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In the vast majority of cases, superconducting transition takes place at exponentially low temperature $T_c$ out of the Fermi liquid regime: \textit{BCS (effective) theory}.

At the level of (effective) weakly interacting theory, the (emergent) BCS physics is perfectly captured by Feynman diagrammatics.

\textbf{The goal:} To utilize Feynman diagrammatics to bridge the (emergent) long-wave BCS physics with (strongly correlated) ultraviolet microscopics.
Cooper instability via linear response: diagrammatic language

Modify the Hamiltonian:

\[ H \rightarrow H + \left( \eta_{12} \psi_1 \psi_2 + \text{H.c.} \right) \]

(infinitesimal) linear response: \( \langle \psi_1 \psi_2 \rangle \)

Diagrammatically:

\[ \langle \psi_1 \psi_2 \rangle = G_{13} G_{24} \eta_{34} + \]

irreducible (in the Cooper channel) four-pole vertex
Singular response: eigenvector-eigenvalue problem

Response diverges (i.e., the critical temperature is reached), when the following eigenvalue becomes equal to 1.

Corresponding eigenvector defines the pairing channel.
Emergent BCS regime: long-wave effective theory

$\Gamma = \lambda \times \ldots$

In this regime:

Green’s function has a Fermi-liquid form (close to the Fermi surface):

$$G(k, \omega_n) \approx \frac{z(\hat{k})}{i\omega_n - v_F(\hat{k}) \cdot [k - k_F(\hat{k})]}$$

The effective four-pole vertex $\Gamma$ is small and independent of temperature and frequency.

The eigenvector is temperature and frequency independent.

Temperature dependence of $\bar{\lambda}$ is essentially due to the Green’s function factor:

$$\bar{\lambda}(T) = g \ln\left(\frac{\#}{T}\right) \Rightarrow T_c = \# e^{-1/g}, \quad g \ll 1$$
Can we deal with the fully microscopic description?
Eigenvalue-eigenvector problem for the gap function $\Delta$

$$\lambda(T) \Delta_{k,\omega_n} = -T \sum_m \int \frac{dp}{(2\pi)^d} \Gamma_{p,\omega_m}^{k,\omega_n} G_{p,m} G_{-p,-m}^{p,-m} \Delta_{p,\omega_m}$$

A crucial obstacle for the (otherwise straightforward) DiagMC:
In the majority of interesting cases, the diagrammatic series for $\Gamma$
is well beyond the convergence radius at $T \sim T_c$.

How about extrapolating $\lambda(T)$ from $T \gg T_c$?

Works with $\bar{\lambda}(T)$ but not with $\lambda(T)$!

(Simple math—and deep physics—associated with this fact will be discussed later.)
Renormalization of the interaction in the Cooper channel

\[-\Gamma = -\Gamma + \Gamma\rightarrow\Gamma\]

excluding the low-energy part

Not doable because of the “curse of multivariableness.”

However, \(\bar{\lambda}(T)\) can be extracted from a modified eigenvalue-eigenvector problem based on bare \(\Gamma\).
A few preliminary steps

matrix-vector notation:

\[ \lambda \vec{\Delta} = - \hat{A} \vec{\Delta}, \quad \vec{\Delta} \equiv \Delta(p, \omega) \]

Introduce low- and high-energy parts with respect to a certain characteristic energy scale \( \Omega_c \):

\[ \vec{\Delta} = \vec{\Delta}^{(1)} + \vec{\Delta}^{(2)} \]

\[ \Delta^{(1)}(p, \omega) \equiv 0 \quad \text{at} \quad \xi_p^2 + \omega^2 > \Omega_c^2 \quad \text{low-energy part} \]

\[ \Delta^{(2)}(p, \omega) \equiv 0 \quad \text{at} \quad \xi_p^2 + \omega^2 \leq \Omega_c^2 \quad \text{high-energy part} \]

\[ \hat{A} = \hat{A}^{(11)} + \hat{A}^{(22)} + \hat{A}^{(21)} + \hat{A}^{(12)} \]

\[
\begin{align*}
\lambda \vec{\Delta}^{(1)} &= - \hat{A}^{(11)} \vec{\Delta}^{(1)} - \hat{A}^{(12)} \vec{\Delta}^{(2)} \\
\lambda \vec{\Delta}^{(2)} &= - \hat{A}^{(22)} \vec{\Delta}^{(2)} - \hat{A}^{(21)} \vec{\Delta}^{(1)}
\end{align*}
\]

(So far, it is just an identical rewriting.)
Implicit-renormalization formulation

Replace

\[
\begin{align*}
\lambda \bar{\Delta}^{(1)} &= - \hat{A}^{(11)} \bar{\Delta}^{(1)} - \hat{A}^{(12)} \bar{\Delta}^{(2)} \\
\lambda \bar{\Delta}^{(2)} &= - \hat{A}^{(22)} \bar{\Delta}^{(2)} - \hat{A}^{(21)} \bar{\Delta}^{(1)} 
\end{align*}
\]

with:

\[
\begin{align*}
\bar{\lambda} \bar{\Delta}^{(1)} &= - \hat{A}^{(11)} \bar{\Delta}^{(1)} - \hat{A}^{(12)} \bar{\Delta}^{(2)} \\
\bar{\Delta}^{(2)} &= - \hat{A}^{(22)} \bar{\Delta}^{(2)} - \hat{A}^{(21)} \bar{\Delta}^{(1)}
\end{align*}
\]

Let us see that $\bar{\lambda}$ and (new) $\bar{\Delta}^{(1)}$ exactly correspond to the renormalized theory.

Substituting $\bar{\Delta}^{(2)} = - [\hat{I} + \hat{A}^{(22)}]^{-1} \hat{A}^{(21)} \bar{\Delta}^{(1)}$ (implied by the second equation) into the first equation, we get:

\[
\bar{\lambda} \bar{\Delta}^{(1)} = - \hat{B} \bar{\Delta}^{(1)}, \quad \hat{B} = \hat{A}^{(11)} - \hat{A}^{(12)} [\hat{I} + \hat{A}^{(22)}]^{-1} \hat{A}^{(21)}
\]

This is exactly the kernel of the effective theory.
Illustrative examples: Models with momentum-independent vertices

\[ \Delta_{p,\omega_n} \equiv \Delta_{\omega_n} \]

\[ G_{p,\omega_n} = 1 / (i\omega_n - \xi_p) \quad \text{with linear dispersion} \]

Integrating over momentum:

\[ \lambda(T)\Delta_{\omega_n} = -\pi T \sum_m V(\omega_m, \omega_n) \frac{\Delta_{\omega_m}}{|\omega_m|} \]
Rietschel-Sham model

Most simple model capturing all the essential physics of BCS s-wave regime with all-repulsive bare interaction.

\[ V_{RS} = \begin{cases} 0, & \text{if } |\omega_n| > E_c, \text{ or } |\omega_m| > E_c \\ g - gf \theta(|\omega_m||)\theta(|\omega_m|) & \text{otherwise} \end{cases} \]

\[ g > 0, \quad 0 < f < 1 \quad \Rightarrow \quad V_{RS} \geq 0 \]

two-parametric piecewise-constant eigenvector:
\[ \Delta_{\omega_n} = [a \theta(|\omega_n| - \Omega) + s \theta(\Omega - |\omega_n|)]\theta(E_c - |\omega_n|) \]

The problem reduces to an algebraic one:
\[ \begin{cases} \lambda_s = -gL - g(1-f)Ls \\ \lambda a = -gL - gLs \end{cases} \]

\[ L = \ln[\Omega / 0.882T], \quad l = \ln[E_c / \Omega] \]

Non-linear dependence of \( \lambda \) on \( L \)!

\textit{Implicit-renormalization formulation:}

\[ \begin{cases} \bar{\lambda}s = -gL - g(1-f)Ls \\ a = -gL - gLs \end{cases} \]

\[ \bar{\lambda}s = g_{\text{eff}}Ls, \quad g_{\text{eff}} = g \left[ f - \frac{1}{1 + g\ell} \right] \]
Case 1:
Cooper instability is present

Case 2:
Cooper instability is absent

\[ L_T = \ln[\Omega / 0.882 T] \]

Behavior of \( \lambda \) at not very large \( L_T \) proves very misleading!
V_{\text{eff}}(\omega) = g\left(1 - \frac{\Omega_a^2}{\omega^2 + \Omega^2}\right) = g\left(\frac{\omega^2 + \Omega_1^2}{\omega^2 + \Omega^2}\right), \quad \omega = \omega_m - \omega_n, \quad \Omega_a \leq \Omega, \quad \Omega_1^2 = \Omega^2 - \Omega_a^2 < \Omega^2
Concluding remarks

• A protocol for extrapolating numerical data towards $T_c$ from higher temperatures—applicable to first-principle description of real metals, as well as strongly interacting models—has to adequately capture the physics of the emergent weakly-interacting effective theory.

• Implicit renormalization protocol provides a simple, efficient, and unbiased way of solving the extrapolation problem. The scheme has a built-in tool of controlling the systematic error of extrapolation—the only systematics of the otherwise numerically exact method.

• The implicit renormalization approach is perfectly compatible with the DiagMC. One can solve the corresponding eigenvalue problem without invoking the matrix inversion or even explicitly calculating the four-point vertex function $\Gamma$.

• The implicit renormalization protocol also allows one to obtain the correct gap function immediately below $T_c$. 