

Higher Level Mathematics Internal Assessment

Optical Illusion: Modelling and Constructing an Ambiguous Cylinder

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Introduction

I have always been interested in puzzles and gadgets, particularly the ones that seem counterintuitive or impossible to solve. Optical illusions are one of my favourites, as they always amaze me with the differences between my perception and the reality. Recently, I visited an exhibition held by the Museum of Illusions in my city. *Ambiguous Cylinder* (Figure 1), one of the installations there, features a cylinder whose top looks like a square while its reflection in the mirror looks like a circle. This interests me as it takes a geometric and mathematical approach to create illusion rather than the more common approach using human's psychological and cognitive weaknesses.

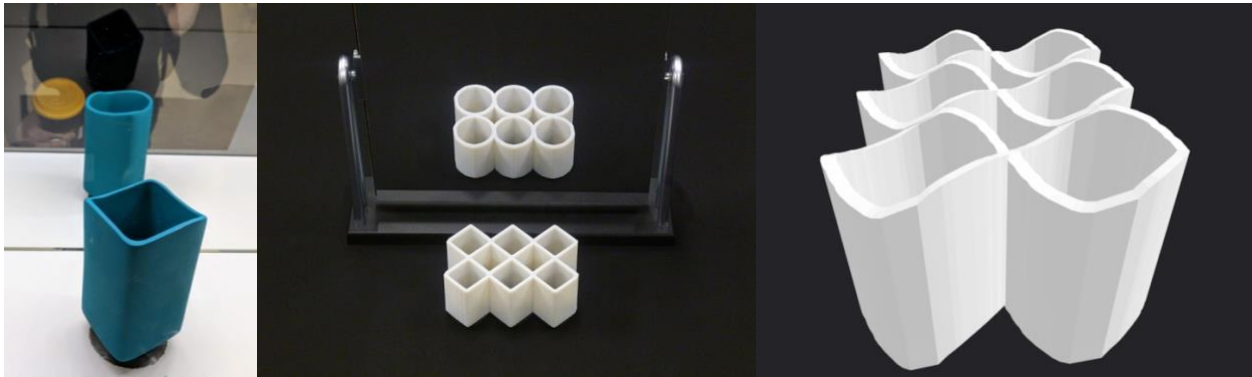


Figure 1. (Left) *Ambiguous Cylinder* installation at the Museum of Illusions.

Figure 2. (Centre) Professor Sugihara's square-circle illusion. From *Ambiguous Cylinder* "Rectangles and Circles", by Kokichi Sugihara, <http://www.isc.meiji.ac.jp/~kokichis/ambiguous/rectcircle.jpg>.

Figure 3. (Right) Side view of Professor Sugihara's 3D model for the illusion. From *Large Model*, by Kokichi Sugihara, <http://www.isc.meiji.ac.jp/~kokichis/ambiguous/rclargebyKSugihara.stl>.

This ambiguous cylinder was first proposed by Professor Kokichi Sugihara from Meiji University in Japan (Figure 2). It creates a shape distinct from itself in the mirror through its unique curvature at the top of the cylinder (Figure 3). This illusion quickly gained popularity since its release in 2016 and many followed to create this kind of ambiguous cylinders. As a 3D-printing enthusiast, I downloaded some 3D models (Svensson, 2018) and printed them on a Prusa i3 MK2 printer.

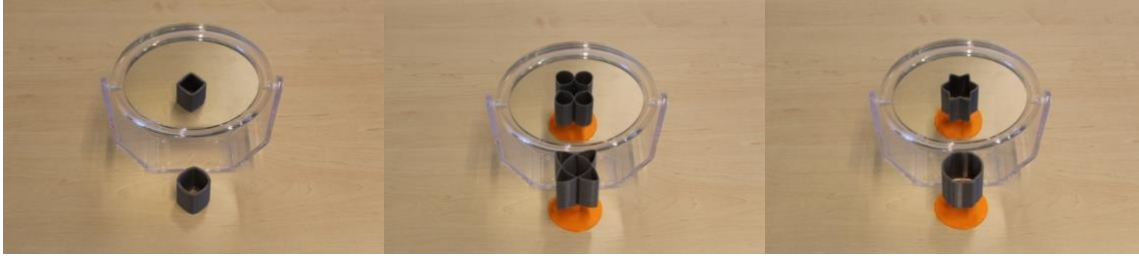


Figure 4. (Left) Circle-square illusion. 3D Model from Impossible Objects, by Mats K. Svensson, https://github.com/Matsemann/impossible-objects/blob/master/3dfiles/circlesquare/old/firsttry_circlesquare_single.stl.

Figure 5. (Centre) Diamond-circles illusion. 3D Model from Impossible Objects, by Mats K. Svensson, <https://github.com/Matsemann/impossible-objects/blob/master/3dfiles/diamond4circles/diamond4circles.stl>.

Figure 6. (Right) Circle-flower illusion. 3D Model from Impossible Objects, by Mats K. Svensson, <https://github.com/Matsemann/impossible-objects/blob/master/3dfiles/flowercircle/flowercircle.stl>.

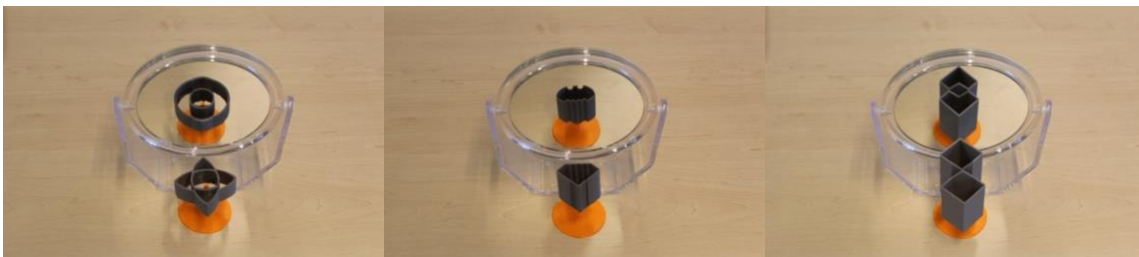


Figure 7. (Left) Diamond-circles illusion. 3D Model from Impossible Objects, by Mats K. Svensson, https://github.com/Matsemann/impossible-objects/blob/master/3dfiles/doublecircle_diamond/diamond_larger_holeforsupport_thinner_Walls.stl.

Figure 8. (Centre) Superman-Batman illusion. 3D Model from Impossible Objects, by Mats K. Svensson, <https://github.com/Matsemann/impossible-objects/blob/master/3dfiles/supermanvsbatman/supermanvsbatman.stl>.

Figure 9. (Right) Adjacent-overlapping squares illusion. 3D Model from Impossible Objects, by Mats K. Svensson, https://github.com/Matsemann/impossible-objects/blob/master/3dfiles/squareoverlapping/overlapping_square_withhole_smaller_width.stl.

Figure 4 to 9 show the 3D models I printed viewed at approximately 45 degree above the table surface. I am amazed by these powerful illusions and even more curious about the mathematics behind them. I wonder if there exists a universal formula for those cylinders that enable them to resemble two shapes when viewed. This exploration aims to mathematically model the curvatures of the ambiguous cylinders with a general framework that applies to all the variations. Then, using the deducted model, the exploration aims to construct an ambiguous cylinder capable of resembling circle and triangle from opposite angles. The deductions and calculations involve the content in the HL Mathematics syllabus such as parametric equation, formula of a vector, and polar coordinates.

Conditions for the Illusion

To understand the illusion, the conditions of the illusion should be considered. The ideal angle to view the illusion is 45 degree above the surface. (Sugihara, n.d.) The image in the mirror is the view 45 degree above the surface from the opposite direction, as shown in Figure 10 and 11.

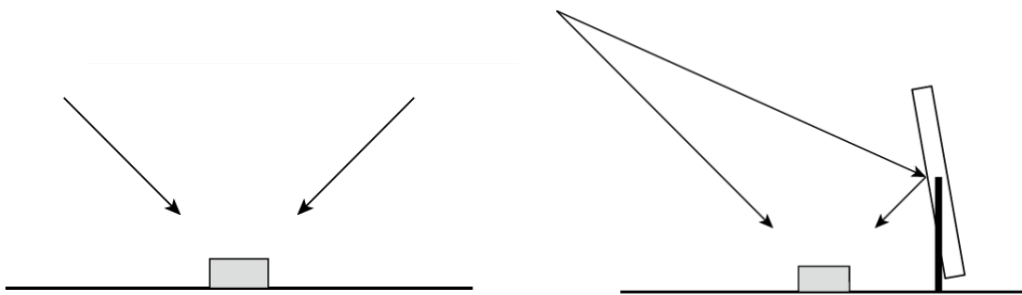


Figure 10. (Left) View in 45 degrees from opposite sides. From Ambiguous Cylinder "Rectangles and Circles", by Kokichi Sugihara, <http://www.isc.meiji.ac.jp/~kokichis/ambiguousc/display1.png>.

Figure 11. (Right) A slightly-tilted mirror allows single viewpoint to view both perspectives. From Ambiguous Cylinder "Rectangles and Circles", by Kokichi Sugihara, <http://www.isc.meiji.ac.jp/~kokichis/ambiguousc/display3.png>.

With this information, the perspectives of the viewer can be modelled. The human observer observes from 45 degrees above the surface, from mutually opposite two sides. The cylinder can be placed at the origin of a 3-dimension cartesian coordinate system. Hence, the observer views the cylinder in two

perspectives, one parallel to vector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and one parallel to vector $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, at the same time. The method of paralleling to vectors is used rather than assuming the coordinates of the viewpoints to be $(0, a, a)$ and $(0, -a, a)$. This is because the points on the curvature have various height and the cylinder itself introduces height; they will cause the actual view angle to deviate from 45 degrees above the surface.

Modelling the Ambiguous Cylinders

This investigation was unable to find or access official methods from Professor Sugihara. However, there are several methods online or in accessible databases that help construct a model. This modelling process will mostly follow the method of Professor David Richeson from the Dickinson College.

(Richeson, 2016)

Continue with the setup in the previous section, the perspective parallel to vector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ can be called perspective 1 and the perspective parallel to vector $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ can be called perspective 2. The cylinder creates ambiguous images in the xy -plane from perspective 1 and 2 because its curve at the top rim varies in height; the curve will be the focus of this model. Assume the width of the top rim is negligible when viewed from a far enough distance. Thus, the curvature can be treated as the path of a single point moving across the 3-dimension coordinate system.

When the curve is observed from perspective 1 and 2, it needs to resemble a shape on a xy -plane. Thus, variations in height are not considered and the shapes can be assumed to have height of 0 to lie on the xy -plane. The shape observed from perspective 1 can be modelled as $y = f(x)$ and the shape observed from perspective 2 can be modelled as $y = g(x)$. Figure 12 demonstrates the case for the Sugihara's original square-circle illusion, where $f(x)$ is the square and $g(x)$ is the circle.

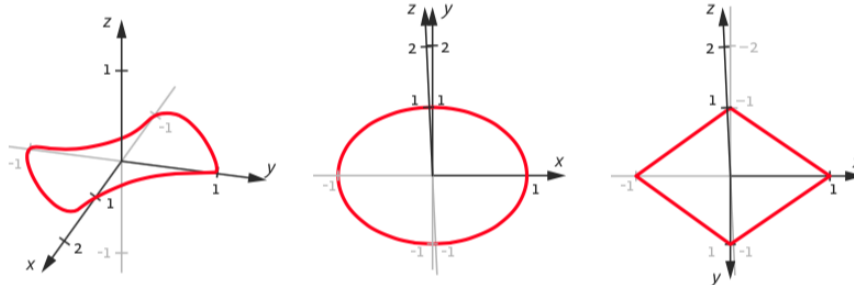


Figure 2. Three views of a closed curve. When we look down from the point $(0, -a, a)$, it looks like a circle in the xy -plane (center). When we look from $(0, a, a)$, it looks like a diamond in the xy -plane (right).

Figure 12. Curvature of top rim viewed from a perspective that is not 1 or 2, perspective 2, perspective

1. Adapted from "DO THE MATH!: Sugihara's Impossible Cylinder," by D. Richeson, 2016, *Math Horizons*, 24.1, p.18-19. Copyright 2016 by Mathematical Association of America.

Note that the curvature for a circle or square, or generally any enclosed shape, cannot be represented by a single function, as no function can have two y values for one x value. During implementation, $f(x)$ and $g(x)$ must be split into various component curves categorized by their domains and ranges.

The observer at perspective 1 and 2 sees the curve $\begin{pmatrix} x \\ f(x) \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x \\ g(x) \\ 0 \end{pmatrix}$ respectively. However, those curves are only projections of the actual curve onto the xy -plane in their perspectives. The actual curve of the rim can be modeled as a parametric curve $\mathbf{r}(t)$ floating in the coordinate system and this is the curve that defines the design of the ambiguous cylinder.

To find the expression for $\mathbf{r}(t)$, the relationship between $\mathbf{r}(t)$, perspectives, and projection curves need to be investigated. Figure 13 is used to help visualize the following explanations

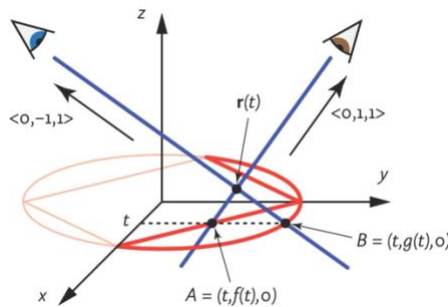


Figure 13. Relationships between $\mathbf{r}(t)$, perspectives, and projection curves. In this graph, the $f(x)$ and $g(x)$ are a unit circle and an inscribed square respectively, resembling Professor Sugihara's original

illusion shown in Figure 2. The eye on the right is perspective 1 and the eye on the left is perspective 2. The ray going into eye on the right can be modelled as $\mathbf{r}_A(s)$ and the ray going into eye on the left can be modelled as $\mathbf{r}_B(s)$. Adapted from "DO THE MATH!: Sugihara's Impossible Cylinder," by D. Richeson, 2016, *Math Horizons*, 24.1, p.18-19. Copyright 2016 by Mathematical Association of America.

Since light travels in a straight line, a point on the actual rim curve $\mathbf{r}(t)$, a point on the projection curve, and the perspective/observer must all be on one line.

In the case of perspective 1, the observer sees a point on $\mathbf{r}(t)$ while perceives a point, represented by position vector \mathbf{A} , on the projection curve $\begin{pmatrix} x \\ f(x) \\ 0 \end{pmatrix}$. In addition, as perspective 1 is parallel to $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, the curve $\mathbf{r}_A(s)$, the parameterizations of the straight-lines parallel to perspective 1 and passes through $\mathbf{r}(t)$ and \mathbf{A} , can be modelled, with s as a parametric variable.

$$\mathbf{r}_A(s) = \mathbf{A} + s \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ (vector equation of a line)}$$

$$\mathbf{r}_A(s) = \begin{pmatrix} t \\ f(t) \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ f(t)+s \\ s \end{pmatrix}$$

Similarly, in the case of perspective 2, the observer sees a point on $\mathbf{r}(t)$ while perceives a point, represented by position vector \mathbf{B} , on the projection curve $\begin{pmatrix} x \\ g(x) \\ 0 \end{pmatrix}$. In addition, as perspective 1 is parallel to $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, the curve $\mathbf{r}_B(s)$, the parameterizations of the straight-lines parallel to perspective 1 and passes through $\mathbf{r}(t)$ and \mathbf{B} , can be modelled, with s as a parametric variable.

$$\mathbf{r}_B(s) = \mathbf{B} + s \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ (vector equation of a line)}$$

$$\mathbf{r}_B(s) = \begin{pmatrix} t \\ g(t) \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ g(t)-s \\ s \end{pmatrix}$$

The parameterizations of straight lines in those two situations, $\mathbf{r}_A(s)$ and $\mathbf{r}_B(s)$, always intersect at $\mathbf{r}(t)$. With this information, $\mathbf{r}(t)$ can be modelled.

At $s = s_0$, $\mathbf{r}_A(s)$ and $\mathbf{r}_B(s)$ intersects at $\mathbf{r}(t)$.

Thus, $\mathbf{r}_A(s_0) = \mathbf{r}_B(s_0)$

$$\begin{pmatrix} t \\ f(t) + s_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} t \\ g(t) - s_0 \\ s_0 \end{pmatrix}$$

$$\therefore f(t) + s_0 = g(t) - s_0$$

$$2s_0 = g(t) - f(t)$$

$$s_0 = \frac{g(t) - f(t)}{2}$$

$$\therefore \mathbf{r}(t) = \mathbf{r}_A(s_0) = \begin{pmatrix} t \\ f(t) + \frac{g(t) - f(t)}{2} \\ \frac{g(t) - f(t)}{2} \end{pmatrix} = \begin{pmatrix} t \\ \frac{g(t) + f(t)}{2} \\ \frac{g(t) - f(t)}{2} \end{pmatrix}$$

This is the general equation of the parametric curve $\mathbf{r}(t)$, curvature of the top rim, regarding the functions of the desired shape, $f(x)$ and $g(x)$, at perspective 1 and 2.

Creating the Functions for Circle-Triangle Illusion

With the general equation for $\mathbf{r}(t)$, this investigation now aims to construct an illusion with customized $f(x)$ and $g(x)$ functions. Specifically, this section is focused on generating a circle-triangle illusion. Triangle is interesting as it is a polygon that is not horizontally symmetrical and has odd numbers of sides, unlike a square. Investigating on circle-triangle illusions will help gain insights in generating circle-polygon illusions that have similar properties (not horizontally symmetrical and have odd numbers of sides).

First, the layout of the triangle and circle in a 2-dimensional coordinate system is drawn, where triangle and circle are centered at the same location. The triangle in this design is an equilateral triangle and is inscribed inside the circle. With this layout, the coordinate of endpoints, the equations of lines and curves, and the domain of variables for the equations can be calculated.

Define $f(x)$ as functions for the sides of a triangle, and $g(x)$ as functions for the shape of a circle.

To model $f(x)$ and $g(x)$, three equations are needed for each to model the sides and the arcs that piece together the full closed shape.

Figure 14 demonstrates the layout of the two shapes and functions that represent the sides and arcs.

These include $f_1(x), f_2(x), f_3(x)$ for three sides of triangles and $g_1(x), g_2(x), g_3(x)$ for three arcs that resembles a circle.

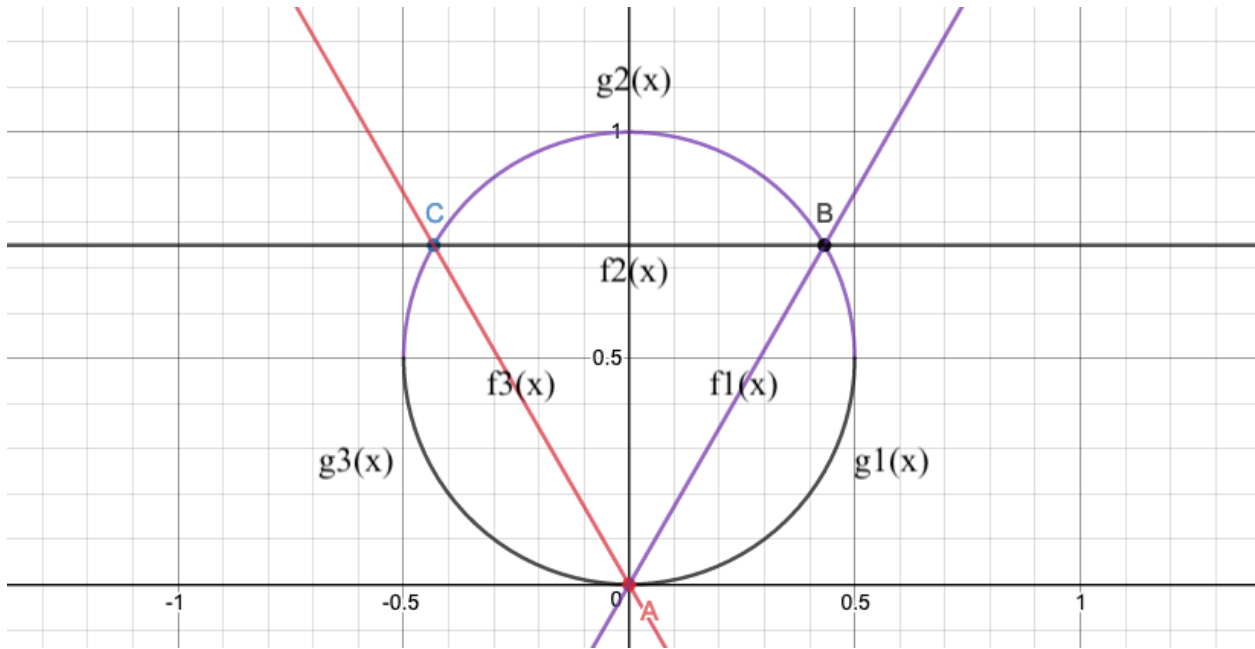


Figure 14. Layout of the shapes and functions in Desmos graphing calculator.

First, the circle can be modelled as a circle cantered at $(0, \frac{1}{2})$ with a radius of $\frac{1}{2}$.

It can be modelled with the polar equation $r = \sin \theta$, where $\theta \in [0, \pi]$.

Then, three segments of equal length can be derived from the equation to form the 3 $g(x)$ required. For now, the equations are written in polar for convenience and will be converted to cartesian forms.

$$g_1(x) \text{ in polar form: } r = \sin \theta, \theta \in \left[0, \frac{\pi}{3}\right]$$

$$g_2(x) \text{ in polar form: } r = \sin \theta, \theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$$

$$g_3(x) \text{ in polar form: } r = \sin \theta, \theta \in \left[\frac{2\pi}{3}, \pi\right]$$

Next, points A, B, C can be located with the polar equations derived. They are the end points of the arcs and the side, thus they are important to determine the equations and domains of functions.

For point A, $\theta = 0$, $r = \sin 0 = 0$, $x = r \cos \theta = 0$, $y = r \sin \theta = 0$. Thus, A has a coordinate of (0,0).

For point B, $\theta = \frac{\pi}{3}$, $r = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $x = r \cos \theta = \frac{\sqrt{3}}{2} \cos \frac{\pi}{3} = \frac{\sqrt{3}}{4}$, $y = r \sin \theta = \frac{\sqrt{3}}{2} \sin \frac{\pi}{3} = \frac{3}{4}$.

Thus, B has a coordinate of $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$.

For point C, $\theta = \frac{2\pi}{3}$, $r = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $x = r \cos \theta = \frac{\sqrt{3}}{2} \cos \frac{2\pi}{3} = -\frac{\sqrt{3}}{4}$, $y = r \sin \theta = \frac{\sqrt{3}}{2} \sin \frac{2\pi}{3} = \frac{3}{4}$.

Thus, C has a coordinate of $\left(-\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$.

With the coordinates for A, B, C, the equations for the 3 sides of the triangles (the three $f(x)$) are:

For $f_1(x)$, end points are A (0,0) and B $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$. $slope = \frac{\frac{3}{4}-0}{\frac{\sqrt{3}}{4}-0} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$$y - y_1 = \sqrt{3}(x - x_1)$$

$$y - 0 = \sqrt{3}(x - 0)$$

$$y = \sqrt{3}x$$

$$\therefore f_1(x) = \sqrt{3}x, x \in \left[0, \frac{\sqrt{3}}{4}\right]$$

For $f_2(x)$, end points are B $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$ and C $\left(-\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$. It is a horizontal line that starts at C and ends at B.

$$\therefore f_2(x) = \frac{3}{4}, x \in \left[-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\right]$$

For $f_3(x)$, end points are C $\left(-\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$ and A (0,0). $slope = \frac{\frac{3}{4}-0}{-\frac{\sqrt{3}}{4}-0} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$

$$y - y_2 = -\sqrt{3}(x - x_2)$$

$$y - 0 = -\sqrt{3}(x - 0)$$

$$y = -\sqrt{3}x$$

$$\therefore f_3(x) = -\sqrt{3}x, x \in \left[-\frac{\sqrt{3}}{4}, 0\right]$$

Before inserting $f(x)$ and $g(x)$ into $\mathbf{r}(t)$, the three $g(x)$ need to be converted from polar to cartesian.

All the $g(x)$ functions have the same equation, just with different domains.

$$g(x) \text{ polar form: } r = \sin \theta$$

$$\text{polar - cartesian relationship, } y = r \sin \theta$$

$$\therefore \sin \theta = \frac{y}{r}$$

$$\therefore \sin \theta = r$$

$$\therefore r = \frac{y}{r}, y = r^2$$

$$\text{polar - cartesian relationship, } r^2 = x^2 + y^2$$

$$\text{with } y = r^2, y = x^2 + y^2$$

Now, the equation needs to be turned into $g(x)$ form.

$$y^2 - y = -x^2$$

$$y^2 - y + \frac{1}{4} = -x^2 + \frac{1}{4}$$

$$\left(y - \frac{1}{2}\right)^2 = \frac{1}{4} - x^2$$

$$y = \frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2}$$

$$\therefore g(x) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2}, x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

This shows $g(x)$ has two forms, one for the upper half of the circle and the bottom half.

That means two cases are needed to make for $g_1(x)$ and $g_3(x)$ as they have portions both in the upper and bottom half of the circle. These cases will be noted with an additional a and b subscript.

For those cases, the domains are needed to be separately treated as well. However, a minor problem will be generated which will be explained further. Table 1 shows the pairs of variations of $f(x)$ and $g(x)$ functions, and their domains.

Pair	$f(x)$	domain for $f(x)$	$g(x)$	domain for $g(x)$
1a	$\sqrt{3}x$	$\left[0, \frac{\sqrt{3}}{4}\right]$	$\frac{1}{2} - \sqrt{\frac{1}{4} - x^2}$	$\left[0, \frac{1}{2}\right]$
1b	$\sqrt{3}x$	$\left[0, \frac{\sqrt{3}}{4}\right]$	$\frac{1}{2} + \sqrt{\frac{1}{4} - x^2}$	$\left[\frac{\sqrt{3}}{4}, \frac{1}{2}\right]$
2	$\frac{3}{4}$	$\left[-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\right]$	$\frac{1}{2} + \sqrt{\frac{1}{4} - x^2}$	$\left[-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\right]$
3a	$-\sqrt{3}x$	$\left[-\frac{\sqrt{3}}{4}, 0\right]$	$\frac{1}{2} - \sqrt{\frac{1}{4} - x^2}$	$\left[-\frac{1}{2}, 0\right]$
3b	$-\sqrt{3}x$	$\left[-\frac{\sqrt{3}}{4}, 0\right]$	$\frac{1}{2} + \sqrt{\frac{1}{4} - x^2}$	$\left[-\frac{1}{2}, -\frac{\sqrt{3}}{4}\right]$

Table 1. The equation and domains for various pairs of $f(x)$ and $g(x)$.

Note that the domain for $f(x)$ and $g(x)$ functions are different. This is because the triangle is inscribed in the circle and the circle has larger range of domain. For the domain to put in $\mathbf{r}(t)$ for parametric variable t , the domain of $g(x)$ is chosen as it will present a complete circle in perspective 2. However, for case 1 and 3, domain for $f(x)$ is now larger and thus the two sides of triangle will become longer and pass the endpoints. It does not affect the formation of triangle on the projected plane in perspective 1, but visually adds an extension to two sides.

$$\text{Recall, } \mathbf{r}(t) = \begin{pmatrix} t \\ \frac{g(t) + f(t)}{2} \\ \frac{g(t) - f(t)}{2} \end{pmatrix}$$

Now all the $f(t)$ and $g(t)$ have been derived and converted in cartesian form. With the domains obtained above, the $\mathbf{r}(t)$ functions can be constructed. There are five pairs of the $\mathbf{r}(t)$ functions with five pairs of $f(t)$, $g(t)$, and domain. With them, the illusion can be created.

$$\mathbf{r}_{1a}(t) = \begin{pmatrix} \frac{t}{2} \\ \frac{g_{1a}(t) + f_1(t)}{2} \\ \frac{g_{1a}(t) - f_1(t)}{2} \end{pmatrix} = \begin{pmatrix} \frac{t}{\frac{1}{2} - \sqrt{\frac{1}{4} - t^2} + \sqrt{3}t}} \\ \frac{t}{2} \\ \frac{t}{\frac{1}{2} - \sqrt{\frac{1}{4} - t^2} - \sqrt{3}t}} \end{pmatrix}, t \in \left[0, \frac{1}{2}\right]$$

$$\mathbf{r}_{1b}(t) = \begin{pmatrix} \frac{t}{2} \\ \frac{g_{1b}(t) + f_1(t)}{2} \\ \frac{g_{1b}(t) - f_1(t)}{2} \end{pmatrix} = \begin{pmatrix} \frac{t}{\frac{1}{2} + \sqrt{\frac{1}{4} - t^2} + \sqrt{3}t}} \\ \frac{t}{2} \\ \frac{t}{\frac{1}{2} + \sqrt{\frac{1}{4} - t^2} - \sqrt{3}t}} \end{pmatrix}, t \in \left[\frac{\sqrt{3}}{4}, \frac{1}{2}\right]$$

$$\mathbf{r}_2(t) = \begin{pmatrix} \frac{t}{2} \\ \frac{g_2(t) + f_2(t)}{2} \\ \frac{g_2(t) - f_2(t)}{2} \end{pmatrix} = \begin{pmatrix} \frac{t}{\frac{1}{2} + \sqrt{\frac{1}{4} - t^2} + \frac{3}{4}} \\ \frac{t}{2} \\ \frac{t}{\frac{1}{2} + \sqrt{\frac{1}{4} - t^2} - \frac{3}{4}} \end{pmatrix}, t \in \left[-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\right]$$

$$\mathbf{r}_{3a}(t) = \begin{pmatrix} \frac{t}{2} \\ \frac{g_{3a}(t) + f_3(t)}{2} \\ \frac{g_{3a}(t) - f_3(t)}{2} \end{pmatrix} = \begin{pmatrix} \frac{t}{\frac{1}{2} - \sqrt{\frac{1}{4} - t^2} - \sqrt{3}t}} \\ \frac{t}{2} \\ \frac{t}{\frac{1}{2} - \sqrt{\frac{1}{4} - t^2} + \sqrt{3}t}} \end{pmatrix}, t \in \left[-\frac{1}{2}, 0\right]$$

$$\mathbf{r}_{3b}(t) = \begin{pmatrix} t \\ \frac{g_{3b}(t) + f_3(t)}{2} \\ \frac{g_{3b}(t) - f_3(t)}{2} \end{pmatrix} = \begin{pmatrix} t \\ \frac{\frac{1}{2} + \sqrt{\frac{1}{4} - t^2} - \sqrt{3}t}{2} \\ \frac{\frac{1}{2} + \sqrt{\frac{1}{4} - t^2} + \sqrt{3}t}{2} \end{pmatrix}, t \in \left[-\frac{1}{2}, -\frac{\sqrt{3}}{4}\right]$$

Results

To see the visual results, the five parametric $\mathbf{r}(t)$ functions with their domains are transferred into GeoGebra commands (Appendix A) and graphed by GeoGebra's GraphingCalculator3D (Figure 15).

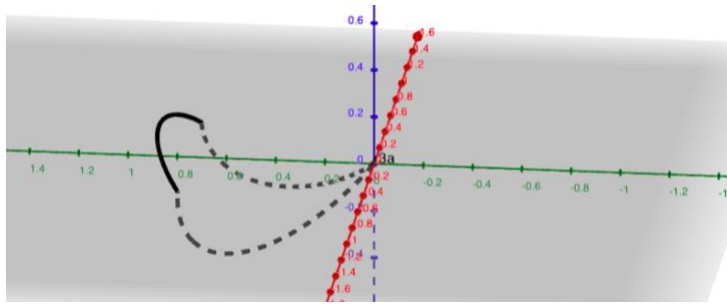


Figure 15. Graph generated viewed not along perspective 1 or 2. Lines are dotted under xy -plane.

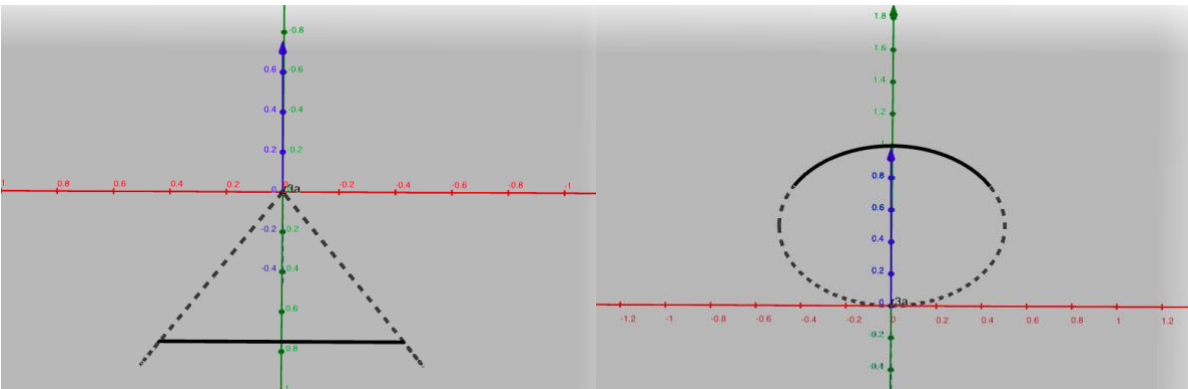


Figure 16. (Left) View from perspective 1 shows a triangle. Lines are dotted under xy -plane.

Figure 17. (Right) View from perspective 2 shows a circle. Lines are dotted under xy -plane.

To generate view of perspective 1 and 2, the angle to view the graph is adjusted. By aligning $(0, -1, 0)$ and $(0, 0, 1)$, perspective 1 is obtained and the desired $f(x)$, a triangle, is shown. As predicted previously, the triangle has two extended sides due to conflict in the domains of functions. By aligning

$(0,1,0)$ and $(0,0,1)$, perspective 2 is obtained and the desired $g(x)$, a circle, is shown. The circle-triangle illusion has been generated successfully.

Conclusion and Reflection

Overall, the investigation effectively modelled the general equation for the curvature of the top rim in an ambiguous cylinder, based on two desired shapes. Afterwards, the investigation successfully modelled a circle-triangle illusion using the previously derived equation. The result shows the method is accurate, despite minor details such as extension of lines due to conflicts in domain while combining functions.

This investigation has its strength in its generalization. With the equation derived, a curve can enable two customized shapes to be seen while viewing from perspective 1 and 2. However, the generalization also adds complexity during implementation. Shapes that are not horizontally symmetrical and polygons with odd number of edges requires more categorization to generate the parametric $\mathbf{r}(t)$ functions. Many shapes will also be easier to map their functions through polar coordinates, as functions in cartesian coordinates sometimes require special considerations (such as the top-half and bottom-half of a circle). In the future, a general equation can be generated in spherical coordinate systems, making polar $f(x)$ and $g(x)$ easier to implement. For more complex shapes such as the Superman-Batman illusion (Figure 8), computer programs will make the process much faster.

Another approach to create the curve is much more intuitive but less theoretical. In a Computer-Aided Design (CAD) software, two cylinders with the desired shape can be created. Then, they are tilted 45 degrees, one pointing towards perspective 1 and one pointing towards perspective 2. After that, their intersection is obtained and thus a curvature of the top rim is generated, as shown in Figure 18.

(Kleiner, 2016) This method relies on software but it is very powerful. It is also easy to design the cylinder and send to 3D-print as it is already in a CAD software. Many of the complicated ambiguous cylinders shown earlier can be created using this technique.

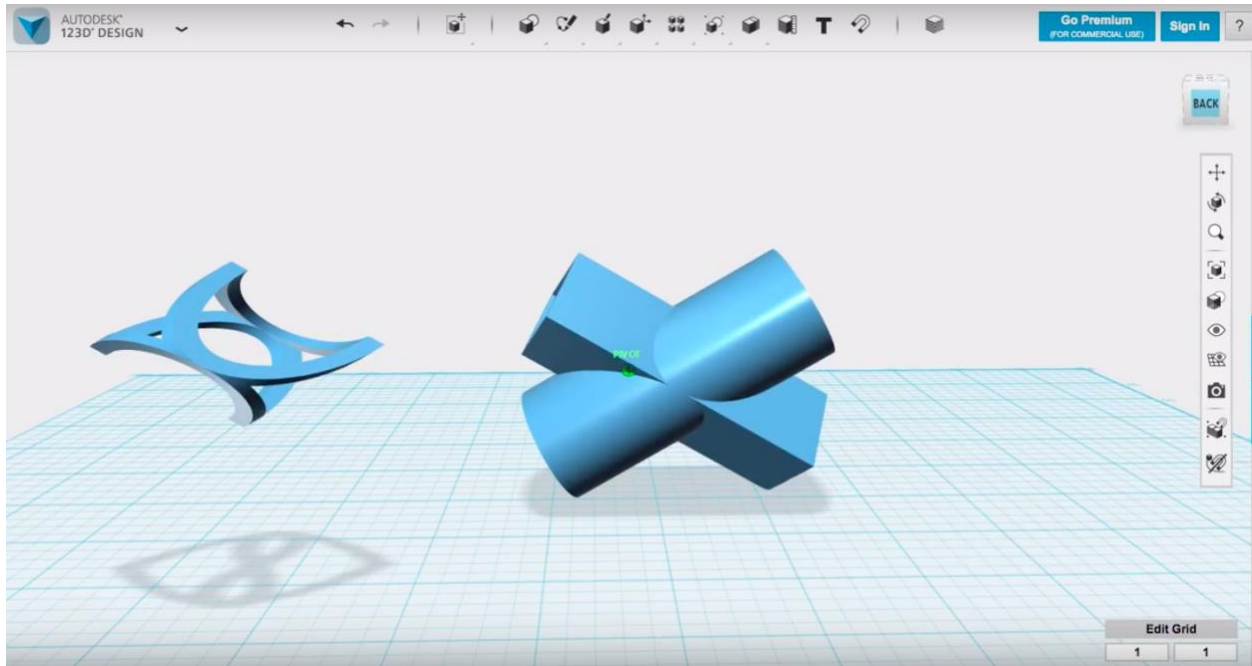


Figure 18. Tilt two cylinders towards perspectives and extract the intersecting curvatures. Adapted from "Ambiguous Cylinder Illusion how it works," Y. Kleiner, 2016, July 04, YouTube. Retrieved February, 24, 2019, <https://www.youtube.com/watch?v=5pOSdZA0298>.

The mathematical knowledge used in the investigation is closely related to the IB Higher Level Mathematics content. Vector equations in 3-dimensional coordinate system help relate the actual curvature, the projected shape, and perspective together. Parametric equation enables the investigation to reach a generalization. Polar coordinates assist modelling the endpoints and equations of curves in a circle inscribing an equilateral triangle.

The next step for this investigation is to construct the circle-triangle illusion in real life. A cylinder can be extruded under the rim $r(t)$ in a CAD software and 3D-printed. The inevitable extra edge of the triangle from perspective 1 can be covered up by smoothing in model or applying lighting techniques, as many illusion displays do. In the future, with the implementation of a spherical coordinate system, an even more general equation can be generated to create ambiguous cylinder of circle-polygon illusion with any number of edges for a regular polygon.

Reference

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Appendix A – GeoGebra Commands

r1a=Curve(t,(1)/(2) ((1)/(2)- sqrt((1)/(4)-t^(2))+ sqrt(3)t),(1)/(2) ((1)/(2)- sqrt((1)/(4)-t^(2))- sqrt(3)t),t,0,(1)/(2))

r1b=Curve(t,(1)/(2) ((1)/(2)+ sqrt((1)/(4)-t^(2))+ sqrt(3)t),(1)/(2) ((1)/(2)+ sqrt((1)/(4)-t^(2))- sqrt(3)t),t,(sqrt(3))/(4),(1)/(2))

r2=Curve(t,(1)/(2) ((1)/(2)+ sqrt((1)/(4)-t^(2))+(3)/(4)),(1)/(2) ((1)/(2)+ sqrt((1)/(4)-t^(2))-(3)/(4)),t,(- sqrt(3))/(4),(sqrt(3))/(4))

r3a=Curve(t,(1)/(2) ((1)/(2)- sqrt((1)/(4)-t^(2))- sqrt(3)t),(1)/(2) ((1)/(2)- sqrt((1)/(4)-t^(2))+ sqrt(3)t),t,(- (1)/(2),0)

r3b=Curve(t,(1)/(2) ((1)/(2)+ sqrt((1)/(4)-t^(2))- sqrt(3)t),(1)/(2) ((1)/(2)+ sqrt((1)/(4)-t^(2))+ sqrt(3)t),t,(- 1)/(2),(-sqrt(3))/(4))