The averagedness of Douglas–Rachford and forward-backward operators

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Introduction
Throughout this talk

\[ X \text{ is a real Hilbert space} \]

with inner product \( \langle \cdot | \cdot \rangle \), and induced norm \( \| \cdot \| \).

- Let \( f : X \to ]-\infty, +\infty] \) and \( g : X \to ]-\infty, +\infty] \) be proper and lower semicontinuous.
- Suppose that \( f + g \) is convex such that \( \text{argmin}(f + g) \neq \emptyset \).
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- Suppose that \( f + g \) is convex such that \( \text{argmin}(f + g) \neq \emptyset \).

- When \( f \) and \( g \) are nice enough, splitting methods (e.g., Douglas–Rachford and forward-backward) can be use to solve the problem:

\[
\text{Find } x \in X \text{ such that } x \text{ minimizes } f + g.
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- Let \(f: X \rightarrow ]-\infty, +\infty]\) and \(g: X \rightarrow ]-\infty, +\infty]\) be proper and lower semicontinuous.
- Suppose that \(f + g\) is convex such that \(\text{argmin}(f + g) \neq \emptyset\).
- When \(f\) and \(g\) are nice enough, splitting methods (e.g., Douglas–Rachford and forward-backward) can be use to solve the problem:

Find \(x \in X\) such that \(x\) minimizes \(f + g\).

- Nice classes of functions in the case of forward-backward method include functions that satisfy the Kurdyka–Łojasiewicz (KL) property (Attouch et al, Bolte et al, Boţ et al, etc ...).
- Nice classes of functions in the case of Douglas–Rachford method include hypoconvex functions (Yuan et al, Dao and Phan).
- The proofs use Fejér monotonicity of the sequence of iterates w.r.t. the set of critical points of \(f + g\).
Nonexpansiveness and related concepts

Let $T : X \to X$ and let $(x, y) \in X \times X$. Recall that

- $T$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$.
- $T$ is $\alpha$-averaged if $T = (1 - \alpha) \text{Id} + \alpha N$, $\alpha \in ]0, 1[$, $N$ is nonexpansive.
- $T$ is firmly nonexpansive if
  \[\|Tx - Ty\|^2 + \|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2.\]
- Let $\beta > 0$. Then $T$ is cocoercive if $\beta T$ is firmly nonexpansive.
Monotone operators

Recall that an operator $A: X \rightrightarrows X$ is monotone if
\[
\{(x, u), (y, v)\} \subseteq \text{gr} \ A \Rightarrow \langle x - y \mid u - v \rangle \geq 0.
\]

Recall also that a monotone operator $A$ is **maximally monotone** if $A$ cannot be properly extended without destroying monotonicity.

Examples: Matrices with positive semidefinite parts, subdifferential operators $\partial f$ of convex functions and skew symmetric operators, e.g.,
\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}.
\]
Let $A: X \rightrightarrows X$ and let $\rho \in \mathbb{R}$. Then

(i) $A$ is $\rho$-monotone if $(\forall (x, u) \in \text{gr } A) (\forall (y, v) \in \text{gr } A)$ we have

$$\langle x - y \mid u - v \rangle \geq \rho \|x - y\|^2.$$
\(\rho\)-monotonicity

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\langle x - y \mid u - v \rangle \geq \rho \|x - y\|^2.
\]

(ii) \(A\) is \textit{maximally \(\rho\)-monotone} if \(A\) is \(\rho\)-monotone and there is no \(\rho\)-monotone operator \(B: X \rightrightarrows X\) such that \(\text{gr} B\) properly contains \(\text{gr} A\), i.e., for every \((x, u) \in X \times X\),

\[
(x, u) \in \text{gr} A \Leftrightarrow (\forall (y, v) \in \text{gr} A) \langle x - y \mid u - v \rangle \geq \rho \|x - y\|^2.
\]
\(\rho\)-monotonicity: Resolvents

**Proposition**

Let \(A: X \Rightarrow X\) be *maximally* \(\rho\)-monotone where \(\rho > -1\). Then the following hold:

(i) \(\text{ran}(\text{Id} + A) = X = \text{dom } J_A\).
Proposition

Let $A: X \rightrightarrows X$ be maximally $\rho$-monotone where $\rho > -1$. Then the following hold:

(i) $\text{ran}(\text{Id} + A) = X = \text{dom } J_A$.

(ii) $J_A$ is single-valued.
Proposition

Let $A: X \rightrightarrows X$ be maximally $\rho$-monotone where $\rho > -1$. Then the following hold:

(i) $\text{ran}(\text{Id} + A) = X = \text{dom } J_A$.
(ii) $J_A$ is single-valued.
(iii) $J_A$ is $(1 + \rho)$-cocoercive.
\( \rho > -1 \) is critical!

Example
Suppose that \( X \neq \{0\} \).

- Let \( C \) be a nonempty closed convex subset of \( X \),
- let \( r \in \mathbb{R}_+ \),
- set \( A = -\text{Id} - rP_C \), and set \( \rho = -(1 + r) \leq -1 \).

Then the following hold:

(i) \( A - \rho \text{Id} = r(\text{Id} - P_C) \) is maximally monotone.

(ii) \( A \) is maximally \( \rho \)-monotone.

(iii) \( \text{ran}(\text{Id} + A) = \text{ran}(-rP_C) = -rC = (\rho + 1)C \).

(iv) \( \text{Id} + A \) is surjective \( \Leftrightarrow [C = X \text{ and } r > 0] \).

(v) \( J_A = (-rP_C)^{-1} \) is at most single-valued \( \Leftrightarrow [C = X \text{ and } r > 0] \).
Theorem (**Minty parametrization**)

Let \( A : X \rightrightarrows X \) be \( \rho \)-monotone where \( \rho > -1 \). Then

\[
gr A = \{ (J_A x, (\text{Id} - J_A)x) \mid x \in \text{ran}(\text{Id} + A) \}.
\]
Minty parametrization

**Theorem (Minty parametrization)**

Let $A: X \ni X$ be $\rho$-monotone where $\rho > -1$. Then

\[
\text{gr } A = \{ (J_A x, (\text{Id} - J_A)x) \mid x \in \text{ran}(\text{Id} + A) \}.
\]

Moreover, $A$ is maximally $\rho$-monotone $\iff$ $\text{ran}(\text{Id} + A) = X$, in which case

\[
\text{gr } A = \{ (J_A x, (\text{Id} - J_A)x) \mid x \in X \}.
\]
Hypoconvex functions (a.k.a. weakly convex functions)

Let $\lambda > 0$. Recall that $f$ is $\lambda$-hypoconvex if for all $(x, y) \in X \times X$ and $\tau \in ]0, 1[$,

$$f((1 - \tau)x + \tau y) \leq (1 - \tau)f(x) + \tau f(y) + \frac{\lambda}{2} \tau (1 - \tau) \|x - y\|^2,$$

or, equivalently,

$$f + \frac{\lambda}{2} \|\cdot\|^2 \text{ is convex.}$$
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An abstract subdifferential $\partial#$ associates a subset $\partial#f(x)$ of $X$ to $f$ at $x \in X$, and it satisfies the following properties:

(i) $\partial#f = \partial f$ if $f$ is a proper lower semicontinuous convex function;

(ii) $\partial#f = \nabla f$ if $f$ is continuously differentiable;

(iii) $0 \in \partial#f(x)$ if $f$ attains a local minimum at $x \in \text{dom} \, f$;

(iv) for every $\beta \in \mathbb{R}$,

$$\partial#\left(f + \beta \frac{\|\cdot - x\|^2}{2}\right) = \partial#f + \beta(\text{Id} - x).$$
Hypoconvex functions

Proposition

Suppose that \( f : X \to ]-\infty, +\infty] \) is a proper lower semicontinuous \( \lambda \)-hypoconvex function. Then

\[
\partial \# f = \partial \left( f + \frac{\lambda}{2} \| \cdot \|^2 \right) - \lambda \text{Id}.
\]

Moreover, we have:

(i) The Clarke–Rockafellar, Mordukhovich, and Fréchet subdifferential operators of \( f \) all coincide, (easy)

(ii) \( \partial \# f \) is maximally \((-\lambda)\)-monotone, (easy)

(iii) \((\forall \mu \in ]0, \lambda[) \) \(\text{Prox}_\mu f = J_\mu \partial \# f = (\text{Id} + \mu \partial \# f)^{-1} \)

(iv) \((\forall \mu \in ]0, \lambda[) \) \(\text{Prox}_\mu f \) is \(\lambda - \mu \lambda\)-cocoercive. (consequence of our earlier results)

---

- For \( \gamma > 0 \), the proximal mapping \( \text{Prox}_{\gamma f} \) is defined at \( x \in X \) by

\[
\text{Prox}_{\gamma f}(x) = \arg\min_{y \in X} \left( f(y) + \frac{1}{2\gamma} \|x - y\|^2 \right).
\]
Hypoconvex functions

Proposition

Suppose that \( f : X \to ]-\infty, +\infty] \) is a proper lower semicontinuous \( \lambda \)-hypoconvex function. Then

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* For \( \gamma > 0 \), the *proximal mapping* \( \text{Prox}_{\gamma f} \) is defined at \( x \in X \) by

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Suppose that \( f : X \to ]-\infty, +\infty] \) is a proper lower semicontinuous \( \lambda \)-hypoconvex function. Then

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Moreover, we have:

(i) The Clarke–Rockafellar, Mordukhovich, and Fréchet subdifferential operators of \( f \) all coincide, (easy)

(ii) \( \partial_\# f \) is maximally \( (-\lambda) \)-monotone, (easy)

(iii) \((\forall \mu \in ]0, \lambda[)\) \( \text{Prox}_{\mu f} = J_{\mu \partial_\# f} = (\text{Id} + \mu \partial_\# f)^{-1} \),

(iv) \((\forall \mu \in ]0, \lambda[)\) \( \text{Prox}_{\mu f} \) is \( \frac{\lambda - \mu}{\lambda} \)-cocoercive. (consequence of our earlier results)

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• For \( \gamma > 0 \), the proximal mapping \( \text{Prox}_{\gamma f} \) is defined at \( x \in X \) by

\[
\text{Prox}_{\gamma f}(x) = \arg\min_{y \in X} \left( f(y) + \frac{1}{2\gamma} \| x - y \|^2 \right).
\]
To conclude this part:

▶ The class of hypomonotone operators ($\rho$-monotone, when $\rho < 0$) is a nice class: Indeed, we obtain single-valuedness, full domain and cocoercivity of the resolvents.

▶ BUT this is a special class.

▶ Question:
  What are other possible/more general classes of operators that have “nice” resolvents?
PART II: On the averagedness of the Douglas–Rachford operator
The Douglas–Rachford operator

Suppose that

\( A \) and \( B \) are maximally monotone operators on \( X \).

The problem:
Find \( x \in X \) such that

\[ x \in \text{zer}(A + B) = (A + B)^{-1}(0). \]

\[ J_A = (\text{Id} + A)^{-1}. \quad R_A = 2J_A - \text{Id}. \]
The Douglas–Rachford operator

Suppose that

\[ A \text{ and } B \text{ are maximally monotone operators on } X. \]

The problem:
Find \( x \in X \) such that

\[ x \in \text{zer}(A + B) = (A + B)^{-1}(0). \]

The Douglas–Rachford algorithm: One successful technique to find a zero of \( A + B \) is via iterating the Douglas–Rachford operator \( T_{A,B} \) defined for the ordered pair \( (A, B) \) by

\[ T = T_{A,B} = \frac{1}{2}(\text{Id} + R_B R_A). \]

\[ \bullet J_A = (\text{Id} + A)^{-1}. \quad \bullet R_A = 2J_A - \text{Id}. \]
Classical convergence results

Let \( x_0 \in X \). Recall that \( T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A) \). When

\[
\text{zer}(A + B) \neq \emptyset
\]

we have, for \( A \) and \( B \) maximally monotone:

▶ Lions–Mercier (1979) and Eckstein–Bertsekas (1992)

\( T \) is firmly nonexpansive.

▶ Krasnosel’kii–Mann (1950s)

\[
x_n = T_{A,B}^n x_0 \xrightarrow{\text{weakly}} \text{some point } \bar{x} \in \text{Fix } T_{A,B} \neq \text{zer}(A + B) \text{ (in general)}.
\]
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- **Lions–Mercier (1979) and Eckstein–Bertsekas (1992)**

  $T$ is firmly nonexpansive.

- **Krasnosel’skiï–Mann (1950s)**

  $x_n = T_{A,B}^n x_0 \xrightarrow{\text{weakly}}$ some point $\bar{x} \in \text{Fix } T_{A,B} \neq \text{zer}(A + B)$ (in general).

- **Combettes (2004)** $J_A(\text{Fix } T_{A,B}) = \text{zer}(A + B)$.
Classical convergence results

Let $x_0 \in X$. Recall that $T_{A,B} = \frac{1}{2} (\text{Id} + R_B R_A)$. When

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we have, for $A$ and $B$ maximally monotone:

- Lions–Mercier (1979) and Eckstein–Bertsekas (1992)
  $$T \text{ is firmly nonexpansive.}$$

- Krasnosel’skii–Mann (1950s)
  $$x_n = T_{A,B}^n x_0 \xrightarrow{\text{weakly}} \text{some point } x \in \text{Fix } T_{A,B} \neq \text{zer}(A + B) \text{ (in general).}$$

- Combettes (2004)
  $$J_A(\text{Fix } T_{A,B}) = \text{zer}(A + B).$$

- Lions–Mercier (1979) and Svaiter (2011)
  $$J_A T_{A,B}^n x \xrightarrow{\text{weakly}} \text{some point in } \text{zer}(A + B).$$
In the absence of monotonicity

- Suppose for instance that $A$ is not monotone. Then $J_A$ (and, in turn, $R_A$ and $T$) is not necessarily single-valued and/or does not necessarily have full domain. 😊
In the absence of monotonicity

- Suppose for instance that $A$ is not monotone. Then $J_A$ (and, in turn, $R_A$ and $T$) is not necessarily single-valued and/or does not necessarily have full domain. 😊

- **BUT**, if $A$ is $\rho$-monotone, and $B$ is nicer than merely monotone, then there is hope for some fun. 😊
Can we show that $T$ is “nice”?

**Lemma**

*Let* $\lambda \in ]0, 1[$. *Suppose that* $T_1 : X \to X$ *and* $T_2 : X \to X$. *Set*

$$T_\lambda = (1 - \lambda) \text{Id} + \lambda (2T_2 - \text{Id})(2T_1 - \text{Id}).$$

*Let* $(x, y) \in X \times X$. 

20

16
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Let $(x, y) \in X \times X$. Then

$$\langle T_\lambda x - T_\lambda y | (\text{Id} - T_\lambda)x - (\text{Id} - T_\lambda)y \rangle$$

$$= (1 - 2\lambda) \langle x - y | (\text{Id} - T_\lambda)x - (\text{Id} - T_\lambda)y \rangle$$

$$+ 4\lambda^2 \langle T_1 x - T_1 y | (\text{Id} - T_1)x - (\text{Id} - T_1)y \rangle$$

$$+ 4\lambda^2 \langle T_2 R_1 x - T_2 R_1 y | (\text{Id} - T_2)R_1 x - (\text{Id} - T_2)R_1 y \rangle.$$
Can we show that $T$ is “nice”?

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Let $\lambda \in ]0, 1[$. Suppose that $T_1 : X \to X$ and $T_2 : X \to X$. Set

$$T_\lambda = (1 - \lambda) \text{Id} + \lambda (2T_2 - \text{Id})(2T_1 - \text{Id}).$$

Let $(x, y) \in X \times X$. Then

$$\langle T_\lambda x - T_\lambda y \mid (\text{Id} - T_\lambda)x - (\text{Id} - T_\lambda)y \rangle = (1 - 2\lambda)\langle x - y \mid (\text{Id} - T_\lambda)x - (\text{Id} - T_\lambda)y \rangle$$

$$+ 4\lambda^2 \langle T_1 x - T_1 y \mid (\text{Id} - T_1)x - (\text{Id} - T_1)y \rangle$$

$$+ 4\lambda^2 \langle T_2 R_1 x - T_2 R_1 y \mid (\text{Id} - T_2)R_1 x - (\text{Id} - T_2)R_1 y \rangle.$$

**Corollary (Eckstein–Bertsekas)**

Suppose that $T_1 : X \to X$ and $T_2 : X \to X$. Set

$$T = \frac{1}{2} (\text{Id} + (2T_2 - \text{Id})(2T_1 - \text{Id})).$$

Then

$$\langle Tx - Ty \mid (\text{Id} - T)x - (\text{Id} - T)y \rangle = \langle T_1 x - T_1 y \mid (\text{Id} - T_1)x - (\text{Id} - T_1)y \rangle$$

$$+ \langle T_2 R_1 x - T_2 R_1 y \mid (\text{Id} - T_2)R_1 x - (\text{Id} - T_2)R_1 y \rangle.$$
Lemma

Let $T : X \rightarrow X$, let $\alpha \in ]0, 1[$ and let $(x, y) \in X \times X$. Then:

$T$ is $\alpha$-averaged $\iff 2\alpha \langle Tx - Ty \mid (\text{Id} - T)x - (\text{Id} - T)y \rangle \geq (1 - 2\alpha) \| (\text{Id} - T)x - (\text{Id} - T)y \|_2$. 
Lemma

Let $T : X \to X$, let $\alpha \in ]0, 1[$ and let $(x, y) \in X \times X$. Then: $T$ is $\alpha$-averaged $\iff$

$$2\alpha \langle Tx - Ty | (\text{Id} - T)x - (\text{Id} - T)y \rangle \geq (1 - 2\alpha) \| (\text{Id} - T)x - (\text{Id} - T)y \|^2.$$
Averagedness of the classical Douglas–Rachford operator when $\mu > \omega \geq 0$

**Theorem**

Let $\mu > \omega \geq 0$ and let $\gamma \in ]0, (\mu - \omega)/(2\mu\omega)[$. Suppose that one of the following holds:

(i) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.
(ii) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

Set

$$T = \frac{1}{2}(Id + R_{\gamma B}R_{\gamma A}) = Id - J_{\gamma A} + J_{\gamma B}R_{\gamma A}, \quad \alpha = \frac{\mu - \omega}{2(\mu - \omega - \gamma \mu \omega)}.$$

Then $\alpha \in ]0, 1[$ and $T$ is $\alpha$-averaged.
Averagedness of the classical Douglas–Rachford operator when $\mu > \omega \geq 0$

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(ii) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

Set

$$T = \frac{1}{2}(\text{Id} + R_{\gamma B}R_{\gamma A}) = \text{Id} - J_{\gamma A} + J_{\gamma B}R_{\gamma A}, \quad \alpha = \frac{\mu - \omega}{2(\mu - \omega - \gamma \mu \omega)}.$$

Then $\alpha \in ]0, 1[$ and $T$ is $\alpha$-averaged.

**Proof.** To verify that $\alpha \in ]0, 1[$ is easy $\checkmark$. We now verify that $T$ is averaged assuming (i) holds.

Note that $\gamma A$ is maximally $(-\gamma \omega)$-monotone and $-\gamma \omega > -1$. Therefore, $J_{\gamma A}$ (and in turn $R_{\gamma A}$ and $T$) is single-valued, full domain and, in fact, $J_{\gamma A}$ is cocoercive.
Our goal is show that $T$ is averaged, i.e., in view of the earlier characterization of averaged operators:

\[
\langle T x - T y | (I - T)x - (I - T)y \rangle = \langle J_{\gamma}A x - J_{\gamma}A y | (I - J_{\gamma}A)x - (I - J_{\gamma}A)y \rangle + \langle J_{\gamma}B R_{\gamma}A x - J_{\gamma}B R_{\gamma}A y | (I - J_{\gamma}B R_{\gamma}A)x - (I - J_{\gamma}B R_{\gamma}A)y \rangle \geq \gamma \mu \|J_{\gamma}B R_{\gamma}A x - J_{\gamma}B R_{\gamma}A y\|_2^2 - \gamma \omega \|J_{\gamma}A x - J_{\gamma}A y\|_2^2 = \gamma \mu \left(\|J_{\gamma}B R_{\gamma}A x - J_{\gamma}B R_{\gamma}A y\|_2^2 - \omega \mu \right) \geq -\gamma \mu \left(\omega / \mu \right) \left(1 - (\omega / \mu) \|A x - B R_{\gamma}A x - (J_{\gamma}A y - J_{\gamma}B R_{\gamma}A y)\|_2^2 \right) = -\gamma \mu \omega \mu - \omega \|A x - B R_{\gamma}A x - (J_{\gamma}A y - J_{\gamma}B R_{\gamma}A y)\|_2^2 > -\frac{1}{2} \|A x - B R_{\gamma}A x - (J_{\gamma}A y - J_{\gamma}B R_{\gamma}A y)\|_2^2.
\]

- $A$ is maximally $(-\omega)$-monotone, $B$ is maximally $\mu$-monotone, $\mu > \omega \geq 0$.
- $I - T = J_{\gamma}A - J_{\gamma}B R_{\gamma}A$
Proof continued

Our goal is to show that $T$ is averaged, i.e., in view of the earlier characterization of averaged operators:

$$
\langle Tx - Ty | (\text{Id} - T)x - (\text{Id} - T)y \rangle
$$

$$
> - \frac{1}{2}\| (\text{Id} - T)x - (\text{Id} - T)y \|^2.
$$

- $A$ is maximally $(-\omega)$-monotone, $B$ is maximally $\mu$-monotone, $\mu > \omega \geq 0$.
- $\text{Id} - T = J_{\gamma A} - J_{\gamma B}R_{\gamma A}$
Proof continued

Our goal is to show that $T$ is averaged, i.e., in view of the earlier characterization of averaged operators:

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= \langle J_{\gamma A}x - J_{\gamma A}y \mid (\text{Id} - J_{\gamma A})x - (\text{Id} - J_{\gamma A})y \rangle \\
+ \langle J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \mid (\text{Id} - J_{\gamma B})R_{\gamma A}x - (\text{Id} - J_{\gamma B})R_{\gamma A}y \rangle
\]

\[
> - \frac{1}{2} \| (\text{Id} - T)x - (\text{Id} - T)y \|^2.
\]

- $A$ is maximally $(-\omega)$-monotone, $B$ is maximally $\mu$-monotone, $\mu > \omega \geq 0$.
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Proof continued

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= \langle J_{\gamma A}x - J_{\gamma A}y \mid (\text{Id} - J_{\gamma A})x - (\text{Id} - J_{\gamma A})y \rangle \\
+ \langle J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \mid (\text{Id} - J_{\gamma B})R_{\gamma A}x - (\text{Id} - J_{\gamma B})R_{\gamma A}y \rangle \\
\geq \gamma \mu \| J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \|^2 - \gamma \omega \| J_{\gamma A}x - J_{\gamma A}y \|^2
\]

\[> - \frac{1}{2} \| (\text{Id} - T)x - (\text{Id} - T)y \|^2.\]

- $A$ is maximally $(-\omega)$-monotone, $B$ is maximally $\mu$-monotone, $\mu > \omega \geq 0$.
- $\text{Id} - T = J_{\gamma A} - J_{\gamma B}R_{\gamma A}$
Proof continued

Our goal is show that $T$ is averaged, i.e., in view of the earlier characterization of averaged operators:

\[
\langle Tx - Ty \mid (\text{Id} - T)x - (\text{Id} - T)y \rangle \\
= \langle J_{\gamma A}x - J_{\gamma A}y \mid (\text{Id} - J_{\gamma A})x - (\text{Id} - J_{\gamma A})y \rangle \\
+ \langle J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \mid (\text{Id} - J_{\gamma B})R_{\gamma A}x - (\text{Id} - J_{\gamma B})R_{\gamma A}y \rangle \\
\geq \gamma \mu \|J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y\|^2 - \gamma \omega \|J_{\gamma A}x - J_{\gamma A}y\|^2 \\
= \gamma \mu (\|J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y\|^2 - \frac{\omega}{\mu} \|J_{\gamma A}x - J_{\gamma A}y\|^2) \\
> - \frac{1}{2} \| (\text{Id} - T)x - (\text{Id} - T)y \|^2.
\]

- $A$ is maximally $(-\omega)$-monotone, $B$ is maximally $\mu$-monotone, $\mu > \omega \geq 0$.
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Proof continued

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+ \langle J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \mid (\text{Id} - J_{\gamma B})R_{\gamma A}x - (\text{Id} - J_{\gamma B})R_{\gamma A}y \rangle
\geq \gamma \mu \| J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \|^2 - \gamma \omega \| J_{\gamma A}x - J_{\gamma A}y \|^2
= \gamma \mu (\| J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \|^2 - \frac{\omega}{\mu} \| J_{\gamma A}x - J_{\gamma A}y \|^2)
\geq - \frac{\gamma \mu (\omega/\mu)}{1 - (\omega/\mu)} \| J_{\gamma A}x - J_{\gamma A}y - (J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y) \|^2
$$

$$
> - \frac{1}{2} \| (\text{Id} - T)x - (\text{Id} - T)y \|^2.
$$

- $A$ is maximally $(-\omega)$-monotone, $B$ is maximally $\mu$-monotone, $\mu > \omega \geq 0$.
- $\text{Id} - T = J_{\gamma A} - J_{\gamma B}R_{\gamma A}$
Proof continued

Our goal is to show that $T$ is averaged, i.e., in view of the earlier characterization of averaged operators:

$$\langle Tx - Ty \mid (\text{Id} - T)x - (\text{Id} - T)y \rangle$$

$$= \langle J_{\gamma A}x - J_{\gamma A}y \mid (\text{Id} - J_{\gamma A})x - (\text{Id} - J_{\gamma A})y \rangle$$

$$+ \langle J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \mid (\text{Id} - J_{\gamma B})R_{\gamma A}x - (\text{Id} - J_{\gamma B})R_{\gamma A}y \rangle$$

$$\geq \gamma \mu \| J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \|^2 - \gamma \omega \| J_{\gamma A}x - J_{\gamma A}y \|^2$$

$$= \gamma \mu (\| J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y \|^2 - \frac{\omega}{\mu} \| J_{\gamma A}x - J_{\gamma A}y \|^2)$$

$$\geq - \frac{\gamma \mu (\omega / \mu)}{1 - (\omega / \mu)} \| J_{\gamma A}x - J_{\gamma A}y - (J_{\gamma B}R_{\gamma A}x - J_{\gamma B}R_{\gamma A}y) \|^2$$

$$= - \frac{\gamma \mu \omega}{\mu - \omega} \| J_{\gamma A}x - J_{\gamma B}R_{\gamma A}x - (J_{\gamma A}y - J_{\gamma B}R_{\gamma A}y) \|^2$$

$$> - \frac{1}{2} \| (\text{Id} - T)x - (\text{Id} - T)y \|^2.$$

---

- $A$ is maximally $(-\omega)$-monotone, $B$ is maximally $\mu$-monotone, $\mu > \omega \geq 0$.
- $\text{Id} - T = J_{\gamma A} - J_{\gamma B}R_{\gamma A}$
Proof continued

Our goal is show that $T$ is averaged, i.e., in view of the earlier characterization of averaged operators:

$$
\langle T x - T y \mid (\text{Id} - T)x - (\text{Id} - T)y \rangle
$$

$$
= \langle J_{\gamma A} x - J_{\gamma A} y \mid (\text{Id} - J_{\gamma A})x - (\text{Id} - J_{\gamma A})y \rangle
$$

$$
+ \langle J_{\gamma B} R_{\gamma A} x - J_{\gamma B} R_{\gamma A} y \mid (\text{Id} - J_{\gamma B}) R_{\gamma A} x - (\text{Id} - J_{\gamma B}) R_{\gamma A} y \rangle
$$

$$
\geq \gamma \mu \|J_{\gamma B} R_{\gamma A} x - J_{\gamma B} R_{\gamma A} y\|^2 - \gamma \omega \|J_{\gamma A} x - J_{\gamma A} y\|^2
$$

$$
= \gamma \mu \left( \|J_{\gamma B} R_{\gamma A} x - J_{\gamma B} R_{\gamma A} y\|^2 - \frac{\omega}{\mu} \|J_{\gamma A} x - J_{\gamma A} y\|^2 \right)
$$

$$
\geq -\frac{\gamma \mu (\omega/\mu)}{1-(\omega/\mu)} \|J_{\gamma A} x - J_{\gamma A} y - (J_{\gamma B} R_{\gamma A} x - J_{\gamma B} R_{\gamma A} y)\|^2
$$

$$
= -\frac{\gamma \mu \omega}{\mu - \omega} \|J_{\gamma A} x - J_{\gamma B} R_{\gamma A} x - (J_{\gamma A} y - J_{\gamma B} R_{\gamma A} y)\|^2
$$

$$
= -\frac{\gamma \mu \omega}{\mu - \omega} \| (\text{Id} - T)x - (\text{Id} - T)y \|^2
$$

$$
> -\frac{1}{2} \| (\text{Id} - T)x - (\text{Id} - T)y \|^2.
$$

---

- $A$ is maximally $(-\omega)$-monotone, $B$ is maximally $\mu$-monotone, $\mu > \omega \geq 0$.
- $\text{Id} - T = J_{\gamma A} - J_{\gamma B} R_{\gamma A}$
Averagedness of the relaxed Douglas–Rachford operator

Theorem

Let \( \mu > \omega \geq 0 \), let \( \lambda \in ]0, 1[ \) and let \( \gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu \omega) [ \). Suppose that one of the following holds:

(i) \( A \) is maximally \((-\omega)\)-monotone and \( B \) is maximally \( \mu \)-monotone.

(ii) \( A \) is maximally \( \mu \)-monotone and \( B \) is maximally \((-\omega)\)-monotone.

Set

\[
T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A}, \quad \text{and} \quad \alpha = \frac{\lambda(\mu - \omega)}{\mu - \omega - \gamma \mu \omega}.
\]
Averagedness of the relaxed Douglas–Rachford operator

Theorem

Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, \frac{(1 - \lambda)(\mu - \omega)}{(\mu \omega)}[$.

Suppose that one of the following holds:

(i) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.

(ii) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

Set

$$T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A}, \quad \text{and} \quad \alpha = \frac{\lambda(\mu - \omega)}{\mu - \omega - \gamma \mu \omega}.$$

Then $\alpha \in ]0, 1[$ and $T$ is $\alpha$-averaged.
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Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu\omega)[$. Suppose that one of the following holds:

(i) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.
(ii) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

Set
\[
T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A}, \quad \text{and} \quad \alpha = \frac{\lambda(\mu - \omega)}{\mu - \omega - \gamma \mu \omega}.
\]

Then $\alpha \in ]0, 1[$ and $T$ is $\alpha$-averaged.

Proof.

- $J_{\gamma A}$ is single-valued and full domain. (easy)
Averagedness of the relaxed Douglas–Rachford operator

Theorem
Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/\mu \omega]$. Suppose that one of the following holds:

(i) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.
(ii) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

Set

$$T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A}, \quad \text{and} \quad \alpha = \frac{\lambda(\mu - \omega)}{\mu - \omega - \gamma \mu \omega}.$$ 

Then $\alpha \in ]0, 1[$ and $T$ is $\alpha$-averaged.

Proof.

- $J_{\gamma A}$ is single-valued and full domain. (easy)
- $R_{\gamma B} R_{\gamma A} = (1 - 2\delta) \text{Id} + 2\delta N$, $N$ is nonexpansive, $\delta = \frac{\mu - \omega}{2(\mu - \omega - \gamma \mu \omega)}$. (proof omitted, it uses the previous result)
Theorem

Let \( \mu > \omega \geq 0 \), let \( \lambda \in ]0, 1[ \) and let \( \gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu \omega) [ \). Suppose that one of the following holds:

(i) \( A \) is maximally \((-\omega)\)-monotone and \( B \) is maximally \( \mu \)-monotone.

(ii) \( A \) is maximally \( \mu \)-monotone and \( B \) is maximally \((-\omega)\)-monotone.

Set

\[
T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A}, \quad \text{and} \quad \alpha = \frac{\lambda(\mu - \omega)}{\mu - \omega - \gamma \mu \omega}.
\]

Then \( \alpha \in ]0, 1[ \) and \( T \) is \( \alpha \)-averaged.

Proof.

\( J_{\gamma A} \) is single-valued and full domain. (easy)

\( R_{\gamma B} R_{\gamma A} = (1 - 2\delta) \text{Id} + 2\delta N \), \( N \) is nonexpansive, \( \delta = \frac{\mu - \omega}{2(\mu - \omega - \gamma \mu \omega)} \).

(proof omitted, it uses the previous result)

Altogether

\[
T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A} = (1 - \delta) \text{Id} + \lambda((1 - 2\delta) \text{Id} + 2\delta N) \\
= (1 - 2\lambda \delta) \text{Id} + 2\lambda \delta N = (1 - \alpha) \text{Id} + \alpha N.
\]
Averagedness of the relaxed Douglas–Rachford operator

Theorem

Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu \omega)[$. Suppose that one of the following holds:

(i) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.

(ii) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

Set

$$T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A}, \quad \text{and} \quad \alpha = \frac{\lambda(\mu - \omega)}{\mu - \omega - \gamma \mu \omega}.$$

Then $\alpha \in ]0, 1[$ and $T$ is $\alpha$-averaged.

Proof.

$\blacktriangleright$ $J_{\gamma A}$ is single-valued and full domain. (easy)

$\blacktriangleright$ $R_{\gamma B} R_{\gamma A} = (1 - 2\delta) \text{Id} + 2\delta N$, $N$ is nonexpansive,

$$\delta = \frac{\mu - \omega}{2(\mu - \omega - \gamma \mu \omega)}.$$

(proof omitted, it uses the previous result)

$\blacktriangleright$ Altogether

$$T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A} = (1 - \delta) \text{Id} + \lambda ((1 - 2\delta) \text{Id} + 2\delta N)$$

$$= (1 - 2\lambda \delta) \text{Id} + 2\lambda \delta N = (1 - \alpha) \text{Id} + \alpha N.$$
Convergence of the shadow sequence

**Theorem**

Let $\mu \geq \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu \omega)[$. Suppose that one of the following holds:

(i) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.
(ii) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.

Set

$$T = (1 - \lambda)\text{Id} + \lambda R_{\gamma B} R_{\gamma A},$$

and let $x_0 \in X$. 

Convergence of the shadow sequence

Theorem

Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu\omega)[$. Suppose that one of the following holds:

(i) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

(ii) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.

Set

$$T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A},$$

and let $x_0 \in X$. Then $\text{zer}(A + B) \neq \emptyset$. 
Convergence of the shadow sequence

**Theorem**

Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega) / (\mu \omega)[$. Suppose that one of the following holds:

(i) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

(ii) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.

Set

$$T = (1 - \lambda) \operatorname{Id} + \lambda R_{\gamma B} R_{\gamma A},$$

and let $x_0 \in X$. Then $\operatorname{zer}(A + B) \neq \emptyset$. Moreover, there exists $\bar{x} \in \text{Fix } T = \text{Fix } R_{\gamma B} R_{\gamma A}$, $\operatorname{zer}(A + B) = \{ J_{\gamma A} \bar{x} \} = \{ J_{\gamma B} R_{\gamma A} \bar{x} \},$
Convergence of the shadow sequence

Theorem

Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu \omega)[$. Suppose that one of the following holds:

(i) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

(ii) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.

Set

$$T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A},$$

and let $x_0 \in X$. Then $\text{zer}(A + B) \neq \emptyset$. Moreover, there exists $\bar{x} \in \text{Fix } T = \text{Fix } R_{\gamma B} R_{\gamma A}$, $\text{zer}(A + B) = \{J_{\gamma A} \bar{x}\} = \{J_{\gamma B} R_{\gamma A} \bar{x}\}$, $T^n x_0 \rightharpoonup \bar{x}$. 
Theorem

Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu \omega)[$. Suppose that one of the following holds:

(i) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

(ii) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.

Set

$$T = (1 - \lambda) \text{Id} + \lambda R_{\gamma B} R_{\gamma A},$$

and let $x_0 \in X$. Then $\text{zer}(A + B) \neq \emptyset$. Moreover, there exists $\bar{x} \in \text{Fix } T = \text{Fix } R_{\gamma B} R_{\gamma A}$, $\text{zer}(A + B) = \{J_{\gamma A} \bar{x}\} = \{J_{\gamma B} R_{\gamma A} \bar{x}\}$, $T^n x_0 \rightharpoonup \bar{x}$, $J_{\gamma A} T^n x_0 \rightharpoonup J_{\gamma A} \bar{x}$.
Convergence of the shadow sequence

**Theorem**

Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu\omega)[$. Suppose that one of the following holds:

(i) $A$ is maximally $\mu$-monotone and $B$ is maximally $(-\omega)$-monotone.

(ii) $A$ is maximally $(-\omega)$-monotone and $B$ is maximally $\mu$-monotone.

Set

$$T = (1 - \lambda)\text{Id} + \lambda R_{\gamma B}R_{\gamma A},$$

and let $x_0 \in X$. Then $\text{zer}(A + B) \neq \emptyset$. Moreover, there exists $\overline{x} \in \text{Fix } T = \text{Fix } R_{\gamma B}R_{\gamma A}$, $\text{zer}(A + B) = \{J_{\gamma A}\overline{x}\} = \{J_{\gamma B}R_{\gamma A}\overline{x}\}$, $T^n x_0 \rightharpoonup \overline{x}$, $J_{\gamma A}T^n x_0 \to J_{\gamma A}\overline{x}$, and $J_{\gamma B}R_{\gamma A}T^n x_0 \to J_{\gamma B}R_{\gamma A}\overline{x}$. 
Convergence of the shadow sequence: proof continued

Claim 1:

\[ \| J_{\gamma A} T^n x_0 - J_{\gamma A} \bar{x} \|^2 - \| J_{\gamma B} R_{\gamma A} T^n x_0 - J_{\gamma B} R_{\gamma A} \bar{x} \|^2 \rightarrow 0. \]
Convergence of the shadow sequence: proof continued

▶ Claim 1:

\[ \| J_{\gamma A} T^n x_0 - J_{\gamma A} \bar{x} \|^2 - \| J_{\gamma B} R_{\gamma A} T^n x_0 - J_{\gamma B} R_{\gamma A} \bar{x} \|^2 \to 0. \]

▶ Claim 2:

\[ \| J_{\gamma A} T^n x_0 - J_{\gamma A} \bar{x} \|^2 - \frac{\omega}{\mu} \| J_{\gamma B} R_{\gamma A} T^n x_0 - J_{\gamma B} R_{\gamma A} \bar{x} \|^2 \to 0. \]

\[ \bar{x} - T \bar{x} = 0 \Rightarrow J_{\gamma A} \bar{x} = J_{\gamma B} R_{\gamma B} \bar{x} \]
Convergence of the shadow sequence: proof continued

▶ Claim 1:

\[
\|J^A T^n x_0 - J^A \bar{x}\|^2 - \|J^B R^A T^n x_0 - J^B R^A \bar{x}\|^2 \\
= \langle J^A T^n x_0 - J^B R^A T^n x_0 | J^A T^n x_0 + J^B R^A T^n x_0 - J^A \bar{x} - J^B R^A \bar{x} \rangle \\
\rightarrow 0.
\]

▶ Claim 2:

\[
\|J^A T^n x_0 - J^A \bar{x}\|^2 - \frac{\omega}{\mu} \|J^B R^A T^n x_0 - J^B R^A \bar{x}\|^2 \\
\rightarrow 0.
\]

\[\bar{x} - T\bar{x} = 0 \Rightarrow J^A \bar{x} = J^B R^B \bar{x}\]
Convergence of the shadow sequence: proof continued

▶ Claim 1:

\[
\|J_\gamma A T^n x_0 - J_\gamma A \bar{x}\|^2 - \|J_\gamma B R_\gamma A T^n x_0 - J_\gamma B R_\gamma A \bar{x}\|^2
\]

\[
= \langle J_\gamma A T^n x_0 - J_\gamma B R_\gamma A T^n x_0 \mid J_\gamma A T^n x_0 + J_\gamma B R_\gamma A T^n x_0 - J_\gamma A \bar{x} - J_\gamma B R_\gamma A \bar{x} \rangle
\]

\[
= \langle \underbrace{T^n x_0 - T^{n+1} x_0} \rightarrow 0 \mid \underbrace{J_\gamma A T^n x_0 + J_\gamma B R_\gamma A T^n x_0 - J_\gamma A \bar{x} - J_\gamma B R_\gamma A \bar{x}} \rangle \rightarrow 0.
\]

▶ Claim 2:

\[
\|J_\gamma A T^n x_0 - J_\gamma A \bar{x}\|^2 - \frac{\omega}{\mu} \|J_\gamma B R_\gamma A T^n x_0 - J_\gamma B R_\gamma A \bar{x}\|^2
\]

\[
\rightarrow 0.
\]

\[
\bullet \bar{x} - T \bar{x} = 0 \Rightarrow J_\gamma A \bar{x} = J_\gamma B R_\gamma B \bar{x}
\]

22
Convergence of the shadow sequence: proof continued

▶ Claim 1:

\[
\left\| J_{\gamma} T^n x_0 - J_{\gamma} \bar{x} \right\|^2 - \left\| J_{\gamma} B R_{\gamma} T^n x_0 - J_{\gamma} B R_{\gamma} \bar{x} \right\|^2 \\
= \langle J_{\gamma} T^n x_0 - J_{\gamma} B R_{\gamma} T^n x_0 \mid J_{\gamma} T^n x_0 + J_{\gamma} B R_{\gamma} T^n x_0 - J_{\gamma} \bar{x} - J_{\gamma} B R_{\gamma} \bar{x} \rangle \\
= \langle T^n x_0 - T^{n+1} x_0 \mid J_{\gamma} T^n x_0 + J_{\gamma} B R_{\gamma} T^n x_0 - J_{\gamma} \bar{x} - J_{\gamma} B R_{\gamma} \bar{x} \rangle \rightarrow 0.
\]

▶ Claim 2:

\[
0 \leftarrow \langle T^{n+1} x_0 - \bar{x} \mid T^n x_0 - T^{n+1} x_0 \rangle - (1 - 2\lambda) \langle T^n x_0 - \bar{x} \mid T^n x_0 - T^{n+1} x_0 \rangle \\
\geq 4\gamma \mu \lambda^2 \left( \left\| J_{\gamma} T^n x_0 - J_{\gamma} \bar{x} \right\|^2 - \frac{\omega}{\mu} \left\| J_{\gamma} B R_{\gamma} T^n x_0 - J_{\gamma} B R_{\gamma} \bar{x} \right\|^2 \right) \\
\rightarrow 0.
\]

\[\bullet \bar{x} - T\bar{x} = 0 \Rightarrow J_{\gamma} \bar{x} = J_{\gamma} B R_{\gamma} \bar{x}\]
Convergence of the shadow sequence: proof continued

Claim 1:

\[ \| J_{\gamma_A} T^n x_0 - J_{\gamma_A} \bar{x} \|^2 - \| J_{\gamma_B} R_{\gamma_A} T^n x_0 - J_{\gamma_B} R_{\gamma_A} \bar{x} \|^2 = \langle J_{\gamma_A} T^n x_0 - J_{\gamma_B} R_{\gamma_A} T^n x_0 \mid J_{\gamma_A} T^n x_0 + J_{\gamma_B} R_{\gamma_A} T^n x_0 - J_{\gamma_A} \bar{x} - J_{\gamma_B} R_{\gamma_A} \bar{x} \rangle \]
\[ = \langle T^n x_0 - T^{n+1} x_0 \mid (J_{\gamma_A} + J_{\gamma_B} R_{\gamma_A} T^n x_0 - J_{\gamma_A} \bar{x} - J_{\gamma_B} R_{\gamma_A} \bar{x}) \rangle \rightarrow 0. \]

Claim 2:

\[ 0 \leftarrow \langle T^{n+1} x_0 - \bar{x} \mid T^n x_0 - T^{n+1} x_0 \rangle - (1 - 2\lambda) \langle T^n x_0 - \bar{x} \mid T^n x_0 - T^{n+1} x_0 \rangle \]
\[ \geq 4\gamma \mu \lambda^2 \left( \| J_{\gamma_A} T^n x_0 - J_{\gamma_A} \bar{x} \|^2 - \frac{\omega}{\mu} \| J_{\gamma_B} R_{\gamma_A} T^n x_0 - J_{\gamma_B} R_{\gamma_A} \bar{x} \|^2 \right) \]
\[ \geq (1 - 2\lambda) \langle T^n x_0 - \bar{x} \mid T^n x_0 - T^{n+1} x_0 \rangle - \frac{\gamma \mu \omega}{\mu - \omega} \| T^n x_0 - T^{n+1} x_0 \|^2 \rightarrow 0. \]

\[ \bullet \bar{x} - T \bar{x} = 0 \Rightarrow J_{\gamma_A} \bar{x} = J_{\gamma_B} R_{\gamma_B} \bar{x} \]
Application to the optimization problems

Theorem
Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu\omega)[$. Suppose that one of the following holds:

(i) $f$ is $\mu$-strongly convex, $g$ is $\omega$-hypoconvex.
(ii) $f$ is $\omega$-hypoconvex, and $g$ is $\mu$-strongly convex.

and that $\text{zer}(\partial f + \partial \# g) \neq \emptyset$ (require sufficient conditions).
Application to the optimization problems

Theorem
Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega)/(\mu \omega)[$. Suppose that one of the following holds:

(i) $f$ is $\mu$-strongly convex, $g$ is $\omega$-hypoconvex.
(ii) $f$ is $\omega$-hypoconvex, and $g$ is $\mu$-strongly convex.

and that $\text{zer}(\partial f + \partial g) \neq \emptyset$ (require sufficient conditions). Set

$$T = (1 - \lambda) \text{Id} + \lambda(2 \text{Prox}_\gamma g - \text{Id})(2 \text{Prox}_\gamma f - \text{Id}), \quad \text{and} \quad \alpha = \frac{\lambda(\mu - \omega)}{\mu - \omega - \gamma \mu \omega},$$

and let $x_0 \in X$. 
Application to the optimization problems

Theorem
Let \( \mu > \omega \geq 0 \), let \( \lambda \in ]0, 1[ \) and let \( \gamma \in ]0, (1 - \lambda)(\mu - \omega) / (\mu \omega) [ \). Suppose that one of the following holds:
(i) \( f \) is \( \mu \)-strongly convex, \( g \) is \( \omega \)-hypoconvex.
(ii) \( f \) is \( \omega \)-hypoconvex, and \( g \) is \( \mu \)-strongly convex.
and that \( \text{zer}(\partial f + \partial \# g) \neq \emptyset \) (require sufficient conditions). Set
\[
T = (1 - \lambda) \text{Id} + \lambda (2 \text{Prox}_\gamma g - \text{Id})(2 \text{Prox}_\gamma f - \text{Id}), \quad \text{and} \quad \alpha = \frac{\lambda(\mu - \omega)}{\mu - \omega - \gamma \mu \omega},
\]
and let \( x_0 \in X \). Then \( \alpha \in ]0, 1[ \), and \( T \) is \( \alpha \)-averaged.
Application to the optimization problems

Theorem
Let $\mu > \omega \geq 0$, let $\lambda \in ]0, 1[$ and let $\gamma \in ]0, (1 - \lambda)(\mu - \omega) / (\mu \omega)[$. Suppose that one of the following holds:

(i) $f$ is $\mu$-strongly convex, $g$ is $\omega$-hypoconvex.

(ii) $f$ is $\omega$-hypoconvex, and $g$ is $\mu$-strongly convex.

and that $\text{zer}(\partial f + \partial \# g) \neq \emptyset$ (require sufficient conditions). Set

$$T = (1 - \lambda) \text{Id} + \lambda (2 \text{Prox}_{\gamma g} - \text{Id})(2 \text{Prox}_{\gamma f} - \text{Id}), \quad \text{and} \quad \alpha = \frac{\lambda(\mu - \omega)}{\mu - \omega - \gamma \mu \omega},$$

and let $x_0 \in X$. Then $\alpha \in ]0, 1[$, and $T$ is $\alpha$-averaged. Moreover, $(\exists \, \overline{x} \in \text{Fix } T)$ such that $T^n x_0 \rightharpoonup \overline{x}$, $\text{argmin}(f + g) = \{\text{Prox}_f \overline{x}\}$, and $\text{Prox}_f T^n x_0 \to \text{Prox}_f \overline{x}$. 


To conclude this part:

▶ **Question:**
What does the algorithm do when \( A + B \) is just monotone?

▶ Recently, Dao and Phan introduced an adaptation of the Douglas–Rachford algorithm to deal with the case when \( A + B \) is monotone. They show that the governing sequence converges to a fixed point of \( T \). **However,** the behaviour of the shadow sequence is not clear. There is a room for progress in this direction.
PART III: On the averagedness of the forward-backward operator
The forward-backward operator

Let $\beta > 0$. Suppose that

\[ A: X \to X \text{ is } \beta\text{-cocoercive,} \]

and that

\[ B: X \rightrightarrows X \text{ is maximally monotone.} \]

The problem: Find $x \in X$ such that

\[ x \in \text{zer}(A + B) = (A + B)^{-1}(0). \]

\[ J_{\gamma}A = (\text{Id} + \gamma A)^{-1} \]

The forward-backward operator

Let $\beta > 0$. Suppose that

\[ A : X \to X \text{ is } \beta\text{-cocoercive,} \]

and that

\[ B : X \rightrightarrows X \text{ is maximally monotone.} \]

The problem: Find $x \in X$ such that

\[ x \in \text{zer}(A + B) = (A + B)^{-1}(0). \]

The forward-backward algorithm: One successful technique to find a zero of $A + B$ is via iterating the forward-backward operator $T$ defined by

\[ T = J_{\gamma B}(\text{Id} - \gamma A), \]

$\gamma \in ]0, 2\beta[.$
Classical convergence results

Let $x_0 \in X$. Recall that $T = A + B$. 
Classical convergence results

Let $x_0 \in X$. Recall that $T = A + B$. When

$$\text{zer}(A + B) \neq \emptyset$$

we have, for $A$ cocoercive and $B$ maximally monotone:

$T$ is averaged.
Classical convergence results

Let $x_0 \in X$. Recall that $T = A + B$. When

$$\text{zer}(A + B) \neq \emptyset$$

we have, for $A$ cocoercive and $B$ maximally monotone:

$\quad$ $\triangleright$

$T$ is averaged.

$\quad$ $\triangleright$ Krasnosel’skiĭ–Mann (1950s)

$$x_n = T^n x_0 \xrightarrow{\text{weakly}} \text{some point } \bar{x} \in \text{Fix } T = \text{zer}(A + B).$$
In the absence of monotonicity

- Suppose for instance that $A$ is not monotone. Then $J_A$ (and, in turn, $R_A$ and $T$) is not necessarily single-valued and/or does not necessarily has full domain. 😊
In the absence of monotonicity

- Suppose for instance that $A$ is not monotone. Then $J_A$ (and, in turn, $R_A$ and $T$) is not necessarily single-valued and/or does not necessarily have full domain. 😃

- BUT, if $A$ is Lipschitz continuous + “nice” and $A + B$ is monotone, then there is more to say. 😁

Let $\mu \geq 0$, $\omega \geq 0$, and $\beta > 0$. We prove that the forward-backward operator is averaged, in each of the following situations:

(i) $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive, $B$ is maximally $(-\omega)$-monotone, and $\mu \geq \omega$.

(ii) $A$ is maximally $(-\omega)$-monotone, $A + \omega \text{Id}$ is $\beta$-cocoercive, $B$ is maximally $\mu$-monotone, and $\mu \geq \omega$.

(iii) $A$ is $(1/\beta)$-Lipschitz continuous, $B$ is maximally $\mu$-monotone, and $\mu \geq 1/\beta$. 
Theorem

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. 

Theorem
Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

1. $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
Theorem

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

- $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
- $B$ is maximally $(\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.
Theorem
Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

- $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
- $B$ is maximally $(-\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Let $\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu \beta)\}[$.
Theorem
Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that
- $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
- $B$ is maximally $(-\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Let $\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu\beta)\}[$. Set

$$T = J_{\gamma B}(\text{Id} - \gamma A),$$
Theorem
Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

- $A$ is maximally $\mu$-monotone, $A - \mu Id$ is $\beta$-cocoercive,
- $B$ is maximally $(-\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Let $\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu\beta)\}[$. Set

$$T = J_{\gamma B}(Id - \gamma A),$$

set $\nu = \gamma / (2\beta(1 - \gamma\mu))$ and set $\delta = (1 - \gamma\mu) / (1 - \gamma\omega)$.
Theorem
Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

- $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
- $B$ is maximally $(-\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Let $\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu\beta)\}[$. Set

$$T = J_{\gamma B}(\text{Id} - \gamma A),$$

set $v = \gamma/(2\beta(1 - \gamma\mu))$ and set $\delta = (1 - \gamma\mu)/(1 - \gamma\omega)$. Then\{v, $\delta$\} $\subseteq ]0, 1[$.
Theorem
Let \( \mu \geq \omega \geq 0 \), and let \( \beta > 0 \). Suppose that

- \( A \) is maximally \( \mu \)-monotone, \( A - \mu \text{Id} \) is \( \beta \)-cocoercive,
- \( B \) is maximally \( (-\omega) \)-monotone
- such that \( \text{zer}(A + B) \neq \emptyset \).

Let \( \gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu\beta)\} [ \). Set

\[
T = J_{\gamma B}(\text{Id} - \gamma A),
\]

set \( \nu = \gamma / (2\beta(1 - \gamma \mu)) \) and set \( \delta = (1 - \gamma \mu) / (1 - \gamma \omega) \). Then \( \{\nu, \delta\} \subseteq ]0, 1[ \). Moreover, the following hold:

(i) \( T \) is \( (1 - \delta(1 - 1/(2 - \nu))) \)-averaged.

(ii) There exists \( \bar{x} \in \text{Fix } T = \text{zer}(A + B) \) such that \( T^nx_0 \to \bar{x} \).
Theorem

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

1. $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
2. $B$ is maximally $(-\omega)$-monotone
3. such that $\text{zer}(A + B) \neq \emptyset$.

Let $\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu\beta)\}[$. Set

$$T = J_{\gamma B}(\text{Id} - \gamma A),$$

set $\nu = \gamma/(2\beta(1 - \gamma\mu))$ and set $\delta = (1 - \gamma\mu)/(1 - \gamma\omega)$. Then $\{\nu, \delta\} \subseteq ]0, 1[$. Moreover, the following hold:

(i) $T$ is $(1 - \delta(1 - 1/(2 - \nu)))$-averaged.
(ii) There exists $\bar{x} \in \text{Fix } T = \text{zer}(A + B)$ such that $T^n x_0 \rightharpoonup \bar{x}$.

Suppose that $\mu > \omega$. Then we additionally have:

(iii) $T$ is Banach contraction with constant $\delta = (1 - \gamma\mu)/(1 - \gamma\omega) < 1$.
(iv) $\text{zer}(A + B) = \{\bar{x}\}$ and $T^n x_0 \to \bar{x}$ with a linear rate $\delta < 1$.
Sketch of the proof

There exist a nonexpansive mapping $N$, such that

$$A - \mu \text{Id} = \frac{1}{2\beta} \text{Id} + \frac{1}{2\beta} N,$$

and

$$J_\gamma B = 1 - \gamma \omega T_1.$$ 

Hence, $T$ is averaged as claimed. □

**Theorem** Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

- $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
- $B$ is maximally $(\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Set $\nu = \gamma/(2\beta(1 - \gamma \mu))$ and set $\delta = (1 - \gamma \mu)/(1 - \gamma \omega)$. Then $T$ is $(1 - \delta(1 - 1/(2 - \nu)))$-averaged.
Sketch of the proof

There exist a nonexpansive mapping $N$, and a firmly nonexpansive mapping $T_1 : X \rightarrow X$, such that

$$A - \mu \text{Id} = \frac{1}{2\beta} \text{Id} + \frac{1}{2\beta} N,$$

and

$$J_{\gamma B} = \frac{1}{1-\gamma \omega} T_1.$$

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and

$$J_{\gamma B} = \frac{1}{1-\gamma \omega} T_1.$$

$$= \delta (T_3 \circ T_2).$$

Hence, $T$ is averaged as claimed.

Theorem Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

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- $B$ is maximally $(-\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Set $\nu = \gamma/(2\beta(1-\gamma\mu))$ and set $\delta = (1-\gamma\mu)/(1-\gamma\omega)$.

Then $T$ is $(1 - \delta(1 - 1/(2 - \nu)))$-averaged.
Sketch of the proof

There exist a nonexpansive mapping $N$, and a firmly nonexpansive mapping $T_1 : X \to X$, such that

$$A - \mu \text{Id} = \frac{1}{2\beta} \text{Id} + \frac{1}{2\beta} N, \quad \text{and} \quad J_{\gamma B} = \frac{1}{1 - \gamma \omega} T_1.$$

$$T = J_{\gamma B}(\text{Id} - \gamma A) = J_{\gamma B}(\text{Id} - \gamma (A - \mu \text{Id} + \mu \text{Id}))$$

$$= \delta (T_3 \circ T_2).$$

Hence, $T$ is averaged as claimed.

Theorem Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

- $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
- $B$ is maximally $(\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Set $\nu = \gamma/(2\beta(1 - \gamma \mu))$ and set $\delta = (1 - \gamma \mu)/(1 - \gamma \omega)$.

Then $T$ is $(1 - \delta(1 - 1/(2 - \nu)))$-averaged.
Sketch of the proof

There exist a nonexpansive mapping $N$, and a firmly nonexpansive mapping $T_1 : X \to X$, such that

$$A - \mu \text{Id} = \frac{1}{2\beta} \text{Id} + \frac{1}{2\beta} N,$$

and

$$J_{\gamma B} = \frac{1}{1 - \gamma \omega} T_1.$$

$$T = J_{\gamma B} (\text{Id} - \gamma A) = J_{\gamma B} (\text{Id} - \gamma (A - \mu \text{Id} + \mu \text{Id}))$$

$$= \frac{1}{1 - \gamma \omega} T_1 \circ (1 - \gamma \mu) \text{Id} \circ (\text{Id} - \frac{\gamma}{1 - \gamma \mu} \left( \frac{1}{2\beta} \text{Id} + \frac{1}{2\beta} N \right))$$

$$= \delta (T_3 \circ T_2).$$

Hence, $T$ is averaged as claimed.

**Theorem** Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

- $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
- $B$ is maximally $(-\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Set $\nu = \gamma / (2\beta (1 - \gamma \mu))$ and set $\delta = (1 - \gamma \mu) / (1 - \gamma \omega)$.

Then $T$ is $(1 - \delta(1 - 1/(2 - \nu)))$-averaged.
Sketch of the proof

There exist a nonexpansive mapping $N$, and a firmly nonexpansive mapping $T_1 : X \to X$, such that

$$A - \mu \text{Id} = \frac{1}{2\beta} \text{Id} + \frac{1}{2\beta} N, \quad \text{and} \quad J_{\gamma B} = \frac{1}{1 - \gamma \omega} T_1.$$ 

$$T = J_{\gamma B} (\text{Id} - \gamma A) = J_{\gamma B} (\text{Id} - \gamma (A - \mu \text{Id} + \mu \text{Id}))$$

$$= \frac{1}{1 - \gamma \omega} T_1 \circ (1 - \gamma \mu) \text{Id} \circ (\text{Id} - \frac{\gamma}{1 - \gamma \mu} \left( \frac{1}{2\beta} \text{Id} + \frac{1}{2\beta} N \right))$$

$$= \frac{1 - \gamma \mu}{1 - \gamma \omega} \frac{1}{1 - \gamma \mu} T_1 \circ (1 - \gamma \mu) \text{Id} \circ \left( (1 - \nu) \text{Id} + \nu (-N) \right)$$

$$= T_3 \text{ (f.n.e.)} = T_2 \text{ (averaged)}$$

$$= \delta (T_3 \circ T_2).$$

Hence, $T$ is averaged as claimed. 

\[\square\]

**Theorem** Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

- $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
- $B$ is maximally $(\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Set $\nu = \gamma / (2\beta(1 - \gamma \mu))$ and set $\delta = (1 - \gamma \mu) / (1 - \gamma \omega)$.

Then $T$ is $(1 - \delta(1 - 1/(2 - \nu)))$-averaged.
Sketch of the proof

There exist a nonexpansive mapping $N$, and a firmly nonexpansive mapping $T_1 : X \to X$, such that

$$A - \mu \text{Id} = \frac{1}{2\beta} \text{Id} + \frac{1}{2\beta} N,$$

and

$$J_{\gamma B} = \frac{1}{1 - \gamma \omega} T_1.$$

$$T = J_{\gamma B} (\text{Id} - \gamma A) = J_{\gamma B} (\text{Id} - \gamma (A - \mu \text{Id} + \mu \text{Id}))$$

$$= \frac{1}{1 - \gamma \omega} T_1 \circ (1 - \gamma) \text{Id} \circ \left( \text{Id} - \frac{\gamma}{1 - \gamma \mu} \left( \frac{1}{2\beta} \text{Id} + \frac{1}{2\beta} N \right) \right)$$

$$= \frac{1 - \gamma \mu}{1 - \gamma \omega} \frac{1}{1 - \gamma \mu} T_1 \circ (1 - \gamma \mu) \text{Id} \circ \left( (1 - \nu) \text{Id} + \nu (-N) \right)$$

$$= \frac{1 - \gamma \mu}{1 - \gamma \omega} T_3 \circ T_2 = \delta (T_3 \circ T_2).$$

Hence, $T$ is averaged as claimed.

\[\Box\]

**Theorem** Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Suppose that

- $A$ is maximally $\mu$-monotone, $A - \mu \text{Id}$ is $\beta$-cocoercive,
- $B$ is maximally $(-\omega)$-monotone
- such that $\text{zer}(A + B) \neq \emptyset$.

Set $\nu = \gamma / (2\beta(1 - \gamma \mu))$ and set $\delta = (1 - \gamma \mu) / (1 - \gamma \omega)$.

Then $T$ is $(1 - \delta(1 - 1/(2 - \nu)))$-averaged.
Lemma

Let $f_1 : X \to \mathbb{R}$, $f_2 : X \to \mathbb{R}$ be a Fréchet differentiable convex functions and let $\delta > \beta > 0$. Suppose that $\nabla f_1$ (respectively $\nabla f_2$) is $\frac{1}{\beta}$-Lipschitz continuous (respectively $\frac{1}{\delta}$-Lipschitz continuous). Then the following hold:

(i) $\nabla f_1 - \nabla f_2$ is $\frac{1}{\beta}$-Lipschitz continuous.

(ii) Suppose that $f_1 - f_2$ is convex. Then $\nabla f_1 - \nabla f_2$ is $\beta$-cocoercive.
Lemma
Let \( f_1 : X \to \mathbb{R}, \ f_2 : X \to \mathbb{R} \) be a Fréchet differentiable convex functions and let \( \delta > \beta > 0 \). Suppose that \( \nabla f_1 \) (respectively \( \nabla f_2 \)) is \( \frac{1}{\beta} \)-Lipschitz continuous (respectively \( \frac{1}{\delta} \)-Lipschitz continuous). Then the following hold:

(i) \( \nabla f_1 - \nabla f_2 \) is \( \frac{1}{\beta} \)-Lipschitz continuous.

(ii) Suppose that \( f_1 - f_2 \) is convex. Then \( \nabla f_1 - \nabla f_2 \) is \( \beta \)-cocoercive.

Lemma
Let \( \mu \geq 0 \), let \( \beta > 0 \) and let \( f : X \to \mathbb{R} \) be a Fréchet differentiable function. Suppose that \( f \) is \( \mu \)-strongly convex with a \( \frac{1}{\beta} \)-Lipschitz continuous gradient. Then the following hold:

(i) \( f - \frac{\mu}{2} \| \cdot \|^2 \) is convex.

(ii) \( \nabla f \) is maximally \( \mu \)-monotone.

(iii) \( \nabla f - \mu \text{Id} \) is \( \beta \)-cocoercive.
Theorem (the forward-backward algorithm when $f$ is $\mu$-strongly convex)

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Let $f$ be $\mu$-strongly convex and Fréchet differentiable with a $\frac{1}{\beta}$-Lipschitz continuous gradient, and let $g$ be $\omega$-hypoconvex. Suppose that $\arg\min (f + g) \neq \emptyset$. Let $\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu \beta)\}[$, and set $\delta = (1 - \gamma \mu)/(1 - \gamma \omega)$. Set

$$T = \text{Prox}_{\gamma g} (\text{Id} - \gamma \nabla f),$$

and let $x_0 \in X$. 


Theorem (the forward-backward algorithm when $f$ is $\mu$-strongly convex)

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Let $f$ be $\mu$-strongly convex and Fréchet differentiable with a $\frac{1}{\beta}$-Lipschitz continuous gradient, and let $g$ be $\omega$-hypoconvex. Suppose that $\text{argmin}(f + g) \neq \emptyset$. Let $\gamma \in \mathbb{R}$, $\min\{1/\mu, 2\beta/(1 + 2\mu \beta)\}$, and set $\delta = (1 - \gamma \mu) / (1 - \gamma \omega)$. Set

$$T = \text{Prox}_{\gamma g}(\text{Id} - \gamma \nabla f),$$

and let $x_0 \in X$. Then the following hold:

(i) There exists $\bar{x} \in \text{Fix } T = \text{zer}(A + B) = \text{argmin}(f + g)$ such that $T^n x_0 \rightharpoonup \bar{x}$. 
Theorem (the forward-backward algorithm when $f$ is $\mu$-strongly convex)

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Let $f$ be $\mu$-strongly convex and Fréchet differentiable with a $\frac{1}{\beta}$-Lipschitz continuous gradient, and let $g$ be $\omega$-hypoconvex. Suppose that $\text{argmin}(f + g) \neq \emptyset$. Let

$$
\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu\beta)\} [,
$$
and set $\delta = (1 - \gamma \mu) / (1 - \gamma \omega)$. Set

$$
T = \text{Prox}_{\gamma g} (\text{Id} - \gamma \nabla f),
$$

and let $x_0 \in X$. Then the following hold:

(i) There exists $\bar{x} \in \text{Fix } T = \text{zer}(A + B) = \text{argmin}(f + g)$ such that $T^n x_0 \to \bar{x}$.

Suppose that $\mu > \omega$. Then, we additionally have:

(ii) $\text{Fix } T = \text{argmin}(f + g) = \{\bar{x}\}$ and $T^n x_0 \to \bar{x}$ with a linear rate $\delta < 1$. 

Proof. $\nabla f$ is maximally $\mu$-monotone, $\nabla f - \mu \text{Id}$ is $\frac{1}{\beta}$-Lipschitz continuous by an earlier result, hence $\beta$-cocoercive. Apply the previous result.
Theorem (the forward-backward algorithm when $f$ is $\mu$-strongly convex)

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Let $f$ be $\mu$-strongly convex and Fréchet differentiable with a $\frac{1}{\beta}$-Lipschitz continuous gradient, and let $g$ be $\omega$-hypoconvex. Suppose that $\text{argmin}(f + g) \neq \emptyset$. Let

$\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu\beta)\}[,$ and set $\delta = (1 - \gamma\mu)/(1 - \gamma\omega)$. Set

$$T = \text{Prox}_{\gamma g}(Id - \gamma \nabla f),$$

and let $x_0 \in X$. Then the following hold:

(i) There exists $\overline{x} \in \text{Fix } T = \text{zer}(A + B) = \text{argmin}(f + g)$ such that $T^n x_0 \rightharpoonup \overline{x}$.

(ii) $\text{Fix } T = \text{argmin}(f + g) = \{\overline{x}\}$ and $T^n x_0 \rightarrow \overline{x}$ with a linear rate $\delta < 1$.

Proof.
$\nabla f$ is maximally $\mu$-monotone,
Theorem (the forward-backward algorithm when $f$ is $\mu$-strongly convex)

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Let $f$ be $\mu$-strongly convex and Fréchet differentiable with a $\frac{1}{\beta}$-Lipschitz continuous gradient, and let $g$ be $\omega$-hypoconvex. Suppose that $\text{argmin}(f + g) \neq \emptyset$. Let $\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu\beta)\}[$, and set $\delta = (1 - \gamma\mu)/(1 - \gamma\omega)$. Set

$$T = \text{Prox}_{\gamma g}(\text{Id} - \gamma \nabla f),$$

and let $x_0 \in X$. Then the following hold:

(i) There exists $\bar{x} \in \text{Fix } T = \text{zer}(A + B) = \text{argmin}(f + g)$ such that $T^n x_0 \rightharpoonup \bar{x}$.

Suppose that $\mu > \omega$. Then, we additionally have:

(ii) $\text{Fix } T = \text{argmin}(f + g) = \{\bar{x}\}$ and $T^n x_0 \to \bar{x}$ with a linear rate $\delta < 1$.

Proof.

$\nabla f$ is maximally $\mu$-monotone, $\nabla f - \mu \text{ Id}$ is $\frac{1}{\beta}$-Lipschitz continuous by an earlier result,
Theorem (the forward-backward algorithm when $f$ is $\mu$-strongly convex)

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Let $f$ be $\mu$-strongly convex and Fréchet differentiable with a $\frac{1}{\beta}$-Lipschitz continuous gradient, and let $g$ be $\omega$-hypoconvex. Suppose that $\argmin(f + g) \neq \emptyset$. Let $\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu \beta)\}[$, and set $\delta = (1 - \gamma \mu) / (1 - \gamma \omega)$. Set

$$T = \text{Prox}_{\gamma g}(\text{Id} - \gamma \nabla f),$$

and let $x_0 \in X$. Then the following hold:

(i) There exists $\bar{x} \in \text{Fix} T = \text{zer}(A + B) = \argmin(f + g)$ such that $T^n x_0 \rightharpoonup \bar{x}$.

Suppose that $\mu > \omega$. Then, we additionally have:

(ii) $\text{Fix} T = \argmin(f + g) = \{\bar{x}\}$ and $T^n x_0 \to \bar{x}$ with a linear rate $\delta < 1$.

Proof.

$\nabla f$ is maximally $\mu$-monotone, $\nabla f - \mu \text{Id}$ is $\frac{1}{\beta}$-Lipschitz continuous by an earlier result, hence $\beta$-cocoercive.
Theorem (the forward-backward algorithm when $f$ is $\mu$-strongly convex)

Let $\mu \geq \omega \geq 0$, and let $\beta > 0$. Let $f$ be $\mu$-strongly convex and Fréchet differentiable with a $\frac{1}{\beta}$-Lipschitz continuous gradient, and let $g$ be $\omega$-hypoconvex. Suppose that $\argmin(f + g) \neq \emptyset$. Let $\gamma \in ]0, \min\{1/\mu, 2\beta/(1 + 2\mu\beta)\}[$, and set $\delta = (1 - \gamma\mu)/(1 - \gamma\omega)$. Set

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(i) There exists $\bar{x} \in \text{Fix } T = \text{zer}(A + B) = \argmin(f + g)$ such that $T^n x_0 \rightharpoonup \bar{x}$.

Suppose that $\mu > \omega$. Then, we additionally have:

(ii) $\text{Fix } T = \argmin(f + g) = \{\bar{x}\}$ and $T^n x_0 \rightarrow \bar{x}$ with a linear rate $\delta < 1$.

Proof.

$\nabla f$ is maximally $\mu$-monotone, $\nabla f - \mu \text{Id}$ is $\frac{1}{\beta}$-Lipschitz continuous by an earlier result, hence $\beta$-cocoercive. Apply the previous result.
Questions:

- What are other possible/more general classes of functions/operators which have “nice” prox operators/resolvents?
- What can we say about splitting operators when applied to problems involving these classes, if any?
For references, please check the preprints below and the references therein.


THANK YOU!!