Chapter 9: Geometry: Transformations, Congruence and Similarity

The goal of 8th grade geometry is to develop student's understanding, intuition and skill with rigid motions and dilations through exploration and experimentation, using the concepts and tools with which students are already familiar: lines, circles, angles, straightedge, compass, and for measurements, ruler and protractor. In 7th grade students drew geometric shapes using straightedge and compass, and copied them; these are called *Euclidean constructions*. In Euclid's geometry two figures are said to be *congruent* if one is a copy of the other by a Euclidean construction. Then measure of line segments and angles are introduced, with ruler and compass as the appropriate devices. Students understood that two line segments are congruent if they have the same length, and two angles with the same measure are congruent. From there the topic moved to that of scale drawings, the point being to reproduce the shape of objects, but not necessarally the size. Such drawing are said to be "to scale" where scale refers to the ratio of corresponding line segments.

In the classical development of geometry (the third century BCE text, "Elements of Geometry" by Euclid) , the foundation of the subject lay in a set of "self-evident" axioms, and "constructions" by straightedge and compass. It is important to recognize that these tools were without measure, and were used to copy points, lines and angles. From there, the point of Euclid's Elements is to deduce all current knowledge from these basics, using only these axioms and tools and Aristotelian logic. The accomplishment was monumental, and formed the basis of instruction for 2000 years. Toward the end of the 19th century CE, there developed, from the application tp contemporary science and engineering, a perceived need to view geometry as defined by the motions or perceptions of objects that preserve shape, and in most cases, size. This was mainly because the perception of the world was no longer planar, but often spherical or hyperbolic (note the connection that the word"hyperbola" has to the meaning "literary excess, extravagance"). This change was put forth by Felix Klein about one hundred years ago as a way to develop geometry based on its dynamical, rather than static, use. This is the approach to geometry followed by this text. In 8th grade this is introduced and explored through explorations and activities; the intent is that students have an intuitive sense of basic geometric facts, that will aid them in its logical development.

In Chapter 2, we introduced translations and and dilations, and observed their properties as transformations of the plane, in order to show that the slope of a line can be calculated using any two points on the line. We used these basic properties of dilations: it has one fixed point (the center of the dilation) and all other points are moved away (or toward) the center. There is a positive number r such that the dilation multiplies any length by r. Note that if $r = 1$, then there is no movement at all: in this case the dilation is called the *identity*: no point moves.

In this chapter we begin to look at transformations of the plane more deeply, in order to get an understanding of the shape and size of a geometric object, no matter where it is positioned on the plane. A *rigid motion* of the plane is a transformation of the plane that takes lines to lines , and preserves lengths of line segments and measures of angles. That is, under a rigid motion, a line segment and its image have the same length, and an angle and

its image have the same measure. A translation (or *shift*) is a rigid motion. There are two other kinds: *reflections* (flips) and *rotations*(turns). Here we consider two figures *congruent* (have the same shape and size) if there is a sequence of rigid motions that takes one to the other. This is a different way of looking at the equivalence of two figures without changing the meaning: if two objects are congruent by way of a Euclidean construction, then there is a sequence of rigid motions that takes one to the other. And if we can move one object onto another by rigid motions, there is a construction taking one to the other. The advantage of working with motions rather than constructions, is that the idea is more directly related the the use of geometry in science and engineering: one does not put a beam on a house by construction in place, but by moving the beam from one place to the other. If we want to create a robot to do that job, we need to conceive of it in terms of rigid motions, not constructions.

In the second section we turn to dilations and scale factors: a dilation preserves lines and angles, but changes the scale of length of line segments. We say that two figures are *similar* (have the same shape) if there is a combination of rigid motions and dilations that takes one to the other.

The focus of 8th grade geometry is to explore the concepts of transformations, congruence and similarity by experimenting with them and gaining familiarity with the correspondence between constructing a new image of an object, and moving the object to its new location. We concentrate on the "what" and "how" of geometry, while high school geometry extends that basis to understanding the "why." In real-life science and industry, people almost constantly draw representations (called *graphics*) of their work, even if it is about medical procedures or finance rather than architecture or construction. In 8th grade we plant the foundations for these skills.

To begin with, students should be given an opportunity to discuss the concepts of "same shape" and "same size and shape," and so it would be good to give several examples like this:

Example 1. The following figure shows several sets of images. In the figure A all the images are of the same shape, and we can move any one to any other one by a rigid motion. Students should try to describe the required set of motions. In the remaining figures, there is no rigid motion taking one figure to the other, and students should be asked to articulate the reasons.

In figure A, students should describe how to move one figure onto another: the first two the third is a translation and rotation, and the first to the second involves a reflection. However, in figure B, the figures are of the same shape, but not of the same size. In figures C and D the figures are neither of the same size nor shape.



Figure A



Figure B



Figure C



Figure D

We recall some basic geometric facts that have been observed in previous grades.

- 1. A line is determined by any two different points on the line, by placing a straight edge against the points and drawing the line.
- 2. Two lines coincide (are the same line) or intersect in precisely one point or do not intersect at all. The issue may come up: what if they do not intersect on my paper, how do I know whether or not they ever intersect? Because the question did come up in the days of Euclid, it generated a controversy that lasted for almost 2000 years.
- 3. Two circles do not intersect, or intersect in a point, or intersect in two points. If they intersect in more than two points, they actually coincide.
- 4. Two lines that do not intersect are said to be *parallel*. If two lines intersect and all the angles at the point of intersection have the same measure, the lines are said to be *perpendicular*.
- 5. The sum of the lengths of any two sides of a triangle is greater than the third.

Section 9.1. Rigid motions and Congruence

Understand congruence in terms of translations, rotations and reflections, (rigid motions) using ruler and compass, physical models, transparencies, geometric software.

Verify experimentally the properties of rotations, reflections and translations:

a) lines are taken to lines, and line segments to line segments of the same length;

b) angles are taken to angles of the same measure;

c) parallel lines are taken to parallel lines. 8G1

A rule that assigns, to each point in the plane another point in the plane is called a *correspondence*. Often a correspondence is defined in terms of coordinates, and written this way: $(x, y) \to (x', y')$, where the values of x', y' are given by the rule.

Example 2.

- 1. The rule may be: "add 3 to the first coordinate and subtract 1 from the second." Then the coordinate rule is $(x, y) \to (x + 3, y - 1)$.
- 2. The rule " double each coordinate" can be written as $(x, y) \to (2x, 2y)$.
- 3. The rule " Multiply the first coordinate by 2 and the second coordinate by 3" is written as $(x, y) \to (2x, 3y)$.

- 4. The rule "square each coordinate" is written as $(x, y) \to (x^2, y^2)$.
- 5. The rule " Multiply teach coordinate by 0" is written as $(x, y) \to (0, 0)$.
- 6. Consider the rule $(x, y) \to (y, x)$. We can get a feel for this rule by drawing a few figures and then, using the coordinate rule to draw their images under the correspondence,

A *mapping* of the plane is a correspondence with this property: can be described in this way: different points go to different points; that is, for two points $P \neq Q$ we must also have $T(P) \neq T(Q)$. This is in fact, just what a map does: it takes a piece of the surface of the earth and represents it, point for point, on the map M. When we study the mappings of objects, it is useful to call the object O the *pre-image* and the set of points to which the points of) are mapped is the *image*, denoted $P(O)$ Of the rules described in example 2, rules 1,2,3,6 are mappings, while rules 4 and 5 are not. We need a little more vocabulary before going on. An object is *fixed* under a mapping, if the mapping takes the object onto itself. When we say that an attribute is *preserved* we mean that if the original object has that attribute, so does the image object.

To be useful, maps must preserve features of interest: so for example, a scale drawing of an object changes only the dimensions, and not the shape, of the object. In this section we are interested in mappings that preserve both the dimension and shape of objects. These are the *rigid motions*. These are mappings of the plane onto itself that takes lines to lines and preserve lengths of line segments and measures of angles. That definition is the starting point of geometry in Secondary 1, but not in 8th grade, where the emphasis is on an intuitive understanding of rigid motions and their action on figures. So, for us, rigid motions are introduced by visualization and activities. Take two pieces of transparent paper with a coordinate grid. Place one on top of the other so that the coordinatizations coincide. A *rigid motion* is given by a motion of the top plane that does not wrinkle or stretch the piece of paper.

Given a figure on the plane, we can track its motion by a rigid motion T in this way. Place a piece of transparent graph paper on top of another so that the coordinate axes coincide. We start with a figure on the bottom piece of paper. Copy that figure on the upper piece of paper. Enact the rigid motion T by moving the upper piece of paper. Now trace the image on the upper piece of paper onto the lower. This figure is the result of moving the given figure by the rigid motion T. Keep in mind that the lower piece of paper is where the action is taking place; the upper piece is the action. Students should experiment with these motions using transparent papers. In the process they will observe that these motions do preserve lines and the measures of line segments and angles. Students may also observe that there are three fundamental kinds of rigid motions characterized by: a) no point is left fixed b) precisely one point is left fixed, c) there is a line all of whose points are left fixed.

Rigid motions include

- Translation: these are the rigid motions T of the plane that preserve "horizontal" and

"vertical": that is, horizontal lines remain horizontal, and vertical lines remain vertical. We can describe this as sliding the top piece of paper over the bottom so that the horizontal and vertical directions remain the same.

- Reflection in a line. Select a line on the plane. This will be the *line of reflection*: all points on the line remain in the same place, and all other points move to their opposite in the line. We effect this with the transparent paper in this way: fold the top transparency (in three dimensions) along that line so that the sides determined by the line exchange places, and so that nothing on the line moves.

- Rotation about a point: these are the rigid motions of a plane that leave one point fixed. This point is called the *center of the rotation*. The center will not move. All other points are moved along a circle with that point as center.

Through experimentation with these moments, students should observe that translations do not leave any points fixed; rotations are rigid motions that leave just one point fixed; for a reflection, all the points on the line of reflection remain fixed. In the next subsections we will work with these motions in detail and collect together their properties.

Example 3. Of the coordinate rules in example 2, which can be represented by a rigid motion?

- 1.This can be realized as a rigid motion: shift the top piece of paper so that the the origin goes to the point (3,-1).This is an example of a translation.
- 2. The second rule stretches the top piece of paper (by a factor of 2) so is not a rigid motion.
- 3. The third rule is even less rigid: it stretches horizontal lengths by 2, and vertical lengths by 3
- 4. The squaring rule does not take distinct points to distinct points: for example (1,1) and (-1,-1) both go to (1,1). In fact, for any positive a and b, all four points $(a, b), (-a, b), (a, -b), (-a, -b)$ go to the same point (a^2, b^2).
- 5. The rule $(x, y) \to (0, 0)$ moves every point to the origin, so is not a mapping.
- 6.The rule $(x, y) \to (y, x)$ is the reflection in the line $y = x$, as we shall see below.

Section 9.1.1 Translations

We have defined *translation* as a rigid motion of the plane that moves horizontal lines to horizontal lines and vertical lines to vertical lines. Let's find the coordinate rule for a translation. Starting with a particular translation, let (a, b) be the coordinates of the image of the origin. We want to show that the pair of numbers (a, b) completely determines the translation; in fact, the coordinate rule is: $(x, y) \to (x+a, y+b)$. For a dynamic visualization, go to http://www.mathopenref.com/translate.html.

For any point (x, y), draw the rectangle with horizontal and vertical sides with one vertex at the origin and the other at (x, y). The translation takes this rectangle to a rectangle with horizontal and vertical sides with one vertex at (a, b) and the other at the image of (x, y). Since the lengths of the sides are preserved, the image rectangle has the same dimensions as the pre-image, and so the vertex across the diagonal from (a, b) has to be $(x + a, y + b)$. But that is the image of (x, y).

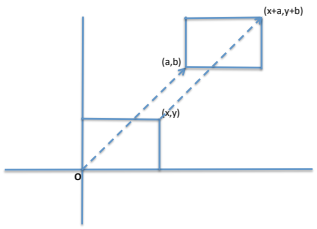


Figure 1

We refer to (a, b) as the *vector* for the translation, for it shows both the direction in which any point is translated, and also the distance it s translated. Now, in our figure, the point (x, y) was chosen to be in the first quadrant; but the same reasoning works for any point (x, y).

This reasoning also shows that a translation preserves the slopes of lines (in particular, any line and its image are parallel). In figure 2 we show the effect of a translation on the line L, denoting its image by L'. Draw the slope triangle AVB for the line L as shown. Now, the translation moves that triangle to the triangle $A'V'B'$ as shown. Since the translation preserves "horizontal" and "vertical," $A'V'B'$ is a slope triangle for the line L'. Now since the translation preserves lengths, the horizontal and vertical legs of the slope triangle on L' have the same lengths as the horizontal and vertical legs of the slope triangle on L, so the lines have the same slope. (Since we have talked about lengths and not signed lengths, why hasn't the sign changed?)

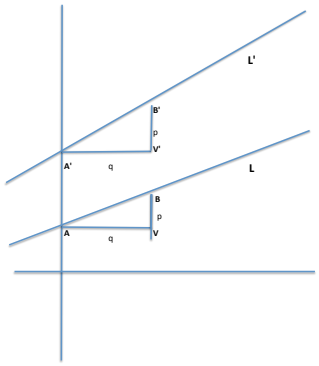


Figure 2

Properties of translations:

- A translation preserves the lengths of line segments and the measures of angles.
- For a translation, there is a pair (a, b), called the *vector* of the translation, such that the image of any point (x, y) is the point $(x + a, y + b)$.
- Under a translation, the image of a line L is a line L' parallel to L Furthermore, translations take parallel lines to parallel lines. This is because a translation does not change the slope of a line.
- A translation that does not leave every point fixed does not leave any point fixed.

Section 9.1.2 Reflections

We have defined a *reflection* by the action of flipping (or folding) along a line L, called the *line of reflection*. A reflection can be described this way: it is a motion that leaves every point on L fixed. If P is a point not on L, and P' is its image under the reflection, then L is the perpendicular bisector of the line segment PP'. We now show that reflections as we visualize them (folding the plane along the line L) have these properties. For a dynamic visualization, go to http://www.mathopenref.com/reflect.html.

First, it is clear, since it is described by a motion that does not stretch our paper in any direction, that a reflection preserves the lengths of line segments and the measure of angles. Let L be the line of reflection for the reflection T, let P be a point on one side of L and P'

the image of P under the reflection. Draw any line through P (not parallel to L), and let Q be the point of intersection of that line with L. This situation is depicted in Figure 3.

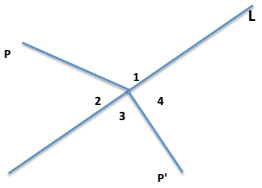


Figure 3

Since a reflection takes lines to lines and leaves the point Q fixed, the image of the segment PQ is the segment $P'Q$. Since the lengths of these segments are the same, we can say that the points P and P' are at the same distance from any point on the line L. Furthermore the image of $\angle 1$ is $\angle 3$, so they have the same measure, and the image of $\angle 2$ is $\angle 4$, so they have the same measure. In particular, if the line PQ is chosen to be perpendicular to L, then $\angle 1$ and $\angle 3$ have the same measure, and thus all these angles have the same measure, so in this case (as shown in Figure 4) since PQ and $P'Q$ have the same measure, L is the perpendicular bisector of the segment PP'.

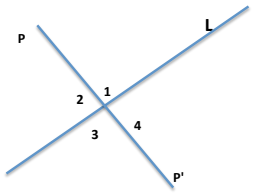
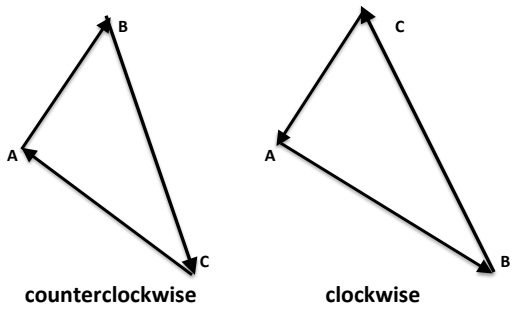


Figure 4

Something special happens with reflections that does not happen with other motions. Notice that for translations and, as we will see, rotations, we do not have to lift the top piece of

transparent paper off the bottom piece; but with reflections we must do so; we execute what we can call a *flip* over the line of the reflection. Here is one effect of this: suppose that A and B are two different points on a line L perpendicular to the line of reflection. Then if, say B is to the left of A to begin with, the image of B is to the right of the image of A. A good illustration of this is the operation of taking the "opposite" of numbers on the real line. If a and b are positive numbers, with $a < b$, then a is to the left of b. Taking the opposite (or what is the same, multiplying by -1), the opposite of a is to the right of the opposite of b (or what is the same, $-b < -a$). We describe this in terms of "orientation", and conclude that "taking the opposite", or more generally, multiplication by a negative number, *changes the orientation* on the real line: it changes "left" and "right."

On the line, orientation is described in terms of an interval and the concepts "left" and "right". When we go to the plane, the basic figure is a triangle - or in fact, any polygon. If we draw a triangle, and label its vertices A, B, C, then there are two senses of going "around the perimeter:" the sequence $A \to B \to C$ traverses the perimeter either "clockwise" or "counterclockwise" (see the figure below).



Orientation can also be described in terms of angles. Consider an angle $\angle AVB$ (with vertex V) determined by the rays VA and VB. Looking out at the angle from V we can say that one of the rays is *clockwise* from the other (in Figure 5A, the ray VB is clockwise from VA). The reverse direction is called *counterclockwise*. Now the point we want to make is that reflections interchange the rays of an angle in the sense of "clockwise." This is depicted in FIgure 5B, representing a reflection in the line L. The angle $\angle AVB$ goes to the angle $A'V'B'$ under the reflection. But, while the ray VB is clockwise to the ray VA, the image ray $V'B'$ is counterclockwise to the image ray $V'A'$. This what is meant when we say that

"reflections change the orientation of the plane." This is what happens when we look in a mirror: our left hand is on the right side of that person in the mirror.

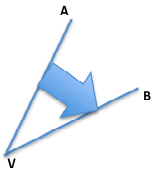


Figure 5A

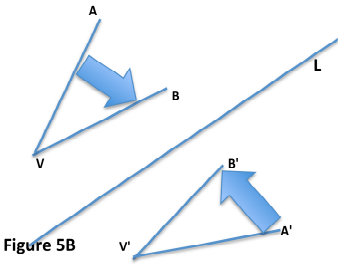


Figure 5B

Properties of Reflections:

- A reflection preserves the lengths of line segments and the measures of angles.
- For a reflection, there is a line L, called the *line of the reflection*, such that for any point P, L is the perpendicular bisector of the line segment joining P to its image.
- A reflection leaves every point on L fixed, and interchanges the two sides of that line. If the image of a point P is P', then, for any point Q on L, the line segments PQ and $P'Q$ are the same.
- A reflection reverses orientation; that is if two rays start at the same point, and ray 2 is clockwise from ray 1, the the image of ray 2 is counterclockwise from that of ray 1.

It is possible to describe all rigid motions by coordinate rules; but at this time it will suffice to show how to do so for these particular reflections: in the coordinate axes and in the lines $y = x$ and $y = -x$.

Example 6. Reflection in the x-axis leaves the x coordinate the same and changes the sign of the y coordinate. For this reflection takes vertical lines to vertical lines, and so the x-coordinate is fixed. For any point (x, y) (with $y \neq 0$), its image is a point (x, y') with $|y| = |y'|$ and $y \neq y'$; the only possibility is that y' is $-y$

Example 7. Reflection in the y-axis leaves the y coordinate the same and changes the sign of the x coordinate. This has the same argument.

Example 8. Reflection in the line $L : y = x$ exchanges the coordinates: a point (x, y) goes to the point (y, x). To show this, let's start with a point $P : (a, b)$ not on the line, so that $a \neq b$. See figure 6 for a proof without words. Nevertheless, here are the words:. draw the

vertical line segment S from (a, b) to the line L: it intersects the line L in the point (a, a). and at the point of intersection, the angle between L and this line segment is 45°. Now, the image of the line segment S will also have an angle of 45° with L and so will be horizontal. The endpoint of the image line segment is the image of P, and has coordinates (u, a) for some number v. Now draw the horizontal line segment S' to the line L, intersecting L in the point (b, b). By the same reasoning, the image of S' is a point with coordinates (b, v). But since this is again the image of P under the reflection we must have $u = b, v = a$, so the image of (a, b) is (b, a)

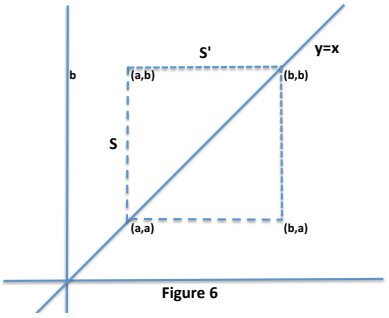


Figure 6

Question. Can you show that reflection in the line $L : y = -x$ can be described in coordinates as $(x, y) \to (-y, -x)$?

Section 9.1.3 Rotations

We have defined a *rotation* as a rigid motion that turns a figure about a fixed point, called the *center* of the rotation. Since lines are mapped into lines and the center C is fixed, any ray with endpoint C is moved to another ray with endpoint C. A rotation can be defined by this property: the angle between any ray with endpoint C and its image is always the same angle α. If $\alpha \neq 0$ the center is the only fixed point of the oration, and if $\alpha > 0$ the rotation is counterclockwise, and if $\alpha < 0$, the rotation is clockwise. For a dynamic visualization, go to http://www.mathopenref.com/rotate.html.

This can be shown using figure 7. Let A be a point on the positive real axis, and B its image under the rotation. Let P be any other point, and Q its image under the rotation (see Figure 7). The angle $\angle ACP$ is transformed into the angle $\angle BCQ$; since the rotation is rigid, these angles are the same. What we want to show is that the angles $\angle ACB$ and $\angle PCQ$ have the

same measure. But

$$\angle ACP = \angle ACB + \angle BCP \quad \text{and} \quad \angle BCQ = \angle BCP + \angle PCQ \ .$$

Since the left hand sides are equal (a rotation preserves angles, we can subtract the second equation from the first to get

$$\angle ACB = \angle PCQ \ .$$

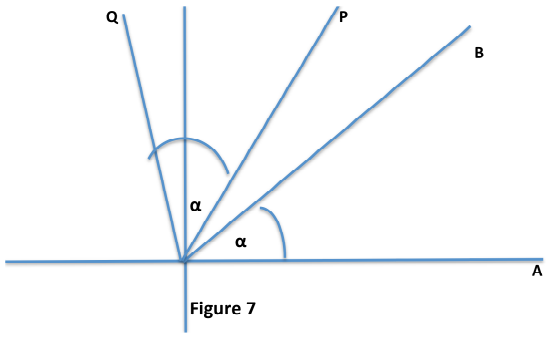


Figure 7

Properties of rotations:

- A rotation preserves the lengths of line segments and the measures of angles.
- For a rotation, there is a point C, called the *center of the rotation*, and an angle α called the *angle of rotation*. For any point P with image Q, the angle $\angle PCQ = \alpha$.
- A rotation preserves orientation; that is, if two rays start at the same point, and the second is clockwise from the first, then the image of the second is also clockwise from the first.

For those angles that are integral multiples of a right angle, we are able to find the coordinate rule of the rotation.

Example 9. Rotation about the origin by 90°. See figure 8, where P is a point in the first quadrant and Q is the image of P. The rotation moves triangle I into the position of triangle II. Note that t the horizontal leg of triangle I is the vertical leg of triangle II, and the vertical leg of triangle I is the horizontal leg of the image, triangle II. Thus the lengths are the same, but since we are in the second quadrant the first coordinate is negative so we must have

$(x, y) \to (-y, x)$ as labeled. Can you show that this rule holds for a point P selected in one of the other quadrants? Can you find the coordinate rule for a clockwise rotation by a right angle (of - 90°)?

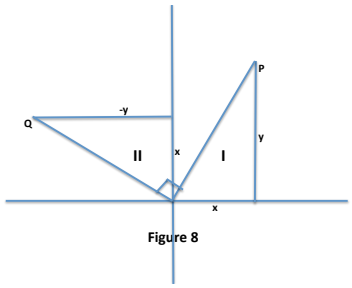


Figure 8

Take a moment to note that a rotation by 90° takes a line into another line perpendicular to it. Thus - as we saw in Chapter 2 - the product of the slope of a line and that of its image, under a rotation by 90° is -1. Note that this is demonstrated in figure 1: the slope of the original line is y/x and that of its image is $-x/y$. This is true whether or not the rotation is clockwise or counterclockwise. Thus the coordinate rule for a clockwise rotation by a right angle (of - 90°) has to be $(x, y \to (y, -x)$ (this can also easily be shown by a diagram).

Example 10. Rotation about the origin by 180°. See figure 9, where P again is in the first quadrant and Q is its image. Since the rotation is by 180°, P and Q lie on the same line through the origin, and the length of the segment CP and CQ are the same. In other words, Q is diametrically opposite to P, so is the point $(-x, -y)$.

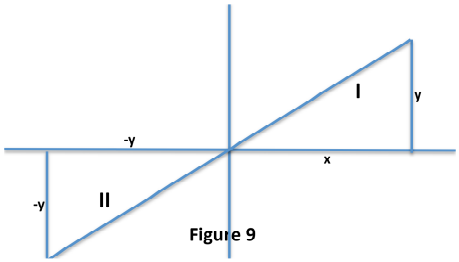


Figure 9

Section 9.1.4. Succession of rigid motions

Now, a rigid motion is a transformation of the plane that takes lines into lines and that preserves lengths of line segments and measures of angles. If we follow one rigid motion by another,we get a third motion which clearly has the same properties: lines go to lines and measures of line segments and angles do not change. We have discussed specific kinds of rigid motions: translations, reflections and rotations. It is a fact that every rigid motion can be viewed as a a succession of motions of one or more of these types; in this section we will look at such examples.

Example 11. Consider two line segments AB and $A'B'$ of the same length as in figure 10a. There is a rigid motion that takes the segment AB onto $A'B'$. First of all, we can translate the point B to the point B' creating $\angle AB'A'$ (see Figure 10a). If we now rotate the line segment around the point B' through the angle $\angle AB'A'$, then the segments AB' and $A'B'$ lie on the same ray. But since the segments have the same length, the point A lands on A', and the succession of the translation by the rotation is the rigid motion taking AB to $A'B'$.

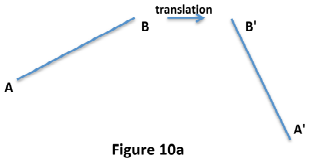


Figure 10a

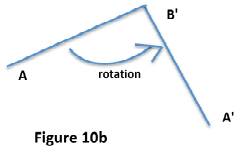


Figure 10b

Example 12. Given two circles of the same radius, there is a rigid motion taking one circle onto the other. In fact, the translation of one center to the other does the trick. Can you explain why? Hint: use the definition of "circle."

Example 13. If two angles have the same measure, there is a rigid motion of one to the other. Let AVB and $A'VB'$ be the two angles. Notice, that the two angles have the same vertex. Initially, they need not, but a translation of one vertex to the other arranges that. Now, look at figure 11. We have already taken another liberty: we have labeled the rays of each angle so that the orientation is consistent: VB is clockwise from VA and VB' is clockwise from VA'. Now rotate with center V so that the ray VA lands on the ray VA'. Since the original angles had the same measure, and we have set up the orientation correctly, the ray VB falls on the ray VB'. The combination of the translation and rotation is the rigid motion landing one angle onto the other. Question: What if we didn't label the angles

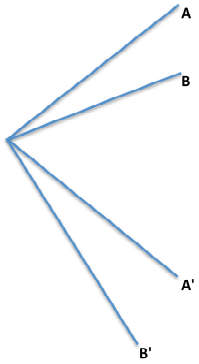


Figure 11

so as to preserve the orientation? Hint: use an inversion to interchange the rays of one of the angles.

Example 14. Under what conditions can we find a rigid motion of one rectangle onto another? First of all, rigid motions preserve lengths and angles, so any rigid motion will alway move a rectangle to another rectangle whose side lengths are the same. So, if there is a rigid motion of rectangle R onto rectangle R', the lengths of corresponding sides must be the same.

But now -to answer the question - if this condition holds, then there is a rigid motion of one rectangle onto the other. We will show this using figure 12 of two rectangles $ABCD$ and $A'B'C'D'$ with corresponding sides of equal lengths. We have labeled the vertices so that the routes $A \to B \to C \to D$ and $A' \to B' \to C' \to D'$ are both clockwise. (Can you check that we can really do that?). By example 13, we can find a rigid motion taking the angle DAB onto the angle $D'A'B'$ (translate A to A' and then rotate). Since the lengths of corresponding sides are equal, that tells us that D lands on D' and B onto B'. Since both figures are rectangles, that forces C onto C', so the rectangles are congruent.

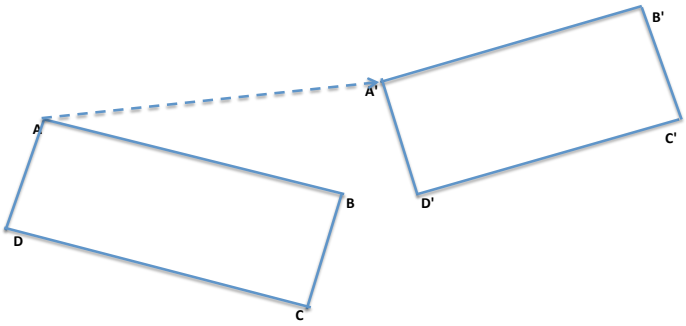


Figure 12

We want to do that so that the ray BA lands on the ray $B'A'$. Well, if it doesn't, then it lands on the ray $B'C'$. We can fix that by a reflection in the line bisecting the angle $A'B'C'$. Now, because lengths are preserved, it must be that the first rectangle has landed on the second.

Since reflections reverse the orientation on the plane, the succession of two reflections preserves orientation, so has to be a pure rotation or a pure translation or a combination of the

two. Let us look at the possibilities. Let R be the reflection in the line L, S the reflection in the line L', and T the combined motion: R followed by S.

a) If the lines L and L' are the same line, then S just undoes what R did, and the succession of R by S leaves every point where it is. So, in this case, the succession of reflections is the *identity* motion - no motion at all.

b) Suppose that the lines L and L' are different, and intersect in a point C . Then C is fixed point for each, so is fixed under the succession T. Suppose T had another fixed point P. Then the first reflection (R) sends P to another point P' and S returns P' to P. Well, that means that P and P' are reflective images in both L and L', and that can only be if the lines are the same. Since they are not, T has only one fixed point: C. But the only such motions are the rotations.

Example 15a. Let R be the reflection in the y axis, and S, reflection in the x axis. Then what is T - the succession of R by S? Do the same problem with either of the lines of reflection replaced by the line $y = x$.

c) Now, if L and L' are different and do not intersect, they are parallel. Show that this tells us that the succession of R by S has *no* fixed points, so must be a translation.

Example15b. Let R be reflection in the line $x = 1$, and S reflection in the line $x = 2$. Describe the translation T.

Both R and S take vertical lines to vertical lines, and take horizontal lines to themselves, so that is true of the succession T. Now, R takes (0,0) to (2,0) and S leaves (2,0) alone, so T takes (0,0) to (0,2). Since a translation does to all points what it does to one, we can say, in coordinates, that T takes (x, y) to $(x + 2, y)$.

Question. Let R be reflection in the line $x = 1$, and S reflection in the line $x = 3$. Describe the translation T. Now replace 1 by a and 3 by b.

Properties of a succession of two rigid motions:

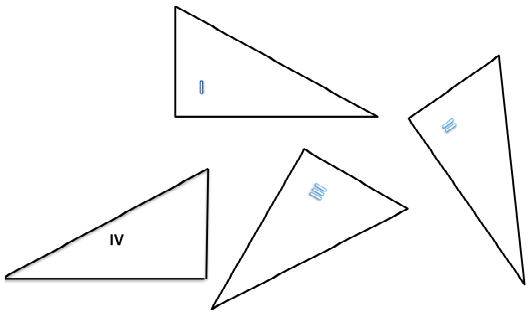
- Rotations: if they have the same center then the succession of the two is a rotation with that center, and whose angle is the sum of the angles of the two given rotations.
- Translations: if T is a translation by (a, b), and T' the translation by (a', b'), then the succession of one after the other is the translation by $(a + a', b + b')$.
- Reflections:
- If the lines of the reflection are the same, we get the identity (that is, there is no motion: every point stays where it is.
- If the lines of the reflection are parallel, we get a translation.
- If the lines of the reflection intersect in a point, we get a rotation about that point.

Section 9.1.5 Congruence

Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. 8G2

Two figures are said to be *congruent* if there is a rigid motion that moves one onto the other. In high school mathematics the topic of congruence will be developed in a coherent, logical way, giving students the tools to answer many geometric questions. In 8th grade we are much more freewheeling, discovering what we can about congruence through experimentation with actual motions. In this section we will list some possible results that the class may discover; many classes will not discover some of these, but instead discover other interesting facts about congruence.

Example 16. Using transparencies, decide, among the four triangles in figure , which are congruent.



Solution. Translating and then rotating appropriately, we can put triangle I on top of triangle II, so these are congruent. When we do the same, moving triangle I to triangle III, we find that the shortest leg of triangle I is shorter than the shortest leg of triangle II; since the short legs must correspond, these triangles are not congruent. Now, as for triangle IV, we can translate the right angle of triangle I to that of triangle !V. Since both short legs are vertical, we see they coincide. If we now reflect in that short leg, we land right on triangle IV. Thus, I and IV are also congruent.

Example 17. All points in the plane are congruent. This may seem unnecessary to point out, but it does state a fact: given any points P and Q in the plane, there is a rigid motion taking P to Q. Actually, there are many. First of all, there is the translation of P to Q, and we can follow that by any rotation about the point Q. There is also a reflection of P to Q: fold the paper in such a way that P lands on top of Q. Then the crease line of the fold is the perpendicular bisector of the line segment PQ, and reflection in this line takes P to Q.

The following have been observed in the preceding sections:

- Two line segments are congruent if they have the same length; otherwise they are not.
- Two angles are congruent if they have the same measure.
- Two rectangles are congruent if their side lengths are the same. In particular, all squares of the same area are congruent, but not all rectangles of the same area are congruent.

Finding and demonstrating criteria for the congruence of triangles is a major topic in Secondary I; as an illustration, here we show how one such criterion, nown as SSS, follows from the properties of rigid motions.

Example 18. Given two triangles, ABC and $A'B'C'$, if corresponding sides have the same length, then the triangles are congruent. Students should be given the opportunity to try to create an example where this fails. They will conclude that this is true, and should provide an explanation, which could be something along these lines; First, since line segments of the same length are congruent, we can move AB onto $A'B'$ by a rigid motion. Now draw a circle with center at A and radius if the length of AC. Draw another circle, with center at B and radius if the length of BC. These two circles intersect in two points. Furthermore, each of C and C' is one of those points. If C and C' are the same point, then we have shown that the triangles are congruent, If they are different, then we can continue with the reflection in the line segment AB; this will place one triangle onto the other, and so the triangles are congruent.

Section 9.2. Dilations and Similarity

Understand similarity in terms of rigid motions and dilations using ruler and compass, physical models, transparencies, geometric software. 8G3,4

Section 9.2.1. Properties of dilations

Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates 8G3. Verify that dilations take lines into lines, takes parallel lines to parallel linesm and that a line and its image under a dilation are parallel.

Recall that in chapter 2, in the section on the slope of a line, we introduced the idea of a dilation in order to show that the slope of a line can be calculated using any two points.First, we review that discussion.

A *dilation* is given by a point C, the *center* of the dilation, and a positive number r, the *factor* of the dilation. The dilation with center C and factor r moves each point P to a point P' on the ray CP so that the ratio of then length of image to the length of original is r:

$|CP'|/|CP| = r$. The important fact about a dilation is that, for *every* line segment, the length of its image is r times the length of the segment.

In order to show the basic fact about slope (that it can be computed as the rise/run quotient using *any* two points on the line), we started with a line L, and drew two right triangles T and T' with horizontal and vertical legs and hypotenuse on the line L (see figure 13).

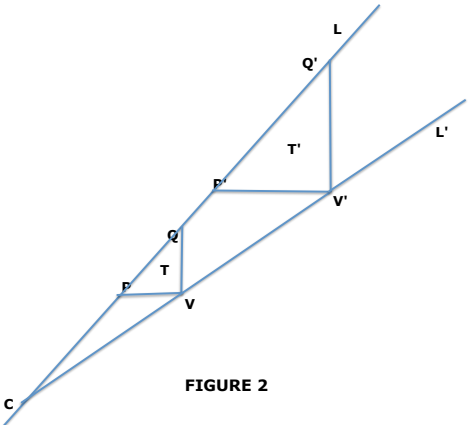


Figure 13

Then we added the line L' through V and V'. If it is parallel to L, then the translation of P to P' carries T to T', so the triangles are congruent: the "rise" and "run" have the same length, so the quotient (slope = rise/run) is the same for both triangle. If the lines L and L' are not parallel, then they intersect in a point C, and the dilation of center C that takes P to P' takes T to T', so the lengths of the sides of T' are all the same multiple of the lengths of the corresponding sides on T. So, again, the rise/run computation is the same, since the common multiple of rise and run cancels.

Dilations are also directly connected to scale changes. Suppose that we start with an image of a rectangle, where the scale is in yards (as in figure 14a). We want more detail, so we create a new image where the scale is in feet. Then, each interval (representing a yard) in the original image must be replicated across three intervals (each of which represents a foot), and so our change of image is a dilation with scale factor 3. The result is Figure 14b. Note that each length has been multiplied by 3, while the area of the larger rectangle is 9 times the area of the of the original rectangle. This will of course be true of every dilation of any figure: lengths are multiplied by the scale factor r, while areas are multiplied by the square of the scale factor, r^2.

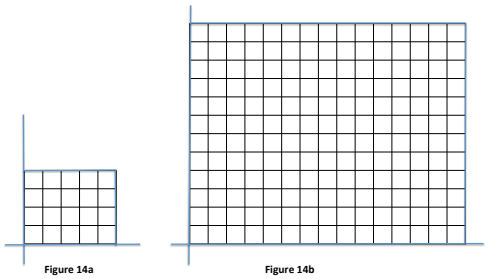


Figure 14a Figure 14b

Now we will find the coordinate rule that expresses a dilation with center the origin. Suppose the factor of the dilation is r. Then the length of any interval is multiplied by r, so that the point $(x, 0)$ goes to the point $(rx, 0)$ and the point $(0, y)$ goes to the point $(0, ry)$. It follows (see figure 16) that $(x, y) \to (rx, ry)$.

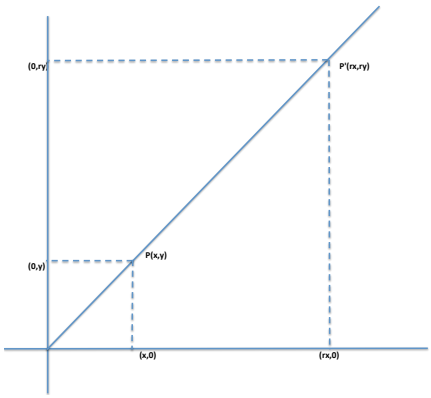


Figure 15

Properties of the dilation with center C and factor r

- If P is moved to P', then $|CP'|/|CP| = r$.
- If P is moved to P' and Q is moved to Q', then $|Q'P'|/|QP| = r$.
- The dilation moves parallel lines to parallel lines.

- Parallel lines are moved to parallel lines.
- An angle and its image have the same measure.

All of these facts, except the last, were demonstrated through examples in Chapter 2. We can show that an angle and its image under a dilation are actually congruent. In figure 16. Suppose that $\angle A'V'B'$ is obtained from $\angle AVB$ by a dilation. First of all, corresponding lines are parallel. Now, translate V to V'. Since corresponding lines under a translation are parallel, the ray VA must go to the ray $V'A'$ and the ray VB to the ray $V'B'$. Well, then the translation T takes the angle $\angle AVB$ to the angle $\angle A'V'B'$, so they have the same measure.

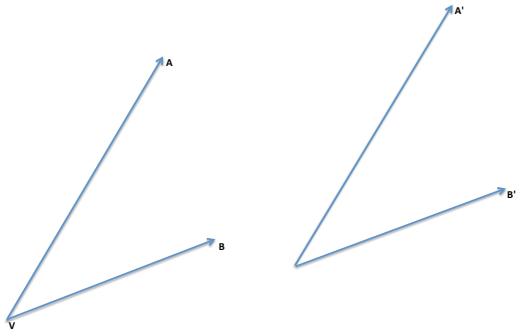


Figure 16

Section 9.2.2. Similarity

Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. 8G4

Two figures are said to be *similar* if there is a sequence of rigid motions and dilations that takes one figure onto the other. So, for example, the triangles in figure 14, the rectangles in figure 15 and the squares and triangles in figure 16 are similar in pairs, since there is a dilation with center the origin that places one on top of the other. Note that, if the dilation that places T onto T' has factor r, then the factor of the dilation placing T' onto T is r^{-1}. Let's list these facts about similarity:

Section 9.2.3 Similar Figures

- Congruent figures are similar. This is because the "sequence of rigid motions and dilations" need not include any dilations, in which case it exhibits a congruence.
- Any two points or angles are similar; because they are congruent.
- Any two line segments or rays are similar. Let AB and CD both be either line segments or rays. First translate A to the point C, and then rotate the image of AB so that it and CD lie on the same line. First let's take the case of rays. Either the rays coincide, or they form the two different rays of the same line. In the second case, another rotation by 180° makes the rays coincide. Now suppose that these are line segments. Let $r = |CD|/|AB|$. Then the dilation with center C and factor r places AB on top of CD.
- Any two circles are similar. Let C be the center of one of the circles and R its radius; and C' the center of the other, and R' its radius. Translate C to C'. Now the circles are concentric. Let $r = R'/R$. Then the dilation of factor r places the first circle on top of the other.
- For two similar triangles, the ratios of corresponding sides are all the same. This is because rigid motions do not change lengths, and dilations change all lengths by the factor of the dilation.
- For two similar triangles, the measure of corresponding angles is the same. This is because rigid motions and dilations do not change the measure of angles.

We end this discussion by showing that the the last statement is actually a criterion for similarity of triangles: if corresponding angles of triangles ABC and $A'B'C'$ have the same measure, the triangles are similar. We'll need another interesting fact for this:

Example 19. If corresponding sides of triangles ABC and $A'B'C'$ are parallel, then the triangles are similar.

Solution. First, translate A to A'. Since a translation takes a line into a line parallel to it, the line of AB is moved to the line of $A'B'$ and the line of AC is moved to the line of $A'C'$, and the image of AB is parallel to $A'B'$, giving us the picture shown in figure 17.

Now, for $r = |AB'|/|AB|$, the dilation with center A and factor r takes ABC onto $A'B'C'$.

Example 20. If corresponding angles of triangles ABC and $A'B'C'$ have the same measure, the triangles are similar.

Solution. Since any two angles of the same measure are congruent, we can find a sequence of rigid motions that takes $\angle CAB$ onto $\angle C'A'B'$. This puts us in the situation of figure 18, except that we do not know that the segments BC and $B'C'$ are parallel. But we do know

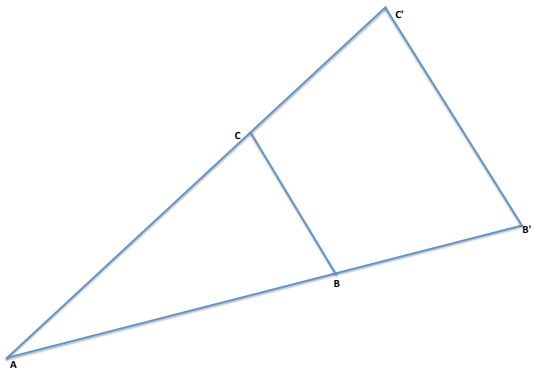


Figure 17

(it is part of the hypothesis) that $\angle ABC$ and $\angle AB'C'$ are equal. That is enough; apply the dilation with center at A and factor $r = |AB'|/|AB|$. This takes B to B', and since the angle doesn't change, the ray BC lands on the ray $B'C'$. So the image of C under that dilation lies on the line of $B'C'$ and the line of AC', so it has to be C'. Thus the first triangle lands on top of the second triangle, and so they are similar,

Summary

Since the properties of the rigid motions and dilations have been gathered in the text, we shall just summarize the coordinate rules for the rigid motions and similarities that we have discussed.

- The coordinate rule for a translation by the vector (a, b) is $(x, y) \to (x + a, y + b)$.
- The coordinate rule for the reflection in the x-axis is $(x, y) \to (x, -y)$.
- The coordinate rule for the reflection in the y-axis is $(x, y) \to (-x, y)$.
- The coordinate rule for the reflection in the line $x = y$ is $(x, y) \to (y, x)$.
- The coordinate rule for the rotation by $90°$ is $(x, y) \to (y, x)$.
- The coordinate rule for the rotation by $180°$ is $(x, y) \to (-x, -y)$.
- The coordinate rule for the dilation with center the origin and factor r is $(x, y) \to (rx, ry)$.