The unitary Fermi gas

Félix Werner

Laboratoire Kastler Brossel
Ecole Normale Supérieure

- K. Van Houcke, FW, T. Ohgoe, N. Prokof’ev, B. Svistunov, *Diagrammatic Monte Carlo algorithm for the resonant Fermi gas*, PRB 2019
- R. Rossi, T. Ohgoe, K. Van Houcke, FW, *Resummation of diagrammatic series with zero convergence radius for strongly correlated fermions*, PRL 2018
- R. Rossi, T. Ohgoe, E. Kozik, N. Prokof’ev, B. Svistunov, K. Van Houcke, FW, *Contact and Momentum Distribution of the Unitary Fermi Gas*, PRL 2018
- T. Ohgoe, R. Rossi, K. Van Houcke, FW, *work in progress*

NOTE: this is an edited version of the slides
Unpublished work was removed
Spin-$\frac{1}{2}$ fermions, 3D continuous space, interactions 
\[
\begin{cases} 
\text{zero range} \\
\text{scattering length } a = \infty
\end{cases}
\]

**Universality hypothesis:**

- Zero-range limit: 
\[
\begin{align*}
\lambda & \gg b \\
\lambda & \equiv \sqrt{\frac{2\pi\hbar^2}{mk_B T}}
\end{align*}
\]

- Properties do not depend on $V(r)$

- $(N_\uparrow = N_\downarrow)$
\[
n(T, \mu) \lambda^3 = \text{universal function of } \beta \mu
\]

**Construction from Hubbard model:**

- $\frac{U}{t} = -7.913552 \ldots$ (appearance of 2-body bound state)
- thermodynamic limit
- filling $\to 0$ with $\frac{T}{T_F}$ fixed (continuum limit)
strongly correlated regime

BCS

-1 0 1

\[ \frac{1}{k_F a} \]

unitary gas \[ a = \infty \]

cold atom experiments
fermionic atom, 2 internal states, Feshbach resonance

accurate comparison

zero-range theory

also relevant in neutron stars
\[ G = G^0 + \left( \Gamma^0 \right) + \ldots \]

Ladder summation:
\[ \Gamma^0 = \cdot + \left( \right) + \left( \right) + \ldots \]

⇒ \( \Gamma^0 \) is well-defined in the continuum limit, which can be taken analytically.

Dyson equation:
\[ G = G^0 + G^0 \Sigma G^0 + \left( \Sigma \right) + \ldots \]

Self-energy:
\[ \Sigma = \left( \right) + \left( \right) + \left( \right) + \ldots \]
sum all diagrams up to order \( \sim 9 \) by Diag MC

\[ G = G^0 + \Gamma^0 + \ldots \]

Ladder summation:

\[ \Gamma^0 = \sum \Gamma^0 \]

\( \Rightarrow \) \( \Gamma^0 \) is well-defined in the continuum limit, which can be taken analytically.

Dyson equation:

\[ G = G^0 + G^0 \Sigma G^0 + G^0 \Sigma G^0 \Sigma G^0 + \ldots \]

Self-energy:

\[ \Sigma = \Sigma + \Sigma + \Sigma + \ldots \]

\( \text{("ladder scheme")} \)
avoid double-counting:

[Diagram showing two arrows pointing in opposite directions with a forbidden sign]
Problem: zero convergence radius
Problem: zero convergence radius
Problem: zero convergence radius

Solution:

\[
Q \overset{?}{=} \sum_{n=0}^{\infty} a_n
\]

\[
a_1 = \text{Gaussian pair fluctuation}
\]

\[
a_9 = + \ldots
\]

construct \( Q(z) \) / \[
\begin{align*}
\text{Taylor } [Q(z)] &\overset{z \to 0+}{=} \sum_{n=0}^{\infty} a_n z^n \\
Q(z = 1) &= Q_{\text{phys}}
\end{align*}
\]

\[
\{ a_n \} \quad \text{resummation} \quad Q(1)
\]
Problem: zero convergence radius

Solution:

\[ Q = \sum_{n=0}^{\infty} a_n \]

\[ a_1 = \]

\[ a_9 = + \ldots \]

\( Q(z) \) constructed

\[ Q(1) \]

\[ Z = \sum_{n=0}^{\infty} a_n z^n \]

- Instanton method \( \Rightarrow \) large-order behavior, branch cuts of \( Q(z) \)
- Nevanlinna theorem
Equation of state

$$\mu = 0 \quad \left( \frac{T}{T_F} \approx 0.6 \right)$$

precision $< 0.1\%$
Measure all particle positions, in a unit volume. Number of pairs of separation \( \sim \epsilon \) as \( \epsilon \to 0 \),

\[
C \sim \epsilon \frac{1}{4\pi}
\]

\[
\langle \hat{n}_\uparrow(\mathbf{r}) \hat{n}_\downarrow(\mathbf{0}) \rangle \sim \frac{C}{(4\pi r)^2}
\]

\[
n_\sigma(\mathbf{k}) \sim \frac{C}{k^4}
\]

\[
C = \frac{4\pi m}{\hbar^2} \left. \frac{\partial p}{\partial (1/a)} \right|_{T,\mu}
\]

\[
C = -\Gamma(\mathbf{r} = \mathbf{0}, \tau = 0^-)
\]

[S. Tan, Ann. Phys. 2008]
\[ C / k_F^4 \]

\[ T_c \]

\[ T / T_F \]

[2] Mukherjee, Patel, Yan, Fletcher, Struck, Zwierlein, PRL 2019
Critical behavior

\[ C / k_F^4 \]

\[ T_c \]

\[ \text{C}_{\text{singular}} (T) \sim \pm A_{\pm} |T - T_c|^{1-\alpha} \quad (T \to T_c^\pm) \]

\[ \alpha \simeq -0.015 \]

\[ \frac{A_+}{A_-} \simeq 1.06 \]

\[ \Rightarrow \text{nearly } \propto (T - T_c) \]

unlikely to see anything
Momentum distribution

\[ n_{\sigma}(k) \]

![Graph showing momentum distribution for different \( T / T_F \) values with \( \beta \mu \) labels.]

- \( \beta \mu = 2.25 \) [\( T / T_F = 0.19 \)]
- \( \beta \mu = 1.5 \) [\( T / T_F = 0.26 \)]
- \( \beta \mu = 1 \) [\( T / T_F = 0.34 \)]
- \( \beta \mu = 0 \) [\( T / T_F = 0.64 \)]

\[ \left( - \frac{dn_{\sigma}(k)}{dk} \right) \bigg|_{k=k_F} \]

- **Unitary Fermi gas**

\[ \left( - \frac{dn_{\sigma}(k)}{dk} \right) \bigg|_{k=k_F} \]

- **Ideal Fermi gas**

\[ T / T_F \]

- **Non Fermi liquid behavior**
Two competing effects:

at higher $T$, maximum is higher

plateau sets in later $k \gg k_F$ and $1/\lambda$
Technical aspect: treatment of singular parts

**Problem:**

\[
\Sigma_\sigma(k, \tau) \sim -4 \sqrt{\frac{\pi}{\tau}} n_{-\sigma}
\]

\[
\Sigma_\sigma(k, \tau) \sim -4 \sqrt{\frac{\pi}{\tau}} n_{-\sigma} e^{-(k^2/4)\tau}, \quad \tau \to 0^+
\]

\[
- e^{-(k^2/2)(\beta-\tau)}, \quad \tau \to \beta^-
\]

comes entirely from \( \Sigma_{bold}^{(1)} = \)

**Solution:**

\[
\Sigma = \left\{ \begin{array}{l}
\Sigma_{bold}^{(1)} \\
\text{analytically}
\end{array} \right\} + \left\{ \begin{array}{l}
\text{other diagrams} \\
\text{MC}
\end{array} \right\}
\]

\[
G^{(N)} = \sum_{M=1}^{N} G^{(0)} \Sigma^{(M)} G^{(N-M)}
\]

\[
\Sigma^{(N)}_{1,bold;\sigma}(r, \tau) = \sum_{M=1}^{N} \Gamma^{(M-1)}(r, \tau) G^{(N-M)}_{-\sigma}(r, -\tau)
\]

\[
\Gamma^{(N)} = \sum_{M=1}^{N} \Gamma^{(0)}(M) \Gamma^{(N-M)}, \quad \text{compute recursively over } N
\]

works also for ladder scheme
Polarized gas
algorithm
OUTLOOK

- Polarized gas:
  compare with Fermi-liquid theory
  FFLO correlations
  $T_c$

- other observables (density-density, spin susceptibility...)

- Analytic continuation $\rightarrow A(k, \omega)$ (radio-frequency)
  $S(k, \omega)$ (Bragg)

- Superfluid phases