Chapter 7

1) (AB/BC, calculator)

The base of a solid is the region in the first quadrant bounded above by the line $y = 2$, below by $y = \sin^{-1} x$, and to the right by the line $x = 1$. For this solid, each cross-section perpendicular to the $x$-axis is a square. What is the volume of the solid?

(a) 1.429

(b) 2

(c) 2.184

(d) 0.766

(e) 4
2. (AB/BC, non-calculator)

Find the area of the region bounded by the y-axis, the line $y = e$, and the graph of the function $y = e^{3x}$.

(a) $\frac{1}{3}$

(b) $e^{3e} - \frac{1}{3}$

(c) $1 - \frac{2}{3}e$

(d) $3 - \frac{8}{3}e$

(e) $\frac{1}{3} + e^2 - e^{3e}$
3. (AB/BC, non-calculator)

A region in the xy-plane is bounded by the curves \( y = 4x - x^2 \) and \( y = 2x - 3 \).

(a) Find the points of intersection of the two curves.

(b) Sketch the region bounded by the curves. Label the bounding curves, show and label all points of intersection, and shade the bounded region.

(c) Find the area of the region, showing all work that leads to your answer.
4. (AB/BC, non-calculator)

A region in the $xy$-plane is bounded by $y = 2x + 2$, $x = \frac{y^2}{2} + 2$, $y = -2$, and $y = 2$.

(a) Sketch the bounded region on a Cartesian axis system. Label each boundary curve and shade the bounded region.

(b) Find the area of the bounded region, showing all work that leads to your answer.
5. (AB/BC, non-calculator)

Consider the region shown above, which is bounded by $f(x) = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 2$.

(a) Find the area of the region.

(b) Find the volume of the solid formed by rotating the region about the x-axis.

(c) The region pictured above is the base of a solid. For this solid, each cross-section perpendicular to the x-axis is an equilateral triangle. Find the volume of this solid.
6. (AB/BC, non-calculator)

Consider the region $R$, bounded by the graphs of $y = x^3$, $y = 8$ and the $y$-axis. The region $S$ is bounded by $y = x^3$, $x = 2$ and the $x$-axis.

(a) Find the area of region $R$.

(b) Find the volume of the solid formed by rotating region $R$ about the $y$-axis.

(c) The region $S$ is the base of a solid. For this solid, each cross-section perpendicular to the $x$-axis is a semi-circle with diameters extending from $y = x^3$ to the $x$-axis. Find the volume of this solid.
7. (BC only, calculator)

Consider the region bounded by the $y$-axis, $y = 10$, and $y = 1 + 6x^{3/2}$.

(a) Set up but do not evaluate an integral equation that will find the value of $k$ so that $x = k$ cuts the region into 2 parts of equal area.

(b) Find the length of the curve $y = 1 + 6x^{3/2}$ on the interval $[0, 1]$.

(c) The region is the base of a solid. For this solid, the cross-sections perpendicular to the $x$-axis are rectangles with a height of 3 times that of its width. Find the volume of this solid.
8. (AB/BC, calculator)

Let $R$ be the region bounded by the graph of $y = \ln x$ and the line $y = 2x - 3$.

(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is rotated about the horizontal line $y = -3$.

(c) Write, but do not evaluate, an expression involving one or more integrals that can be used to find the volume of the solid generated when $R$ is revolved about the $y$-axis.
9. (AB/BC, calculator)

Let \( R \) be the region bounded by the graphs of \( y = x^2 - 1 \) and the graph of \( x = y^2 \).

(a) Find the area of \( R \).

(b) Find the volume of the solid generated when \( R \) is rotated about the vertical line \( x = 2 \).

(c) Write, but do not evaluate, an expression involving one or more integrals to find the volume of the solid generated when \( R \) is rotated about the horizontal line \( y = -1 \).
10. (AB/BC, non-calculator)

Consider the region bounded by the graphs of \( f(x) = \sqrt{x} \), \( y = 0 \), and \( x = 2 \).

(a) Find the volume of the solid formed by rotating the region about the \( x \)-axis.

(b) Find the volume of the solid formed by rotating the region about the \( y \)-axis.

(c) Write, but do not evaluate, the volume integral of the solid formed by rotating the region about the line \( y = -2 \).
Chapter 7 (Solutions)

Questions 1-2

1. c \[ \int_{0}^{1} \left( 2 - \sin^{-1} x \right)^2 dx \approx 2.184 \]

2. a \[ \int_{0}^{1/3} \left( e - e^{3x} \right) dx = \left[ ex - \frac{1}{3} e^{3x} \right]_{0}^{1/3} = \left( \frac{e}{3} - \frac{e}{3} \right) - \left( 0 - \frac{1}{3} \right) = \frac{1}{3} \]
**Question 3**

A region in the $xy$-plane is bounded by the curves $y = 4x - x^2$ and $y = 2x - 3$.

(a) Find the points of intersection of the two curves.

(b) Sketch the region bounded by the curves. Label the bounding curves, show and label all points of intersection, and shade the bounded region.

(c) Find the area of the region, showing all work that leads to your answer.

| (a) $4x - x^2 = 2x - 3 \Rightarrow x = -1, x = 3$ |
| $y(-1) = 2(-1) - 3 = -5$ |
| $y(3) = 2(3) - 3 = 3$ |
| The points of intersection are $(-1, -5)$ and $(3, 3)$. |

2: points of intersection

4: graph showing shading and points of intersection

| (c) $\int_{-1}^{3} \left(4x - x^2 - (2x - 3)\right) dx = \frac{32}{3}$ |

3: $\int$ integrand

1: limits

1: answer
Question 4

A region in the $xy$-plane is bounded by $y = 2x + 2$, $x = \frac{y^2}{2} + 2$, $y = -2$, and $y = 2$.

(a) Sketch the bounded region on a Cartesian axis system. Label each boundary curve and shade the bounded region.

(b) Find the area of the bounded region, showing all work that leads to your answer.

(a)

(b) $\int_{-2}^{2} \left( \left( \frac{y^2}{2} + 2 \right) - \left( \frac{y-2}{2} \right) \right) dy = \frac{44}{3}$
Consider the region shown above, which is bounded by $f(x) = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 2$.

(a) Find the area of the region.

(b) Find the volume of the solid formed by rotating the region about the x-axis.

(c) The region pictured above is the base of a solid. For this solid, each cross-section perpendicular to the x-axis is an equilateral triangle. Find the volume of this solid.

<table>
<thead>
<tr>
<th>Global limit point</th>
<th>1: correct limits in an integral in (a), (b), or (c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Area $= \int_0^2 \sqrt{x} , dx = \frac{2}{3} \left[ x^\frac{3}{2} \right]_0^2 = \frac{4\sqrt{2}}{3}$</td>
<td>1: integrand</td>
</tr>
<tr>
<td>(b) Volume $= \pi \int_0^2 (\sqrt{x})^2 , dx = \pi \left[ \frac{x^2}{2} \right]_0^2 = 2\pi$</td>
<td>3: 1: antiderivative</td>
</tr>
<tr>
<td>(c) Volume $= \frac{\sqrt{3}}{4} \int_0^2 x , dx = \frac{\sqrt{3}}{4} \left[ \frac{x^2}{2} \right]_0^2 = \frac{\sqrt{3}}{2}$</td>
<td>1: answer</td>
</tr>
</tbody>
</table>
Consider the region \( R \), bounded by the graphs of \( y = x^3 \), \( y = 8 \) and the \( y \)-axis. The region \( S \) is bounded by \( y = x^3 \), \( x = 2 \) and the \( x \)-axis.

(a) Find the area of region \( R \).

(b) Find the volume of the solid formed by rotating region \( R \) about the \( y \)-axis.

(c) The region \( S \) is the base of a solid. For this solid, each cross-section perpendicular to the \( x \)-axis is a semi-circle with diameters extending from \( y = x^3 \) to the \( x \)-axis. Find the volume of this solid.

(a) \[
\int_{0}^{2} (8 - x^3) \, dx = 8x - \frac{x^4}{4} \bigg|_{0}^{2} = 16 - 4 = 12
\]

(b) Volume = \[
\pi \int_{0}^{8} \left( \sqrt{y} \right)^2 \, dy = \pi \left[ \frac{3}{5} y^{\frac{5}{3}} \right]_{0}^{8} = \frac{96}{5} \pi
\]

(c) Volume = \[
\frac{\pi}{2} \int_{0}^{2} \left( \frac{x^3}{2} \right)^2 \, dx = \frac{\pi}{8} \int_{0}^{2} x^6 \, dx = \frac{\pi}{8} \left[ \frac{x^7}{7} \right]_{0}^{2} = \frac{16}{7} \pi
\]
Consider the region bounded by the $y$-axis, $y = 10$, and $y = 1 + 6x^{3/2}$.

(a) Set up but do not evaluate an integral equation that will find the value of $k$ so that $x = k$ cuts the region into 2 parts of equal area.

(b) Find the length of the curve $y = 1 + 6x^{3/2}$ on the interval $[0, 1]$.

(c) The region is the base of a solid. For this solid, the cross-sections perpendicular to the $x$-axis are rectangles with a height of 3 times that of its width. Find the volume of this solid.

(a) Let $a$ represent the $x$-coordinate of point of intersection: $a \approx 1.31037$

Area $= \int_0^a \left( 10 - (1 + 6x^{3/2}) \right) dx \approx 7.076$

Equation: $\int_0^a \left( 10 - (1 + 6x^{3/2}) \right) dx \approx \frac{7.076}{2}$

(b) $\text{length} = \int_0^1 \sqrt{1 + \left( 9x^{1/2} \right)^2} dx = \int_0^1 \sqrt{1 + 81x} dx \approx 6.103$

(c) $\text{Volume} = 3 \int_0^a \left( 10 - (1 + 6x^{3/2}) \right)^2 dx \approx 143.3$
Question 8

Let \( R \) be the region bounded by the graph of \( y = \ln x \) and the line \( y = 2x - 3 \).

(a) Find the area of \( R \).

(b) Find the volume of the solid generated when \( R \) is rotated about the horizontal line \( y = -3 \).

(c) Write, but do not evaluate, an expression involving one or more integrals that can be used to find the volume of the solid generated when \( R \) is revolved about the \( y \)-axis.

\[
\ln x = 2x - 3 \Rightarrow A = 0.05565, \quad B = 1.79154
\]

(a) \[
\int_a^b \left( \ln x - (2x - 3) \right) dx \approx 1.471
\]

(b) Volume = \[
\pi \int_a^b \left( (\ln x + 3)^2 - (2x)^2 \right) dx = 18.783
\]

(c) Volume = \[
\pi \int_{2A-3}^{2B-3} \left( \left( \frac{y + 3}{2} \right)^2 - \left( e^y \right)^2 \right) dy
\]
Let $R$ be the region bounded by the graphs of $y = x^2 - 1$ and the graph of $x = y^2$.

(a) Find the area of $R$.

(b) Find the volume of the solid generated when $R$ is rotated about the vertical line $x = 2$.

(c) Write, but do not evaluate, an expression involving one or more integrals to find the volume of the solid generated when $R$ is rotated about the horizontal line $y = -1$.

$(A, B)$ and $(S, T)$, where $A \approx 0.5249$, $B \approx -0.7245$, $S \approx 1.4902$, $T \approx 1.2207$ are the points of intersection.

(a) $\int_{B}^{T} (\sqrt{y+1} - y^2) \, dy = 1.377$

(b) $V = \pi \int_{B}^{T} \left( (2 - y^2)^2 - (2 - \sqrt{y+1})^2 \right) \, dy = 11.501$

(c) The volume equals

$\pi \int_{0}^{A} \left( (\sqrt{x} + 1)^2 - (-\sqrt{x} + 1)^2 \right) \, dx + \pi \int_{A}^{S} \left( (\sqrt{x} + 1)^2 - (x^2)^2 \right) \, dx$
Question 10

Consider the region bounded by the graphs of \( f(x) = \sqrt{x} \), \( y = 0 \), and \( x = 2 \).

(a) Find the volume of the solid formed by rotating the region about the \( x \)-axis.

(b) Find the volume of the solid formed by rotating the region about the \( y \)-axis.

(c) Write, but do not evaluate, the volume integral of the solid formed by rotating the region about the line \( y = -2 \).

(a)
\[
\pi \int_0^2 (\sqrt{x})^2 \, dx = \pi \int_0^2 x^2 \, dx = \frac{2\pi}{3} \approx 2.094
\]

(b)
\[
\pi \int_0^4 (4 - y^4) \, dy = \pi \int_0^4 4 - y^4 \, dy = \left[ \frac{4y}{1} - \frac{y^5}{5} \right]_0^4 = \frac{16\sqrt{2}}{5} \pi
\]

(c) Volume = \( \pi \int_0^2 \left[ \left( \sqrt{x} + 2 \right)^2 - 4 \right] \, dx \)