4.6 INFINITE LIMITS AND ASYMPTOTES

When you turn on an automobile or a light bulb many things happen, and some of them are uniquely part of the start up of the system. These "transient" things occur only during start up, and then the system settles down to its steady–state operation. The start up behavior of systems can be very important, but sometimes we want to investigate the steady–state or long term behavior of the system: how is the system behaving "after a long time?" In this section we consider ways of investigating and describing the long term behavior of functions and the systems they may model: how is a function behaving "when x (or –x) is arbitrarily large?"

Limits As X Becomes Arbitrarily Large ("Approaches Infinity")

The same type of questions we considered about a function f as x approached a finite number can also be asked about f as x "becomes arbitrarily large," "increases without bound," and is eventually larger than any fixed number.

Example 1: What happens to the values of $f(x) = \dfrac{5x}{2x+3}$ (Fig. 1) and $g(x) = \dfrac{\sin(7x+1)}{3x}$ as x becomes arbitrarily large, as x increases without bound?

Solution: One approach is numerical: evaluate f(x) and g(x) for some "large" values of x and see if there is a pattern to the values of f(x) and g(x). Fig. 1 shows the values of f(x) and g(x) for several large values of x. When x is very large, it appears that the values of f(x) are close to 2.5 = 5/2 and the values of g(x) are close to 0. In fact, we can guarantee that the values of f(x) are as close to 5/2 as someone wants by taking x to be "big enough." The values of $f(x) = \dfrac{5x}{2x+3}$ may or may not ever equal 5/2 (they never do), but if x is "large," then f(x) is "close to" 5/2. Similarly, we can guarantee that the values of g(x) are as close to 0 as someone wants by taking x to be "big enough." The graphs of f and g are shown in Fig. 2 for "large" values of x.

x	$\dfrac{5x}{2x+3}$	$\dfrac{\sin(7x+1)}{3x}$
10	2.17	0.03170
100	2.463	–0.00137
1000	2.4962	0.00033
10,000	2.4996	0.000001

Fig. 1

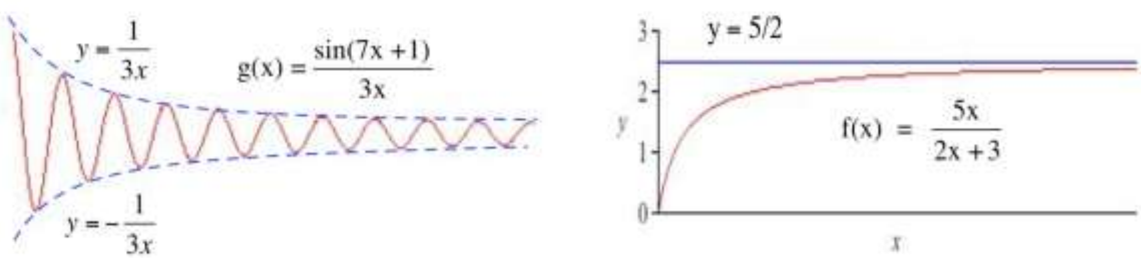

Fig. 2

Practice 1: What happens to the values of $f(x) = \frac{3x+4}{x-2}$ and $g(x) = \frac{\cos(5x)}{2x+7}$ as x becomes arbitrarily large?

The answers for Example 1 can be written as limit statements:

"As x becomes arbitrarily large,
the values of $\frac{5x}{2x+3}$ approach $\frac{5}{2}$ " can be written " $\lim_{x\to\infty} \frac{5x}{2x+3} = \frac{5}{2}$ " and

"the values of $\frac{\sin(7x+1)}{3x}$ approach 0." can be written " $\lim_{x\to\infty} \frac{\sin(7x+1)}{3x} = 0$."

The symbol " $\lim_{x\to\infty}$ " is read "the limit as x approaches infinity" and means "the limit as x becomes arbitrarily large" or as x increases without bound. (During this discussion and throughout this book, we do not treat "infinity" or "∞", as a number, but only as a useful notation. "Infinity" is not part of the real number system, and we use the common notation "$x\to\infty$" and the phrase "x approaches infinity" only to mean that "x becomes arbitrarily large." The notation "$x \to -\infty$," read as "x approaches negative infinity," means that the values of $-x$ become arbitrarily large.)

Practice 2: Write your answers to Practice 1 using the limit notation.

The $\lim_{x\to\infty} f(x)$ asks about the behavior of f(x) as the values of x get larger and larger without any bound, and one way to determine this behavior is to look at the values of f(x) at some values of x which are "large". If the values of the function get arbitrarily close to a single number as x gets larger and larger, then we will say that number is the limit of the function as x approaches infinity. A definition of the limit as "$x\to\infty$" is given at the end of this section.

x	$\frac{6x+7}{3-2x}$	$\frac{\sin(3x)}{x}$
10		
200		
5000		
20,000		

Fig. 3

Practice 3: Fill in the table in Fig. 3 for $f(x) = \frac{6x+7}{3-2x}$ and $g(x) = \frac{\sin(3x)}{x}$, and then use those values to estimate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$.

Example 2: How large does x need to be to guarantee that $f(x) = \frac{1}{x} < 0.1$? 0.001? $< E$ (assume $E>0$)?

Solution: If $x > 10$, then $\frac{1}{x} < \frac{1}{10} = 0.1$ (Fig. 4). If $x > 1000$, then $\frac{1}{x} < \frac{1}{1000} = 0.001$.

In general, if E is any positive number, then we can guarantee that $| f(x) | < E$ by picking only values of $x > \frac{1}{E} > 0$: if $x > \frac{1}{E}$, then $\frac{1}{x} < E$.

From this we can conclude that $\lim_{x\to\infty} \frac{1}{x} = 0$.

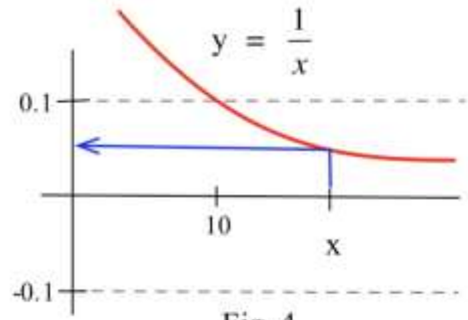

Fig. 4

Practice 4: How large does x need to be to guarantee that $f(x) = \frac{1}{x^2} < 0.1$? 0.001? $< E$ (assume $E>0$)?

Evaluate $\lim_{x\to\infty} \frac{1}{x^2}$.

The Main Limit Theorem (Section 1.2) about limits of combinations of functions is true if the limits as "$x\to a$" are replaced with limits as "$x\to\infty$", but we will not prove those results.

Polynomials occur commonly, and we often need the limit, as $x \to \infty$, of ratios of polynomials or functions containing powers of x. In those situations the following technique is often helpful:

(i) factor the highest power of x in the denominator from both the numerator and the denominator, and

(ii) cancel the common factor from the numerator and denominator.

The limit of the new denominator is a constant, so the limit of the resulting ratio is easier to determine.

Example 3: Determine $\lim_{x\to\infty} \frac{7x^2+3x-4}{3x^2-5}$ and $\lim_{x\to\infty} \frac{9x+2}{3x^2-5x+1}$.

Solutions: $\lim_{x\to\infty} \frac{7x^2+3x-4}{3x^2-5} = \lim_{x\to\infty} \frac{x^2(7+3/x-4/x^2)}{x^2(3-5/x^2)}$ factoring out x^2

$= \lim_{x\to\infty} \frac{7+3/x-4/x^2}{3-5/x^2} = \frac{7}{3}$ canceling x^2 and noting $\frac{3}{x}, \frac{4}{x^2}, \frac{5}{x^2} \to 0$.

Similarly, $\lim_{x\to\infty} \frac{9x+2}{3x^2-5x+1} = \lim_{x\to\infty} \frac{x^2(9/x-2/x^2)}{x^2(3-5/x+1/x^2)}$

$$= \lim_{x\to\infty} \frac{9/x - 2/x^2}{3 - 5/x + 1/x^2} = \frac{0}{3} = 0 .$$

If we have a difficult limit, as $x \to \infty$, it is often useful to algebraically manipulate the function into the form of a ratio and then use the previous technique.

If the values of the function oscillate and do not approach a single number as x becomes arbitrarily large, then the function does not have a limit as x approaches infinity: the limit Does Not Exist.

Example 4: Evaluate $\lim_{x\to\infty} \sin(x)$ and $\lim_{x\to\infty} x - [x]$.

Solution: $f(x) = \sin(x)$ and $g(x) = x - [x]$ do not have limits as $x \to \infty$. As x grows without bound, the values of $f(x) = \sin(x)$ oscillate between -1 and $+1$ (Fig. 5), and these values of $\sin(x)$ do not approach a single number. Similarly, $g(x) = x - [x]$ continues to take on values between 0 and 1, and these values are not approaching a single number.

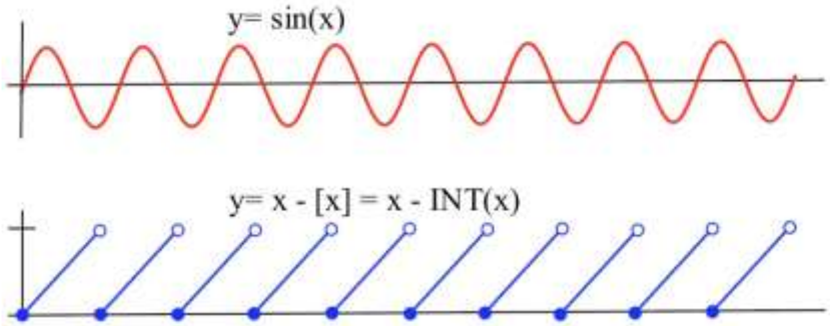

Fig. 5

Using Calculators To Help Find Limits as "$x \to \infty$" or "$x \to -\infty$"

Calculators only store a limited number of digits of a number, and this is a severe limitation when we are dealing with extremely large numbers.

Example: The value of $f(x) = (x + 1) - x$ is clearly equal to 1 for all values of x, and your calculator will give the right answer if you use it to evaluate f(4) or f(5). Now use it to evaluate f for a big value of x, say $x = 10^{40}$. $f(10^{40}) = (10^{40} + 1) - 10^{40} = 1$, but most calculators do not store 40 digits of a number, and they will respond that $f(10^{40}) = 0$ which is **wrong**. In this example the calculator's error is obvious, but the same type of errors can occur in less obvious ways when very large numbers are used on calculators.

You need to be careful with and somewhat suspicious of the answers your calculator gives you.

Calculators can still be helpful for examining some limits as "$x \to \infty$" and "$x \to -\infty$" as long as we do not place too much faith in their responses.

Even if you have forgotten some of the properties of natural logarithm function ln(x) and the cube root function $\sqrt[3]{x}$, a little experimentation on your calculator can help you determine that $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}} = 0$.

The Limit is Infinite

The function $f(x) = \frac{1}{x^2}$ is undefined at $x = 0$, but we can still ask about the behavior of f(x) for values of x "close to" 0. Fig. 6 indicates that if x is very small, close to 0, then f(x) is very large. As the values of x get closer to 0, the values of f(x) grow larger and can be made as large as we want by picking x to be close enough to 0. Even though the values of f are not approaching any number, we use the "infinity" notation to indicate that the values of f are growing without bound, and write

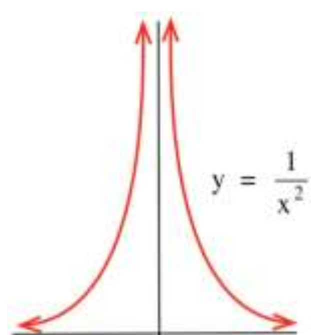

Fig. 6

$$\lim_{x\to 0} = \infty .$$

The values of $\frac{1}{x^2}$ do not equal "infinity:" $\lim_{x\to 0} = \infty$ means that the values of $\frac{1}{x^2}$ can be made arbitrarily large by picking values of x very close to 0.

The limit, as $x \to 0$, of $\frac{1}{x}$ is slightly more complicated. If x is close to 0, then the value of $f(x) = 1/x$ can be a large positive number or a large negative number, depending on the sign of x.

The function $f(x) = 1/x$ does not have a (two–sided) limit as x approaches 0, but we can still ask about one–sided limits:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty .$$

Example 5: Determine $\lim_{x\to 3^+} \frac{x-5}{x-3}$ and $\lim_{x\to 3^-} \frac{x-5}{x-3}$.

Solution: (a) As $x \to 3^+$, then $x - 5 \to -2$ and $x - 3 \to 0$. Since the denominator is approaching 0 we cannot use the Main Limit Theorem, and we need to examine the functions more carefully. If $x \to 3^+$, then $x > 3$ so $x - 3 > 0$. If x is close to 3 and slightly larger than 3, then the ratio of $x - 5$ to $x - 3$ is the ratio $\frac{\text{a number close to } -2}{\text{small positive number}}$ = large negative number. As $x > 3$ gets closer to 3, $\frac{x-5}{x-3}$ is $\frac{\text{a number closer to } -2}{\text{positive and closer to 0}}$ = larger negative number. By taking $x > 3$ closer to 3, the denominator gets closer to 0 but is always positive, so the ratio gets arbitrarily large and negative: $\lim_{x\to 3^+} \frac{x-5}{x-3} = -\infty$.

(b) As $x \to 3^-$, then $x - 5 \to -2$ and $x - 3$ gets arbitrarily close to 0, and $x - 3$ is negative. The value of the ratio $\frac{x-5}{x-3}$ is $\frac{\text{a number close to } -2}{\text{arbitrarily small negative number}}$ = arbitrarily large positive number: $\lim_{x\to 3^-} \frac{x-5}{x-3} = +\infty$.

Practice 5: Determine $\lim_{x\to 2^+} \frac{7}{2-x}$, $\lim_{x\to 2^+} \frac{3x}{2x-4}$, $\lim_{x\to 2^+} \frac{3x^2-6x}{x-2}$

Horizontal Asymptotes

The limits of f, as "$x \to \infty$" and "$x \to -\infty$," give us information about horizontal asymptotes of f.

> **Definition:** The line $y = K$ is a **horizontal asymptote** of f if $\lim_{x\to\infty} f(x) = K$ or $\lim_{x\to-\infty} f(x) = K$.

Example 6: Find any horizontal asymptotes of $f(x) = \dfrac{2x + \sin(x)}{x}$.

Solution: $\lim\limits_{x\to\infty} \dfrac{2x+\sin(x)}{x} = \lim\limits_{x\to\infty} \dfrac{2x}{x} + \dfrac{\sin(x)}{x} = 2 + 0 = 2$ so the line $\mathbf{y = 2}$ is a horizontal asymptote of f. The limit, as "$x \to -\infty$," is also 2 so $y = 2$ is the only horizontal asymptote of f. The graphs of f and $y = 2$ are given in Fig. 7 . A function may or may not cross its asymptote.

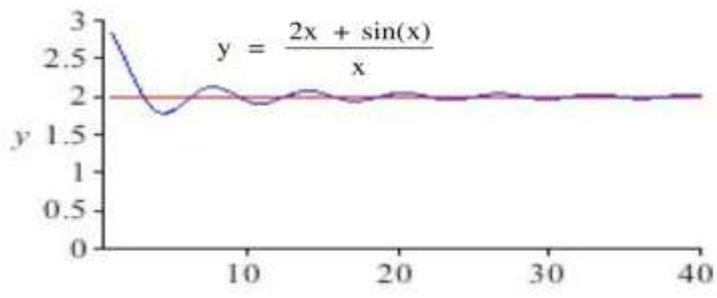

Fig. 7

Vertical Asymptotes

> **Definition:** The vertical line $x = a$ is a **vertical asymptote** of the graph of f if either or both of the one–sided limits, as $x \to a^-$ or $x \to a^+$, of f is infinite.

If our function f is the ratio of a polynomial P(x) and a polynomial Q(x), $f(x) = \dfrac{P(x)}{Q(x)}$, then the only **candidates** for vertical asymptotes are the values of x where $Q(x) = 0$. However, the fact that $Q(a) = 0$ is **not** enough to guarantee that the line $x = a$ is a vertical asymptote of f ; we also need to evaluate P(a). If $Q(a) = 0$ and $P(a) \neq 0$, then the line $x = a$ is a vertical asymptote of f. If $Q(a) = 0$ and $P(a) = 0$, then the line $x = a$ may or may not be a vertical asymptote.

Example 7: Find the vertical asymptotes of $f(x) = \dfrac{x^2 - x - 6}{x^2 - x}$ and $g(x) = \dfrac{x^2 - 3x}{x^2 - x}$.

Solution: $f(x) = \dfrac{x^2 - x - 6}{x^2 - x} = \dfrac{(x - 3)(x + 2)}{x(x - 1)}$ so the only values which make the denominator 0 are $x = 0$ and $x = 1$, and these are the only candidates to be vertical asymptotes.

$\lim\limits_{x\to 0^+} f(x) = +\infty$ and $\lim\limits_{x\to 1^+} f(x) = -\infty$ so $x = 0$ and $x = 1$ are both vertical asymptotes of f.

$g(x) = \dfrac{x^2 - 3x}{x^2 - x} = \dfrac{x(x - 3)}{x(x - 1)}$ so the only candidates to be vertical asymptotes are $x = 0$ and $x = 1$.

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} \frac{x(x-3)}{x(x-1)} = \lim_{x\to 1^+} \frac{x-3}{x-1} = -\infty$$ so x = 1 is a vertical asymptote of g.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{x(x-3)}{x(x-1)} = \lim_{x\to 0} \frac{x-3}{x-1} = 3 \neq \infty$$ so x = 0 is **not** a vertical asymptote.

Practice 6: Find the vertical asymptotes of $f(x) = \dfrac{x^2 + x}{x^2 + x - 2}$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$.

Other Asymptotes as "$x \to \infty$" and "$x \to -\infty$"

If the limit of f(x) as "$x \to \infty$" or "$x \to -\infty$" is a constant K, then the graph of f gets close to the horizontal line y = K , and we said that y = K was a horizontal asymptote of f. Some functions, however, approach other lines which are not horizontal.

Example 8: Find all asymptotes of $f(x) = \dfrac{x^2 + 2x + 1}{x} = x + 2 + \dfrac{1}{x}$.

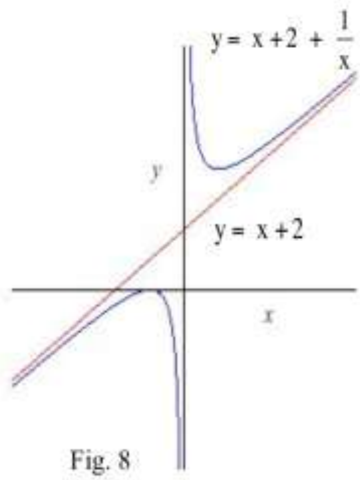

Fig. 8

Solution: If x is a large positive number or a large negative number,

then $\dfrac{1}{x}$ is very close to 0, and the graph of f(x) is very close to the line y = x + 2 (Fig. 8). The line **y = x + 2** is an asymptote of the graph of f.

If x is a large positive number, then 1/x is positive, and the graph of f is slightly above the graph of y = x + 2. If x is a large negative number, then 1/x is negative, and the graph of f will be slightly below the graph of y = x + 2. The 1/x piece of f never equals 0 so the graph of f never crosses or touches the graph of the asymptote y = x + 2.

The graph of f also has a vertical asymptote at x = 0 since $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to 0^-} f(x) = -\infty$.

Practice 7: Find all asymptotes of $g(x) = \dfrac{2x^2 - x - 1}{x + 1} = 2x - 3 + \dfrac{2}{x + 1}$.

Some functions even have **nonlinear asymptotes**, asymptotes which are not straight lines. The graphs of these functions approach some nonlinear function when the values of x are arbitrarily large.

Example 9: Find all asymptotes of $f(x) = \frac{x^4 + 3x^3 + x^2 + 4x + 5}{x^2 + 1} = x^2 + 3x + \frac{x + 5}{x^2 + 1}$.

Solution: When x is very large, positive or negative, then $\frac{x + 5}{x^2 + 1}$ is very close to 0, and the graph of f is very close to the graph of $g(x) = x^2 + 3x$. The function $\mathbf{g(x) = x^2 + 3x}$ is a nonlinear asymptote of f. The denominator of f is never 0, and f has no vertical asymptotes.

Practice 8: Find all asymptotes of $f(x) = \frac{x^3 + 2\sin(x)}{x} = x^2 + \frac{2\sin(x)}{x}$.

If f(x) can be written as a sum of two other functions, $f(x) = g(x) + r(x)$, with $\lim_{x\to\pm\infty} r(x) = 0$, then the graph of f is asymptotic to the graph of g , and g is an asymptote of f.

> Suppose $f(x) = g(x) + r(x)$ and $\lim_{x\to\pm\infty} r(x) = 0$:
>
> if $g(x) = K$, then f has a horizontal asymptote $y = K$;
>
> if $g(x) = ax + b$, then f has a linear asymptote $y = ax + b$; or
>
> if $g(x) =$ a nonlinear function, then f has a nonlinear asymptote $y = g(x)$.

Definition of $\lim_{x\to\infty} f(x) = K$

The following definition states precisely what is meant by the phrase "we can guarantee that the values of f(x) are arbitrarily close to K by using sufficiently large values of x."

Definition: $\lim_{x\to\infty} f(x) = K$

means

for every given $\varepsilon > 0$, there is a number N so that

if x is larger than N

then f(x) is within ε units of K.

(equivalently; $| f(x) - K | < \varepsilon$ whenever $x > N$.)

Example 10: Show that $\lim_{x\to\infty} \frac{x}{2x+1} = \frac{1}{2}$.

Solution: Typically we need to do two things. First we need to find a value of N, usually depending on ε. Then we need to show that the value of N we found satisfies the conditions of the definition.

(i) Assume that $| f(x) - K |$ is less than ε and solve for $x > 0$.

If $\varepsilon > \left| \frac{x}{2x+1} - \frac{1}{2} \right| = \left| \frac{2x-(2x+1)}{2(2x+1)} \right| = \left| \frac{-1}{4x+2} \right| = \frac{1}{4x+2}$, then $4x + 2 > \frac{1}{\varepsilon}$ and

$x > \frac{1}{4}\left(\frac{1}{\varepsilon} - 2 \right)$. For any $\varepsilon > 0$, take $N = \frac{1}{4}\left(\frac{1}{\varepsilon} - 2 \right)$.

(ii) For any $\varepsilon > 0$, take $N = \frac{1}{4}\left(\frac{1}{\varepsilon} - 2 \right)$. (Now we can just reverse the order of the steps in part (i).)

If $x > 0$ and $x > N = \frac{1}{4}\left(\frac{1}{\varepsilon} - 2 \right)$,

then $4x + 2 > \frac{1}{\varepsilon}$ so $\varepsilon > \frac{1}{4x+2} = \left| \frac{x}{2x+1} - \frac{1}{2} \right| = | f(x) - K |$.

We have shown that "for every given ε, there is an N" that satisfies the definition.

PROBLEMS

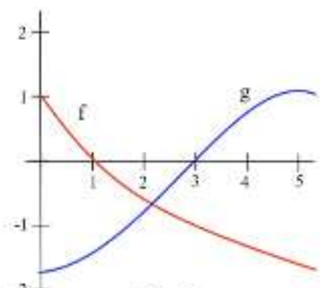

Fig. 9

1. Fig. 9 shows f(x) and g(x) for $0 \le x \le 5$. Let $h(x) = \frac{f(x)}{g(x)}$.
   (a) At what value of x does h(x) have a root?
   (b) Determine the limits of h(x) as $x \to 1^+$, $x \to 1^-$, $x \to 3^+$, and $x \to 3^-$.
   (c) Where does h(x) have a vertical asymptote?

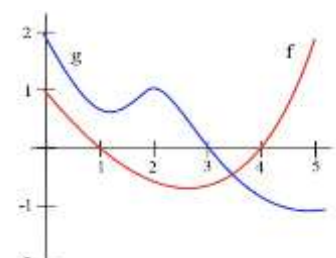

Fig. 10

2. Fig. 10 shows f(x) and g(x) for $0 \le x \le 5$. Let $h(x) = \frac{f(x)}{g(x)}$.
   (a) At what value(s) of x does h(x) have a root?
   (b) Where does h(x) have vertical asymptotes?

3. Fig. 11 shows f(x) and g(x) for $0 \le x \le 5$. Let $h(x) = \frac{f(x)}{g(x)}$, and determine the limits of h(x) as $x \to 2^+$, $x \to 2^-$, $x \to 4^+$, and $x \to 4^-$.

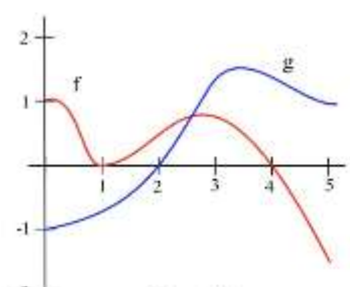

Fig. 11

For problems 4 – 24, calculate the limit of each expression as "$\mathbf{x} \to \infty$."

4. $\frac{6}{x+2}$ 5. $\frac{28}{3x-5}$

6. $\frac{7x+12}{3x-2}$ 7. $\frac{4-3x}{x+8}$

8. $\dfrac{5\sin(2x)}{2x}$

9. $\dfrac{\cos(3x)}{5x-1}$

10. $\dfrac{2x-3\sin(x)}{5x-1}$

11. $\dfrac{4+x\cdot\sin(x)}{2x-3}$

12. $\dfrac{x^2-5x+2}{x^2+8x-4}$

13. $\dfrac{2x^2-9}{3x^2+10x}$

14. $\dfrac{\sqrt{x+5}}{\sqrt{4x-2}}$

15. $\dfrac{5x^2-7x+2}{2x^3+4x}$

16. $\dfrac{x+\sin(x)}{x-\sin(x)}$

17. $\dfrac{7x^2+x\cdot\sin(x)}{3-x^2+\sin(7x^2)}$

18. $\dfrac{7x^{143}+734x-2}{x^{150}-99x^{83}+25}$

19. $\dfrac{\sqrt{9x^2+16}}{2+\sqrt{x^3+1}}$

20. $\sin\left(\dfrac{3x+5}{2x-1}\right)$

21. $\cos\left(\dfrac{7x+4}{x^2+x+1}\right)$

22. $\ln\left(\dfrac{3x^2+5x}{x^2-4}\right)$

23. $\ln(x+8)-\ln(x-5)$

24. $\ln(3x+8)-\ln(2x+1)$

25. Salt water with a concentration of 0.2 pounds of salt per gallon flows into a large tank that initially contains 50 gallons of pure water.
   (a) If the flow rate of salt water into the tank is 4 gallons per minute, what is the volume $V(t)$ of water and the amount $A(t)$ of salt in the tank t minutes after the flow begins?
   (b) Show that the salt concentration $C(t)$ at time t is $C(t) = \dfrac{.8t}{4t+50}$.
   (c) What happens to the concentration $C(t)$ after a "long" time?
   (d) Redo parts (a) – (c) for a large tank which initially contains 200 gallons of pure water.

26. Under certain laboratory conditions, an agar plate contains $B(t) = 100\left(2 - \dfrac{1}{e^t}\right) = 100(2-e^{-t})$ bacteria t hours after the start of the experiment.
   (a) How many bacteria are on the plate at the start of the experiment ($t = 0$)?
   (b) Show that the population is always increasing. (Show $B'(t) > 0$ for all $t > 0$.)
   (c) What happens to the population $B(t)$ after a "long" time?
   (d) Redo parts (a) – (c) for $B(t) = A\left(2-\dfrac{1}{e^t}\right) = A(2-e^{-t})$.

For problems 27 – 41 , calculate the limits.

27. $\lim\limits_{x \to 0} \frac{x+5}{x^2}$

28. $\lim\limits_{x \to 3} \frac{x-1}{(x-3)^2}$

29. $\lim\limits_{x \to 5} \frac{x-7}{(x-5)^2}$

30. $\lim\limits_{x \to 2^+} \frac{x-1}{x-2}$

31. $\lim\limits_{x \to 2^-} \frac{x-1}{x-2}$

32. $\lim\limits_{x \to 2} \frac{x-1}{x-2}$

33. $\lim\limits_{x \to 4^+} \frac{x+3}{4-x}$

34. $\lim\limits_{x \to 1^-} \frac{x^2+5}{1-x}$

35. $\lim\limits_{x \to 3^+} \frac{x^2-4}{x^2-2x-3}$

36. $\lim\limits_{x \to 2} \frac{x^2-x-2}{x^2-4}$

37. $\lim\limits_{x \to 0} \frac{x-2}{1-\cos(x)}$

38. $\lim\limits_{x \to \infty} \frac{x^3+7x-4}{x^2+11x}$

39. $\lim\limits_{x \to 5} \frac{\sin(x-5)}{x-5}$

40. $\lim\limits_{x \to 0} \frac{x+1}{\sin^2(x)}$

41. $\lim\limits_{x \to 0^+} \frac{1+\cos(x)}{1-e^x}$

In problems 42 – 50, write the **equation** of each asymptote for each function and state whether it is a vertical or horizontal asymptote.

42. $f(x) = \frac{x+2}{x-1}$

43. $f(x) = \frac{x-3}{x^2}$

44. $f(x) = \frac{x-1}{x^2-x}$

45. $f(x) = \frac{x+5}{x^2-4x+3}$

46. $f(x) = \frac{x+\sin(x)}{3x-3}$

47. $f(x) = \frac{x^2-4}{x^2+1}$

48. $f(x) = \frac{\cos(x)}{x^2}$

49. $f(x) = 2 + \frac{3-x}{x-1}$

50. $f(x) = \frac{x \cdot \sin(x)}{x^2-x}$

In problems 51 – 59, write the **equation** of each asymptote for each function.

51. $f(x) = \frac{2x^2+x+5}{x}$

52. $f(x) = \frac{x^2+x}{x+1}$

53. $f(x) = \frac{1}{x-2} + \sin(x)$

54. $f(x) = x + \frac{x}{x^2+1}$

55. $f(x) = x^2 + \frac{x}{x^2+1}$

56. $f(x) = x^2 + \frac{x}{x+1}$

57. $f(x) = \frac{x \cdot \cos(x)}{x-3}$

58. $f(x) = \frac{x^3-x^2+2x-1}{x-1}$

59. $f(x) = \sqrt{\frac{x^2+3x+2}{x+3}}$

Section 4.6 **PRACTICE Answers**

Practice 1: As x becomes arbitrarily large, the values of f(x) approach 3 and the values of g(x) approach 0.

Practice 2: $\lim_{x\to\infty} \frac{3x+4}{x-2} = 3$ and $\lim_{x\to\infty} \frac{\cos(5x)}{2x+7} = 0$

Practice 3: The completed table is shown in Fig. 12.

Practice 4: If $x > \sqrt{10} \approx 3.162$, then $f(x) = \frac{1}{x^2} < 0.1$.

If $x > \sqrt{1000} \approx 31.62$, then $f(x) = \frac{1}{x^2} < 0.001$.

If $x > \sqrt{1/E}$, then $f(x) = \frac{1}{x^2} < E$.

Practice 5: (a) $\lim_{x\to 2^+} \frac{7}{2-x} = -\infty$.

As $x \to 2^+$ the values $2 - x \to 0$, and $x > 2$ so $2 - x < 0$: $2 - x$ takes small negative values.

Then the values of $\frac{7}{2-x} = \frac{7}{\text{small negative values}}$ are large negative values so we represent the limit as "$-\infty$."

(b) $\lim_{x\to 2^+} \frac{3x}{2x-4} = +\infty$.

As $x \to 2^+$ the values of $2x - 4 \to 0$, and $x > 2$ so $2x - 4 > 0$: $2x - 4$ takes small positive values. As $x \to 2^+$ the values of $3x \to +6$.

Then the values of $\frac{3x}{2x-4} = \frac{\text{values near } +6}{\text{small positive values}}$ are large positive values so we represent the limit as "$+\infty$."

(c) $\lim_{x\to 2^+} \frac{3x^2-6x}{x-2} = 6$.

As $x \to 2^+$, the values of $3x^2 - 6x \to 0$ and $x - 2 \to 0$ so we need to do more work. The numerator can be factored $3x^2 - 6x$

x	$\frac{6x+7}{3-2x}$	$\frac{\sin(3x)}{x}$
10	**–3.94117647**	**–0.09880311**
200	**–3.04030227**	**0.00220912**
5000	**–3.00160048**	**0.00017869**
20,000	**–3.00040003**	**0.00004787**
	$\downarrow$ **–3**	$\downarrow$ **0**

Fig. 12

$= 3x(x-2)$ and then the rational function can be reduced (since $x \to 2$ we know $x \neq 2$):

$$\lim_{x \to 2^+} \frac{3x^2 - 6x}{x-2} = \lim_{x \to 2^+} \frac{3x(x-2)}{x-2} = \lim_{x \to 2^+} 3x = 6 .$$

Practice 6: (a) $f(x) = \dfrac{x^2 + x}{x^2 + x - 2} = \dfrac{x(x+1)}{(x-1)(x+2)}$.

f has vertical asymptotes at $x = 1$ and $x = -2$.

(b) $g(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{x-1}$.

The value $x = 1$ is not in the domain of g. If $x \neq 1$, then $g(x) = x + 1$.

g has a "hole" when $x = 1$ and no vertical asymptotes.

Practice 7: $g(x) = 2x - 3 + \dfrac{2}{x+1}$.

g has a vertical asymptote at $x = -1$.

g has no horizontal asymptotes.

$\displaystyle\lim_{x \to \infty} \frac{2}{x+1} = 0$ so g has the linear asymptote $y = 2x - 3$.

Practice 8: $f(x) = x^2 + \dfrac{2 \cdot \sin(x)}{x}$.

f is not defined at $x = 0$, so f has a vertical asymptote or a "hole" when $x = 0$.

$\displaystyle\lim_{x \to 0} x^2 + \frac{2 \cdot \sin(x)}{x} = 0 + 2 = 2$ so f has a "hole" when $x = 0$.

$\displaystyle\lim_{x \to \infty} \frac{2 \cdot \sin(x)}{x} = 0$ so f has the nonlinear asymptote $y = x^2$.

Appendix: MAPLE, infinite limits and limits as "$x \to \infty$"

command	output	comment
limit(1/x, x = 0);	undefined	Maple uses the convention "x=0" even though what we really mean is "x approaches 0"
limit(1/x, x=0, right);	∞	"x=0, right" means $0 \le x$
limit(1/x. x=0, left);	$-\infty$	"x=0, left" means $x \ge 0$
limit(sin(x)/x, x = infinity);	0	
limit(2*x, x= infinity);	∞	
limit(5-3*x, x= infinity);	$-\infty$	
limit((1=cos(x))/(1-exp(x)). x = 0, right);	$-\infty$	