1. Introduction and motivation

- Modern theory of bulk orbital magnetization:
  - Space formula involving Berry connections of Bloch functions
  - Valid for insulators, metals, Chern insulators
  \[ M_{\text{bulk}} = \frac{e}{2h} \sum_{\text{states}} \int \frac{d^3 k}{(2\pi)^3} \text{Im} \epsilon_{\alpha \beta} \langle \partial_{\alpha} n_{\beta} \rangle H_k + E_{\text{F}} \langle \partial_{\alpha} n_{\beta} \rangle \]

- Recent efforts are focused on deriving surface analog of bulk quantities:
  - Surface anomalous Hall conductivity
  - Edge polarizations and surface quadrupoles

- For systems with no bulk orbital magnetization:
  - When is a surface orbital magnetization defined?
  - How may we compute it?

- Hinge current as signature of facet magnetization:
  - Magnetizations of adjacent surface facets generate hinge current
  - Generally, only differences of surface magnetizations are well-defined

2. Symmetry Analysis

A symmetry analysis is conducted to determine when magnetizations on individual surface facets are well-defined.

Given a bulk symmetry group \( G \), we perform the series of steps outlined on the right:

- n-fold rotation about \( z \)
- Time-reversed n-fold rotation about \( z \)
- Time-reversed 4-fold screw axis about \( z \)

3. Local marker for magnetization

- Background:
  - Magnetization may be expressed via a trace involving the single-particle density matrix (ground state projector) \( P \) (\( Q = 1 - P \))
  - Tracing over position space generates a local marker for magnetization:

\[ M = \frac{\hbar}{i} \text{Im} \text{Tr} \left[ P x Q H y P Q \right] - \frac{\hbar}{i} \text{Im} \text{Tr} \left( Q x P H y Q \right) + \frac{2 \hbar}{i} \text{Im} \text{Tr} \left( Q y P H x \right) \]

- Chern marker

\[ \mathcal{M}(\mathbf{r}) = \frac{\hbar}{i} \text{Im} \left( \text{Tr} \left[ P x Q H y P + \cdots \right] \right) \]

- Can the local marker be used to compute surface magnetization?

4. Tight-binding model and calculations

Layers of topologically trivial Haldane models with alternating signs of magnetization are coupled by interlayer hoppings between identical sublattices.

\[ \theta_{CS} \] is quantized by tuning \( t'_3 \) to match \( t_3 \).

\[ t'_3 = t_3; \quad \theta_{CS} \text{ quantized by } \langle E \rangle_{c/2} \]

\[ t'_3 \neq t_3; \quad \theta_{CS} \text{ not quantized} \]

Below, \( t_3 \) is set to 0.1

5. Summary

- Surface magnetization is well-defined for systems with a quantized Chern-Simons axion coupling:

\[ \theta_{CS} \equiv \frac{1}{i} \int_{\text{Bdry}} \epsilon_{\alpha \beta} \text{Tr} \left[ A_\alpha \partial_\beta A_\beta A_\alpha A_\beta A_\alpha A_\beta \right] \]

- \( \theta_{CS} \) is invariant modulo 2\( \pi \) under unitary mixing of occupied bands

- In pseudo-scalar symmetric systems \( \theta_{CS} \) is quantized, i.e., \( \theta_{CS} = 0, \pi \)

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