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Amol Amol
University of Minnesota and Federal Reserve Bank of Minneapolis

Erzo G. J. Luttmer
University of Minnesota and Federal Reserve Bank of Minneapolis

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Unique Implementation of Permanent Primary Deficits?

Amol Amol and Erzo G.J. Luttmer
University of Minnesota

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Abstract

In an economy with incomplete markets and consumers who are sufficiently risk averse, we show that the government can uniquely implement a permanent primary deficit using nominal debt and continuous Markov strategies for primary deficits and payments to debtholders. But this result fails if there are also useless pieces of paper (bitcoin for short) that can be traded. If there is trade in bitcoin, then there is no continuous Markov strategy for the government that leads to unique implementation. Instead, there is a continuum of equilibria with distinct real allocations in which the price of bitcoin converges to zero. And there is a balanced budget trap: continuous government policies designed for a permanent primary deficit cannot eliminate an alternative steady state in which $r - g = 0$ and the government is forced to balance its budget. A legal prohibition against bitcoin can restore unique implementation of permanent primary deficits, and so can a tax on bitcoin at the rate $-(r - g) > 0$.

1 Introduction

Imagine an economy, specified in continuous time, in which the government issues stock and pays a flow of non-negative dividends. If a unit of government stock is used as the numeraire, then the price level in this economy is the price of consumption in units of government stock. And the nominal interest rate is the dividend yield on government stock. Agents in this economy who hold government stock are, in effect, holding a nominal bank account at the Treasury that pays a certain nominal interest rate.

Depending on the precise policies followed by the government, this nominal interest rate may be a policy instrument or an equilibrium outcome. The government can mechanically fix the nominal interest rate by paying what in corporate finance is called a stock dividend.[1] A stock dividend fixes the dividend yield by construction. Or the government can follow a policy of distributing a certain flow of consumption goods to the current holders of its stock. In that case, the dividend yield on government stock, and therefore the nominal interest rate, will be an equilibrium outcome, implied by the equilibrium price of government stock.[2]

All of this follows the interpretation of the fiscal theory of the price level proposed by Cochrane [2005]. It is well understood how, in many settings, this theory leads to a uniquely determined price level when the present value of the future primary surpluses of the government is always strictly positive.[3] But, as Bassetto and Cui [2018] have emphasized, this uniqueness may well fail when the government always balances its budget, or when it attempts to run a permanent primary deficit. And non-uniqueness has real consequences. A classic example is the continuum of Pareto-ranked equilibria that arises in the overlapping generations model of money introduced by Samuelson [1958]. On the contrary, Brunnermeier, Merkel, and Sannikov [2023] present an economy in which the price level is uniquely determined because the government uses a trigger strategy that switches to a permanent primary surplus as soon as the price of government stock is inconsistent with its targeted permanent primary deficits.

We believe there are good reasons to question the practicality and credibility of such trigger strategies. Prices that are briefly off track would put the government on a sur-

[1] "For example, if the firm pays a stock dividend of 5%, it sends each shareholder 5 extra shares for every 100 shares currently owned" (p. 445 of *Principles of Corporate Finance, Ninth Edition* by Brealey, Myers, and Allen (2008)). The distinction between stock dividends and regular dividends will play an important role in this paper.

[2] Our economy will have no aggregate uncertainty. But these two paragraphs also apply if government stock follows an Ito process and delivers a stochastic *flow* of dividends. The risk-free rate in units of stock will still be the dividend yield on government stock.

[3] We will not attempt to review the literature. Cochrane [2023] is an excellent source.

plus trajectory forever. This then raises the question: are there other policies that the government can use to ensure that the price level is uniquely determined even when the government runs a permanent primary deficit?

We answer this question in the context of a standard AK economy with idiosyncratic returns to capital and incomplete markets. There is continuous trading in capital and government stock, and possibly in "bitcoin" as well. We use bitcoin as a metaphor for a private-sector security that is in fixed supply and that is not a claim to any real resources.[4] Consumers are infinitely lived and they all have the same homothetic preferences. Their intertemporal elasticity of substitution is equal to one, and their coefficient of relative risk aversion can be any positive number. Provided the coefficient of relative risk aversion is large enough, this economy has competitive equilibria in which the price of bitcoin is zero and the government runs a permanent primary deficit that grows with the size the economy. There is a finite upper bound on the feasible ratio of primary deficits over consumption.

Over time, the government pays non-negative dividend flows and trades government stock for consumption. When it trades, the government has to take the price of its stock as given. In particular, if the market price of government stock is zero at some point in time, then the government cannot run a primary deficit at that time.[5] But the government is a large agent and so its trading plans will influence the equilibrium prices that it faces. We require these trading plans be Markov, in the sense that they only depend on the current physical state of the economy and on current prices. In addition, we require the dependence on current prices to be continuous. We call policies that meet these requirements admissible. We take these policies as given and do not consider the political economy that might give rise to them.

Purely nominal policies are an interesting class of admissible policies that can lead to multiple equilibria. These are policies that specify a trajectory for stock dividends and sales of government stock (that is, nominal interest rates and sales of nominal debt). The net of government stock sold and stock dividend paid, if positive, amounts to nominal revenue that the government can use to pay for consumption goods not already paid for by tax revenues. The market price of government stock then determines the real value of the primary deficit of the government. Nominal policies of this kind can allow the

[4]There is a large literature on the phenomenon of cryptocurrencies. The determinacy question we study also features prominently in the crypto papers by Garrett and Wallace [2018], Fernandez-Villaverde and Sanchez [2019], and Schilling and Uhlig [2019]. These papers do not consider the possibility that the design of off-equilibrium fiscal policies can rule out multiple equilibria.

[5]This is in the spirit of Bassetto [2003]. But we do not use the Shapley-Shubik trading post game. Instead, the government trades continuously by submiting history-dependent schedules that say how much stock it is willing to trade for current consumption.

government to run primary deficits on a permanent basis. But it is immediate that these policies also admit an equilibrium in which the price of government stock is zero at all times. In that case, the government is unable to raise any real revenues by selling stock, and so all its purchases must be paid for by taxes. The resulting balanced budgets make the zero price of government stock self-fulfilling. In other words, purely nominal policies designed to cover a primary deficit imply a balanced budget trap.

To see if it is possible to avoid this indeterminacy, we begin by focusing on an economy without bitcoin and a government that targets a permanent primary deficit that is a constant fraction of aggregate consumption. For this economy, we show that the government can remove the balanced budget trap by running a strictly positive primary surplus when the price of its stock is zero, and running the targeted primary deficit when the price of its stock is consistent with the associated steady state. In between, the government can vary its primary surplus continuously in a way that makes the targeted steady state the only equilibrium. The basic reason this policy works is that the price of government stock can never escape from zero along an equilibrium trajectory—that would be an arbitrage. Taking the primary surplus to be strictly positive when the price of government stock is zero gives the government the ability to pay a strictly positive flow of dividends. The prospect of such dividend flows means that the price of government stock can never actually reach zero, eliminating the balanced budget trap.[6]

Now add bitcoin and again consider a government that targets a permanent primary deficit that is a constant fraction of aggregate consumption. Government policy is now a continuous function of the price of government stock and the price of bitcoin. It is easy to ensure that policy is consistent with a targeted steady state primary deficit. But we can now prove that, no matter what continuous Markov policy the government uses to rule out a zero price of its stock, there is always another steady state. In that steady state $r - g = 0$ and the market values of government stock and bitcoin are both strictly positive. The fact that $r - g = 0$ means that the government must again be balancing its budget. So the presence of a private-sector security that, just like government stock, is a claim to no real resources introduces a new balanced budget trap. The economy will have at least two steady states. In addition, there are equilibrium trajectories that start near the balanced budget trap and converge to the targeted steady state.

The fact that we restrict attention to Markov policies that are continuous in prices is crucial for this result. Without the continuity assumption, the government could adopt a policy that uniquely implements a targeted permanent primary deficit whenever the

[6]Unlike Obstfeld and Rogoff [1983] we do not assume the government has a real long-lived asset that it can use to redeem its outstanding stock when that stock trades at a low price.

price of bitcoin is zero, and switch, abruptly, to a primary surplus policy whenever the price of bitcoin is positive. That switch would turn government stock into a Lucas tree and force $r - g > 0$, which is inconsistent with bitcoin trading at a positive price. This rules out any equilibrium in which the price of bitcoin is positive. Another way to put this is: discontinuous Markov strategies can mimic the trigger strategy proposed by Brunnermeier, Merkel, and Sannikov [2023]. The discontinuity has to be right at the targeted steady state, or else there will be a continuum of equilibria with trajectories that converge to the steady state.

Given a need to finance government purchases equal to a certain fraction of aggregate consumption, the policy that maximizes utility in our economy (and also its growth rate) is for the government to charge very large consumption taxes. This implies very large permanent primary surpluses and a unique equilibrium, and it turns government stock into a very large Lucas tree that eliminates almost all idiosyncratic risk.[7] But large consumption taxes may not be feasible, and then a permanent primary deficit may be the best the government can do, provided the equilibrium does indeed deliver the targeted steady state. To achieve this, the government could simply make bitcoin illegal.[8]

Our final result says that, short of full prohibition, the government could use a continuous Markov policy and combine it with a tax on bitcoin. Let $r - g < 0$ be the difference between the real interest rate and the growth rate of the economy in the steady state that allows the government to run its targeted permanent primary deficit. We show that a flow tax on every unit of bitcoin equal to or in excess of $-(r - g)$ times the price of bitcoin rules out all equilibria in which bitcoin trades at a positive price. The underlying reason is that $-(r - g)$ times the market value of bitcoin is precisely the steady state flow of income that a private sector entity can earn when it issues new bitcoin in the same way as the government issues new stock to finance a primary deficit.

These results suggest that a legal prohibition of bitcoin or a tax on bitcoin are forms of financial repression that may be useful when the ability of the government to use consumption taxes is limited.

[7] Amol and Luttmer [2022] prove this result for an economy with exogenous market incompleteness that nests the model we use in this paper. Amol [2024] shows that this policy is optimal in a wide range of settings with endogenous labor supply, idiosyncratic capital income risk, and collateral constraints.

[8] Wallace [1983] uses legal restrictions to explain rate of return dominance between Federal Reserve Notes and Treasury bills. All risk-free securities in our paper earn the same return.

2 An AK Economy with Incomplete Markets

There is a unit measure of infinitely lived consumers who have perfect foresight. They can accumulate physical capital using a linear technology with returns that are subject to idiosyncratic risk. There are no markets that allow consumers to share this risk. But there is government stock that is risk-free. And there is a supply of useless pieces of paper (perhaps old dollar bills, or maybe non-fungible tokens) for which we will use the shorthand bitcoin.[9] [10] We only consider competitive equilibria in which bitcoin is also risk-free. If consumers are sufficiently risk averse, they are going to be willing to hold government stock and bitcoin even when the real return on these securities is very low.[11]

Outline of this Section We first describe what competitive equilibrium trajectories must look like for more or less arbitrary trajectories of the fiscal variables controlled by the government. The definition of competitive equilibrium we use is standard. But because the government is a large agent, that definition means that government policy can cause the price of its stock to be strictly positive by running a primary deficit. We then restrict attention to government policies that imply a balanced budget or a surplus whenever the price of government stock is zero.

2.1 The Economy

Subject to the constraint $K_{j,t} \geq 0$, individual consumers $j \in [0,1]$ can accumulate capital according to

$$\mathrm{d}K_{j,t} = (\mu K_{j,t} - Y_{j,t})\,\mathrm{d}t + \sigma K_{j,t}\mathrm{d}Z_{j,t} + \mathrm{d}I_{j,t},$$

where μ and σ are positive, $Y_{j,t} \geq 0$ is the flow of output sold, and $I_{j,t}$ is the cumulative amount of capital purchased by consumer j. The standard Brownian motions $Z_{j,t}$ average out to zero. Output can be consumed by consumers or the government. Trades in capital have to add up to zero,

$$0 = \int_0^1 I_{j,t}\mathrm{d}j,$$

[9] We could have referred to these useless pieces of paper as money. But that term is usually interpreted as government money. We are interested in an economy with useless pieces of paper that are not necessarily supplied by the government.

[10] There could be several types of useless pieces of paper. This would introduce additional sources of indeterminacy along the lines of Kareken and Wallace [1981]. We will abstract fom this possibility.

[11] The version of this economy described in Amol and Luttmer [2022] has overlapping generations and perpetual youth, as in Blanchard [1985]. And it includes labor as a fixed factor.

at all times. The aggregate capital stock and the aggregate supply of output are

$$K_t = \int_0^1 K_{j,t} \mathrm{d}j, \quad Y_t = \int_0^1 Y_{j,t} \mathrm{d}j.$$

As a result, the aggregate capital stock evolves according to

$$\mathrm{d}K_t = (\mu K_t - Y_t)\,\mathrm{d}t. \tag{1}$$

Both K_t and Y_t have to remain non-negative. Aggregate consumption C_t and government purchases G_t are non-negative and must satisfy the resource constraint

$$Y_t = C_t + G_t.$$

The government imposes consumption taxes at a rate $\tau > 0$. The resulting tax revenues are equal to $T_t = \tau C_t$. Consumers pay the consumption taxes, and so their consumption expenditures are $E_t = (1 + \tau)C_t$. The primary surplus of the government, relative to consumption expenditures, is equal to

$$\mathcal{S}_t = \frac{T_t - G_t}{E_t}. \tag{2}$$

The non-negativity constraint on government purchases means that this surplus ratio must satisfy $\mathcal{S}_t \in (-\infty, \tau/(1+\tau)]$. As a matter of accounting, note that $Y_t = (1 - \mathcal{S}_t)E_t$. We are interested in policies that implement $\mathcal{S}_t = \mathcal{S}_*$ for some feasible $\mathcal{S}_* < 0$.

2.2 Risk-Free Securities

Consumers cannot borrow. In addition to holding capital subject to their own idiosyncratic risk, consumers can also choose to hold non-negative quantities of government stock and bitcoin.[12] The aggregate supplies of these securities are $D_t \geq 0$ and $B > 0$, respectively. We will only consider government policies that guarantee $D_t > 0$ at all times. So the supplies of both government stock and bitcoin are always strictly positive. The government pays a flow of dividends equal to $d_t \geq 0$ units of consumption per unit of stock. Bitcoin pays no dividends. The price of a unit of government stock is $s_t \geq 0$, and the price of a unit of bitcoin is $p_t \geq 0$, both expressed in units of consumption.

We only consider equilibrium price trajectories $\{(s_t, p_t)\}_{t \geq 0} \subset \mathbb{R}^2_+$ that are deterministic

[12]Short positions can lead to bankruptcies when foresight turns out to be less than perfect. We want to avoid having to specify bankruptcy procedures or imposing artifical restrictions on how prices can evolve.

and almost always differentiable. If $s_t > 0$ and $p_t > 0$, then the trajectories for s_t and p_t must satisfy

$$\mathrm{d}s_t = (r_t s_t - d_t)\,\mathrm{d}t, \qquad \mathrm{d}p_t = r_t p_t \mathrm{d}t, \tag{3}$$

for some common risk-free rate of return r_t. If $s_t = 0$ along some trajectory of equilibrium prices, then it must be that $s_u = 0$ for all $u \geq t$. Otherwise the prospect of a positive price of government stock implies an arbitrage opportunity. The same is true for the price of bitcoin.

To emphasize, for both government stock and bitcoin, a zero price must be an absorbing state along any equilibrium trajectory $\{(s_t, p_t)\}_{t\geq 0}$ of prices. In the case of government stock, a zero price is possible only if the subsequent dividends are almost always equal to zero, or else there would be another arbitrage opportunity.

2.3 Government Trades

As already noted, we restrict attention to government policies that keep the supply of government stock strictly positive. We do not let the government accumulate capital or claims on the private sector.

Over time, the government can trade subject to the constraint

$$s_t \mathrm{d}D_t = (d_t D_t + G_t - T_t)\,\mathrm{d}t. \tag{4}$$

Another way to write this is

$$\frac{s_t D_t}{E_t} \times \frac{1}{D_t}\frac{\mathrm{d}D_t}{\mathrm{d}t} = \mathcal{D}_t - \mathcal{S}_t,$$

where $\mathcal{S}_t$ is defined in (2) and

$$\mathcal{D}_t = \frac{d_t D_t}{E_t} \tag{5}$$

is the ratio of total government dividends over aggregate consumption expenditures. The surplus ratio $\mathcal{S}_t$ and the dividend ratio $\mathcal{D}_t$ are pinned down by the $G_t \geq 0$ and $d_t \geq 0$ set by the government and the equilibrium value of aggregate consumption expenditures. Together with the equilibrium value of $s_t D_t / E_t$, this determines the growth rate of the supply of government stock. Note that $\mathcal{D}_t \geq 0$ and that the government can ensure that the supply of government stock is non-decreasing by taking $\mathcal{D}_t \geq \max\{0, \mathcal{S}_t\}$.

Primary deficits are only possible if $s_t > 0$, so that the government can raise revenues by issuing new stock. If $s_t = 0$, then the constraint (4) on government trades forces $d_t D_t + G_t - T_t = 0$, and hence $T_t - G_t \geq 0$. Since $D_t > 0$, the government can still select $G_t \in$

$[0, T_t)$ and pay strictly positive dividends, no matter what $s_t \geq 0$. In other words, $D_t > 0$ ensures that nothing ever prevents the government from running a primary surplus and paying a strictly positive dividend.

As long as $s_t > 0$, and therefore $\mathrm{d}s_t = (r_t s_t - d_t)\,\mathrm{d}t$, the market value of the outstanding supply of government stock evolves according to

$$\begin{aligned} \mathrm{d}\left[s_t D_t\right] &= D_t \mathrm{d}s_t + s_t \mathrm{d}D_t \\ &= D_t\left(\mathrm{d}s_t + d_t \mathrm{d}t\right) + \left(G_t - T_t\right)\mathrm{d}t = \left(r_t \times s_t D_t - \left(T_t - G_t\right)\right)\mathrm{d}t. \end{aligned} \tag{6}$$

A Modigliani-Miller type result applies: the dynamics of $s_t D_t > 0$ does not depend on how much of the return r_t comes from dividends or capital gains. As already argued, $s_t = 0$ is an absorbing state for any equilibrium trajectory of prices. It follows that $s_t D_t = 0$ is also an absorbing state.

2.3.1 No Bitcoin Companies

As we shall see, the government will be able to run permanent primary deficits when consumers are sufficiently risk averse. One can imagine a company that uses its stock exactly the way the government does, with the only exception that the company has no tax revenues. Suppose this company is owned by a single person, named X (who is not one of the unit continuum of consumers). The analog of government purchases is then the flow of income that goes to X.

In such an economy, there can be competitive equilibria in which both the government and the bitcoin company run a permanent primary deficit. X consumes a constant and strictly positive fraction of aggregate consumption, forever. The control rights held by X cannot be a traded security—the value of that security would be infinite. Clearly, there are very strong incentives to set up these kinds of companies. But laws against counterfeiting and private-sector ponzi schemes make them illegal in the United States and many other countries. We will not consider them in this paper.

2.4 Consumer Choices

Consumers all have the same Epstein-Zin preferences, with a subjective discount rate $\rho > 0$, an intertemporal elasticity of substitution equal to 1, and a coefficient of relative risk aversion equal to $\xi > 0$. The price of capital must be equal to 1 in any equilibrium.

At time t, consumer-j wealth is

$$W_{j,t} = K_{j,t} + s_t D_{j,t} + p_t B_{j,t}.$$

Consumers all solve the same Merton problem. Write $C_{j,t}$ for the consumption of consumer j at time t, and let $E_{j,t} = (1+\tau)C_{j,t}$ be the resulting consumption expenditures.

If the price of at least one of the two risk-free securities is strictly positive, then consumers can earn a risk-free return r_t. They then use the decision rules

$$\frac{E_{j,t}}{W_{j,t}} = \rho, \qquad \frac{K_{j,t}}{W_{j,t}} = \psi_t, \tag{7}$$

where

$$\psi_t = \frac{\mu - r_t}{\xi\sigma^2}. \tag{8}$$

Consumers are indifferent between holding government stock and bitcoin when both are trading at a positive price. Let $\phi_{j,t} = s_t D_{j,t}/W_{j,t}$ be the portfolio share of government stock for consumer j. Because consumers are not allowed to hold any negative positions, these portfolio shares must satisfy $\phi_{j,t} \in [0, 1-\psi_t]$ for all j. The aggregate portfolio share of government stock is then

$$\phi_t = \frac{\int_0^1 \phi_{j,t} W_{j,t} \mathrm{d}j}{\int_0^1 W_{j,t} \mathrm{d}j}.$$

Since the decision rules (7)-(8) do not depend on j, and since all consumers earn the same rate of return r_t on the risk-free securities they happen to hold, we will not have to keep track of the distribution of wealth or the individual portfolio shares $\phi_{j,t}$.

The decision rules (7)-(8) imply that consumer-j wealth evolves according to

$$\mathrm{d}W_{j,t} = W_{j,t} \left(g_t \mathrm{d}t + \psi_t \sigma \mathrm{d}Z_{j,t} \right),$$

where

$$g_t = r_t + \psi_t(\mu - r_t) - \rho$$

is the common expected growth rate of everyone's consumption and wealth. Individual consumption growth will be risky, with diffusion coefficient $\psi_t\sigma$.

If $s_t = 0$ and $p_t = 0$, then consumers only hold capital, and that capital is subject to their own idiosyncratic risk. Their consumption expenditures are simply $E_{j,t} = \rho K_{j,t}$, and hence $g_t = \mu - \rho$. That is, the above wealth dynamics applies even if risk-free securities prices are zero.

Every consumer is assumed to have some capital at the initial date. Everyone therefore has strictly positive initial wealth. Given their preferences, consumers will not let their wealth reach zero in finite time. Everyone will always be able to consume a positive amount and pay the associated consumption taxes. As a consequence, the consumption tax revenues of the government will always be strictly positive.

2.5 Market Clearing

If at least one of the two risk-free securities trades at a positive price, then there is a well-defined risk-free rate r_t and market clearing will force $\psi_t \in (0,1)$. The decision rule (8) then implies $r_t = \mu - \xi\sigma^2\psi_t$. So the risk-free rate of return can be inferred from $\psi_t \in (0,1)$, and that rate of return will be bounded. If both risk-free securities trade at a zero price, then $\psi_t = 1$ and the risk-free rate r_t is not defined. It will be convenient to extend the definition of r_t to include the scenario in which $s_t = 0$ and $p_t = 0$,

$$r_t = \mu - \xi\sigma^2\psi_t, \qquad \psi_t \in (0,1]. \tag{9}$$

The resulting risk-free returns are in $[\mu - \xi\sigma^2, \mu)$, and $r_t = \mu - \xi\sigma^2$ can be interpreted as a shadow rate of return that consumers face when $s_t = 0$ and $p_t = 0$.

Given aggregate capital $K_t > 0$, the supply of government stock $D_t > 0$, and the supply $B > 0$ of bitcoin, aggregate wealth must be given by $W_t = K_t + s_tD_t + p_tB$. This is strictly positive, and the individual decision rules imply that aggregate consumption expenditures are $E_t = \rho W_t$. The definitions of ψ_t and ϕ_t imply $\psi_t = K_t/W_t \in (0,1]$ and $\phi_t = s_tD_t/W_t \in [0, 1-\psi_t]$ in any equilibrium.

The individual shocks $Z_{j,t}$ average out across all consumers. Adding up the individual wealth dynamics across consumers therefore gives $\mathrm{d}W_t = g_tW_t\mathrm{d}t$. Using (9) to eliminate r_t from $g_t = r_t + \psi_t(\mu - r_t) - \rho$ then yields

$$g_t = \mu - \rho - \xi\sigma^2(1-\psi_t)\psi_t. \tag{10}$$

So aggregate consumption will grow at a rate $g_t \leq \mu - \rho$ that only depends on $\psi_t \in (0,1]$. At $s_t = 0$ and $p_t = 0$ we have $\psi_t = 1$ and hence $g_t = \mu - \rho$, consistent with (10). Observe that the growth rate of this economy is U-shaped in ψ_t. Intermediate values of ψ_t are bad for growth.

2.5.1 The Implied Euler Equation

As an aside, note that (9) and (10) imply

$$r_t = \rho + g_t - \xi\sigma^2\psi_t^2 \tag{11}$$

and recall that $\sigma\psi_t$ is the diffusion coefficient of individual consumption. So this is the usual Euler equation, given our assumption that consumers have an intertemporal elasticity of substitution equal to 1 and a coefficient of relative risk aversion equal to ξ. For future reference, note that $(r_t - g_t)/\rho = 1 - (\xi\sigma^2/\rho)\psi_t^2$. Clearly, this can only take on negative values if the parameters satisfy $\xi\sigma^2/\rho > 1$.

2.6 Competitive Equilibrium Trajectories

Fix some policy trajectory $\{\mathcal{S}_t, \mathcal{D}_t\}_{t\geq 0}$, where $\mathcal{S}_t \in (-\infty, \tau/(1+\tau)]$ and $\mathcal{D}_t \geq \max\{0, \mathcal{S}_t\}$. Conjecture that the policy trajectory $\{\mathcal{S}_t, \mathcal{D}_t\}_{t\geq 0}$ is such that the economy has a competitive equilibrium—consumers maximize and markets clear. For a given policy trajectory, competitive equilibria may fail to exist, or there may be more than one equilibrium. Here we show how competitive equilibria, when they exist, can be constructed.

At any point in time, the state $[K_t, D_t] \in \mathbb{R}^2_{++}$ is pre-determined. Together with prices $s_t \geq 0$ and $p_t \geq 0$, one can compute $W_t = K_t + s_t D_t + p_t B > 0$, $\psi_t = K_t/W_t \in (0, 1]$, and $\phi_t = s_t D_t/W_t \in [0, 1-\psi_t]$. Conversely, from $[K_t, D_t] \in \mathbb{R}^2_{++}$, $\psi_t \in (0, 1]$, and $\phi_t \in [0, 1-\psi_t]$, one can infer

$$W_t = \frac{K_t}{\psi_t}, \quad s_t D_t = \phi_t W_t, \quad p_t B = (1 - \psi_t - \phi_t) W_t. \tag{12}$$

In other words, given the physical state $[K_t, D_t] \in \mathbb{R}^2_{++}$ of the economy, there is a one-to-one relation between the prices s_t and p_t on the one hand, and the portfolio shares ψ_t and ϕ_t on the other. Constructing a trajectory $\{s_t, p_t\}_{t\geq 0}$ for equilibrium prices is the same as constructing a trajectory for $\{\psi_t, \phi_t\}_{t\geq 0}$.

2.6.1 Equilibrium Trajectories that Start With $\phi_0 = 0$

Suppose that $\phi_0 = 0$, and hence $s_0 = 0$. We know that this must be an absorbing state. In turn, $s_t = 0$ for all $t \geq 0$ means that $d_t = 0$ has to be true for all $t \geq 0$, or else there is an arbitrage opportunity. So $\phi_0 = 0$ can be part of an equilibrium trajectory only if $\mathcal{D}_t = 0$ for all $t \geq 0$. This forces $\mathcal{S}_t \leq 0$ for all $t \geq 0$. But $\mathcal{S}_t < 0$ would require the government to raise revenues by selling stock, and that is only possible if $s_t > 0$. So it has to be the case that $\mathcal{S}_t = 0$ and $\mathcal{D}_t = 0$ for all $t \geq 0$. This proves the following lemma.

Lemma 1 *Equilibrium trajectories with $\phi_0 = 0$ are possible only if $\mathcal{S}_t = 0$ and $\mathcal{D}_t = 0$ for all $t \geq 0$.*

Given $\phi_0 = 0$, and a government that simply sets G_t equal to its tax revenues and pays no dividends, there are now two cases to consider: the initial price of bitcoin is also zero, or it is positive.

Zero Bitcoin Prices If $\psi_0 = 1$, then $1 - \psi_0 - \phi_0 = 0$ and hence $p_0 = 0$ as well. This must also be an absorbing state, and so the equilibrium trajectory must be $[\psi_t, \phi_t] = [1, 0]$ for all $t \geq 0$. The fact that $\mathcal{S}_t = 0$ means that $Y_t = E_t = \rho K_t$, and so the capital stock evolves according to $\mathrm{d}K_t = (\mu - \rho) K_t \mathrm{d}t$. The supply of government stock is constant and worth nothing, just like bitcoin.

Positive Bitcoin Prices The other possibility is $\psi_0 \in (0, 1)$, and so $1 - \psi_0 - \phi_0 \in (0, 1)$. That is, bitcoin trades at a positive price. From (3) we know that $\mathrm{d}p_t = r_t p_t \mathrm{d}t$. Together with $\mathrm{d}W_t = g_t W_t \mathrm{d}t$ and $1 - \psi_t = p_t B / W_t$ this implies $\mathrm{d}(1 - \psi_t) = (r_t - g_t)(1 - \psi_t)\mathrm{d}t$. The Euler condition (11) then gives

$$\mathrm{d}\psi_t = -\rho F(\psi_t)\mathrm{d}t,$$

where $F(\cdot)$ is defined by

$$F(\psi) = \left(1 - \frac{\xi\sigma^2}{\rho} \times \psi^2\right)(1 - \psi). \tag{13}$$

This is a third-order polynomial that satisfies $F(0) = 1$ and $F(1) = 0$.

If $\xi\sigma^2/\rho \in (0, 1)$, then $F(\cdot)$ is a strictly positive and decreasing function for all $\psi \in (0, 1)$. This implies $\mathrm{d}\psi_t < -\rho F(\psi_0)\mathrm{d}t < 0$ for all $\psi_t \in (0, 1)$. In turn, this means that ψ_t reaches 0 in finite time. This cannot be part of an equilibrium trajectory, because the capital stock must remain positive. When consumers are not very risk averse, $r_t - g_t$ is positive and bounded away from zero. This would imply that the market value of bitcoin explodes relative to the size of the economy, which is impossible. So it is not possible to have $\psi_0 \in (0, 1)$. The only equilibrium trajectory is one in which $\psi_t = 1$ for all $t \geq 0$.

If $\xi\sigma^2/\rho > 1$, then there is a unique $\psi_* \in (0, 1)$ so that $F(\psi_*) = 0$. Specifically,

$$1 - \frac{\xi\sigma^2}{\rho} \times \psi_*^2 = 0. \tag{14}$$

This means that $\psi_0 = \psi_*$ is a steady state. In that steady state, $r_t - g_t = 0$ and the market value of bitcoin is constant relative to the size of the economy. Furthermore, $F(\psi) > 0$ for all $\psi \in (0, \psi_*)$ and $F(\psi) < 0$ for all $\psi \in (\psi_*, 1)$. This means that $\psi_0 = \psi_*$ is an unstable steady state while $\psi_0 = 1$ is a stable steady state. This implies a continuum of equilibrium trajectories indexed by $\psi_0 \in [\psi_*, 1]$. All trajectories in the interior of this interval converge to 1. As we know from a simple two-period overlapping generations exchange economy, pure bubble steady states come with a continuum of equilibria in which the value of the bubble converges to zero.

2.6.2 Pure Bubble Equilibria

It is not difficult to leverage our results for $\phi_0 = 0$ and $\psi_0 \in (0, 1]$ to prove the following proposition.

Proposition 1 *Suppose the government sets $\mathcal{S}_t = 0$ and $\mathcal{D}_t = 0$ for all $t \geq 0$. If $\xi\sigma^2/\rho > 1$, then the economy has a two-dimensional continuum of equilibria, parameterized by an initial $\psi_0 \in [\psi_*, 1]$ and an initial $\phi_0 \in [0, 1 - \psi_0]$. And $\psi_0 \in (\psi_*, 1)$ means that ψ_t converges monotonically to 1. If $\xi\sigma^2/\rho \in (0, 1]$ then the only equilibrium is $\psi_0 = 1$.*

If the government simply balances its budget at all times, then government stock and bitcoin are just two different types of useless pieces of paper. As Kareken and Wallace [1981] pointed out long ago, the relative price of two such bubble assets is indeterminate. For every bitcoin-only equilibrium, one can construct a continuum of additional equilibria in which the price of bitcoin in units of government stock is constant and in $[0, \infty)$.

2.6.3 Equilibrium Trajectories that Start with $\phi_0 \in (0, 1)$

At time t, given the state $[K_t, D_t] \in \mathbb{R}^2_{++}$ and some $[\psi_t, \phi_t] \in (0, 1) \times (0, 1 - \psi_t]$, aggregate wealth $W_t > 0$ and the prices $s_t > 0$ and $p_t \geq 0$ are implied by (12). In turn, this pins down $E_t = \rho W_t > 0$, and hence $T_t = \tau E_t/(1 + \tau)$, $Y_t = (1 - \mathcal{S}_t) E_t$, and $T_t - G_t = \mathcal{S}_t E_t$. By construction, this clears goods markets at time t. The dividends on government stock follow from $d_t D_t = \mathcal{D}_t E_t$.

The physical state of the economy then evolves according to

$$\mathrm{d}K_t = (\mu K_t - (1 - \mathcal{S}_t) E_t) \, \mathrm{d}t, \tag{15}$$

$$s_t \mathrm{d}D_t = (\mathcal{D}_t - \mathcal{S}_t) E_t \mathrm{d}t. \tag{16}$$

Note that we are using $s_t > 0$ in (16). It remains to determine how ψ_t and ϕ_t evolve.

The definitions $\psi_t = K_t/W_t$ and $\phi_t = s_t D_t/W_t$, together with (1), (6), and $\mathrm{d}W_t = g_t W_t \mathrm{d}t$, give

$$\begin{aligned}
\mathrm{d}\psi_t &= \frac{\mathrm{d}K_t}{W_t} - \frac{K_t}{W_t}\frac{\mathrm{d}W_t}{W_t} = \left(\psi_t\mu - \frac{Y_t}{E_t}\frac{E_t}{W_t}\right)\mathrm{d}t - \psi_t g_t \mathrm{d}t, \\
\mathrm{d}\phi_t &= \frac{\mathrm{d}\left[s_t D_t\right]}{W_t} - \frac{s_t D_t}{W_t}\frac{\mathrm{d}W_t}{W_t} = \left(\phi_t r_t - \frac{T_t - G_t}{E_t}\frac{E_t}{W_t}\right)\mathrm{d}t - \phi_t g_t \mathrm{d}t.
\end{aligned}$$

Using (9) and (10) to eliminate r_t and g_t, as well as $E_t/W_t = \rho$, $Y_t/E_t = 1 - \mathcal{S}_t$, and the definition of $\mathcal{S}_t$, gives

$$\mathrm{d}\psi_t = \rho\left(\mathcal{S}_t - \left(1 - \frac{\xi\sigma^2}{\rho} \times \psi_t^2\right)(1 - \psi_t)\right)\mathrm{d}t, \tag{17}$$

$$\mathrm{d}\phi_t = \rho\left(\left(1 - \frac{\xi\sigma^2}{\rho} \times \psi_t^2\right)\phi_t - \mathcal{S}_t\right)\mathrm{d}t. \tag{18}$$

This defines a differential equation for $\psi_t \in (0,1)$ and $\phi_t \in (0, 1-\psi_t]$. The initial value $[\psi_0, \phi_0] \in (0,1) \times (0, 1-\psi_0]$ defines an equilibrium trajectory if the $\{\psi_t, \phi_t\}_{t\geq 0}$ implied by (17)-(18) satisfies $\psi_t \in (0,1)$ and $\phi_t \in (0, 1-\psi_t]$ at all times. It also defines an equilibrium trajectory if the solution to the differential equation (17)-(18) reaches $[\psi_T, \phi_T] \in (0,1] \times \{0\}$ at some $T \in (0,\infty)$ and $\{[\psi_t, 0]\}_{t\geq T}$ is an equilibrium trajectory starting at time T. Because of Lemma 1, this can only happen if $\mathcal{S}_t = 0$ and $\mathcal{D}_t = 0$ for all $t \geq T$.

Note that (17) and (18) are the same equation if $\phi_t = 1-\psi_t$ identically. We can therefore restrict attention to (17) if the portfolio share of bitcoin is zero. In general, adding up (17) and (18) yields

$$\mathrm{d}\left(1 - \psi_t - \phi_t\right) = \rho\left(1 - \frac{\xi\sigma^2}{\rho} \times \psi_t^2\right)\left(1 - \psi_t - \phi_t\right)\mathrm{d}t. \tag{19}$$

Recall that $1 - (\xi\sigma^2/\rho)\,\psi_t^2 = (r_t - g_t)/\rho$. So (19) says that the portfolio share of bitcoin grows at the rate $r_t - g_t$. Fundamentally, this is why bitcoin cannot trade at a positive price if $r_t - g_t > 0$ and bounded away from zero, and why bitcoin trading at a positive price becomes a possibility when $r_t - g_t \leq 0$.

2.7 Admissible Policies and Equilibrium

So far, we have adopted the standard definition of a competitive equilibrium. The standard definition describes a vector of prices. A vector of prices is an equilibrium if the quantities chosen optimally given those prices clear markets. What consumers or the government might do at prices other than a given vector of equilibrium prices is irrelevant. In particular, if the policy trajectory $\{\mathcal{S}_t, \mathcal{D}_t\}_{t\geq 0}$ specifies $\mathcal{S}_t < 0$ at some time t, then $s_t = 0$ will simply not be part of an equilibrium trajectory.

Debreu [1982, p.708] shows that a competitive equilibrium with free disposal can also be interpreted as a Nash equilibrium of a game in which market participants submit budget-feasible excess demand schedules and a fictitious auctioneer chooses non-negative prices to maximize the market value of the aggregate excess demand. An essential ingredient for this equivalence is that market participants are small. In the standard definition, market participants cannot manipulate prices by assumption. But in the demand schedule game, large players can use their demand schedules to manipulate prices.

In what follows, we are going to take the Nash equilibrium approach of Debreu, adapted to the fact that our economy has a sequence of markets. Concretely, we impose on the government that it uses policies for $\mathcal{S}_t$ and $\mathcal{D}_t$ that, taking into account $E_t = \rho W_t$ and (12), are feasible in the sense that (16) holds at any $s_t \geq 0$ and $p_t \geq 0$ that the auctioneer might call out. We want to allow for the possibility —central to oft-expressed concerns about the sustainability of primary deficits— that the government's attempt to auction off more stock might fail.

2.7.1 Markovian, Continuous, and Budget-Feasible Policies

Consider government policies of the form

$$\mathcal{S}_t = \mathcal{S}\left(\psi_t, \phi_t\right), \quad \mathcal{D}_t = \mathcal{D}\left(\psi_t, \phi_t\right), \tag{20}$$

where $\mathcal{S}(\cdot,\cdot) \in (-\infty, \tau/(1+\tau)]$ and $\mathcal{D}(\cdot,\cdot) \in \mathbb{R}_+$ are defined on $\{[\psi, \phi] \in \mathbb{R}_{++} \times \mathbb{R}_+ : \psi + \phi \leq 1\}$. Together with (12), the policy (20) specifies G_t and d_t as a function of the physical state $[K_t, D_t] \in \mathbb{R}^2_{++}$ and the prices s_t and p_t. Policies of the form (20) can be viewed as Markov policies.[13]

Define a Markov policy to be admissible if $\mathcal{S}(\cdot,\cdot)$ and $\mathcal{D}(\cdot,\cdot)$ are continuous and satisfy

$$\mathcal{S}\left(\psi, 0\right) = \mathcal{D}\left(\psi, 0\right), \quad \psi \in (0, 1]. \tag{21}$$

This condition ensures that the government meets the constraint (16) on its trades when $s_t = 0$. In an economy in which trade in bitcoin is ruled out by legal restrictions, we take admissibility to require only continuity on the line segment $[\psi, \phi] = [\psi, 1-\psi]$, together

[13] The policy (20) does rule out, for example, policies that depend on some arbitrary threshold values for K_t or D_t. But preferences are homothetic, the technology is linear in capital, and D_t is a nominal quantity. This means that at any point in time t, one can re-normalize the economy to $K_t = 1$ and $D_t = 1$ and recover an economy that is effectively the same as before.

with $\mathcal{S}(1,0) = \mathcal{D}(1,0)$.

As long as they are well defined, the growth rate of the supply of government stock and the nominal interest rate implied by admissible Markov policies are also functions only of ψ_t and ϕ_t. If $\phi_t > 0$, then the growth rate of the supply of government stock is

$$\frac{1}{D_t}\frac{\mathrm{d}D_t}{\mathrm{d}t} = \frac{\mathcal{D}(\psi_t, \phi_t) - \mathcal{S}(\psi_t, \phi_t)}{\phi_t/\rho}. \tag{22}$$

This follows from (16), $\phi_t = s_t D_t / W_t$, and $E_t = \rho W_t$. Primary deficits imply a growing supply of government stock, and paying dividend on government stock will only increase that growth rate. If we strengthen our continuity assumption to differentiability, then (21) implies that (22) has a finite limit as ϕ_t goes to zero.

When its price is strictly positive, government stock can be used as the numeraire. The price of consumption in units of government stock is then $P_t = 1/s_t$. The nominal interest rate for this numeraire is simply $i_t = P_t d_t = d_t/s_t$. Since $\mathcal{D}(\psi_t, \phi_t) = d_t D_t / E_t$ and $\phi_t = s_t D_t / W_t$ and $E_t = \rho W_t$, this gives

$$i_t = \frac{\mathcal{D}(\psi_t, \phi_t)}{\phi_t/\rho}. \tag{23}$$

Importantly, this may or may not explode as ϕ_t converges to zero.

Adding a Stock Dividend If the policies $\mathcal{S}(\psi, \phi)$ and $\mathcal{D}(\psi, \phi)$ satisfy (21), then so do the policies $\mathcal{S}(\psi, \phi)$ and $\mathcal{D}(\psi, \phi) + \delta \times \phi/\rho$ for any $\delta \geq 0$. The government can add a stock dividend with non-negative incremental dividend yield δ without changing the admissibility of its underlying policy. It is immediate from (22) and (23) that this simply increases the growth rate of the supply of government stock and the nominal interest rate, by an additive term δ.

2.7.2 Definition of Equilibrium

Given initial values $[K_0, D_0] \in \mathbb{R}^2_{++}$ and Markov policies $\mathcal{S}(\cdot,\cdot)$ and $\mathcal{D}(\cdot,\cdot)$ that satisfy the admissibility requirement (21), an equilibrium is a competitive equilibrium trajectory constructed for a policy trajectory $\{\mathcal{S}_t, \mathcal{D}_t\}_{t\geq 0}$ that satisfies (20).

The structure of the equilibrium conditions is block-recursive. Replacing $\mathcal{S}_t$ in (17)-(18) by $\mathcal{S}_t = \mathcal{S}(\psi_t, \phi_t)$ turns (17)-(18) into an autonomous differential equation for ψ_t and ϕ_t. Given a solution to this autonomous differential equation, one can then use (12) and (15)-(16) to back out trajectories for K_t, D_t, s_t, and p_t.

Observe that the autonomous differential equation (17)-(18) for ψ_t and ϕ_t only depends on the surplus policy $\mathcal{S}(\cdot,\cdot)$ and not on the dividend policy $\mathcal{D}(\cdot,\cdot)$, at least not directly. But admissibility requires $\mathcal{S}(\psi,0) = \mathcal{D}(\psi,0)$. Therefore, if $\mathcal{S}(\psi,0) = \mathcal{D}(\psi,0) > 0$, then Lemma 1 rules out equilibrium trajectories for which ϕ_t reaches 0 in finite time.

3 Unique Implementation when Bitcoin is Illegal

When government stock is the only risk-free security that is traded, we can take $\phi_t = 1-\psi_t$ everywhere and focus on the differential equation (17) for policies that are admissible on the domain $\phi_t = 1-\psi_t$. This leads to

$$\mathrm{d}\psi_t = \rho\left(\mathcal{S}(\psi_t, 1-\psi_t) - F(\psi_t)\right)\mathrm{d}t, \tag{24}$$

for all $\psi_t \in (0,1)$, where $F(\cdot)$ is the third-order polynomial that was defined in (13). There is no initial condition for $\psi_0 \in (0,1]$ and $\psi_0 = 1$ can only be an absorbing state.

The interpretation is again $\rho F(\psi_t) = (r_t - g_t)(1-\psi_t)$. The equation (24) is now an autonomous differential equation defined for trajectories that satisfy $\{\psi_t\}_{t\geq 0} \subset (0,1)$. If ψ_t reaches 1 in finite time, then $s_t = 0$ and we already know that this can only be an absorbing state. If there is indeed an equilibrium trajectory that gets absorbed at $\psi_t = 1$, then Lemma 1 implies that the admissible policy has to be of the form $\mathcal{S}(1,0) = \mathcal{D}(1,0) = 0$.

Recall that (13) implies $F(0) = 1$ and $F(1) = 0$. If $\xi\sigma^2/\rho \in (0,1]$ then $F(\cdot)$ is everywhere decreasing on $[0,1]$, and hence $F(\psi) \geq 0$ for all $\psi \in (0,1]$. So there can be no steady state with a permanent primary deficit. Permanent primary deficits are only possible if $F(\cdot)$ is negative somewhere on $(0,1)$. This will be the case if $\xi\sigma^2/\rho > 1$, so that $(r-g)/\rho = 1-(\xi\sigma^2/\rho)\psi^2 < 0$ for all $\psi \in (0,1]$ close enough to 1.

3.1 The Feasibility of Permanent Primary Deficits

With $F(\cdot)$ extended to all of $(-\infty,\infty)$, the local extrema of the cubic polynomial equation $0 = F(\psi)$ are at

$$\psi_{\pm} = \frac{1}{3}\left(1 \pm \sqrt{1+\frac{3}{\xi\sigma^2/\rho}}\right).$$

Clearly, $\psi_- < 0 < \psi_+$. It is easy to verify that $\psi_+ < 1$ if and only if $\xi\sigma^2/\rho > 1$. And then $\min_{\psi\in[0,1]} F(\psi) = F(\psi_+) < 0$.

Suppose that, indeed, $\xi\sigma^2/\rho > 1$, so that $\psi_+ \in (0,1)$. Consider a target permanent primary deficit ratio $\mathcal{S}_*$ that satisfies

$$F(\psi_+) < \mathcal{S}_* < 0.$$

The first inequality means that the target deficit is strictly below the largest feasible permanent primary deficit. Because $F(0) = 1$, $F(1) = 0$, and $F(\psi)$ is a cubic that attains a negative local minimum at $\psi_+ \in (0, 1)$, there will then be two possible steady states in $[0, 1]$,

$$\mathcal{S}_* = F(\psi) \Leftrightarrow \psi \in \{\psi_{\mathrm{L}}, \psi_{\mathrm{H}}\},$$

where $0 < \psi_* < \psi_{\mathrm{L}} < \psi_+ < \psi_{\mathrm{H}} < 1$. Recall that ψ_* was defined in (14) as the unique portfolio share that implies $r - g = 0$. The blue curve in Figure 1 gives an example of $\mathcal{S}_* - F(\psi)$.

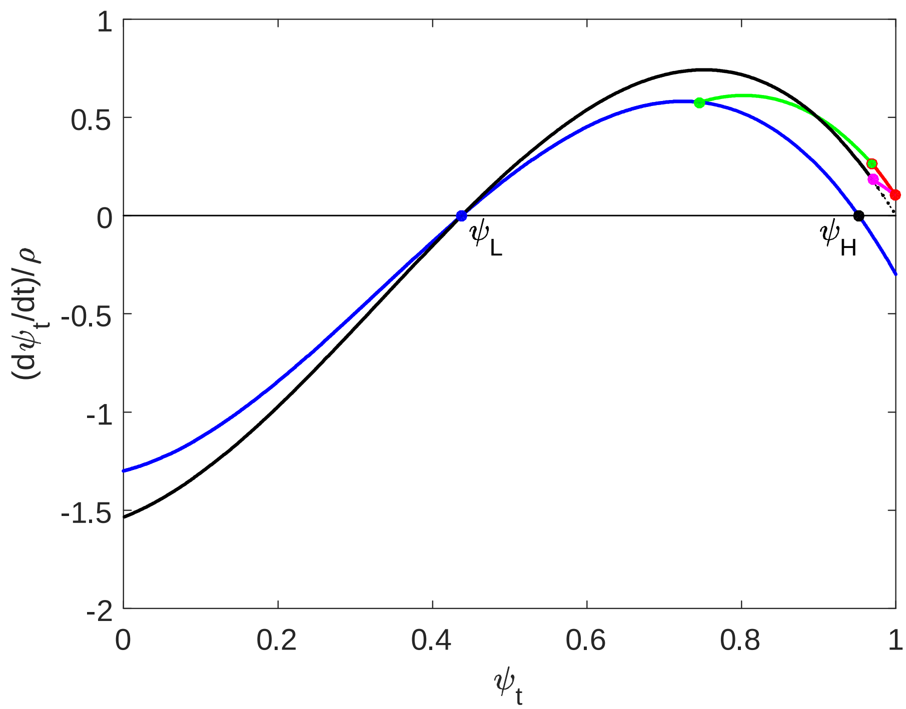

Figure 1 *Alternative surplus policies.*

The Epstein-Zin preferences we have adopted imply that the steady state utilities of consumers are strictly decreasing in the steady state portfolio share ψ of capital. Therefore, holding fixed $\mathcal{S}_*$, the steady state at ψ_{L} implies a better outcome for consumers than the steady state at ψ_{H}. In everything that follows, we will focus on implementing ψ_{L}.

3.1.1 The Balanced Budget and Primary Surplus Cases

The constant policy $\mathcal{S}(\psi, \phi) = \mathcal{S}_*$ is inadmissible for $\mathcal{S}_* < 0$ but admissible for $\mathcal{S}_* \geq 0$. Everything else in this section applies when $\mathcal{S}_* = 0$. This gives $\psi_{\mathrm{L}} = \psi_*$ and $\psi_{\mathrm{H}} = 1$, and ψ_{L} is still the preferred steady state. If $\mathcal{S}_* > 0$ then $\mathcal{S}_* = F(\psi)$ and $\psi \in (0, 1]$ implies $\psi = \psi_{\mathrm{L}} \in (0, \psi_*)$.

3.2 Purely Nominal Policies

Consider the following policy

$$\mathcal{S}(\psi, \phi) = \frac{i - m}{\rho} \times \phi, \quad \mathcal{D}(\psi, \phi) = \frac{i}{\rho} \times \phi, \tag{25}$$

as in Brunnermeier, Merkel, Sannikov [2023]. Here, i and m are two non-negative real numbers. Since $\rho(\mathcal{D}(\psi_t, 1 - \psi_t) - \mathcal{S}(\psi_t, 1 - \psi_t)) / (1 - \psi_t) = m$, this says that the supply of government stock grows at the rate m, no matter what the value is of $\psi_t \in (0, 1]$. Since $\mathcal{D}(\psi_t, 1 - \psi_t) = d_t D_t / E_t$ and $(1 - \psi_t)/\rho = s_t D_t / E_t$, we have $d_t / s_t = i$, again no matter what $\psi_t \in (0, 1]$. So the government is fixing two nominal parameters: the nominal growth rate of the supply of government stock, and a stock dividend.

This policy is clearly admissible. The surplus and dividend ratios are continuous functions of $\psi_t \in (0, 1]$, and both are equal to zero at $\psi_t = 1$. To ensure that ψ_{L} is a steady state, we simply have to take $i - m$ so that $\mathcal{S}(\psi_{\mathrm{L}}, 1 - \psi_{\mathrm{L}}) = F(\psi_{\mathrm{L}})$. This is the same as

$$i - m = \rho - \xi \sigma^2 \psi_{\mathrm{L}}^2.$$

From the Euler equation, the right-hand side is just the value of $r - g$ associated with the steady state ψ_{L}. So the government simply has to let the gap between the nominal interest rate and the growth rate of the supply of government stock match the value of $r - g < 0$ that corresponds to the desired permanent primary deficit.

In Figure 1, the black curve and its dotted extension show the function $\mathcal{S}(\psi, 1 - \psi) - F(\psi)$ for this policy. Note that $(\mathcal{S}(\psi, 1 - \psi) - F(\psi))/(1 - \psi)$ is linear in $\psi \in (0, 1)$. Solving the steady state condition $\mathcal{S}(\psi, 1 - \psi) = F(\psi)$ for $\psi \in (0, 1)$ gives $\psi = \psi_{\mathrm{L}}$, by construction. This means that there is a unique steady state value of $\psi \in (0, 1)$ for which the government can run a permanent primary deficit.

The trouble with any of these purely nominal policies is that $\psi_t = 1$ is also a steady state equilibrium. This corresponds to $s_t = 0$ identically in t. Since $i \times s_t = 0$, paying a nominal interest rate (a stock dividend) does not result in a transfer of consumption goods to the holders of government stock. That makes $s_t = 0$ self-fulfilling. In this equilibrium, the government is forced to balance its budget.

3.3 Unique Implementation

A purely nominal policy generates a balanced budget trap at $s_t = 0$. A constant surplus ratio $\mathcal{S}(\psi, 1-\psi) = \mathcal{S}_* < 0$ is not an admissible policy and would, in any case, also produce two steady states. Here we show how these policies can be modified to uniquely implement the desired surplus ratio at ψ_{L}. The idea is simply to make sure that $\mathcal{S}(1,0) = \mathcal{D}(1,0) > 0$ and then apply Lemma 1.[14] [15] The flaw of a purely nominal policy is that $\mathcal{D}(\psi, 1-\psi) \downarrow 0$ automatically as $\psi \uparrow 1$.

3.3.1 Paying Real Dividends When s_t is at or Near Zero

Consider a policy defined by

$$\mathcal{S}(\psi,\phi) = \alpha + \beta\phi, \quad \mathcal{D}(\psi,\phi) = \max\{0, \mathcal{S}(\psi,\phi)\}, \tag{26}$$

for all $[\psi, \phi, 1-\psi-\phi]$ in the unit simplex. Here we focus on the line $\phi = 1-\psi$. Take $\alpha \in (0, \tau/(1+\tau))$ and determine $\beta < 0$ so that $\alpha + \beta \times (1-\psi_{\mathrm{L}}) = \mathcal{S}_*$. This policy implies $\mathcal{S}(1,0) = \mathcal{D}(1,0) = \alpha > 0$. So it is an admissible policy, and, in contrast to a purely nominal policy (26), it implies strictly positive dividends even when the auctioneer calls out a zero price for government stock. In Figure 1, the black curve and its magenta extension show the function $\mathcal{S}(\psi, 1-\psi) - F(\psi)$ for this policy.

The condition for a steady state is $\mathcal{S}(\psi, 1-\psi) = F(\psi)$. Specifically,

$$\alpha + \beta \times (1-\psi) = \rho\left(1 - \frac{\xi\sigma^2}{\rho} \times \psi^2\right)(1-\psi).$$

Clearly, ψ_{L} is a solution, and $\psi = 1$ is no longer a solution because $\alpha > 0$. Dividing both sides by $1-\psi$ makes the left-hand side increasing in $\psi \in (0,1)$ and the right-hand side decreasing in $\psi \in (0,1)$. So the solution $\psi = \psi_{\mathrm{L}}$ must be unique, and we have $\mathcal{S}(\psi, 1-\psi) < F(\psi)$ for $\psi \in (0, \psi_{\mathrm{L}})$ on the one hand, and $\mathcal{S}(\psi, 1-\psi) > F(\psi)$ for $\psi \in (\psi_{\mathrm{L}}, 1]$ on the other. This not only confirms uniqueness of the steady state, it also

[14]Policies of essentially the same form as will be described here also imply unique implementation of permanent primary deficits in the overlapping generations exchange economy that was used by Bassetto and Cui [2018] to point out the indeterminacy associated with naive policies. Explicit calculations and diagrams are available at `www.luttmer.org/research`.

[15]It will not be difficult to verify that the policies considered here also uniquely implement the preferred steady state $\psi_{\mathrm{L}} = \psi_*$ if $\mathcal{S}_* = 0$.

implies that the steady state is unstable. This implies that $\psi_0 = \psi_{\mathrm{L}}$ is the only possible equilibrium. Other values of ψ_0 would lead to trajectories that hit 0 or 1 in finite time. Such trajectories cannot be part of an equilibrium. The capital stock hits zero at $\psi_t = 0$. At $\psi_t = 1$, Lemma 1 requires that subsequent dividends are zero. But they are not, because $\mathcal{D}(1,0) = \alpha > 0$.

The nominal interest rate and the growth rate of government stock implied by this policy are

$$i_t = \rho \max\left\{0, \frac{\alpha}{1-\psi_t} + \beta\right\}, \quad \frac{1}{D_t}\frac{\mathrm{d}D_t}{\mathrm{d}t} = \rho \max\left\{0, -\left(\frac{\alpha}{1-\psi_t} + \beta\right)\right\}, \tag{27}$$

respectively. Note that $i_t = 0$ and $(\mathrm{d}D_t/\mathrm{d}t)/D_t > 0$ at $\psi_t = \psi_{\mathrm{L}}$. For $\alpha = 0$, this would be a purely nominal policy with $i = 0$ and $m > 0$ that leads to a balanced budget trap at $\psi_t = 1$. But because $\alpha > 0$, the government will run a surplus and pay a strictly positive dividend when ψ_t approaches 1 (and the price of government stock approaches zero). As a result, the nominal interest rate explodes.

3.3.2 A Not So Skittish Policy

The policy (26) immediately begins to adjust government purchases when the price of government stock is lower than implied by the desired steady state. Our next example serves to emphasize that there is really no need for deficits or dividends on government stock to be responsive to prices at or anywhere near the desired steady state.

To illustrate, consider piecewise linear surplus and dividend policies of the form

$$\mathcal{S}(\psi, 1-\psi) = \max\left\{0, \min\left\{\frac{b-x}{b-a}, 1\right\}\right\}\varepsilon + \min\left\{\frac{x}{b}, 1\right\}\mathcal{S}_*, \tag{28}$$

$$\mathcal{D}(\psi, 1-\psi) = \max\left\{0, \min\left\{\frac{b-x}{b-a}, 1\right\}\right\}\varepsilon, \tag{29}$$

where x is defined as

$$x = \frac{1-\psi}{\rho},$$

for all $\psi \in (0,1]$. So x measures $s_t D_t/E_t$. The parameter ε, and the thresholds a and b are taken to satisfy

$$\varepsilon \in \left(0, \frac{\tau}{1+\tau}\right), \quad a \in [0,b), \quad b \in \left(\frac{1-\psi_{\mathrm{H}}}{\rho}, \frac{1-\psi_{\mathrm{L}}}{\rho}\right). \tag{30}$$

The parameter $\varepsilon > 0$ is a cap on the dividend ratio that will be paid when the price of government stock is close to zero. That cap itself can be arbitrarily close to zero. The upper bound is necessary to ensure that $\mathcal{S}(1,0) = \mathcal{D}(1,0) < \tau/(1+\tau)$, so that government purchases are always positive. Note that a and b can be interpreted as thresholds for $s_t D_t/E_t$.

The function $\mathcal{S}(\psi, 1-\psi) - F(\psi)$ implied by (28)-(29) is shown in Figure 1, by the blue curve up to $\psi = 1 - \rho b$, by the green curve on the interval $[1-\rho b, 1-\rho a]$, and by the red curve on $(1-\rho b, 1]$. The actual policies (28)-(29) are shown in the first two panels of Figure 2. Panels three and four in Figure 2 show the implied growth rate $(\mathrm{d}D_t/\mathrm{d}t)/D_t$ and the implied nominal interest rate. In both figures, the blue and black dots are at $\psi = \psi_{\mathrm{L}}$ and $\psi = \psi_{\mathrm{H}}$, respectively. The green dots in Figure 2 are at $\psi \in \{1-\rho b, 1-\rho a\}$. The red dots serve to emphasize the continuity of these policies at $\psi = 1$.

According to this policy, the government runs the targeted primary deficit $-\mathcal{S}_*$ at ψ_{L} and pays a zero nominal interest rate in that steady state. It simply prints new stock to finance the primary deficit. The policy is constructed to deliver strictly positive primary surplus and dividend ratios at $\psi = 1$. The government only begins to transition from a primary deficit to a primary surplus when $s_t D_t/E_t$ falls below $b > 0$. This threshold is chosen high enough to eliminate ψ_{H} as a possible steady state, and low enough to preserve ψ_{L} as a steady state. The policy caps the dividend ratio at $\varepsilon > 0$, and that cap is reached when $s_t D_t/E_t$ reaches $a \geq 0$. As a result, the nominal interest rate $i_t = \rho \mathcal{D}(\psi_t, 1-\psi_t)/(1-\psi_t)$ explodes as $s_t D_t/E_t$ approaches zero.

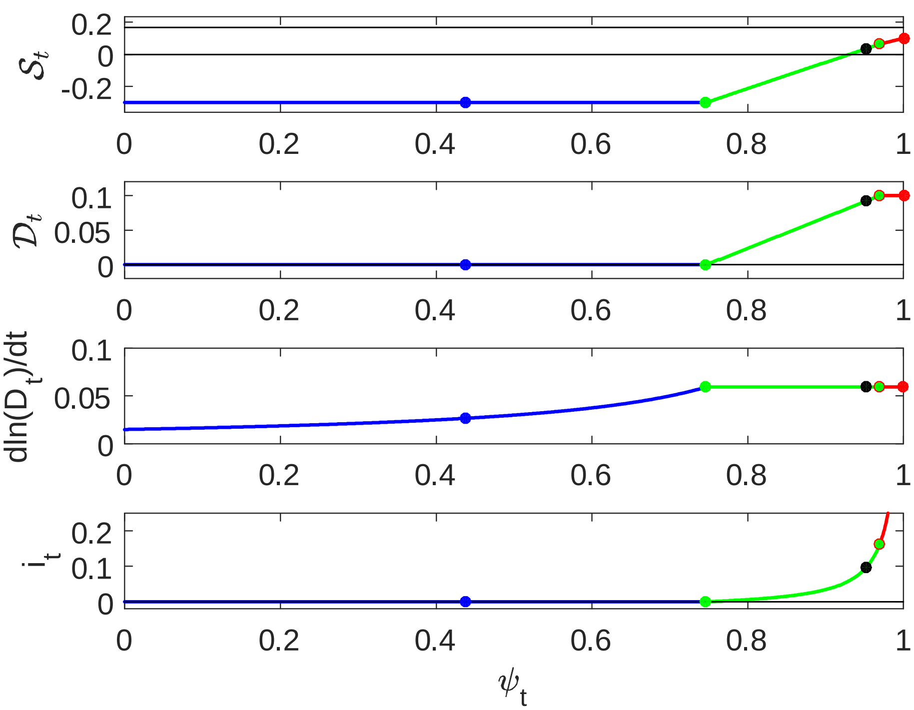

Figure 2 *A piecewise linear policy.*

In Figure 2, the consumption tax parameter is $\tau = 0.2$. This means that the surplus ratio is bounded above by $1/6$, as indicated by the horizontal line in the first panel. The targeted primary surplus is $\mathcal{S}_* = -0.3$. This is quite large, but the parameter $\xi\sigma^2/\rho = 10$ implies $F(\psi_+) \approx -0.88$, and so $-\mathcal{S}_*$ is well below the upper bound on permanent primary deficits.[16]

4 Government Stock and Bitcoin Implies Indeterminacy

Now remove the legal prohibition and suppose bitcoin can be traded. Recall that the supply is fixed at some $B > 0$. We are going to show that it is no longer possible to uniquely implement a primary surplus ratio $\mathcal{S}_* < 0$ at ψ_{L}. Our requirement that surplus and dividend ratios are continuous functions of market prices automatically creates a balanced budget trap in which both government stock and bitcoin are claims to nothing and nevertheless trade at a positive price. Furthermore, smooth policies also imply a local indeterminacy near the desired steady state.

4.1 The Unavoidable Balanced Budget Trap

To prove this, suppose the policy $\mathcal{S}(\cdot,\cdot)$ admits $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}] \in (0,1)\times(0,1)$ as a steady state,

$$0 > \mathcal{S}_* = \mathcal{S}(\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}) = F(\psi_{\mathrm{L}}) = \left(1 - \frac{\xi\sigma^2}{\rho} \times \psi_{\mathrm{L}}^2\right)(1-\psi_{\mathrm{L}})\,.$$

Of course, this implies $(r-g)/\rho = 1 - (\xi\sigma^2/\rho)\times\psi_{\mathrm{L}}^2 < 0$. Recall from (14) that $\psi_* \in (0,\psi_{\mathrm{L}})$ is the unique $\psi \in (0,1)$ that solves $F(\psi) = 0$, and that $r - g = 0$ at $\psi = \psi_*$.

At the very least, unique implementation would require a unique equilibrium for trajectories that satisfy $[\psi_t, \phi_t] = [\psi_t, 1-\psi_t]$ at all times. That is, there should be no equilibria other than $[\psi_t, 1-\psi_t] = [\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$ if the price of bitcoin is identically zero. This requires that

$$\mathrm{d}\psi_t = \rho\left(\mathcal{S}(\psi_t, 1-\psi_t) - F(\psi_t)\right)\mathrm{d}t < 0$$

for all $\psi_t \in (0,\psi_{\mathrm{L}})$.

To verify this result, suppose that $\mathcal{S}(\psi, 1-\psi) - F(\psi) > 0$ for some $\psi \in (0,\psi_{\mathrm{L}})$. If this is true for all $\psi \in (0,\psi_{\mathrm{L}})$, then all trajectories starting with $\psi_0 \in (0,\psi_{\mathrm{L}})$ and $\phi_0 = 1-\psi_0$ converge to $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$. On the other hand, if there are also $\psi \in (0,\psi_{\mathrm{L}})$ with $\mathcal{S}(\psi, 1-\psi) - F(\psi) > 0$, then the continuity of $\mathcal{S}(\cdot,\cdot) - F(\cdot)$ implies the existence of another

[16] The level of government spending, the size of the primary deficit, and the portfolio share of government stock are massive in this example. This makes Figures 1 and 2 easier to read.

steady state. Either way, the desired steady state $[\psi_{\rm L}, 1-\psi_{\rm L}]$ would not be unique. So we certainly must have $\mathcal{S}(\psi, 1-\psi) - F(\psi) \leq 0$ for all $\psi \in (0, \psi_{\rm L})$. Moreover, this cannot hold with equality for any $\psi \in (0, \psi_{\rm L})$, or we would have another steady state.

Applying this result to $\psi_t = \psi_* \in (0, \psi_{\rm L})$ gives $\mathrm{d}\psi_t < 0$. Since $F(\psi_*) = 0$, this says that government policy must be such that $\mathcal{S}(\psi_*, 1-\psi_*) < 0$. But we also know that admissibility requires that $\mathcal{S}(\psi_*, 0) \geq 0$. The continuity of $\mathcal{S}(\psi_*, \phi)$ as a function of $\phi \in [0, 1-\psi_*]$ then implies the existence of a $\phi_* \in [0, 1-\psi_*)$ so that

$$\mathcal{S}(\psi_*, \phi_*) = 0.$$

It is now easy to check that the differential equation (17)-(18) implies $\mathrm{d}\psi_t = 0$ and $\mathrm{d}\phi_t = 0$ at $[\psi_t, \phi_t] = [\psi_*, \phi_*]$. So this is a proper steady state. Since $1 - (\psi_* + \phi_*) > 0$, the price of bitcoin is positive. And it remains constant relative to the size of the economy because $(r-g)/\rho = 1 - (\xi\sigma^2/\rho)\psi_*^2 = 0$. Since $\mathcal{S}(\psi_*, \phi_*) = 0$, the government is balancing its budget, contrary to the desired outcome $\mathcal{S}_* < 0$. So the economy is in a balanced budget trap.

Proposition 2 *Suppose the continuous policy $\mathcal{S}(\cdot,\cdot)$ is admissible and satisfies $\mathcal{S}(\psi_{\rm L}, 1-\psi_{\rm L}) = \mathcal{S}_* < 0$. Then there is also a balanced budget trap at $[\psi_*, \phi_*]$, where $(r-g)/\rho = 1-(\xi\sigma^2/\rho)\psi_* = 0$ and $\mathcal{S}(\psi_*, \phi_*) = 0$, for some $\phi_* \in [0, 1-\psi_*)$.*

The government can easily ensure that the price of government stock will be strictly positive, by taking $\mathcal{S}(\psi_*, 0) > 0$, so that $\mathcal{S}(\psi_*, \phi_*) = 0$ forces $\phi_* > 0$. In that case, (22) and (23) imply that

$$i_t = \frac{1}{D_t}\frac{\mathrm{d}D_t}{\mathrm{d}t} = \frac{\mathcal{D}(\psi_*, \phi_*)}{\phi_*/\rho} \geq 0$$

in the balanced budget trap. Government stock is, like bitcoin, a claim to nothing. While the government may be paying a dividend, it does so not by distributing a primary surplus but by simply printing more claims to nothing at some rate that will be the nominal interest rate. Taking government stock as the numeraire, the inflation rate in this balanced budget trap is $i_t - r_t = i_t - g_t$ while the price of bitcoin p_t/s_t grows at the nominal interest rate (or, in units of consumption, at the rate $r_t = g_t$).

4.1.1 Eliminating a Kareken-Wallace Indeterminacy

Knowing only that government policy satisfies $\mathcal{S}(\psi_*, 1-\psi_*) < 0 \leq \mathcal{S}(\psi_*, 0)$ leaves open the possibility that there are many balanced budget traps. But if government policy is such that $\phi = \phi_*$ is the unique solution to $\mathcal{S}(\psi_*, \phi) = 0$, then the prices of government stock and bitcoin are uniquely determined within the balanced budget trap. This is in contrast to the Kareken-Wallace indeterminacy described in Proposition 1. There the underlying policy was $\mathcal{S}(\psi, \phi) = 0$ for all $\psi \in (0,1]$ and $\phi \in [0, 1-\psi]$. Although admissible government policies cannot avoid the balanced budget trap, they can be used to eliminate steady state indeterminacies within that trap.

4.2 The Unavoidable Local Indeterminacy

The balanced budget trap constructed in Proposition 2 is a steady state that is distinct from $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$. So it represents a global indeterminacy. There is also a local indeterminacy near the desired steady state $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$. This local indeterminacy is implied by the dynamics of the bitcoin portfolio share given in (19). Locally, near the desired steady state, the dynamics of that portfolio share does not depend on the specifics of government policy. So there is nothing the government can do about it.

For a quick way to see this, consider the differential equation (19) for the bitcoin portfolio share $1-\psi_t-\phi_t$. Taking a derivative with respect to ψ_t and ϕ_t gives

$$\frac{\partial}{\partial[\psi_t, \phi_t]} \frac{\mathrm{d}\left(1-\psi_t-\phi_t\right)}{\mathrm{d}t} = -\rho\left[\frac{\xi\sigma^2}{\rho} \times 2\psi_t\left(1-\psi_t-\phi_t\right) + 1 - \frac{\xi\sigma^2}{\rho} \times \psi_t^2, 1 - \frac{\xi\sigma^2}{\rho} \times \psi_t^2\right].$$

At the targeted steady state $[\psi_t, \phi_t] = [\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$, we have $1-\psi_t-\phi_t = 0$, and then these derivatives simplify to

$$\frac{\partial}{\partial[\psi_t, \phi_t]}\left(\frac{\mathrm{d}\psi_t}{\mathrm{d}t} + \frac{\mathrm{d}\phi_t}{\mathrm{d}t}\right) = [1,1]\rho\left(1 - \frac{\xi\sigma^2}{\rho} \times \psi_{\mathrm{L}}^2\right) = [1,1](r-g).$$

The left-hand side is simply $[1,1]$ times the Jacobian of the differential equation (17)-(18) for $[\psi_t, \phi_t]$, evaluated at the steady state $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$. Therefore, this result says that $[1,1]$ is a left eigenvector of that Jacobian, and that $r-g<0$ is the associated eigenvalue. No matter how the surplus policy $\mathcal{S}(\psi_t, \phi_t)$ responds to deviations of ψ_t and ϕ_t from the targeted steady state, one of the eigenvalues of the Jacobian is always the same, and equal to $r-g<0$. The fact that this eigenvalue is negative implies an unavoidable indeterminacy.

Assume $\mathcal{S}(\cdot,\cdot)$ and $\mathcal{D}(\cdot,\cdot)$ to be differentiable, not just continuous. Then we can give a more precise account of the local properties near the desired steady state of (17)-(18) with $\mathcal{S}_t = \mathcal{S}(\psi_t, \phi_t)$ and $\mathcal{D}_t = \mathcal{D}(\psi_t, \phi_t)$. Evaluated at $[\psi_t, \phi_t] = [\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$, the Jacobian of this differential equation is

$$\frac{\partial}{\partial[\psi_t,\phi_t]}\begin{bmatrix}\frac{\mathrm{d}\psi_t}{\mathrm{d}t}\\ \frac{\mathrm{d}\phi_t}{\mathrm{d}t}\end{bmatrix} = \rho\begin{bmatrix}1-\frac{\xi\sigma^2}{\rho}\times\psi_{\mathrm{L}}^2+\frac{\xi\sigma^2}{\rho}\times 2\psi_{\mathrm{L}}\left(1-\psi_{\mathrm{L}}\right) & 0\\ -\frac{\xi\sigma^2}{\rho}\times 2\psi_{\mathrm{L}}(1-\psi_{\mathrm{L}}) & 1-\frac{\xi\sigma^2}{\rho}\times\psi_{\mathrm{L}}^2\end{bmatrix} + \begin{bmatrix}1\\ -1\end{bmatrix}\rho\mathrm{D}\mathcal{S}(\psi_{\mathrm{L}},1-\psi_{\mathrm{L}}). \tag{31}$$

The $(1,1)$ element of the first matrix on the right is $-\rho\mathrm{D}F(\psi_{\mathrm{L}})$, and the $(1,2)$ element is zero because $\mathrm{d}\psi_t/\mathrm{d}t$ only depends on ϕ_t via $\mathcal{S}(\psi_t,\phi_t)$. This Jacobian is of the form

$$\frac{\partial}{\partial[\psi_t,\phi_t]}\begin{bmatrix}\frac{\mathrm{d}\psi_t}{\mathrm{d}t}\\ \frac{\mathrm{d}\phi_t}{\mathrm{d}t}\end{bmatrix} = \begin{bmatrix}p & 0\\ q-p & q\end{bmatrix} + \begin{bmatrix}1\\ -1\end{bmatrix}\begin{bmatrix}a & b\end{bmatrix} = \begin{bmatrix}p+a & b\\ q-(p+a) & q-b\end{bmatrix}. \tag{32}$$

As already argued, one of the left eigenvectors of this Jacobian is $[1,1]$, with the associated eigenvalue $q=r-g<0$. The right eigenvectors and associated eigenvalues are

$$\begin{bmatrix}1\\ -1\end{bmatrix}\leftrightarrow p+a-b, \quad \begin{bmatrix}b\\ q-(p+a)\end{bmatrix}\leftrightarrow q. \tag{33}$$

Observe that $p=-\rho\mathrm{D}F(\psi_{\mathrm{L}})>0$, and that $p+a-b=-\rho\mathrm{D}F(\psi)+\rho\mathrm{d}\mathcal{S}(\psi,1-\psi)/\mathrm{d}\psi$ evaluated at $\psi=\psi_{\mathrm{L}}$. This is the slope $\partial(\mathrm{d}\psi_t/\mathrm{d}t)/\partial\psi_t$ evaluated at $\psi_t=\psi_{\mathrm{L}}$ when $\phi_t=1-\psi_t$. The corresponding eigenvector simply defines the line segment $[\psi,\phi]=[\psi,1-\psi]$, $\psi\in(0,1]$.

If the price of bitcoin is identically equal to zero, then a minimal requirement for the steady state $[\psi_{\mathrm{L}},1-\psi_{\mathrm{L}}]$ to be locally unique is $p+a-b\geq 0$. Or else there would be a continuum of equilibrium trajectories $[\psi_t,\phi_t]=[\psi_t,1-\psi_t]$ that all converge to $[\psi_{\mathrm{L}},1-\psi_{\mathrm{L}}]$.

If $p+a-b>0$, then both eigenvalues are different from zero. This says that $[\psi_{\mathrm{L}},1-\psi_{\mathrm{L}}]$ is a hyperbolic steady state. We can therefore apply the Hartman-Grobman theorem (Guckenheimer and Holmes [1983, p.13]) to conclude that there must be a stable manifold that is tangent to the eigenspace associated with the stable eigenvalue $q<0$. Since $p+a-b>0>q$, and therefore $b+q-(p+a)<0$, we know that this stable manifold is not tangent to the line $\psi+\phi=1$. So there will be trajectories that start with $\psi_0+\phi_0<1$ and converge to $[\psi_{\mathrm{L}},1-\psi_{\mathrm{L}}]$ over time. These are all equilibrium trajectories. In all of them, the initial price of bitcoin is strictly positive and converges to zero at the approximate rate $r-g<0$. In other words, any policy that implies $p+a-b>0$ means that there is a continuum of equilibria, with positive bitcoin prices that converge to zero.

It remains to consider policies that imply $p+a-b=0$. This need not necessarily imply an equilibrium indeterminacy when the price of bitcoin is identically zero. The steady state $[\psi_{\rm L}, 1-\psi_{\rm L}]$ could be an inflection point that satisfies $(\mathrm{d}\psi_t/\mathrm{d}t)(\psi_t - \psi_{\rm L}) > 0$ for all $\psi_t \neq \psi_{\rm L}$. But, because one of the eigenvalues is now zero, this steady state is no longer hyperbolic, and we can no longer apply the Hartman-Grobman theorem. Nevertheless, the Central Manifold Theorem (Guckenheimer and Holmes [1983, p.127]) implies that, because $q < 0$, there is still a stable manifold that is tangent to the eigenspace associated with $q < 0$. Since $p + a - b = 0$ is the same as $q - (p + a) = q - b$, that eigenspace is spanned by the vector $[b, q - b]$ and $q < 0$ ensures that this cannot coincide with the line $[\psi, 1 - \psi]$. Again, it follows that there is a continuum of equilibria in which bitcoin prices are positive but converge to zero eventually.

Proposition 3 *Assume the policy given by $\mathcal{S}(\cdot,\cdot)$ and $\mathcal{D}(\cdot,\cdot)$ is admissible, differentiable, and satisfies $\mathcal{S}(\psi_{\rm L}, 1-\psi_{\rm L}) = \mathcal{S}_*$. Suppose this policy is such that the unique steady state is $[\psi_{\rm L}, 1-\psi_{\rm L}]$ when the price of bitcoin is identically zero. Then there is still a continuum of equilibria with bitcoin prices that are strictly positive and converge to zero over time.*

There are no smooth policies that can be used to rule out equilibria in which bitcoin prices are strictly positive. Any attempt by the government to manipulate its primary surplus or dividends to rule out $\psi_t + \phi_t < 1$ must fail. The best the government can do is take $b = \mathrm{D}_2\mathcal{S}(\psi_{\rm L}, 1 - \psi_{\rm L}) = 0$, so that deviations of ψ_t from $\psi_{\rm L}$ will be of second order in $1 - \psi_{\rm L} - \phi_t$.

4.3 Two Examples

To illustrate the multiplicity of equilibria implied by smooth policies, we consider two explicit examples. The first is one we have already examined in the absence of bitcoin. The second example shows that it is possible to construct policies for which real quantities deviate from the government's target for only a finite amount of time.

4.3.1 A Continuum of Non-Steady State Equilibria with $\mathcal{S}(\psi_0, \phi_0) \in [\mathcal{S}_*, 0]$

Consider again the policy (26), but now for all $[\psi, \phi, 1 - \psi - \phi]$ in the unit simplex. As before, because of the positive intercept $\mathcal{S}(\psi, 0) = \mathcal{D}(\psi, 0) = \alpha > 0$ and Lemma 1, $\phi_t = 0$ can never be part of an equilibrium trajectory. There can be no balanced budget trap at $[\psi, \phi] = [1, 0]$. The implicit policies for the nominal interest rate and the growth rate of government stock are (27), with $1 - \psi_t$ replaced by ϕ_t.

By construction, $[\psi_{\rm L}, 1 - \psi_{\rm L}]$ is a steady state for this policy. At this steady state, the eigenvalues of the Jacobian are $r - g < 0$ and $-\mathrm{D}F(\psi_{\rm L}) - \beta > 0$. As we argued in

Proposition 3, the desired steady state is a saddle point. The balanced budget trap $[\psi_*, \phi_*]$ predicted by Proposition 1 is easy to calculate from $1 - (\xi\sigma^2/\rho) \times \psi_*^2 = 0$ and $\alpha + \beta\phi_* = 0$. By construction, this gives $r - g = 0$ and $\mathcal{S}(\psi_*, \phi_*) = 0$.

Figure 3 shows the phase diagram for this economy. The blue dot represents the desired steady state and the interior red dot is the balanced budget trap. It is not difficult to verify that both eigenvalues of the Jacobian at this balanced budget trap are real and strictly positive (even though $r - g = 0$ in this balanced budget trap). So the balanced budget trap is completely unstable. The desired steady state and the balanced budget trap are connected by a smooth manifold, shown in black. Trajectories that start on this manifold are equilibrium trajectories that move away from the balanced budget trap and converge to the desired steady state. In contrast, the trajectories that start very close to this manifold eventually violate $\psi_t > 0$ or $\phi_t \geq 0$. The trajectory that starts on the other side of the balanced budget trap eventually violates $\phi_t \geq 0$.

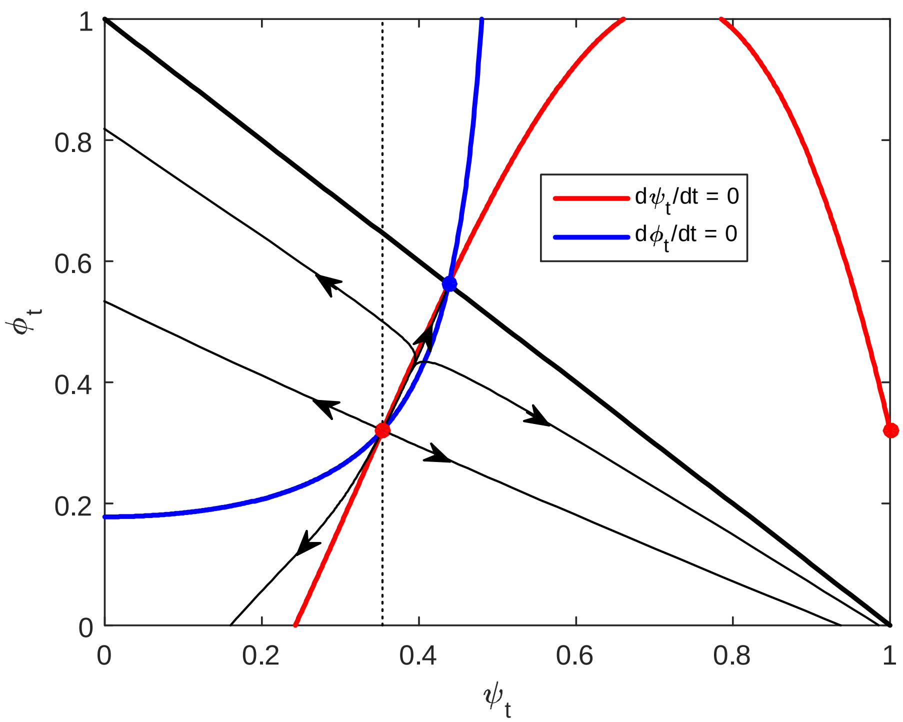

Figure 3 *A balanced budget trap and a continuum of equilibria.*

Clearly, the equilibria that start with $[\psi_0, \phi_0]$ away from the desired steady state do not implement the desired steady state. But they do converge to that steady state in the long run.

The Balanced Budget Scenario is Special Recall that $\mathcal{S}_* \uparrow 0$ implies $\psi_{\mathrm{L}} \to \psi_*$. Since the slope β is chosen so that $\alpha + \beta(1 - \psi_{\mathrm{L}}) = \mathcal{S}_*$ and since the balanced budget trap value ϕ_*

is determined by $\alpha + \beta\phi_* = 0$, it follows that $[\psi_*, \phi_*] \to [\psi_\mathrm{L}, 1 - \psi_\mathrm{L}]$. That is, the balanced budget trap for this class of policies converges to the desired steady state as $\mathcal{S}_* \uparrow 0$.

The balanced budget trap for $\mathcal{S}_* < 0$ shown in Figure 3 has eigenvalues that are both strictly positive precisely because $\phi_* \in [0, 1 - \psi_*)$. One of these eigenvalues converges to zero as $\phi_* \uparrow 1 - \psi_*$ together with $\mathcal{S}_* \uparrow 0$. The limiting eigenvalues are $-\mathrm{D}F(\psi_*) - \beta > 0$ and 0. This makes $[\psi_*, \phi_*] = [\psi_\mathrm{L}, 1 - \psi_\mathrm{L}]$ a non-hyperbolic fixed point in the $\mathcal{S}_* = 0$ economy, with stability properties that cannot be deduced from its eigenvalues alone. The $\mathrm{d}\psi_t/\mathrm{d}t = 0$ and $\mathrm{d}\phi_t/\mathrm{d}t = 0$ curves in the $\mathcal{S}_* = 0$ economy are tangent and upward sloping at $[\psi_*, \phi_*] = [\psi_\mathrm{L}, 1 - \psi_\mathrm{L}]$. The tangent line separates the regions where $\mathrm{d}\psi_t/\mathrm{d}t > 0$ and $\mathrm{d}\phi_t/\mathrm{d}t > 0$, unlike what happens in the $\mathcal{S}_* < 0$ case shown in Figure 3, where there is an upward sloping trajectory $[\psi_t, \phi_t]$ that can reach $[\psi_\mathrm{L}, 1 - \psi_\mathrm{L}]$. It is easy to use the resulting phase diagram to show that $[\psi_\mathrm{L}, 1 - \psi_\mathrm{L}]$ is, in fact, the only equilibrium.

In other words, the government can uniquely implement a balanced budget equilibrium in which the price of bitcoin is zero. Along the equilibrium trajectory, government stock and bitcoin are both claims to nothing. What makes government stock different from bitcoin is that consumers know that the government will begin to run a primary surplus and pay a positive dividend on its stock whenever the price of its stock dips below its equilibrium trajectory. This is enough to eliminate the Kareken-Wallace indeterminacy we highlighted in Proposition 1.

4.3.2 A Continuum of Non-Steady State Equilibria with $\mathcal{S}(\psi_0, \phi_0) = \mathcal{S}_*$

Consider the policy

$$\mathcal{S}(\psi, \phi) = \mathcal{S}_* + (\varepsilon - \mathcal{S}_*) \max\left\{0, 1 - \frac{\phi}{\delta}\right\}, \quad \mathcal{D}(\psi, \phi) = \max\left\{0, \mathcal{S}(\psi, \phi)\right\},$$

for some $\varepsilon \in (0, \tau/(1+\tau))$ and $\delta \in (1 - \psi_\mathrm{H}, 1 - \psi_\mathrm{L})$. The cap $\varepsilon > 0$ on $\mathcal{S}(\psi, \phi)$ can be arbitrarily close to zero. The bounds on δ ensure that ψ_L is a steady state but ψ_H is not. The government only begins to run a primary surplus and pay dividends when the portfolio share of government stock gets too close to zero. Within the class of equilibria in which the price of bitcoin is identically equal to zero, this leads to a unique equilibrium, at the desired steady state.

As implied by Proposition 2, there is still the balanced budget trap when bitcoin is allowed to trade at a positive price. An interesting feature of the equilibrium trajectories that approach the desired steady state is that the government implements its desired primary deficit at ψ_L most of the time, except possibly for an initial startup period. Figure 4 shows the phase diagram. As before, the blue dot is the desired steady state and the red dot is the balanced budget trap. The black dot represents ψ_{H}. As before, the thin black curves in Figure 4 show trajectories that are not equilibria because they violate $\psi_t > 0$ or $\phi_t > 0$ eventually.

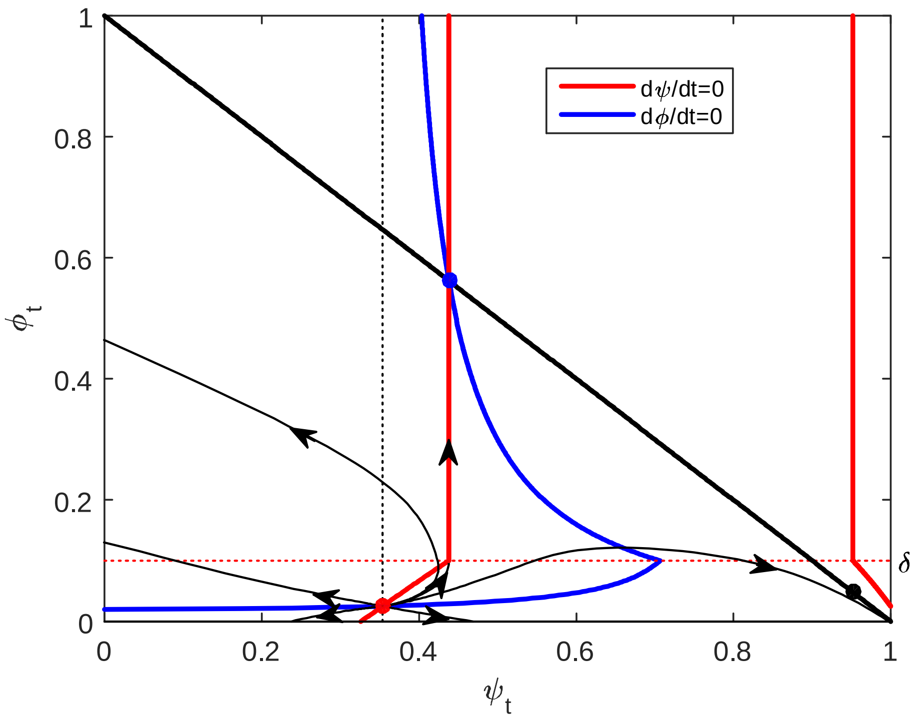

Figure 4 *A continuum of equilibria with $\psi_t = \psi_{\mathrm{L}}$ most of the time.*

The balanced budget trap and the desired steady state are connected by a manifold that is depicted using a black curve and a vertical red line, where $\mathrm{d}\phi_t/\mathrm{d}t = 0$. Trajectories that start with $[\psi_{0,}, \phi_0]$ on the black curve are equilibrium trajectories that will have $\mathrm{d}\psi_t/\mathrm{d}t > 0$ $\mathrm{d}\phi_t/\mathrm{d}t > 0$ for a finite period of time, until $[\psi_t, \phi_t]$ reaches $[\psi_{\mathrm{L}}, \delta]$. Subsequently, ψ_t remains at ψ_{L} and ϕ_t increases mononotically from δ to $1 - \psi_{\mathrm{L}}$ over time. During this second phase, the $r - g < 0$ is at its desired steady state value and the market value of bitcoin converges to zero at the rate $r - g$.

4.4 The Power of Discontinuous Policies

It is very easy to rule out all positive bitcoin equilibria when the government can follow a policy that is discontinuous in $[\psi_t, \phi_t]$.

4.4.1 A Discontinuous Markov Policy

Consider a policy with two components. For all $[\psi_t, \phi_t] = [\psi_t, 1 - \psi_t]$, the government follows a policy that leads to a unique equilibrium with $\mathcal{S}(\psi_{\mathrm{L}}, 1 - \psi_{\mathrm{L}}) = \mathcal{S}_* < 0$, as in all our examples. The other component is

$$\mathcal{S}(\psi_t, \phi_t) = \varepsilon, \qquad \phi_t \in [0, 1 - \psi_t),$$

for some $\varepsilon \in (0, \tau/(1+\tau))$. In other words, the government jumps to a surplus whenever the price of bitcoin is positive. The phase diagram for this policy is shown in Figure 5.

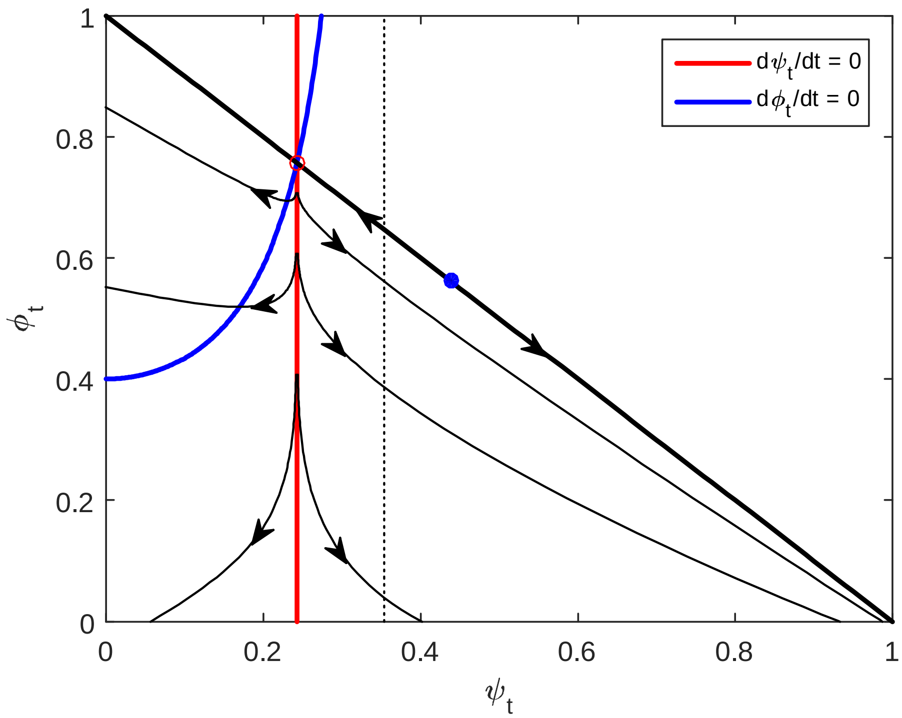

Figure 5 *A discontinuous policy with a unique equilibrium.*

If the policy for $\phi \in [0, 1-\psi)$ were extended to all $[\psi, \phi, 1-\psi-\phi] \in \mathbb{R}^3_+$, this policy would imply a unique equilibrium with a permanent primary surplus equal to ε. All other trajectories, such as the ones shown in the phase diagram, eventually violate $\psi_t > 0$ or $\phi_t \geq 0$. The unique equilibrium for this alternative policy would satisfy $\phi = 1 - \psi$ and $1 - (\xi\sigma^2/\rho)\psi^2 > 0$, and therefore $\psi \in (0, \psi_*) \subset (0, \psi_{\mathrm{L}})$. But that combination of $[\psi, \phi]$ is on the line where the actual policy implies $\mathcal{S}(\psi, \phi) < 0$. So it cannot be the equilibrium.

This example shows that the trigger strategies used by Brunnermeier, Merkel, and Sannikov [2023], which switch to a permanent primary surplus when the price of bitcoin is positive, are not necessary for uniqueness. Markov policies that only have the government switch to a surplus while the price of bitcoin is positive can also do the job. But these Markov policies must be discontinuous to avoid a balanced budget trap.

4.4.2 An Open Market Operation

An even simpler way to uniquely implement $\mathcal{S}_* < 0$ at ψ_L is for the government to buy up all of the supply of bitcoin.

Suppose there is an ex ante trading date 0_- and write $D_{0-} > 0$ for the initial supply of government stock, and $B_{0-} > 0$ for the initial supply of bitcoin. The government can trade its stock for bitcoin subject to $s_{0-}(D_0 - D_{0-}) + p_{0-}(B_0 - B_{0-}) = 0$, and the goal is $B_0 = 0$. If $s_{0-} = 0$, simply take $D_0 = D_{0-}$ and $B_0 = B_{0-}$. If $s_{0-} > 0$, take $D_0 - D_{0-} = p_{0-}B_{0-}/s_{0-}$ and $B_0 = 0$. Note that $s_{0-}(D_0 - D_{0-})$ is discontinuous in $s_{0-} \geq 0$ at $s_{0-} = 0$ when $p_{0-} > 0$.

We know that government policy can guarantee that $s_t > 0$ for all $t \geq 0$ even if it does not buy up all of the supply of bitcoin. So $s_{0-} = 0$ would be an arbitrage opportunity and cannot be part of an equilibrium. For any $s_{0-} > 0$, the government can print the amount of stock it will take to buy up all bitcoin. From $t = 0$ onwards, we then have an economy without bitcoin, and we already know that the government can uniquely implement its targeted primary deficit in that case. If we interpret this continuation equilibrium to mean that not only $s_t > 0$ but also $p_t = 0$ for all $t \geq 0$, then it is optimal for consumers to sell all of their bitcoin holdings in the ex ante trading period. So any combination of $s_{0-} > 0$ and $p_{0-} \geq 0$ will be part of an equilibrium, and subsequently the government will be able to uniquely implement its desired steady state.

Of course, any other type of useless pieces of paper still lying around could now take on the same role as bitcoin.

5 A Tax on Bitcoin

Suppose the government imposes a flow tax equal to θp_t on every owner of a unit of bitcoin. This tax can be viewed as a negative dividend. Whenever $p_t > 0$, the price of bitcoin must evolve according to $\mathrm{d}p_t = (r_t + \theta)p_t\mathrm{d}t$. Government tax revenues are now

$$T_t = \tau C_t + \theta p_t B.$$

As before, via $E_t = \rho W_t$ and the definition of aggregate wealth, these tax revenues are a function only of the physical state $[K_t, D_t] \in \mathbb{R}^2_{++}$ and the prices $s_t \geq 0$ and $p_t \geq 0$. The resource constraint $Y_t = C_t + G_t$ then implies

$$\begin{aligned}\frac{Y_t}{E_t} &= \frac{C_t + T_t}{E_t} - \frac{T_t - G_t}{E_t} \\ &= 1 + \frac{\theta}{\rho} \times \frac{p_t B}{W_t} - \frac{T_t - G_t}{E_t} = 1 - \mathcal{S}_t + \frac{\theta}{\rho} \times (1 - \psi_t - \phi_t)\,.\end{aligned}$$

Using this expression for Y_t/E_t exactly as in the construction of the differential equation (17)-(18) gives

$$\mathrm{d}\psi_t = \rho\left(\mathcal{S}_t - \left(1 - \frac{\xi\sigma^2}{\rho} \times \psi_t^2\right)(1-\psi_t) - \frac{\theta}{\rho} \times (1-\psi_t-\phi_t)\right)\mathrm{d}t, \tag{34}$$

$$\mathrm{d}\phi_t = \rho\left(\left(1 - \frac{\xi\sigma^2}{\rho} \times \psi_t^2\right)\phi_t - \mathcal{S}_t\right)\mathrm{d}t. \tag{35}$$

As before, $(r - g_t)/\rho = 1 - (\xi\sigma^2/\rho)\psi_t^2$. Adding up (34)-(35) therefore gives $\mathrm{d}(1-\psi_t-\phi_t) = (r_t - g_t + \theta)(1-\psi_t-\phi_t)\mathrm{d}t$. If consumers are willing to hold bitcoin, its price grows at the rate $r_t + \theta$, and so the portfolio share of bitcoin grows at the rate $r_t - g_t + \theta$. For $\theta = 0$ we then have $r_t - g_t + \theta < 0$ in the targeted steady state, and this is what causes the local indeterminacy of that steady state. But now $\theta > 0$ can be used to flip the sign of $r_t - g_t + \theta$. As we shall see, this is precisely what can eliminate bitcoin equilibria from this economy.

5.1 Removing the Second Steady State

Given an admissible policy defined by $\mathcal{S}_t = \mathcal{S}(\psi_t, \phi_t)$ and $\mathcal{D}_t = \mathcal{D}(\psi_t, \phi_t)$, the steady state conditions for (34)-(35) can be summarized by

$$0 = \left(1 - \frac{\xi\sigma^2}{\rho} \times \psi^2 + \frac{\theta}{\rho}\right)(1-\psi-\phi), \tag{36}$$

$$\mathcal{S}(\psi,\phi) = \left(1 - \frac{\xi\sigma^2}{\rho} \times \psi^2\right)\phi, \tag{37}$$

where $\psi \in (0,1]$ and $\phi \in [0, 1-\psi]$. Condition (36) says that $0 = (r - g + \theta)(1 - \psi - \phi)$, and (37) is the familiar steady state condition for the portfolio share of government stock. As before, assume that $\mathcal{S}(\psi,\phi)$ is such that imposing $\phi = 1 - \psi$ in (37) implies $\psi = \psi_{\mathrm{L}}$. Then $[\psi, \phi] = [\psi_{\mathrm{L}}, 1 - \psi_{\mathrm{L}}]$ obviously satisfies (36) as well, and so policy uniquely implements the desired steady state when bitcoin is illegal. As before, $r - g < 0$ in this steady state.

Suppose that the tax on bitcoin satisfies $\rho + \theta \in [\rho, \xi\sigma^2)$. Then another way to satisfy (36) is to solve $1 - (\xi\sigma^2/\rho)\psi^2 + \theta/\rho = 0$ for $\psi \in (0,1]$. This gives $\psi \in [\psi_*, 1)$, and $\psi \in (\psi_*, 1)$ if the bitcoin tax rate is strictly positive. Holding fixed this $\psi \in [\psi_*, 1)$, the steady state condition (37) must be solved for $\phi \in [0, 1-\psi]$.

If $\theta = 0$, then we know from Proposition 2 that this can be done, resulting in a second steady state given by the balanced budget trap at $\psi = \psi_*$ (which solves (36)) and some $\phi_* \in [0, 1 - \psi_*)$ that solves $\mathcal{S}(\psi_*, \phi_*) = 0$ (which corresponds to (37)).

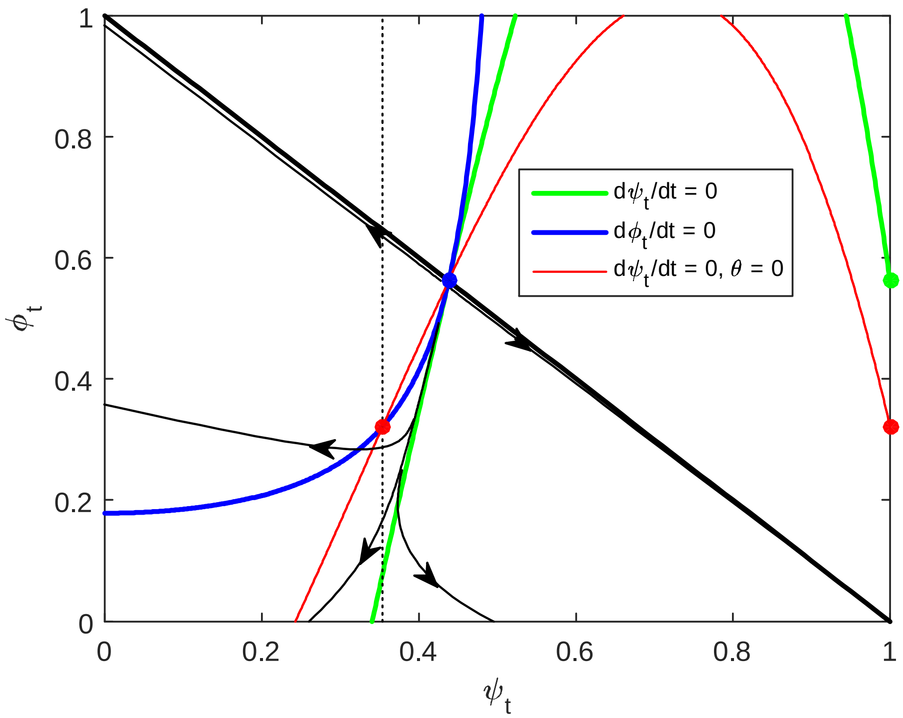

Figure 6 *The phase diagram for* $\theta = -(r - g)$

Such a second steady state can easily emerge again when the bitcoin tax is strictly positive but not too large. To show this, consider the policy (26). By construction, this policy satisfies $\left(1 - \frac{\xi\sigma^2}{\rho} \times \psi_{\mathrm{L}}^2\right)(1 - \psi_{\mathrm{L}}) = \alpha + \beta(1 - \psi_{\mathrm{L}})$ for various combinations of $\alpha > 0 > \beta$. Note that this construction implies $1 - (\xi\sigma^2/\rho)\psi^2 - \beta > 0$ for all $\psi \in [\psi_*, \psi_{\mathrm{L}}]$. Solving (36) using $1 - (\xi\sigma^2/\rho)\psi^2 + \theta/\rho = 0$ and then imposing (37) gives a ψ and ϕ determined by

$$1 - \frac{\xi\sigma^2}{\rho} \times \psi^2 + \frac{\theta}{\rho} = 0, \qquad \phi = \frac{\alpha}{1 - \frac{\xi\sigma^2}{\rho} \times \psi^2 - \beta}.$$

At $\theta = 0$ this gives $[\psi, \phi] = [\psi_*, \phi_*]$. Increasing the bitcoin tax rate then causes both ψ and ϕ to increase. When the bitcoin tax rate reaches $\theta = -\rho\left(1 - (\xi\sigma^2/\rho)\psi_{\mathrm{L}}^2\right)$ the solution will be $[\psi, \phi] = [\psi_{\mathrm{L}}, 1 - \psi_{\mathrm{L}}]$. Since (37) does not depend on θ, the combinations of $[\psi, \phi]$ generated by varying $\theta \in [0, -\rho\left(1 - (\xi\sigma^2/\rho)\psi_{\mathrm{L}}^2\right)]$ correspond to the segment between $[\psi_*, \phi_*]$ and $[\psi_{\mathrm{L}}, 1 - \psi_{\mathrm{L}}]$ of the curve $\mathrm{d}\phi_t/\mathrm{d}t = 0$ shown in Figure 6. For larger values of θ, the resulting $[\psi, \phi]$ lie entirely in the area where $\phi > 1 - \psi$. That means there is no longer a second steady state. In other words, the steady state $[\psi_{\mathrm{L}}, 1 - \psi_{\mathrm{L}}]$ is unique for all $\theta \geq -(r - g)$ where $r - g = \rho - \xi\sigma^2\psi_{\mathrm{L}}^2$. The phase diagram for $\theta = -(r - g)$ is shown in Figure 6.

5.2 Near the Targeted Steady State

To study the local properties of the steady state $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$, recall (31)-(33) and observe that adding a bitcoin tax generates a Jacobian at $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$ of the form

$$\frac{\partial}{\partial[\psi_t,\phi_t]}\begin{bmatrix}\frac{\mathrm{d}\psi_t}{\mathrm{d}t}\\ \frac{\mathrm{d}\phi_t}{\mathrm{d}t}\end{bmatrix}=\begin{bmatrix}\theta+p & \theta\\ q-p & q\end{bmatrix}+\begin{bmatrix}1\\ -1\end{bmatrix}\begin{bmatrix}a & b\end{bmatrix}=\begin{bmatrix}\theta+p+a & \theta+b\\ q-(p+a) & q-b\end{bmatrix},$$

with the same interpretation as in (31)-(33). The resulting eigenvectors and associated eigenvalues are

$$\begin{bmatrix}-1\\ 1\end{bmatrix}\leftrightarrow p+a-b, \qquad \begin{bmatrix}b+\theta\\ q-(p+a)\end{bmatrix}\leftrightarrow q+\theta,$$

where $q=r-g$. Nothing changes on the line $\phi=1-\psi$, and we already know that $\mathcal{S}(\psi,\phi)$ can be chosen so that $p+a-b>0$, as is the case for the policy (26). For $\theta=0$ we recover the local indeterminacy of Proposition 3 implied by $r-g<0$.

But for $\theta>-(r-g)$, the steady state $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$ is a hyperbolic steady state, with two strictly positive eigenvalues. The Grobman-Hartman theorem implies that all trajectories in a neighborhood of $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$ move away from that steady state.

In the boundary case $\theta=-(r-g)$, the fixed point $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$ is non-hyperbolic and so more detailed information is needed. The fact that $\theta=-(r-g)$ means that the curves $\mathrm{d}\psi_t/\mathrm{d}t=0$ and $\mathrm{d}\phi_t/\mathrm{d}t=0$ are tangent at $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$. For the policy (26) it is easy to show that the regions $\mathrm{d}\psi_t/\mathrm{d}t>0$ and $\mathrm{d}\phi_t/\mathrm{d}t>0$ are separated by the common tangent of $\mathrm{d}\psi_t/\mathrm{d}t=0$ and $\mathrm{d}\phi_t/\mathrm{d}t=0$ at $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$. Figure 6 shows the phase diagram for the policy (26) and $\theta=-(r-g)$. As the figure shows, all trajectories that start with $\phi_0\in(0,1-\psi_0)$ eventually violate $\psi_t>0$ or $\phi_t\geq 0$.

Proposition 4 *Suppose the tax rate on bitcoin satisfies $\theta\geq-(r-g)=-(\rho-\xi\sigma^2\psi_{\mathrm{L}}^2)$. Then it is possible to construct admissible policies $\mathcal{S}(\cdot,\cdot)$ and $\mathcal{D}(\cdot,\cdot)$ that uniquely implement the steady state $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$.*

For $\mathcal{S}_*=0$, the targeted steady state has $r-g=0$ and then Proposition 4 says that it is possible to uniquely implement $[\psi_{\mathrm{L}}, 1-\psi_{\mathrm{L}}]$ without a bitcoin tax. As described earlier, this is the scenario in which the balanced budget trap and the desired steady state coincide.

6 Conclusion

When there are laws against private-sector bubble assets, it is easy for the government to design policies that uniquely implement a permanent primary deficit, assuming there is enough idiosyncratic risk to make such deficits possible in the first place. An outright ban is not necessary if the government can tax private-sector bubble assets at the rate $-(r-g)>0$. Without such taxes, Markov strategies can only implement permanent primary deficits if they are discontinuous. Discontinuous Markov strategies can mimic the trigger strategies proposed by Brunnermeier, Merkel, and Sannikov [2023].

For continuous Markov strategies, and absent a large enough tax on bitcoin, unique implementation is not possible. There is always a balanced budget trap and a continuum of equilibria near the targeted steady state. The determinacy result of the fiscal theory of the price level comes with caveats when the government plans to run a permanent primary deficit.

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