lin’s Theorem

**Theorem (Lin ’95)**

*For any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $n \in \mathbb{N}$ and any two contractive self-adjoint matrices $a, b \in \mathbb{M}_n$ with $\|[a, b]\| < \delta$ there exist commuting self-adjoints $a', b' \in \mathbb{M}_n$ with $\|a - a'\| + \|b - b'\| < \varepsilon$. Here, $\mathbb{M}_n$ denotes the $n \times n$ matrices with complex entries, and $[x, y] := xy - yx$.***
Lin’s Theorem

Equivalent sequential statement:

Theorem (Lin ’95)

For any sequence of pairs of self-adjoint contractive matrices $a_k, b_k \in M_{n(k)}$ such that

$$\lim_{k \to \infty} ||[a_k, b_k]|| = 0$$

then there exists a sequence of commuting pairs of self-adjoints $a'_k, b'_k \in M_{n(k)}$ such that

$$\lim_{k \to \infty} ||a_k - a'_k|| + ||b_k - b'_k|| = 0.$$
Early results

Rosenthal ‘69: Are almost commuting matrices near commuting matrices?

Luxemburg-Taylor ‘70: “Yes” for arbitrary matrices with fixed dimension.

Halmos < Bastian-Harrison ‘74: “Yes” for self-adjoints with fixed dimension.
Halmos '76: Independent of dimension, are almost normal matrices near normal matrices?

Lin '95: Yes, because $a^*a - aa^* = 2i(\text{Re}(a)\text{Im}(a) - \text{Im}(a)\text{Re}(a))$. 
Berg-Olsen ’81: “No” for general operators (used weighted shifts)

Voiculescu ’81: “No” for triples of self-adjoints.

Davidson ’85: “No” for a self-adjoint and a normal. “All evidence points toward a negative solution.”
Voiculescu ’83: “No” for unitaries.

**Example**

\[
\begin{align*}
    s_n &= \begin{pmatrix}
        0 & \cdots & \cdots & 1 \\
        1 & 0 & \cdots & \cdots \\
        \vdots & \ddots & \ddots & \ddots \\
        1 & 0 & \cdots & 0
    \end{pmatrix}
    \quad \text{and} \quad
    u_n &= \begin{pmatrix}
        \omega_n & \omega_n^2 & \cdots \\
        \omega_n^2 & \cdots & \ddots & \vdots \\
        \cdots & \ddots & \ddots & \omega_n^n
    \end{pmatrix}
\end{align*}
\]

where \( \omega_n = e^{\frac{2\pi i}{n}} \).
Dimension-independent negative results

Choi ’88: “No” for arbitrary matrices.

**Example (Exel-Loring ’89)**

$s_n, u_n$ as before.

- $||[s_n, u_n]|| = |1 - \omega_n|$.
- For any pair of commuting matrices $x, y \in \mathbb{M}_n$, we have

  $$||s_n - x|| + ||u_n - y|| \geq \sqrt{2 - |1 - \omega_n|} - 1.$$
Friis-Rørdam ’96: We can replace $\mathbb{M}_n$ in Lin’s Theorem with any $C^*$-algebra satisfying property (IR). E.g. simple von Neumann algebras.
Hadwin’s Hilbert-Schmidt version

**Theorem (Hadwin ’97)**

*For any* $n \in \mathbb{N}$ *and any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that for any finite factor von Neumann algebra* $M$ *with tracial state* $\tau$, *if* $a_1, \ldots, a_n \in M_{\leq 1}$ *with* $\|[a_j^*, a_j]\|_{2,\tau}, \|[a_i, a_j]\|_{2,\tau} < \delta$ *for every* $1 \leq i, j \leq n$,

**THEN**

*there exists a commuting family of normals* $\{b_1, \ldots, b_n\} \subset M$ *such that*

$$\sum_{j=1}^{n} \|[a_j - b_j]\|_{2,\tau} < \varepsilon.$$  

*Furthermore,* $\|[a_j^* - a_j]\|_{2,\tau} < \delta \Rightarrow b_j$’s self-adjoint, *and*  

$\|[a_j^* a_j - 1_M]\|_{2,\tau} < \delta \Rightarrow b_j$’s unitary. *Here* $\|x\|_{2,\tau} = \sqrt{\tau(x^*x)}$. 

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Arzhantseva-Păunescu ’15: Almost commuting permutations are near commuting permutations with respect to the Hamming metric:

\[ d(\sigma, \rho) = \frac{|\{j | \sigma(j) \neq \rho(j)\}|}{n} \]

(independent of dimension, any size tuple).

Elek-Grabowski ’17: Almost commuting self-adjoint matrices are near commuting self-adjoint matrices with respect to the rank metric (independent of dimension, any size tuple).
Definition

Let $\mathcal{U} \in \beta \mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. For each $k \in \mathbb{N}$, let $\mathcal{A}_k$ be a unital $C^*$-algebra with tracial state $\tau_k$. We define the *tracial ultraproduct*, denoted $(\mathcal{A}_k, \tau_k)^\mathcal{U}$, to be the following quotient algebra.

$$(\mathcal{A}_k, \tau_k)^\mathcal{U} := \left(\prod_k \mathcal{A}_k\right) / \mathcal{I}_\mathcal{U}$$

where $\mathcal{I}_\mathcal{U} := \left\{ (a_k) \in \prod_k \mathcal{A}_k : \lim_{k \to \mathcal{U}} \|a_k\|_{2, \tau_k} = 0 \right\}$. Think $\ell^\infty / c_0$. 

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Tracial stability

**Definition (Hadwin-Shulman '18)**

Let \( \mathcal{C} \) denote a class of \( C^* \)-algebras closed under \(*\)-isomorphism. A unital \( C^* \)-algebra \( \mathcal{A} \) is \( \mathcal{C} \)-tracially stable if for any unital \(*\)-homomorphism \( \pi : \mathcal{A} \to (\mathcal{A}_k, \tau_k)^U \) with \( \mathcal{A}_k \in \mathcal{C} \), there exist unital \(*\)-homomorphisms \( \pi_k : \mathcal{A} \to \mathcal{A}_k \) such that \( \pi(a) = (\pi_k(a))^U \) for every \( a \in \mathcal{A} \). We say \( \pi \) is approximately liftable. Let \( \text{fvNa} \) denote the class of finite von Neumann algebras.

**Example (Hadwin-Shulman '18)**

\( C(X) \) is \( \text{fvNa} \)-tracially stable.
Proof of Hadwin’s Hilbert-Schmidt version

- By way of contradiction, suppose that for every $k \in \mathbb{N}$, there is a finite factor $M_k$ with tracial state $\tau_k$ and elements $a_1,k, \ldots, a_n,k$ such that $\|[a_i,k, a^*_i,k]\|_{2,\tau_k}, \|[a_i,k, a_j,k]\|_{2,\tau_k} < \frac{1}{k}$ with the property that for any commuting family of normal elements $\{b_1,k, \ldots, b_n,k\}$ in $M_k$, we have

$$\sum_{i=1}^{n} \|[a_i,k - b_i,k]\|_{2,\tau_k} > \varepsilon. \quad (1)$$

- Consider the ultraproduct $(M_k, \tau_k)^U$. Let $x_j = (a_j,k)_U, 1 \leq j \leq n$.

- $C^*(1, x_1, \ldots, x_n)$ is a unital abelian $C^*$-algebra.
Proof of Hadwin’s Hilbert-Schmidt version, continued

- Consider the inclusion $\ast$-homomorphism denoted $\pi: C^*(1, x_1, \ldots, x_n) \to (M_k, \tau_k)^U$.

- For each $k \in \mathbb{N}$ there is a $\ast$-homomorphism $\pi_k: C^*(1, x_1, \ldots, x_n) \to M_k$ such that $(\pi_k(a))^U = \pi(a)$ for every $a \in C^*(1, x_1, \ldots, x_n)$.

- Let $b_{j,k} = \pi_k(x_j)$.

- $\{b_{1,k}, \ldots, b_{n,k}\}$ is a collection of commuting normal elements in $M_k$ with $(a_{j,k})^U = (b_{j,k})^U$ for every $1 \leq j \leq n$, a contradiction.
More examples

Example

If $A_1, \ldots, A_n$ are $C$-tracially stable, then $*_{j=1}^n A_j$ is $C$-tracially stable.

Example (Hadwin-Shulman ’18)

If $A$ is $fvNa$-tracially stable, then $A \otimes C(X)$ is $fvNa$-tracially stable.

$fvNa$-tracial stability is preserved by free products and tensor products with abelian $C^*$-algebras...What about graph products?
Graph products

Let $\Gamma$ be a finite simplicial graph. To each $v \in V\Gamma$, assign a unital $C^*$-algebra $A_v$. Form the graph product $\star_{\Gamma} A_v$ as follows.

- If $(v, w) \in E\Gamma$ then $A_v$ and $A_w$ commute in $\star_{\Gamma} A_v$.
- If $(v, w) \notin E\Gamma$ then $A_v$ and $A_w$ have no relations in $\star_{\Gamma} A_v$.

$\Gamma$ edgeless $\Rightarrow \star_{\Gamma} A_v = \ast_{V\Gamma} A_v$.

$\Gamma$ complete $\Rightarrow \star_{\Gamma} A_v = \otimes_{V\Gamma} A_v$.

So $\Gamma$ encodes selective commuting relations.
Pincushions

Pincushion class of finite simplicial graphs \( \supset \) edgeless graphs, complete graphs, trees...

Theorem (A. ’18)

If \( \Gamma \) is a pincushion, then \( \star_{\Gamma} C(X_v) \) is \( \text{fvNa} \)-tracially stable.
Selective version of Lin’s Theorem

**Theorem (A ’18)**

Let $\Gamma$ be a pincushion, and let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that for any finite von Neumann algebra $M$ with tracial state $\tau$, if for every $v \in V\Gamma$, $a_v \in M_{\leq 1}$ such that $\|[a_v, a_v^*]\|_{2,\tau} < \delta$ for every $v \in V\Gamma$ and $\|[a_v, a_w]\|_{2,\tau} < \delta$ whenever $(v, w) \in E\Gamma$, THEN there is a family $\{b_v\}_{v \in V\Gamma}$ of normal elements in $M$ commuting according to $\Gamma$ such that

$$\sum_{v \in V\Gamma} \|[a_v - b_v]\|_{2,\tau} \leq \varepsilon.$$ 

Furthermore, $\|[a_v - a_v^*]\|_{2,\tau} < \delta \rightsquigarrow b_v$’s self-adjoint; $\|[a_v^* a_v - 1_M]\|_{2,\tau} < \delta \rightsquigarrow b_v$’s unitary.
THANKS!