Some recent results on graph products

Scott Atkinson

Vanderbilt University

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Fix $\Gamma = (V, E)$ a simplicial (undirected, no single-vertex loops, at most one edge between vertices) graph.

Assign an object (group, algebra, etc.) $A_v$ to each vertex $v \in V$.

- If $(v, w) \in E$ then $A_v$ and $A_w$ commute.
- If $(v, w) \notin E$ then $A_v$ and $A_w$ have no relations.

Edgeless graphs $\sim$ free products

Complete graphs $\sim$ tensor/direct products

**Example**

- Right-angled Artin groups ($\star_{\Gamma} \mathbb{Z}$) [Baudisch 1981]
- Right-angled Coxeter groups ($\star_{\Gamma} (\mathbb{Z}/2\mathbb{Z})$) [Chiswell 1986]
Reduced words

Bookkeeping is done by considering words with letters from $V$.

**Definition**

A word $v = (v_1, \ldots, v_n)$ is *reduced* if whenever $v_k = v_l$, $k < l$, there exists a $p$ with $k < p < l$ such that $(v_k, v_p) \notin E$.

For each $v \in V$ let $A_v$ be a unital $C^*$-algebra and fix $\varphi_v \in S(A_v)$. Put $\mathring{A}_v = \ker(\varphi_v)$.

**Definition**

A *reduced word* in $\star_{\Gamma} \mathring{A}_v$ is an element $a \in \star_{\Gamma} \mathring{A}_v$ of the form $a = a_1 \cdots a_n$ where $a_k \in \mathring{A}_{v_k}$ and $v_a = (v_1, \ldots, v_n)$ is reduced. Scalar multiples of the unit are reduced by convention.
Graph product of ucp maps

Let $B$ be a unital $C^*$-algebra. For each $v \in V$, let $A_v$ be a unital $C^*$-algebra and $\theta_v : A_v \to B$ be a unital completely positive (ucp) map such that the images $\theta_v(A_v)$ commute according to $\Gamma$.

Densely define $\bigstar_{\Gamma} \theta_v : \bigstar_{\Gamma} A_v \to B$ as follows. If $a = a_1 \cdots a_n$ is reduced with $a_k \in \hat{A}_{v_k}$, then set

$$\bigstar_{\Gamma} \theta_v(a) = \theta_{v_1}(a_1) \cdots \theta_{v_n}(a_n)$$

and extend linearly.

**Theorem (A. 2017)**

$\bigstar_{\Gamma} \theta_v$ is ucp.
Theorem (A. 2017)

Let $\mathcal{H}$ be a Hilbert space, and let $\{T_v\}_{v \in V} \subset B(\mathcal{H})$ be contractions such that if $(v, w) \in E$ then $T_v$ and $T_w$ doubly commute ($[T_v, T_w] = [T_v^*, T_w] = 0$). Then there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and unitaries $U_v \in B(\mathcal{K})$ for each $v \in V$ such that the unitaries doubly commute according to $\Gamma$ and for any polynomial $p \in \mathbb{C}\langle \{X_v\}_{v \in V} \rangle$ in $|V|$ non-commuting indeterminates we have

$$p(\{T_v\}_{v \in V}) = P_\mathcal{H}p(\{U_v\}_{v \in V})|_\mathcal{H};$$

and thus

$$\|p(\{T_v\}_{v \in V})\| \leq \|p(\{X_v\}_{v \in V})\|_{C^*(\Gamma \mathbb{Z})}.$$
The Haagerup property for group actions

Consider $G$ a discrete group, $\mathcal{A}$ a unital $C^*$-algebra, and $\alpha : G \curvearrowright \mathcal{A}$ an action.

**Definition (Anantharaman-Delaroche 1987)**

A function $h : G \to \mathbb{Z}(\mathcal{A})$ is a positive-definite multiplier with respect to the action $\alpha$ if for every sequence of group elements $g_1, \ldots, g_n \in G$, we have $[\alpha_{g_j}(h_{g_i^{-1}g_j})]_{ij} \geq 0$ in $\mathbb{M}_n(\mathcal{A})$.

**Definition (Dong-Ruan 2012)**

A group action $\alpha : G \curvearrowright \mathcal{A}$ has the Haagerup property if there exists a sequence of positive-definite multipliers \( \{h_n\} \subset C_0(G, \mathcal{A}) \) converging to 1 pointwise.
Graph products of actions

Fix a simplicial graph $\Gamma = (V, E)$ and a unital $C^*$-algebra $\mathcal{A}$. For each $v \in V$, let $G_v$ be a discrete group and $\alpha_v : G_v \curvearrowright \mathcal{A}$ be an action. If whenever $g \in G_v$, $h \in G_w$ and $(v, w) \in E$ we have $\alpha_{v,g} \circ \alpha_{w,h} = \alpha_{w,h} \circ \alpha_{v,g}$, then we can form the graph product action

$$\star_{\Gamma} \alpha_v : \star_{\Gamma} G_v \curvearrowright \mathcal{A}.$$
Graph products of multipliers

As in the previous slide, let \( \{ \alpha_v : G_v \curvearrowright \mathcal{A} \}_{v \in V} \) be a collection of group actions commuting according to \( \Gamma \).

**Definition (A. 2018)**

Suppose that for each \( v \in V \), there is a unital positive-definite multiplier \( h_v : G_v \rightarrow \mathcal{Z}(\mathcal{A}) \) such that whenever \( g \in G_v, \ h \in G_w \) and \( (v, w) \in E \) we have \( \alpha_{v,g}(h_w,h) = h_{w,h} \), then we can form the graph product multiplier \( h := \star_{\Gamma} h_v : \star_{\Gamma} G_v \rightarrow \mathcal{Z}(\mathcal{A}) \) defined as follows.

Given a reduced word \( s_1 \cdots s_n \in \star_{\Gamma} G_v \) with \( s_j \in G_{v_j} \) for \( 1 \leq j \leq n \), put

\[
h_{s_1 \cdots s_n} = \alpha_{p_1}^{-1}(h_{v_1,s_1}) \cdots \alpha_{p_{n-1}}^{-1}(h_{v_{n-1},s_{n-1}}) h_{v_n,s_n}
\]

where \( p_j = s_{j+1} \cdots s_n \) for \( 1 \leq j \leq n - 1 \) and \( \alpha \) denotes the graph product action \( \star_{\Gamma} \alpha_v \).
Results

Theorem (A. 2018)

The map \( \star \Gamma h_v : \star \Gamma G_v \to \mathcal{Z} (\mathcal{A}) \) is a positive-definite multiplier with respect to the action \( \star \Gamma \alpha_v : G_v \curvearrowright \mathcal{A} \).

Theorem (A. 2018)

If for each \( v \in V, \alpha_v : G_v \to \mathcal{A} \) has the Haagerup property and the witnessing multipliers commute according to \( \Gamma \), then \( \star \Gamma \alpha_v \) has the Haagerup property.
Tracial Stability

Motivation: great question, ask at the end!

**Definition (Hadwin-Shulman 2018)**

Given a class of tracial unital $C^*$-algebras $\mathcal{C}$, a unital separable $C^*$-algebra $A$ is $\mathcal{C}$-tracially stable if for any sequence $(C_k, \tau_k) \in \mathcal{C}$ with $\tau_k \in T(C_k)$, we have that every unital $*$-homomorphism $\pi : A \to \prod_{\mathcal{U}} (C_k, \tau_k)$ “lifts” so that for each $k$ there is a unital $*$-homomorphism $\pi_k : A \to C_k$ with $\pi(a) = (\pi_k(a))_{\mathcal{U}}$ for every $a \in A$. 
Examples

1. For most $C$, $C(X)$, $X$ compact Hausdorff;
2. Free products of $C$-tracially stable algebras;
3. If $A$ is $C$-tracially stable, then $A \otimes C(X)$ is also $C$-tracially stable (Hadwin-Shulman 2018)

Question: Given any RAAG $A$, is

$$C^*(A) = C^*(\bigstar_{\Gamma} \mathbb{Z}) \cong \bigstar_{\Gamma} C^*(\mathbb{Z}) \cong \bigstar_{\Gamma} C(\mathbb{T})$$

$C$-tracially stable where $C = \text{ff}$ is the class of all finite factors?

More general: are graph products of unital abelian $C^*$-algebras $\text{ff}$-tracially stable?
There is a class $\mathcal{G}'$ of graphs for which the corresponding graph product of abelian $C^*$-algebras is \textbf{ff}-tracially stable.
There is a class $\mathcal{G}'$ of graphs for which the corresponding graph product of abelian $C^*$-algebras is $\mathfrak{ff}$-tracially stable.
Partial results

**Theorem (A. 2018)**

If $\Gamma \in \mathcal{G}'$, then $\star_{\Gamma} C(X_{\nu})$ is ff-tracially stable.

**Corollary (A. 2018)**

"Selective Lin’s theorem" for tuples of operators in a finite factor that nearly commute according to $\Gamma \in \mathcal{G}'$ in the Hilbert-Schmidt norm.

Non-pin cushion graphs? Great question, ask at the end!
Thanks!

Smallest non-member of $\mathcal{G}'$

Wide open: $C^*(\star \square \mathbb{Z}) \cong C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ RFD? ($\iff$ Connes Embedding)