COUNCIL ROCK HIGH SCHOOL

MATHEMATICS

A Note Guideline of Algebraic Concepts

Designed to assist students in

A Summer Review of Algebra

[A teacher prepared compilation of the 37 Algebraic concepts deemed most important for mastery by the student entering Accelerated/Honors Analysis revised 2014]

Solving Equations in One Variable

A **linear equation** in one variable has a single unknown quantity called a **variable** represented by a letter. Eg: 'x', where 'x' is always to the power of 1. This means there is no ' x^2 ' or ' x^3 ' in the equation. To solve linear equations, you add, subtract, multiply and divide both sides of the equation by numbers and variables, so that you end up with a single variable on one side and a single number on the other side.

EX. Solve for x.

$2x+1=17$ Subtract 1

$2x=16$ Divide by 2

$x=8$

$5-\frac{1}{8}x=-1$ Subtract 5

$-\frac{1}{8}x=-6$ Multiply by -8

$x=48$

$\frac{6x-2(x-4)}{3}=8$ Multiply by 3

$6x-2(x-4)=24$ Distribute -2

$6x-2x+8=24$ Combine like terms

$4x+8=24$ Subtract 8

$4x=16$ Divide by 4

$x=4$

Solving Inequalities in One Variable

A. Inequality Symbols

> means "is greater than"

< means "is less than"

≥ means "is greater than or equal to"

≤ means "is less than or equal to"

B. Steps for Solving Inequalities with One Variable

Perform the distributive property on each side.

Combine like terms on each side.

Add or subtract to get the variable terms on the same side.

Add or subtract to move the number terms to the other side.

Multiply or divide to move the coefficient.

Graph the solution

> **Remember:** If you multiply or divide both sides of an inequality by a **negative number**, the **inequality symbol changes direction.**

C. <u>Steps for Graphing the Solutions to Inequalities with One Variable</u>

** Make sure the variable is on the LEFT in all solutions.

A solution with >:

Graph an open circle on the number. (The number is not part of the solution.)

A dark bar with an arrow goes to the right of the circle.

A solution with <:

Graph an open circle on the number. (The number is not part of the solution.)

A dark bar with an arrow goes to the left of the circle.

A solution with ≥:

Graph a solid circle on the number. (The number is part of the solution.)

A dark bar with an arrow goes to the right of the circle.

A solution with ≤:

Graph a solid circle on the number. (The number is part of the solution.)

A dark bar with an arrow goes to the left of the circle.

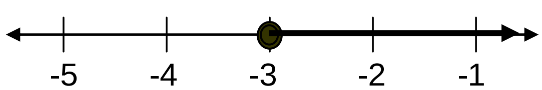
EX. Solve and Graph

$-13 - 7n \leq 8$ add 13

$-7n \leq 21$ divide by -7

$\frac{-7n}{-7} \leq \frac{21}{-7}$

$n \geq -3$ change inequality sign



$-x + 4 > 2(x - 8)$ distribute

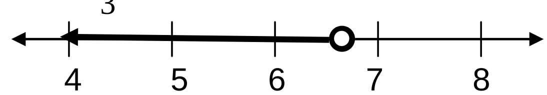
$-x + 4 > 2x - 16$ subtract 2x

$-3x + 4 > -16$ subtract 4

$-3x > -20$

$\frac{-3n}{-3} > \frac{-20}{-3}$ divide by -3

$n < \frac{20}{3}$ change inequality sign



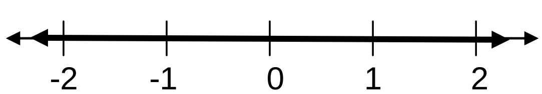
$2(3x - 2) < 6x + 8$ distribute

$6x - 4 < 6x + 8$ substract 6x

$-4 < 8$ variables drop out

True statement

∴All Real Numbers



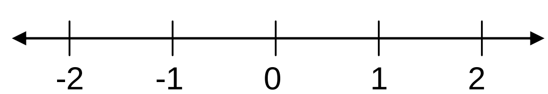
$2(x - 1) \geq 2x + 7$ distribute

$2x - 2 \geq 2x + 7$ subtract 2x

$-2 \geq 7$ variables drop out

False statement

∴No Solution



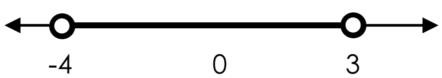
Solving Combined Inequalities

Solving combined inequalities is just like solving normal inequalities, but with two separate inequalities combined into one. The main concept to know is the difference between the words "AND" and "OR," which control or limit the answers. "AND" means the solution will be an intersection where both inequalities hold true. "OR" means the solution will be a union where one OR the other inequality holds true.

A. Combined Inequalities with "AND"

EX. Solve and Graph $-6 < 2x + 2 < 8$

$-6 < 2x + 2$ **AND** $2x + 2 < 8$	Separate the combined inequality into two inequalities
$-8 < 2x$ **AND** $2x < 6$	Solve each inequality for the variable.
$-4 < x$ **AND** $x < 3$	
$-4 < x < 3$	Write the answer in combined inequality form.



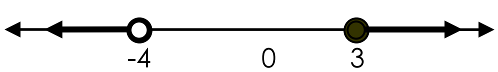
Graph the answers on a number line

Use open dots at -4 and 3 and shade between them since all solutions lie BETWEEN -4 and 3

B. Combined Inequalities with "OR"

EX. Solve and Graph $-6 > 2x + 2$ or $2x + 2 > 8$

$-6 > 2x + 2$ **OR** $2x + 2 > 8$	Combined inequality with "OR" are already separated.
$-8 > 2x$ **OR** $2x > 6$	Solve each inequality for the variable.
$-4 > x$ **OR** $x > 3$	
$x < -4$ **OR** $x > 3$	Write the answer in combined inequality form.



Graph the answers on a number line

Use an open dot at -4 and a closed dot at 3 and shade beyond them with arrow pointing outward to infinity. There are no solutions between -4 and 3.

Solving Absolute Value Equations

Absolute Value

- The **absolute value** of a number x, written $|x|$, is the number's distance from zero on a number line.
- The absolute value equation $|ax + b|$ **=** c, where c is positive, is equivalent to the compound statement ax + b = **c** **or** ax + b = **– c** .

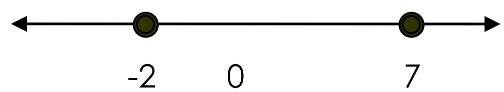
Ex: Solve and Graph $|2x - 5| = 9$.

Step 1: Set up 2 equations using "OR". $2x - 5 = \mathbf{9}$ or $2x - 5 = \mathbf{-9}$

Step 2: Solve each equation. $2x = 14$ $\qquad$ $2x = -4$

$x = 7$ or $x = -2$

Step 3: Graph the solutions.



Solving Absolute Value Inequalities

- The inequality $|ax + b|$ **<** c, where c is positive, means that ax + b is **between –c and c**. This is a compound "**and**" inequality.
  It is equivalent to $\mathbf{-c < ax + b < c}$.

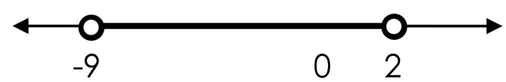
  Ex: Solve and Graph $|2x + 7| < 11$.

  Step 1: Set up a compound inequality. $-11 < 2x + 7 < 11$

  Step 2: Solve. $-18 < 2x < 4$

  $-9 < x < 2$

  Step 3: Graph the solutions.



- The inequality $|ax + b|$ **>** c, where c is positive, means that ax + b is **beyond –c and c**. This is an "**or**" compound inequality.
  It is equivalent to $\mathbf{ax + b < -c}$ **or** $\mathbf{ax + b > c}$.

  Alert: Make sure to <u>isolate the absolute value on one side</u> of the equation or inequality <u>before</u> setting up the 2 equations/inequalities.

  EX: Solve and Graph $4 + 2|3x - 5| \geq 10$

  Step 1. Isolate the absolute value. $4 + 2|3x - 5| \geq 10$

  by subtracting 4, then dividing by 2. $2|3x - 5| \geq 6$

  $|3x - 5| \geq 3$

  Step 2: Set up the compound inequality: $3x - 5 \geq 3$ **or** $3x - 5 \geq -3$

  Step 3: Solve $3x \geq 8$ **or** $3x \geq 2$

  $x \geq 8/3$ **or** $x \geq 2/3$

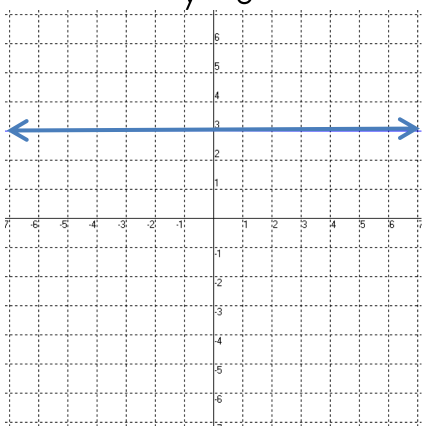
  Step 4: Graph the solutions.

The Slope of a Line

Slope of a line, *m* = $\frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

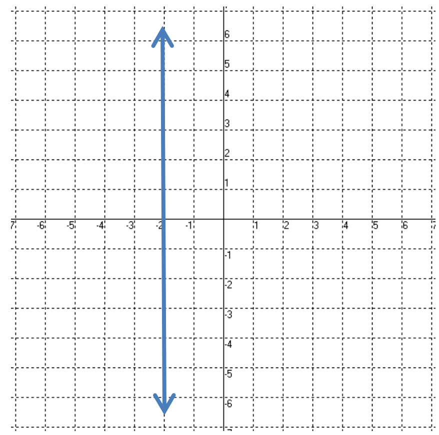
Horizontal lines have 0 slope.

y = 3



Vertical Lines have no slope.

x = -2



Graphs of Linear Equations in Two Variables

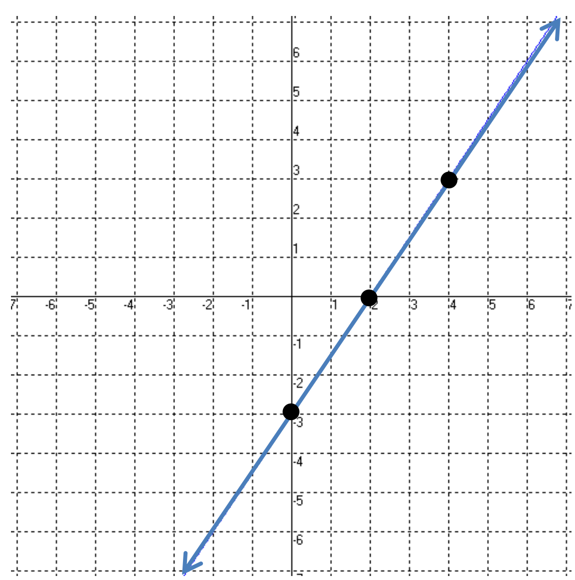
Slope Intercept Form: **y = mx + b**

(where *m* represents the slope and *b* represents the y intercept)

To graph a line:

1.) Write the equation in slope intercept form
2.) Plot the y-intercept
3.) Use the slope to plot at least 2 more points
4.) Draw a line through the points

EX: Graph 3x – 2y = 6



Solution

1.) Solve for y.

3x – 2y = 6 Subtract 3x from both sides

-2y = -3x + 6 Divide by each term by -2

$$y = \frac{3}{2}x - 3$$

2.) Plot (0, -3)

3.) Plot at least 2 more points using the slope

(Up 3, Right 2)

Finding the Equation of a Line

Point-Slope Form: **$y - y_1 = m(x - x_1)$**

To write an equation of a line:

1.) Substitute the point in for $\boldsymbol{x_1}$ and $\boldsymbol{y_1}$ and the slope in for ***m***.

2.) Solve for y.

EX: Write the equation of the line that passes through (3, -4) and has a slope of $\frac{1}{3}$.

1.) $y - y_1 = m(x - x_1)$ Substitute (**3, -4**) in for (**x_1, y_1**) and $\frac{1}{3}$ in for ***m***

2.) $y - (-4) = \frac{1}{3}(x - 3)$ Distribute

3.) $y + 4 = \frac{1}{3}x - 1$ Subtract 4 from both sides

4.) $y = \frac{1}{3}x - 5$

Standard Form: **Ax + By = C**

- *A*, *B* and *C* must be integers (no fractions)
- *A* must be positive

EX: Rewrite the answer in the previous example in standard form.

1.) $y = \frac{1}{3}x - 5$ Multiple each term by 3

2.) 3y = x – 15 Subtract x from both sides

3.) -x + 3y = -15 Divide each term by -1

4.) x – 3y = 15

<u>Parallel lines</u> have the same slope.

Ex: $y = 2x + 5$ & $y = 2x - 4$ are parallel lines

<u>Perpendicular lines</u> have slopes that are opposite reciprocals.

Ex: $y = 2x + 5$ & $y = -\frac{1}{2}x + 1$ are perpendicular lines

EX: Write the equation in standard form that is parallel to $y = -2x - 6$ and passes through (-3, 5)

1.) $y - y_1 = m(x - x_1)$ Substitute (-3, 5) in for (x_1, y_1) and -2 in for m (parallel lines have the same slope)

2.) $y - 5 = -2(x - (-3))$ Simplify

3.) $y - 5 = -2(x + 3)$ Distribute

4.) $y - 5 = -2x - 6$ Add 2x

5.) $2x + y - 5 = -6$ Add 5

6.) $2x + y = -1$

Algebraic Solving of Systems of Linear Equations in Two Variables

To solve a system of equations algebraically, use substitution or elimination methods.
EX: Solve the system using elimination.

$8x+3y=5$ Multiply first equation by -4 $\rightarrow$ $-32x+-12y=-20$
$5x+4y=18$ and second equation by 3 $\rightarrow$ $15x+12y=54$

Add to eliminate $-17x = 34$
Solve for x $x = -2$
Substitute into original equation to find y $8(-2)+3y=5$
Solve for y $-16+3y=5$
$3y=21$
$y=7$
Solution is an ordered pair Answer (-2, 7)

EX: Solve the system using substitution.
3x + 2y = 12
– 7 = – x – 3y

Solve the bottom equation for x: **x = -3y + 7**

Substitute expression into top equation for x: 3**(-3y + 7)** + 2y = 12
Solve for y. -9y + 21 + 2y = 12
-7y = -9
Sometimes the values aren't pretty y = 9/7

Substitute the value for y into **x = -3y + 7** to find x. x = -3 **(9/7)** + 7
Get common denominators x = -27/7 + 49/7
Solution is an ordered pair x = 22/7
Answer (22/7, 9/7)

Relations and Functions

Relation: Set of ordered pairs
Domain (D) of the function is the set of x values.
Range (R) of the function is the set of y values.
Function: A relation where there is correspondence between the two sets, ***D*** and ***R***, that assigns to each member of ***D*** exactly one element of ***R***.
EX: Given $f(x)=x^2-3$ and D = $\{-2,\ 0,\ 2\}$ find the range. Is the relation a function?

$f(-2)=(-2)^2-3=4-3=1$

$f(0)=(0)^2-3=0-3=-3$

$f(2)=(2)^2-3=4-3=1$

Range: ***R*** = $\{1,-3\}$
Relation is $\{(-2,1),\ (0,-3)\ (2,1)\}$
This is a function since each value of x is paired with exactly one value of y.

Graphs of Linear Inequalities in Two Variables

Linear Inequalities may have infinite solutions. To show these infinite solutions, give a graph as your solution. To graph a linear inequality:

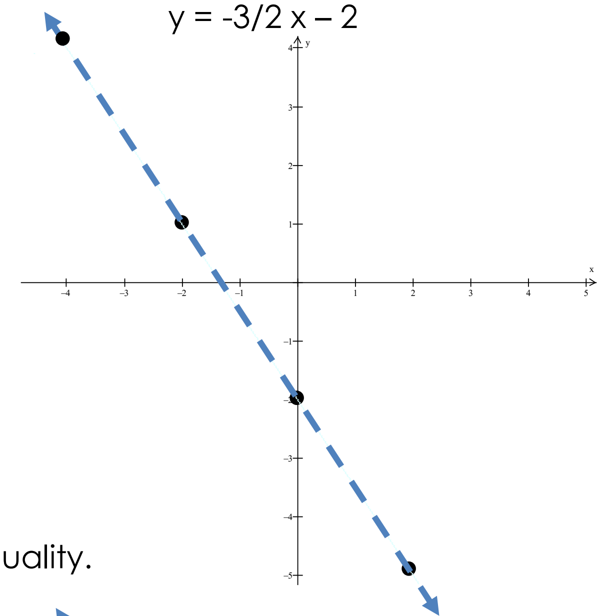
- **Replace** the inequality sign with **=** and solve for y
- The graph of the **EQUALITY** will serve as your boundary line for your inequality. Decide if this boundary should be DASHED or SOLID:
  $>$ *or* $<$ => DASHED $\geq$ *or* $\leq$ => SOLID
- Choose a TEST POINT…(0,0) is a great choice…unless of course it's on the boundary line…you may never choose a test point on the boundary line.
- SUBSTITUTE your test point into the original INEQUALITY.
  Simplify both sides and evaluate the statement:
  TRUE = SHADE the portion of the graph that **DOES include** the test point
  FALSE = SHADE the portion of the graph that **does NOT include** the test point

EX. Graph the solutions of 3x + 2y + 4 > 0

Replace the inequality sign with **=** and solve for y

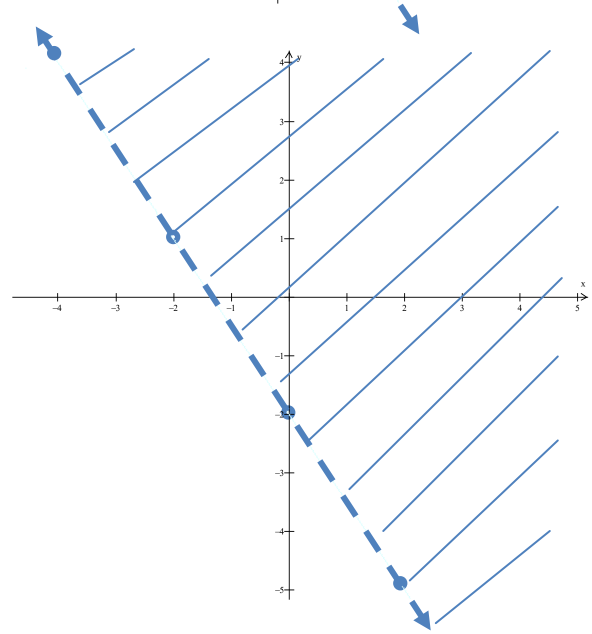
$3x + 2y + 4 = 0$
$2y = -3x - 4$
$y = -3/2\,x - 2$

Graph the boundary line using dashed line (>)



Choose a test point not on the boundary line.
Let's use (0, 0). Substitute into the original inequality.

$3x + 2y + 4 > 0$
$3(0) + 2(0) + 4 > 0$
$4 > 0$
True, so shade to include (0, 0).



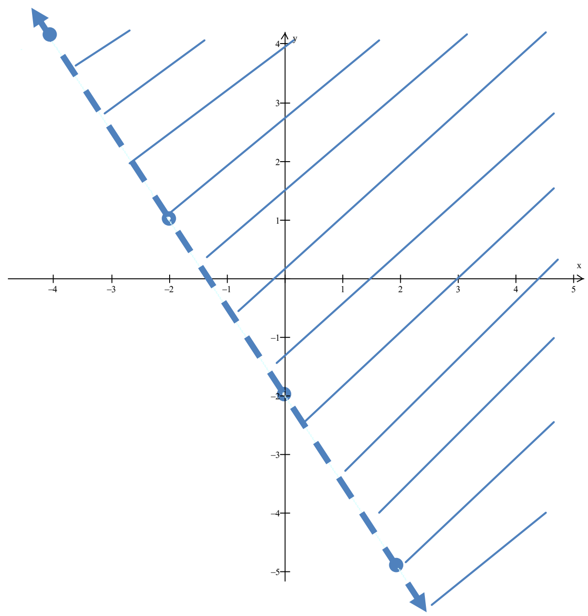
Graphs of Systems of Linear Inequalities in Two Variables

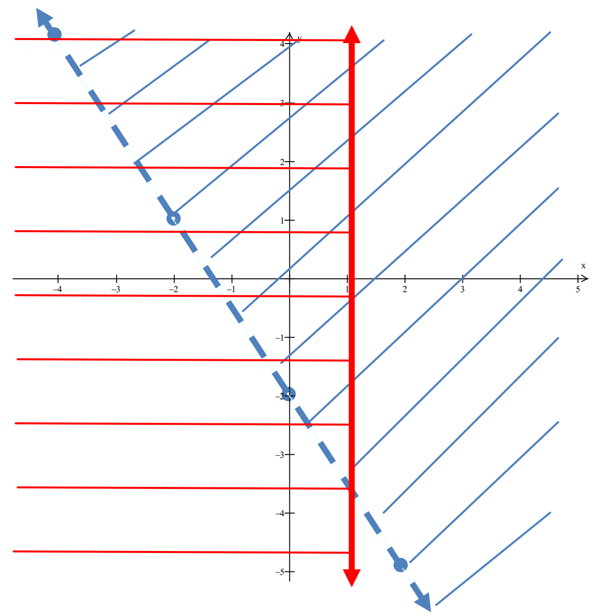
To graph the solution to a system of linear inequalities, graph each inequality on the same coordinate plane, and shade the area where the two inequality graphs overlap.

EX. Graph the solutions of the system: $3x + 2y + 4 > 0$
$x \leq 1$

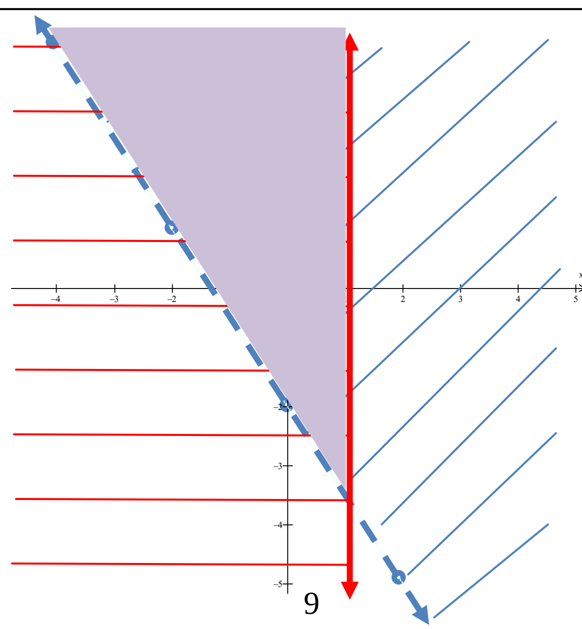
To graph $3x + 2y + 4 > 0$, remove the inequality sign and solve for y to get the dashed boundary line. Choose a test point and shade.
(see last note section for complete work)

To graph, $x \leq 4$, remove the inequality sign and graph the solid vertical line x = 1.
Shade to the left for all of the value of x less than 1.





Darken the area of overlap for the solution to the system of inequalities.



Addition and Scalar Multiplication of Matrices

A matrix is a rectangular array of numbers. The dimensions of the matrix are given by the number of rows (horizontal) and the number of columns (vertical).

EX: $\begin{bmatrix} 2 & 6 & 8 \\ -3 & 1 & 5 \end{bmatrix}$ is a 2 x 3 matrix (2 rows by 3 columns).

- To add matrices, the matrices must have the same dimensions. Then, add the corresponding elements.

  EX: $\begin{bmatrix} 4 & 7 \\ 0 & -2 \\ 1 & -6 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ -3 & 2 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -3 & 0 \\ 1 & -10 \end{bmatrix}$

- To subtract matrices, change to addition of the opposite of each element in the second matrix. Then, add the corresponding elements.

  $\begin{bmatrix} -2 & 5 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ -7 & -5 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix}$

  add the opposite

- Remember, the dimensions of the matrices must match in order to add or subtract.

  EX: $\begin{bmatrix} -4 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 10 \\ -9 \end{bmatrix}$ = Not Possible because the dimensions are not the same.

- Scalar Multiplication is when a real number is outside a matrix. To find the scalar product, multiply each element in the matrix by the scalar.

  EX: $3\begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ -6 & 12 \end{bmatrix}$

- Sometimes more than one operation is performed. Follow order of operations. Scalar **multiplication first**, then **add/subtraction second**.

  EX: $6\begin{bmatrix} 7 & 3 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -3 \\ -7 & -6 \end{bmatrix} = \begin{bmatrix} 42 & 18 \\ -12 & 6 \end{bmatrix} + \begin{bmatrix} -5 & 3 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 37 & 21 \\ -5 & 12 \end{bmatrix}$

- Two matrices are **equal** if and only if their dimensions are the same **and** each corresponding element is the same.

  EX: $\begin{bmatrix} 3x & 2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & -y & 4 \end{bmatrix}$ means that 3x =9 and 2 = -y, so we can solve and get x = 3 and y = -2

Laws of Exponents

When **multiplying** terms with like bases, **add** the exponents. $a^m \cdot a^n = a^{m+n}$

Ex. $x^3 \bullet x^5 \bullet x = x^{3+5+1} = x^9$ Ex. $2^2 \bullet 2^3 = 2^{2+3} = 2^5 = 32$

When **dividing** terms with like bases, **subtract** the exponents. $\frac{a^m}{a^n} = a^{m-n}$

Ex. $x^6 \div x^4 = x^{6-4} = x^2$ Ex. $\frac{5^7}{5^4} = 5^{7-4} = 5^3 = 125$

When **raising** a term **to a power**, **multiply** the exponents. $(a^m)^n = a^{mn}$

Ex. $(x^3)^5 = x^{15}$ Ex. $(2^3)^2 = 2^6 = 64$

When **raising** a term composed of different bases **to a power**, **multiply** the exponents for each term. $(ab)^m = a^m b^m$

Ex. $(x^3y^2)^5 = x^{15}y^{10}$ Ex. $(2^3 \bullet 3)^2 = 2^6 \bullet 3^2 = 64 \bullet 9 = 576$

When **raising** a fraction **to a power**, **multiply** the exponents for each term in the numerator and the denominator. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \quad b \neq 0$

Ex. $\left(\frac{x}{y^3}\right)^4 = \frac{x^4}{y^{12}}$ Ex. $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$

Any term **to zero power** equals **1**. $a^o = 1, \quad a \neq 0$

Ex. $(xy^2)^0 = 1$ Ex. $\left(\frac{7}{3}\right)^0 = 1$

Any term **to a negative power** means **reciprocal**. $a^{-n} = \frac{1}{a^n}, \quad a \neq 0$

Ex. $x^{-1}y^{-2}z = \frac{z}{xy^2}$ Ex. $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

Factoring

A. **GCF**: Factor out what is common to all terms.
Write as a product of the GCF and a polynomial.

EX. $-6m^9n^5p - 2m^7n^2p + 4m^5p$

$\mathbf{-2m^5p\,(3m^4n^5 + m^2n^2 - 2)}$

B. **Difference of two Squares**: $a^2 - b^2 = (a + b)(a - b)$

EX. $x^2 - 4y^6$

$\mathbf{(x + 2y^3)(x - 2y^3)}$

C. **Sum of two Cubes**: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of two Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

EX. $x^3 - 64$ — $\mathbf{(x - 4)(x^2 + 4x + 16)}$

$8x^3 + 27$ — $\mathbf{(2x + 3)(4x^2 - 6x + 9)}$

D. **Perfect Square Trinomials**: $a^2 + 2ab + b^2 = (a + b)^2$
$a^2 - 2ab + b^2 = (a - b)^2$

EX. $p^2 - 10p + 25$ — $\mathbf{(p - 5)^2}$

$16t^2 + 56t + 49$ — $\mathbf{(4t + 7)^2}$

E. **General Trinomials**: Guess and check by FOIL-ing

EX. $3a^2 - 4ab - 4b^2$

(3a   ) (a   )

STEP #1
Try pairs that multiply to be the first term $3a^2$:
$3a \bullet 1a$
There's only one pair since 3 is prime.
(3a   ) (a   )

STEP #2
Try pairs that multiply to be the last term $-4b^2$:
$-4b \bullet 1b$
$4b \bullet -1b$
$-2b \bullet 2b$

$\mathbf{(3a + 2b)(a - 2b)}$

+2ab

– 6ab

STEP #3
Try all combinations checking inner and outer terms of your factors to see if they add to be the middle term of the trinomial: -4ab

F. **<u>Factor by Grouping</u>**:

> Group terms into <u>two groups</u>. The groups must be <u>added</u> together. You <u>may</u> re-arrange the erms

EX1 . $a^2 - 2ab + a - 2b$

$a^2 + a - 2ab - 2b$	rearrange(optional)
$(a^2 + a) + (-2ab - 2b)$	group
$a(a + 1) - 2b\,(a + 1)$	factor out GCF for each group
$(a + 1)\,(a - 2b)$	factor out GCF quantity

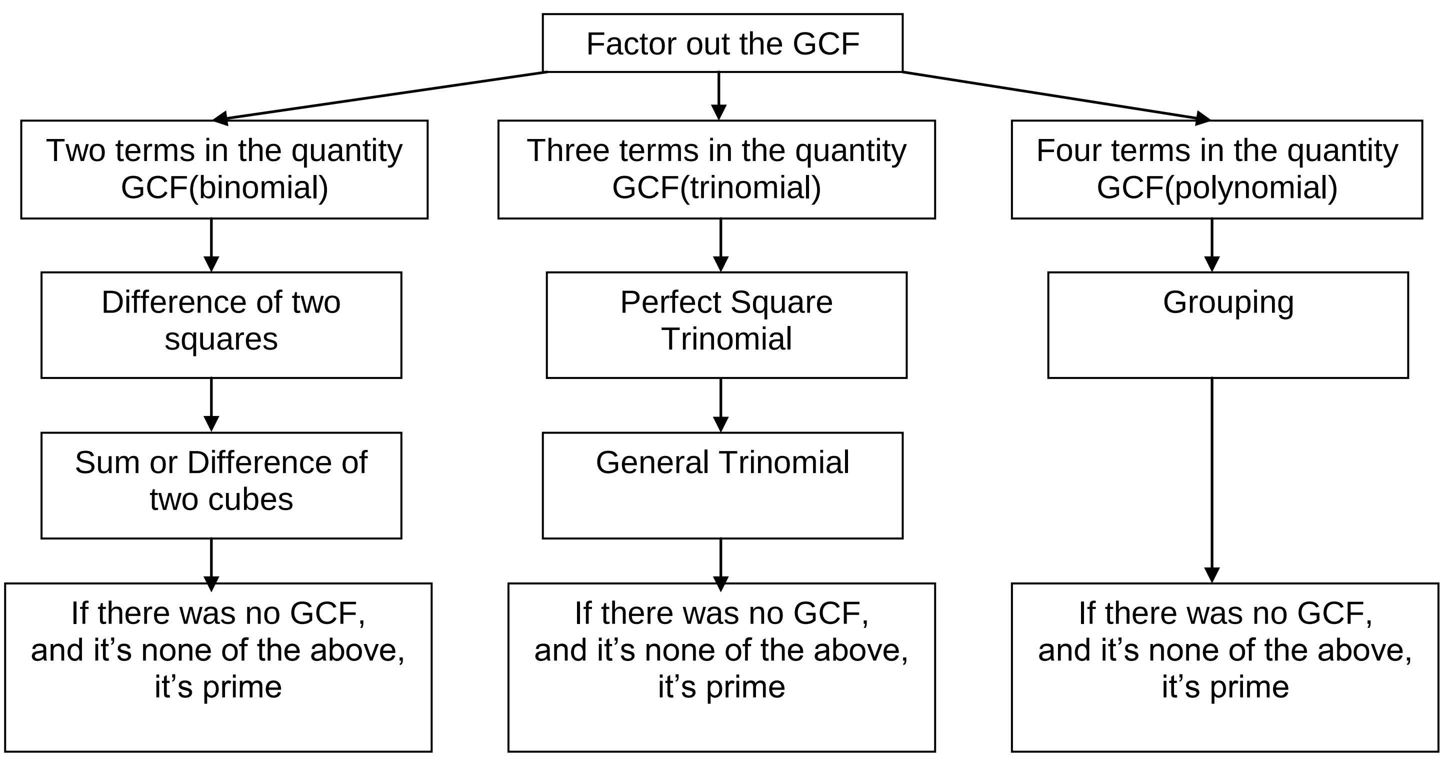
> To factor out the common quantities, the quantities must be

> The quantities are not the same… but, they **are** opposites.
> So, change 5y to -5y and (7b – 2a) to (2a – 7b) so the quantity factors match

EX2 . $4ax - 14bx + 35by - 10ay$

group	$(4ax - 14bx) + (35by - 10ay)$
factor out GCF for each group	$2x(2a - 7b) + 5y\,(7b - 2a)$
make 5y and (7b – 2a) opposites	$2x(2a - 7b) - 5y\,(2a - 7b)$
factor out GCF quantity	$(2a - 7b)\,(2x - 5y)$

G. **<u>Factor Completely</u>:** Follow the flow chart to factor completely.



Solve/Find the Roots by Factoring

.

The directions will be "solve" or "find the roots." One method is "factoring."
When an equation has a variable with an exponent higher than one, you must get all terms on one side of the equation so that it is equal to zero. Then, you can factor. Set each individual factor equal to zero and solve. The answers are called "solutions" or "roots." The number of solutions/roots you get is equal to the degree (highest variable exponent) of the equation (ex. Leading term is x^2…two solutions exist, x^3… three solutions exist, etc)

EX1.

$-5a^2 - 70a = 245$	
$-5a^2 - 70a - 245 = 0$	Set equation equal to zero
$-5(a^2 + 14a + 49) = 0$	Factor out GCF
$-5(a + 7)^2 = 0$	Factor perfect square trinomial
$-5 = 0$ or $a+7 = 0$	Set each factor equal to zero
$\varnothing$ $a = -7$	

Since -5 = 0 is false, it does not give you a solution

Final solution:
a = -7 (dbl rt)

Since a+7 was squared, you are really getting the solution a = – 7 twice… No need to do the solving twice, but it counts as two solutions… we call it a double root.

Notice that "a" is to the **second** power. Therefore, there are **2 solutions**.

EX2. $2x^3 + 5x^2 - 3x = 0$

$2x^3 + 5x^2 - 3x = 0$	Already set equal to zero
$x(2x^2 + 5x - 3) = 0$	Factor out GCF
$x(2x - 1)(x + 3) = 0$	Factor the general trinomial
$x = 0$ or $2x - 1 = 0$ or $x + 3 = 0$	Set each factor equal to zero
$x = ½$ $a = -3$	

Final solution:
x = -3, 0, ½

Notice that "x" is to the **third** power. Therefore, there are **3 solutions**.

Find the Zeros by Factoring

Sometimes the equation has two variables, x and y. The directions will be "find the zeros." Zeros are the values of x when $y = 0$. So, simply substitute zero in for y and solve for x as above.

EX1.

$y = (x + 3)(x - 3) - 40$	
$0 = (x + 3)(x - 3) - 40$	Substitute zero for y and solve for x
$0 = x^2 - 9 - 40$	Foil and combine like terms
$x^2 - 49 = 0$	
$(x + 7)(x - 7) = 0$	Factor difference of two squares
$(x + 7) = 0$ or $(x - 7) = 0$	Set each factor equal to zero
$x = -7$ $\quad$ $x = 7$	Solve

The roots are:
$x = \pm 7$

Sometimes the equation in two variables is written in function form: $f(x)$ instead of y. Since $f(x)$ is y, simply substitute zero in for $f(x)$ and solve for x as above.

EX 2.

$f(x) = (x - 4)^3$	
$0 = (x - 4)^3$	Substitute zero for $f(x)$ and solve for x
	This is already factored
$x - 4 = 0$	Since the three factors are the same, set the factor equal to zero
$x = 4$	Solve for the triple root

The roots are:
$x = 4$ (triple root)

Simplifying Rational Algebraic Expressions/Determining Domain

Domain: the set of x values for which a function is defined.

EX. Simplify $\frac{x^2 - 25}{x+5}$ and state its domain.

To simplify, factor the numerator and dominator completely.
Then cancel out identical factors.

$$\frac{x^2 - 25}{x+5} = \frac{\cancel{(x+5)}(x-5)}{\cancel{(x+5)}} = x - 5$$

To determine the domain, remember the **original** denominator $\neq 0$.

$$x + 5 \neq 0$$

So, $x \neq -5$

This means the domain is all real numbers except -5 or D: $\{\Re, x \neq -5\}$.

EX. Simplify $f(x) = \frac{2x^2 + 3x - 9}{x^2 + 3x}$ and state its domain.

To simplify, factor the numerator and dominator completely.
Then cancel out identical factors.

$$\frac{2x^2 + 3x - 9}{x^2 + 3x} = \frac{(2x-3)\cancel{(x+3)}}{x\cancel{(x+3)}} = \frac{2x-3}{x}$$

To determine the domain, remember the **original** denominator $\neq 0$.

$$x^2 + 3x \neq 0$$

$$x(x+3) \neq 0$$

So, $\boldsymbol{x \neq -3,}\ 0$

This means the domain is all real numbers except -3 and 0 or D: $\{\Re, x \neq -3, 0\}$.

EX. Determine the zeros and the domain of $f(x) = \frac{x^2 + 3x - 10}{x+2}$.

Zeros: the x values that will make a function zero.

Begin by factoring the numerator and dominator completely.

$$\frac{x^2 + 3x - 10}{x+2} = \frac{(x+5)(x-2)}{x+2}$$

To find the zeros, set f(x) = 0 and solve.

$$\frac{(x+5)(x-2)}{x+2} = 0$$

$$\cancel{(x+2)}\frac{(x+5)(x-2)}{\cancel{x+2}} = 0(x+2)$$

$$(x+5)(x-2) = 0$$

$$x = -5,\ 2$$

The zeros are - 5 and 2.

To determine the domain, remember the original denominator $\neq 0$.

$$x + 2 \neq 0$$

$$x \neq -2$$

The domain is all real numbers except - 2 or D : $\{\Re, x \neq -2\}$

Products and Quotients of Rational Expressions

EX. Simplify $\frac{x^2 - x}{(x-2)^2} \div \frac{(x-1)^2}{x-2}$.

Remember, division is multiplication by the reciprocal.
So begin by re-writing the given problem as multiplication.

$$\frac{x^2 - x}{(x-2)^2} \cdot \frac{x-2}{(x-1)^2}$$

Then, factor each numerator and denominator completely.

$$\frac{x(x-1)}{(x-2)(x-2)} \cdot \frac{(x-2)}{(x-1)(x-1)}$$

Next, cancel out identical factors, as shown.

$$\frac{x\cancel{(x-1)}}{\cancel{(x-2)}(x-2)} \cdot \frac{\cancel{(x-2)}}{(x-1)\cancel{(x-1)}}$$

This makes $\frac{x}{(x-2)(x-1)}$ your final answer.

Adding or Subtracting Rational Expressions

EX. Simplify $\frac{3}{x^2 + x - 6} - \frac{2}{x^2 - 3x + 2}$.

Begin by determining the LCD. To do this, first factor each denominator.

$$\frac{3}{(x-2)(x+3)} - \frac{2}{(x-1)(x-2)}$$

So, the LCD is $(x-2)(x+3)(x-1)$.

The first fraction will need to be multiplied by $\frac{x-1}{x-1}$ and

the second fraction will need to be multiplied by $\frac{x+3}{x+3}$.

This gives $\frac{3(x-1)}{(x-1)(x-2)(x+3)} - \frac{2(x+3)}{(x-1)(x-2)(x+3)}$.

Don't forget to distribute the negative.

So, we have $\frac{3(x-1) - 2(x+3)}{(x-1)(x-2)(x+3)} = \frac{3x - 3 - 2x - 6}{(x-1)(x-2)(x+3)} = \boxed{\frac{x-9}{(x-1)(x-2)(x+3)}}$

Simplifying Complex Fractions

EX. Simplify $\dfrac{1+\dfrac{1}{x-1}}{1-\dfrac{1}{x+1}}$.

Begin by determining the LCD for all of the fractions.
In this case, the LCD is $(x-1)(x+1)$.

Multiply the numerator and denominator by this LCD.

$$\frac{\left(1+\dfrac{1}{x-1}\right)}{\left(1-\dfrac{1}{x+1}\right)}\cdot\frac{(x-1)(x+1)}{(x-1)(x+1)}=\frac{(x-1)(x+1)+(x+1)}{(x-1)(x+1)-(x-1)}=\frac{x^2-1+x+1}{x^2-1-x+1}=\frac{x^2+x}{x^2-x}=\frac{x(x+1)}{x(x-1)}=\boxed{\frac{x+1}{x-1}}$$

Don't forget to distribute the negative.

Solving Fractional Equations

EX. Solve $\dfrac{2}{x+2}+\dfrac{x^2}{x^2-4}=\dfrac{1}{x-2}$.

Begin by determining the LCD. To do this, first factor each denominator.

$$\frac{2}{x+2}+\frac{x^2}{(x-2)(x+2)}=\frac{1}{x-2}$$

Next multiply every term by the LCD. This will clear out the fractions ☺.

$$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{x^2}{(x-2)(x+2)}\right)=\left(\frac{1}{x-2}\right)(x+2)(x-2)$$

$2(x-2)+x^2=x+2$

$2x-4+x^2=x+2$

$x^2+x-6=0$ Solve this by factoring.

$(x+3)(x-2)=0$

So, x = -3, 2.

HOWEVER, $x\neq 2$ because this value would make the denominator zero. So, it must be excluded from the domain.
x = 2 is called an **extraneous solution**. Although we used correct algebra to determine it, it does not create a true statement when it is plugged back into the original problem.
The final answer is $\boxed{x = -3}$

EX. Solve $\frac{3}{x+1}-\frac{1}{x-2}=\frac{1}{x^2-x-2}$.

Begin by determining the LCD. To do this, first factor each denominator.

$\frac{3}{x+1}-\frac{1}{x-2}=\frac{1}{(x+1)(x-2)}$ The LCD = $(x+1)(x-2)$.

Next multiply every term by the LCD. This will eliminate the fractions ☺.

$$(x+1)(x-2)\left(\frac{3}{x+1}-\frac{1}{x-2}\right)=\left(\frac{1}{(x+1)(x-2)}\right)(x+1)(x-2)$$

$$3(x-2)-(x+1)=1$$
$$3x-6-x-1=1$$
$$2x-7=1$$
$$2x=8$$
$$\boxed{x=4}$$

Remember, be sure x = 4 is in the domain of the original function before you "box it out" as your final answer.

Solving Fractional Inequalities

EX. Solve $\frac{3x}{5}-\frac{5}{6}<\frac{x}{10}$.

Begin by determining the LCD. The LCD of 5, 6, and 10 is 30.
Multiply every term in your inequality by the LCD. This will eliminate the fractions ☺.

$$30\left(\frac{3x}{5}-\frac{5}{6}\right)<\left(\frac{x}{10}\right)\cdot 30$$

$$6(3x)-5(5)<(x)\cdot 3$$
$$18x-25<3x$$
$$-25<-15x$$
$$\frac{-25}{-15}>x$$
$$\boxed{x<\frac{5}{3}}$$

Don't forget to flip the inequality when you divide by a negative number.

Roots of Real Numbers

Every positive real number has two even roots, **one positive and one negative**.

$\sqrt{25} = \pm 5$ $\sqrt[4]{81} = \pm 3$ $\sqrt[6]{64} = \pm 2$

The positive root is called the **principal root**.
When taking an even root:

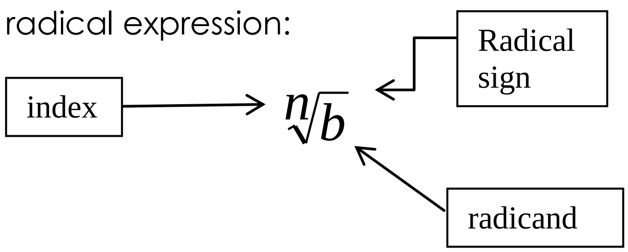
- $\pm\sqrt{\ }$ *means give both positive and negative values* $ex. \pm\sqrt{25} = \pm 5$
- $-\sqrt{\ }$ *means give only the negative value* $ex. -\sqrt{25} = -5$
- $\sqrt{\ }$ *means give only the positive value or the PRINCIPAL VALUE* $ex. \sqrt{25} = 5$

Every real number has one odd root, **either positive or negative**, taking the number's sign.

$\sqrt[3]{125} = 5$ $\sqrt[5]{-32} = -2$ $\sqrt[7]{1} = 1$ $\sqrt[7]{-1} = -1$ $\sqrt[3]{10^{-9}} = \sqrt[3]{\frac{1}{10^9}} = \frac{1}{10^3}$

*Remember, negative exponents mean reciprocal. *Remember, negative exponents mean reciprocal

$\sqrt[n]{b}$ is called a radical expression:



When the index (n) is even, the radicand(b) must be positive to get two real answers (±).

When the index is odd, the radicand can be positive or negative to get one real answer(+or -).

Find the real roots of the equation.
When solving equations, you must give all possible solutions.

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$\sqrt{x^2} = \sqrt{4}$$
$$x = \pm 2$$

$$x^2 + 16 = 0$$
$$x^2 = -16$$
No Real Roots

Since both 2 and -2 make the equation true, there are two solutions.

Since square roots cannot be negative, there are no real roots.

Properties of Radicals

Product Property $\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}; x > 0, y > 0$ for even integers n

Quotient Property $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}; x > 0, y > 0$ for even integers n

Theorems for indexes $\sqrt[n]{\sqrt[m]{x}} = \sqrt[(n \cdot m)]{x}$ $\sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

EX. Simplify.

$\sqrt[3]{25} \bullet \sqrt[3]{10} = \sqrt[3]{5^2} \bullet \sqrt[3]{2 \bullet 5} = \sqrt[3]{5^3 \bullet 2} = 5\sqrt[3]{2}$

Since neither radicand is a perfect cube, factor under each radical.
Since neither radical can be simplified, combine factors under one cube root.
Simplify.

$\sqrt[3]{\frac{81}{8}} = \frac{\sqrt[3]{81}}{\sqrt[3]{8}} = \frac{3\sqrt[3]{3}}{2}$

Since the fraction under the radical cannot be simplified, use the quotient property to split into two separate radicals…one in the numerator and one in the denominator.
Simplify.

$\frac{\sqrt{60}}{\sqrt{5}} = \sqrt{\frac{60}{5}} = \sqrt{12} = 2\sqrt{3}$

Since 60 is divisible by 5, use the quotient property to make a fraction under one radical.
Divide. Simplify.

EX. Simplify with Variables.

$\sqrt[3]{\frac{27a}{4b^4}} = \frac{\sqrt[3]{27a}}{\sqrt[3]{4b^4}} = \frac{\sqrt[3]{3^3 a}}{\sqrt[3]{2^2 b^4}} = \frac{3\sqrt[3]{a}}{b\sqrt[3]{2^2 b}} \bullet \frac{\sqrt[3]{2b^2}}{\sqrt[3]{2b^2}} = \frac{3\sqrt[3]{2ab^2}}{b\sqrt[3]{2^3 b^3}} = \frac{3\sqrt[3]{2ab^2}}{2b^2}$

Since the fraction under the radical cannot be simplified, use the quotient property to split into two separate radicals…one in the numerator and one in the denominator.
Factor and Simplify.
Rationalize the denominator…multiply numerator and denominator by $\sqrt[3]{2b^2}$ to create a perfect cube in the denominator. Simplify.

$\sqrt{x^2 + 6x + 9} = \sqrt{(x+3)^2} = x + 3$

Factor…notice the radicand is a perfect square trinomial.
Simplify.

Sums/Differences of Radicals

To add or subtract radicals, first simplify each radical term, then combine like radicals. **Like radicals** are radicals with the same index(n) and the same radicant(b). Like radicals are combined by adding/subtracting the coefficients.

EX. Simplify.

$$\sqrt{8}+\sqrt{98}=2\sqrt{2}+7\sqrt{2}=9\sqrt{2}$$

Simplify each radical
Add like radicals

$$\sqrt{\frac{32}{3}}-\sqrt{\frac{8}{3}}=\frac{4\sqrt{2}}{\sqrt{3}}-\frac{2\sqrt{2}}{\sqrt{3}}=\frac{2\sqrt{2}}{\sqrt{3}}\bullet\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{6}}{3}$$

Use quotient property/simplify
Subtract like radicals
Rationalize denominator

$$\sqrt{6}\left(\sqrt{2}+\sqrt{3}\right)=\sqrt{12}+\sqrt{18}=2\sqrt{3}+3\sqrt{2}$$

Distribute/Use product property
Simplify each radical
Unlike radicals cannot be combined

$$\frac{\sqrt{21}-\sqrt{15}}{\sqrt{3}}=\sqrt{\frac{21}{3}}-\sqrt{\frac{15}{3}}=\sqrt{7}-\sqrt{5}$$

Divide/Use quotient property
Simplify under each radical
Unlike radicals cannot be combined

$$\sqrt{12x^5}-x\sqrt{3x^3}+5x^2\sqrt{3x}=2x^2\sqrt{3x}-x^2\sqrt{3x}+5x^2\sqrt{3x}=6x^2\sqrt{3x}$$

Simplify each radical. Combine like radicals.

Binomials Containing Radicals

You can FOIL binomials containing radicals just the way you would multiply any binomial.

EX. Simplify.

$$\left(4+\sqrt{7}\right)\left(3+2\sqrt{7}\right)=12+8\sqrt{7}+3\sqrt{7}+2(7)=12+11\sqrt{7}+14=26+11\sqrt{7}$$

FOIL. Remember $\sqrt{7}\bullet\sqrt{7}=\sqrt{49}=7$. Combine like radicals and constants.

$$\left(2\sqrt{3}-\sqrt{6}\right)^2=4\bullet3-4\sqrt{18}+6=12-12\sqrt{2}+6=18-12\sqrt{2}$$

Binomial Squared: Square the first, multiple the terms and double, square the last. Combine like radicals and constants.

$$\frac{3+\sqrt{5}}{3-\sqrt{5}} \bullet \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{9+6\sqrt{5}+5}{9-5} = \frac{14+6\sqrt{5}}{4} = \frac{7+3\sqrt{5}}{2}$$

To rationalize the denominator, multiply the numerator and denominator by the conjugate of the denominator. Combine like terms. Simplify by a factor or 2.

The Imaginary Number *i*

In the domain of real numbers, the radicant of an even index radical must be positive. When the radicant is negative, and the index is even (square rt, fourth rt, sixth rt, etc), the expression is not real, but imaginary.

The imaginary number $i = \sqrt{-1}$. When simplifying, substitute *i* for $\sqrt{-1}$.

EX. Simplify.

$$\sqrt{-50} = \sqrt{-1} \bullet \sqrt{50} = i\sqrt{50} = 5i\sqrt{2}$$

EX. Addition/Subtractraction**.**

$$i\sqrt{2} + 3i\sqrt{2} = 4i\sqrt{2}$$

$$\sqrt{-16} - \sqrt{-49} = 4i - 7i = -3i$$

EX. Multiplication Since $i = \sqrt{-1}$, $i^2 = -1$. Whenever i^2 appears, replace it with -1.

$$\sqrt{-4} \bullet \sqrt{-25} = 2i \bullet 5i = 10i^2 = 10(-1) = -10$$

$$i\sqrt{2} \bullet i\sqrt{3} = i^2\sqrt{6} = (-1)\sqrt{6} = -\sqrt{6}$$

EX. Division
You may not leave an imaginary number in the denominator.
To rationalize the denominator, multiply the numerator and denominator by *i*.
Of course, you still need to rationalize radical denominators, too.

$$\frac{2}{3i} \bullet \frac{i}{i} = \frac{2i}{3i^2} = \frac{2i}{3(-1)} = -\frac{2i}{3}$$

$$\frac{6}{\sqrt{-2}} = \frac{6}{i\sqrt{2}} \bullet \frac{i\sqrt{2}}{i\sqrt{2}} = \frac{6i\sqrt{2}}{2i^2} = \frac{6i\sqrt{2}}{2(-1)} = -3i\sqrt{2}$$

EX. Simplify with Variables

$$\sqrt{-9x^2} + \sqrt{-x^2} = 3ix + ix = 4ix$$

$$\sqrt{-6y} \bullet \sqrt{-2y} = i\sqrt{6y} \bullet i\sqrt{2y} = i^2\sqrt{12y^2} = -2y\sqrt{3}$$

EX. Solving Equations with Non-Real Solutions.

$2x^2 + 19 = 3$

$2x^2 = -16$

$x^2 = -8$

$\sqrt{x^2} = \sqrt{-8}$

$x = \pm 2i\sqrt{2}$ *Don't forget the* $\pm$

Complex Numbers

A complex number is a number in the form a + b*i*.

If b = 0, the number is real. Ex. 5+0*i*…usually written 5
If b ≠ 0, the number is imaginary. Ex. 2 – 3*i*
If b ≠ 0 and a = 0, the number is pure imaginary. Ex 0+ 2*i*…usually written 2*i*

EX. Addition/Subtractraction.
To add/subtract complex numbers, combine the real parts and combine the imaginary parts.

(3 + 6*i*) – (4 – 2*i*)
3 + 6*i* – 4 + 2*i*
-1 + 8*i*

EX. Multiplication FOIL. Whenever i^2 appears, replace it with -1. Simplify.

(3 + 4*i*) (5 + 2*i*)
15 +6*i* +20*i* +8i^2
15 + 26*i* + 8(-1)
15 + 26*i* – 8
7 + 26*i*

EX. Division
You may not leave a binomial imaginary number in the denominator.
To rationalize the denominator, multiply the numerator and denominator by the complex conjugate of the denominator. **Reminder:** a + b*i* and a – b*i* are complex conjugates

Simplify: $\frac{5-i}{2+3i}$

$$\frac{5-i}{2+3i} \bullet \frac{2-3i}{2-3i} = \frac{10-15i-2i+3i^2}{4-9i^2} = \frac{10-17i+3(-1)}{4-9(-1)} = \frac{10-17i-3}{4+9} = \frac{7-17i}{13} = \frac{7}{13} - \frac{17}{13}i$$

Multiply the numerator and denominator by the complex conjugate of the denominator.	Simplify	Separate final answer into a + b*i* form.

The Quadratic Formula

A quadratic equation is an equation with degree of 2. For example, $3x^2 + x = 1$
The quadratic formula is used to find the solutions or roots to quadratic equations in one variable. To use the quadratic formula, the quadratic equation must be in the form:

$ax^2 + bx + c = 0$ where $a \neq 0$

The quadratic formula is: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute the values of a, b, and c from the equation into the formula.
Simplify to find the solutions/roots of the equation.

As an analysis student, you **must have the Quadratic Formula memorized**.

Ex 1. Find the roots of $3x^2 + x = 1$ using the quadratic formula.

$3x^2 + x = 1$

$3x^2 + x - 1 = 0$ — Rewrite the equation in the form **$ax^2 + bx + c = 0$**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Substitute a = 3, b = 1, c = -1 into the quadratic formula

$$x = \frac{-1 \pm \sqrt{1^2 - 4(3)(-1)}}{2(3)}$$ Simplify to find the two roots.

$$x = \frac{-1 \pm \sqrt{1 + 12}}{6}$$

$$x = \frac{-1 \pm \sqrt{13}}{6}$$

The roots are: $x = \frac{-1 + \sqrt{13}}{6}$, $x = \frac{-1 - \sqrt{13}}{6}$

Incorporating the two roots using ± is acceptable.

$$x = \frac{-1 \pm \sqrt{13}}{6}$$

This equation has two real irrational roots.

Ex 2. Find the roots of $5a^2 = 6a - 3$ using the quadratic formula.

$5a^2 + 3 = 6a$ — Rewrite the equation in the form **$ax^2 + bx + c = 0$**

$5a^2 - 6a + 3 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Substitute a = 5, b = -6, c = 3 into the quadratic formula

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(5)(3)}}{2(5)}$$ Simplify to find the two roots.

$$x = \frac{6 \pm \sqrt{36-60}}{10}$$

$$x = \frac{6 \pm \sqrt{-24}}{10}$$

$$x = \frac{6 \pm 2i\sqrt{6}}{10}$$

$$x = \frac{3 \pm i\sqrt{6}}{5}$$

The roots are: $x = \frac{3+i\sqrt{6}}{5}$, $x = \frac{3-i\sqrt{6}}{5}$

**Imaginary roots are usually written in the form a + b*i*, so you should split each root into

two separate fractions: $x = \frac{3}{5} + \frac{i\sqrt{6}}{5}$, $x = \frac{3}{5} - \frac{i\sqrt{6}}{5}$.

Incorporating the two roots using ± is acceptable.

$$x = \frac{3}{5} \pm \frac{i\sqrt{6}}{5}$$

This equation has two imaginary roots.

EX3. The quadratic formula can be used when finding the zeros of an equation.

$f(x) = x^4 - 16$	
$0 = x^4 - 16$	Substitute zero for f(x)
$(x^2 + 4)(x^2 - 4) = 0$	No GCF exists/factor difference of two squares
$(x^2 + 4)(x + 2)(x - 2) = 0$	Factor difference of two squares again
$x^2 + 4 = 0$ or $x + 2 = 0$ or $x - 2 = 0$	Set each factor equal to zero
$x = -2$ $x = 2$	Solve

Solve using quadratic formula or square roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{\pm\sqrt{-16}}{2}$$

$$x = \frac{\pm 4i}{2}$$

$$x = \pm 2i$$

$$x^2 + 4 = 0$$

$$x^2 = -4$$

$$\sqrt{x^2} = \sqrt{-4}$$

$$x = \pm 2$$

The roots are:
x= ±2, ±2*i*

Notice that "x" is to the **fourth** power. Therefore, there are **4 roots**.

The Discriminant

The discriminant is the portion of the quadratic formula found beneath the radical:

$$b^2 - 4ac$$

By calculating **only** $b^2 - 4ac$, you can tell **the nature of the roots**. That is, you can tell what kind of roots you have without solving for the roots. Here are the possibilities:

Two real rational roots

Two real irrational roots

One real rational double root

Two imaginary conjugate roots

$b^2 - 4ac$ is called the **discriminant**, because it discriminates (tells the difference) among the four natures (types) of roots.

If the discriminant ($b^2 - 4ac$) is:	The nature of the roots is:
A positive perfect square	Two real rational roots
A positive non-perfect square	Two real irrational roots
Zero (a neutral perfect square)	One real rational double root
Negative	Two imaginary conjugate roots

Directions will be: **Find the nature of the roots.**

Expectation: Use $b^2 - 4ac$ **only**. Show work. Give one of the four descriptions based upon the value of the discriminant.

Find the nature of the roots.

EX 1. $x^2 + 6x + 5 = 0$

$b^2 - 4ac$
$6^2 - 4(1)(5)$
$36 - 20$
16

Since 16 is positive real number and a perfect square, answer:
Two real rational roots

EX 2. $x^2 + 6x - 2 = 0$

$b^2 - 4ac$
$6^2 - 4(1)(-2)$
$36 + 8$
44

Since 44 is positive real number and not a perfect square, answer:
Two real irrational roots

EX 3. $x^2 + 6x + 9 = 0$

$b^2 - 4ac$
$6^2 - 4(1)(9)$
$36 - 36$
0

Since 0 is a neutral real number and a perfect square, answer:
One real rational double root

EX 4. $x^2 + 6x + 10 = 0$

$b^2 - 4ac$
$6^2 - 4(1)(10)$
$36 - 40$
-4

Since -4 is a negative number, answer:

Two imaginary conjugate roots

Just to show that $b^2 - 4ac$ works, let's solve each example using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

EX 1

$x = \frac{-6 \pm \sqrt{16}}{2}$

$x = \frac{-6 \pm 4}{2}$

$x = \frac{-6+4}{2}, \frac{-6-4}{2}$

$x = \frac{-2}{2}, \frac{-10}{2}$

$x = -1 \; or \; -5$

Two real rational roots

EX 2

$x = \frac{-6 \pm \sqrt{44}}{2}$

$x = \frac{-6 \pm 2\sqrt{11}}{2}$

$x = -3 \pm \sqrt{11}$

$x = -3 + \sqrt{11}, -3 - \sqrt{11}$

Two real irrational roots

EX 3

$x = \frac{-6 \pm \sqrt{0}}{2}$

$x = \frac{-6 \pm 0}{2}$

$x = \frac{-6}{2}, \frac{-6}{2}$

$x = -3 \; or \; -3$

One real rational double root

EX 4

$x = \frac{-6 \pm \sqrt{-4}}{2}$

$x = \frac{-6 \pm 2i}{2}$

$x = -3 \pm i$

Two imaginary conjugate roots

Graphing Quadratic Functions

A quadratic function is an equation in the form: $f(x) = ax^2 + bx + c$

When graphing a function, it is often helpful to replace f(x) with y: $\mathbf{y} = ax^2 + bx + c$
The graph of a quadratic function is a parabola.
The first step is to find the vertex point (x, y).

To find the x-coordinate of the vertex, use the formula: $x = \frac{-b}{2a}$.
To find the y- coordinate of the vertex, substitute the x-coordinate into the equation and solve for y.

Ex. Graph $y = x^2 + 6x + 5$

Step 1: Find the **vertex**.

Find the **x-coordinate**.
$$x = \frac{-b}{2a} = \frac{-6}{2(1)} = \frac{-6}{2} = -3$$

Find the **y-coordinate**.
$y = x^2 + 6x + 5$
$y = (-3)^2 + 6(-3) + 5$
$y = 9 - 18 + 5$
$y = -4$

Vertex point:
(-3, -4)

Step 2: Make a table of values choosing x-values on either side of – 3 (x-coord. of the vertex)
Choose enough values to cross the x-axis.

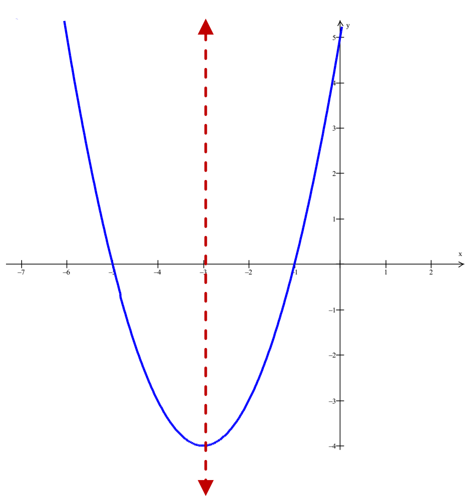
x	$x^2 + 6x + 5$	y
-6	36 – 36 + 5	5
-5	25 – 30 + 5	0
-4	16 – 24 + 5	-3
-3	VERTEX	-4
-2	4 – 12 + 5	-3
-1	1 – 6 +5	0
0	0 + 0 + 5	5

Notice the **symmetry** of the y-coordinates

Notice the **zeros of the function.**
Zeros are the values of x where y = 0.

The zeros are: -1 and -5.

Step 3: Plot the points and connect with a smooth curve.



Notice the equation for the **axis of symmetry** is the vertical line x = -3 (the x-coordinate of the vertex)

You may recall, when *a* is **positive**, the parabola **opens up** and when *a* is **negative**, the parabola **opens down.**

Dividing Polynomials (Long Division)

Division Algorithm:

$$\frac{Dividend}{Divisor} = Quotient + \frac{Remainder}{Divisor}$$

Think back to long division from 3rd grade.

- How many times does the divisor go into the dividend? Put that number on top.
- Multiply that number by the divisor and put the result under the dividend.
- Subtract and bring down the next number in the dividend. Repeat until you have used all the numbers in the dividend.

EX: $(x^3 - 5x + 2) \div (x - 2)$

First, we need to write this expression in long division form. You write it the same way you do normal long division, as if they are only simple numbers. **You need to use place holders if you have powers of x that are missing.** The dividend, the thing on the inside being divided, needs to have all powers of x represented in decreasing order. The first term must be of the highest power of x that is in the expression. The second term must be one lower power of x. If there is none of that power, you put a zero. In this first example, there is no x^2, so put a zero in to hold the place.

$$x-2\overline{\big)\,x^3+0x^2-5x+2}$$

Look at the divisor, the term outside of the division symbol, (in this case, x – 2). Only at the highest power of x (in this example, x.) The coefficient is just 1. What times 'x' is equal to the first term of the dividend? In other words, what times x equals x^3? The answer is x^2, and this is the first term of the quotient. Write this on top of the division bar, exactly above the first term of the dividend.

$$\begin{array}{r} x^2\text{.......................} \\ x-2\overline{\big)\,x^3+0x^2-5x+2} \end{array}$$

Now, like in normal long division, we multiply this first part of our answer by the divisor, the expression on the right side. Then, you subtract that from the dividend! After this subtraction, you bring down the next term in the dividend, and repeat the process.

$$\begin{array}{r} x^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x-2\overline{)\ldots x^3+0x^2-5x+2} \\ -(x^3-2x^2)\quad\quad \\ \hline 2x^2-5x\quad \end{array}$$

Subtract → ; Bring down the next term

Now, repeat the process until the first term of the divisor can no longer "go into" the term in the dividend. Look at the first term of the divisor, the x. Of this new expression, $2x^2 - 5x$, look at the first term. What times x equals $2x^2$? Then continue doing what you just did.

$$\begin{array}{r} x^2+2x\ldots\ldots\ldots\ldots \\ x-2\overline{)\ldots x^3+0x^2-5x+2} \\ -(x^3-2x^2)\quad\quad \\ \hline \ldots 2x^2-5x\quad \\ -(2x^2-4x)\quad \\ \hline -x+2 \end{array}$$

Bring down the next term

Now, one more time, what times x equals $-x$?

$$\begin{array}{r} x^2+2x-1\ldots\ldots\ldots \\ x-2\overline{)\ldots x^3+0x^2-5x+2} \\ -(x^3-2x^2)\quad\quad \\ \hline \ldots 2x^2-5x\quad \\ -(2x^2-4x)\quad \\ \hline -x+2 \\ -(-x+2) \\ \hline 0 \end{array}$$

There is no remainder.

The quotient is x^2+2x-1.

Another example: $\dfrac{4x^3+2x^2-10x}{2x^2-4}$ **(Pay attention to lining up like terms as you multiply.)**

$$\begin{array}{r l}
 & 2x+1 \\
2x^2-4 \,\big) & 4x^3+2x^2-10x \\
 & -(4x^3 \qquad\quad -8x) \\
\hline
 & \qquad 2x^2-2x+0 \\
 & \quad -(2x^2 \qquad -4) \\
\hline
 & \qquad\quad -2x+4
\end{array}$$

$-2x+4$ Since $2x^2$ does not go into -2x, this is the remainder.

The answer is written: $2x+1+\dfrac{-2x+4}{2x^2-4}$

Synthetic Division

To use synthetic division:

- There must be a coefficient for every possible power of the variable.
- The divisor must have a leading coefficient of 1.

For an example of synthetic division, consider $\mathbf{3x^3 - 6x + 2}$ divided by $\mathbf{x - 2}$.
First, if a power of x is missing from the polynomial, a term with that power and a zero coefficient must be inserted into the correct position in the polynomial. In this case the x^2 term is missing, so we must add $\mathbf{0x^2}$ between the cubic and linear terms:

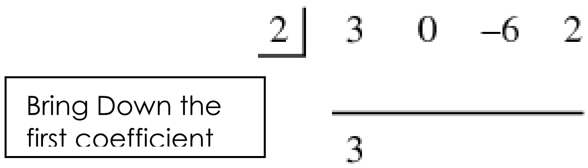
$$\mathbf{3x^3 + 0x^2 - 6x + 2}$$

Next, all the variables and their exponents (x^3, x^2, x) are removed, leaving only a list of the coefficients: 3, 0, -6, 2. These numbers form the dividend. We form the divisor for the synthetic division using only the constant term (2) of the linear factor x - 2.
Note: If the divisor were x + 2 we would put it in the format x – (-2), resulting in a divisor of -2.

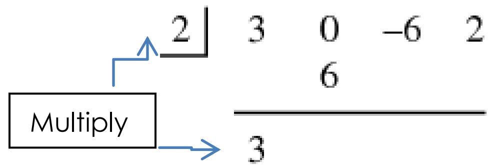
The numbers representing the divisor and the dividend are placed into a division-like configuration:

$$\begin{array}{c|cccc}
2 & 3 & 0 & -6 & 2 \\
 & & & & \\
\hline
\end{array}$$

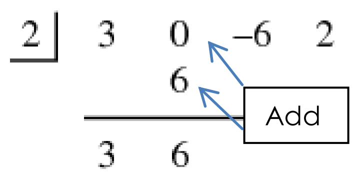
The first number in the dividend (3) is put into the first position of the result area (below the horizontal line). This number is the coefficient of the x^3 term in the original polynomial:



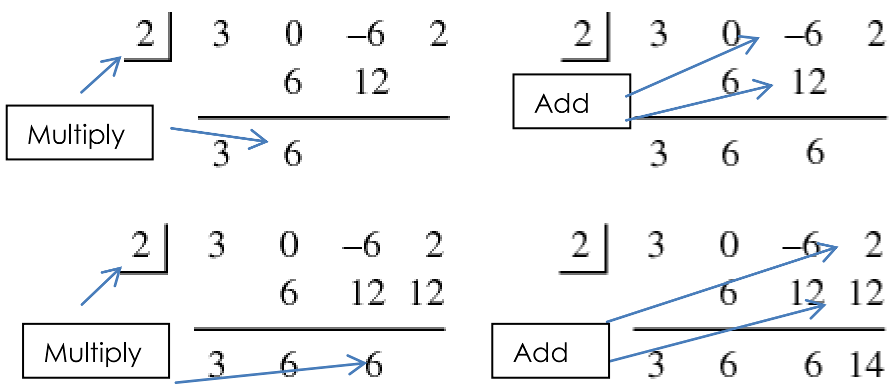
Then this first entry in the result (3) is multiplied by the divisor (2) and the product is placed under the next term in the dividend (0):



Next the number from the dividend and the result of the multiplication are added together and the sum is put in the next position on the result line:



This process is continued for the remainder of the numbers in the dividend:



The result is the list 3, 6, 6, 14. All numbers except the last become the coefficients of the quotient polynomial. Since we started with a cubic polynomial and divided it by a linear term, the quotient is a 2nd degree polynomial: $3x^2 + 6x + 6.$

The last entry in the result list (14) is the remainder. The quotient and remainder can be combined into one expression:

$$3x^2 + 6x + 6 + \frac{14}{x-2}.$$

(Note that no division operations were performed to compute the answer to this division problem.)

Rational Exponents

The word "rational" refers to any number that can be written as a fraction. Therefore, rational exponents are fractional exponents (or exponents that can be written as fractions).

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} \text{ or } \left(\sqrt[n]{b}\right)^m$$

The denominator of the exponent is the index of the root.
The numerator of the exponent is: the exponent on base b in the radicand OR
the exponent on the entire radical expression.

EX. $16^{\frac{3}{4}} = \sqrt[4]{16^3} = \sqrt[4]{4096} = \sqrt[4]{8^4} = 8$

OR

$16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$

Choose the arrangement that makes the work easier…
In this case, the second option $\left(\sqrt[4]{16}\right)^3$ is easier.

Negative exponents mean reciprocal.

EX. $25^{-\frac{3}{2}} = \frac{1}{\left(\sqrt{25}\right)^3} = \frac{1}{(5)^3} = \frac{1}{125}$

Many decimal exponents can be written as fractions

EX. $9^{2.5} = 9^{\frac{5}{2}} = \left(\sqrt{9}\right)^5 = (3)^5 = 243$

Write in exponential form.

EX. $\sqrt[3]{\frac{a^5b^3}{c^2}} = \frac{a^{\frac{5}{3}}b^{\frac{3}{3}}}{c^{\frac{2}{3}}} = a^{\frac{5}{3}}bc^{-\frac{2}{3}}$

Multiply.

EX.

To multiply radicals with different indexes, work in exponential form.

$\sqrt{8} \bullet \sqrt[3]{4} = \sqrt{2^3} \bullet \sqrt[3]{2^2}$ — Factor the radicands

$= 2^{\frac{3}{2}} \bullet 2^{\frac{2}{3}}$ — Write in exponential form

$= 2^{\frac{9}{6}} \bullet 2^{\frac{4}{6}}$ — Get common denominators to add exponents

$= 2^{\frac{13}{6}}$ — Multiply by adding exponents on like bases

$= \sqrt[6]{2^{13}}$ — Change to radical form.

$= 4\sqrt[6]{2}$ — Simplify.

To solve equations in exponential form, raise each side to the reciprocal power.

EX. Solve $5x^{\frac{-1}{3}} = 20$

$$5x^{\frac{-1}{3}} = 20$$
$$x^{\frac{-1}{3}} = 4$$
$$\left(x^{\frac{-1}{3}}\right)^{-3} = (4)^{-3}$$
$$x = \frac{1}{4^3}$$
$$x = \frac{1}{64}$$

EX. Solve $(x-1)^{\frac{3}{2}} = 8$

$$(x-1)^{\frac{3}{2}} = 8$$
$$\left((x-1)^{\frac{3}{2}}\right)^{\frac{2}{3}} = (8)^{\frac{2}{3}}$$
$$x-1 = \left(2^3\right)^{\frac{2}{3}}$$
$$x-1 = 2^2$$
$$x-1 = 4$$
$$x = 5$$

Composition of Functions

The functions f(x) and g(x) can be combined to form a composition function. f(g(x)), read "*f* of *g* of x," is one composition. g(f(x)) read "*g* of *f* of x," is the other composition depending on the order in which you work. They are two different compositions generally yielding two different values.

EX. Find both compositions: f(g(3)) and g(f(3)), given f(x) = x^2 and g(x) = 2x.

To find f(g(3)), start from the inside and work out.

In this example, x = 3. To find f(g(3)), substitute x = 3 into function g: g(3) = 2(3) = 6.
Now, substitute the value of g(3) into function f: f(g(3)) = f(6) = 6^2 = 36.
So, the composition f(g(3)) equals 36.

Now, find g(f(3)). Here, x is still 3. But, the order of the functions has changed.
Since we always work from the inside out, substitute x = 3 into function f first:
f(3) = 3^2 = 9. Substitute the value of f(3) into function g: g(f(3)) = g(9) = 2(9) = 18.
So, the composition g(f(3)) equals 18.

EX. Find both compositions: f(g(x)) and g(f(x)), given f(x) = x^2 and g(x) = 2x.

To find f(g(x)), start from the inside and work out.

In this example, x = x. So, g(x) = 2x.
Now, substitute the value of g(x) into function f: f(g(x)) = f(2x) = $(2x)^2$ = $4x^2$.
So, the composition f(g(x)) equals $4x^2$.

Now, find g(f(x)). Here, x is still x. But, the order of the functions has changed.
Since we always work from the inside out, f(x) = x^2.
Substitute the value of f(x) into function g: g(f(x)) = g(x^2) = 2(x^2) = $2x^2$.
So, the composition g(f(x)) equals $2x^2$.

EX. Find both compositions: f(g(x)) and g(f(x)).

Given f(x) = $\sqrt{x}$ and g(x) = 3x + 2, find both compositions.

f(g(x)) = *f(*3x+2*)*= $\sqrt{3x+2}$
g(f(x)) = $f(\sqrt{x}) = 3\sqrt{x}+2$

Inverses of Functions

Inverse functions are two functions, *f* and f^{-1} (read "f inverse") that have the following characteristics: NOTE: f^{-1} is not an exponent…it is a notation or way of writing "inverse of function f"

1. Every point (a, b) on function *f* corresponds to a point (b, a) on the inverse function f^{-1}. In other words, all of the coordinate pairs are reversed.
2. The domain of function f is the range of function f^{-1}. And, the range of function f is the domain of function f^{-1}. (That should make sense as the coordinate pairs are all reversed, so the domain and range are reversed.)
3. To find the inverse equation of a function, reverse the x and the y in the equation, then solve for y.
4. Because of this reversal, the graphs of a function and its inverse are reflections over the line y = x.
5. When you do both compositions, $f(f^{-1}(x))$ and $f^{-1}(f(x))$, you get x for both. This is a good algebraic way to see if two equations are inverses…just do both compositions and see if you get x for both. If you do, the equations are inverses. If you do not, the equations are not inverses.

EX. Find the inverse of f(x) = 2x – 4.

To find the inverse function, switch places with x and y.
It is helpful to change f(x)= to y=.

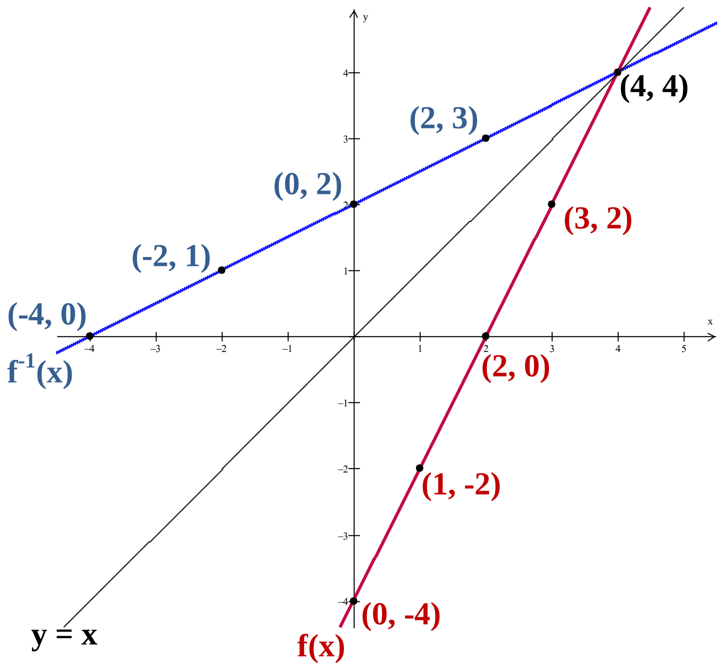
Function: f(x) = 2x – 4

or y = 2x – 4 → Inverse function: x = 2y – 4
Solve for y: x + 4 = 2y
y = ½ x +2
Use function Notation: $f^{-1}(x)$ = ½ x +2

Therefore, f(x) = 2x – 4 and $f^{-1}(x)$ = ½ x +2 are inverse functions.

EX. Graph f(x) and $f^{-1}(x)$ on a coordinate plane.

$f(x) = 2x - 4$ $\qquad$ $f^{-1}(x) = \frac{1}{2}x + 2$



<u>Notice</u> that all of the coordinate pairs for f(x) are reversed for $f^{-1}(x)$.

<u>Notice</u> that the graphs of f(x) and $f^{-1}(x)$ are a reflection over the line y = x.

Since f(x) and $f^{-1}(x)$ are inverse functions, **both compositions should equal x.**
Let's verify that the functions are indeed inverses by showing that $f(f^{-1}(x))$ and $f^{-1}(f(x))$ both equal x.

$f(x) = 2x - 4$ $\qquad$ $f^{-1}(x) = \frac{1}{2}x + 2$

$$f(f^{-1}(x)) = f(\tfrac{1}{2}x + 2) = 2(\tfrac{1}{2}x + 2) - 4 = x + 4 - 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(2x - 4) = \tfrac{1}{2}(2x - 4) + 2 = x - 2 + 2 = x$$

<u>Conclusion</u>:

$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, therefore f(x) and $f^{-1}(x)$ are indeed inverses.