4.5 APPLIED MAXIMUM AND MINIMUM PROBLEMS

We have used derivatives to help find the maximums and minimums of some functions given by equations, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme values. More typically, someone will describe a problem and ask your help in maximizing or minimizing something: "What is the largest volume package which the post office will take?"; "What is the quickest way to get from here to there?"; or "What is the least expensive way to accomplish some task?" Usually these problems have some restrictions or constraints on what is allowed, and sometimes neither the problem nor the constraints are clearly stated.

Before we can use calculus or other mathematical techniques to solve the max/min problem, we need to **understand** what is really being asked. We need to **translate** the problem into a mathematical form which we can **solve**, and we need to **check** our mathematical solution to see if it is really a solution of the original problem. Often, the hardest parts of the problem are understanding the problem and translating it into a mathematical form.

In this section we examine some problems which require understanding, translation, solution, and checking. Most of these problems are not as complicated as those a working scientist, engineer or economist needs to solve, but they represent a step in developing the required skills.

Example 1: The company you own has a large supply of 8 inch by 15 inch rectangular pieces of tin, and you decide to make them into boxes by cutting a square from each corner and folding up the sides (Fig. 1). For example, if you cut a 1 inch square from each corner the resulting 6 inch by 13 inch by 1 inch box has a volume of 78 cubic inches. The amount of money you get for a box depends on how much the box holds, so you want to make boxes with the largest possible volumes. How large a square should you cut from each corner?

Solution: First we need to understand the problem, and a diagram can be very helpful. Then we need to translate it into a mathematical problem:

* **identify the variables**,
* **label the variable and constant parts of the diagram**, and
* **represent the quantity to be maximized as a function.**

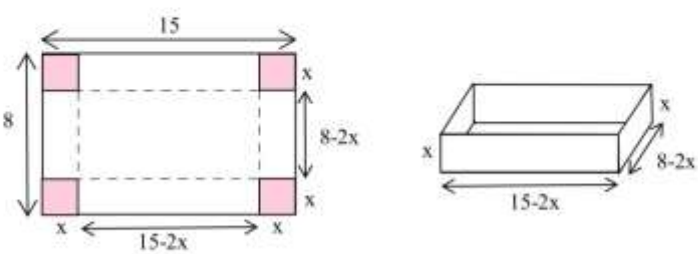

Fig. 1

If we label the side of the square as x inches, then the box is x inches high, $8 - 2x$ inches wide, and $15 - 2x$ inches long, so the volume is (length)·(width)·(height) $= (15 - 2x)\cdot(8 - 2x)\cdot(x)$ $= 4x^3 - 46x^2 + 120x$ cubic inches. Now we have a mathematical problem, maximize $V(x) = 4x^3 - 46x^2 + 120x$, and we can use the calculus techniques from the previous sections.

$V'(x) = 12x^2 - 92x + 120$, and we need to find the critical points. (i) We can find where $V'(x) = 0$ by factoring or using the quadratic formula: $V'(x) = 12x^2 - 92x + 120 = 4(3x - 5)(x - 6) = 0$ if $x = 5/3$ or $x = 6$, so $x = 5/3$ and $x = 6$ are critical points of V. (ii) $V'(x)$ is a polynomial so it is always defined and there are no critical points from an undefined derivative. (iii) What are the endpoints for x in this problem? A square cannot have a negative length so $x \geq 0$. We cannot remove more than half of the width, so $8 - 2x \geq 0$ and $x \leq 4$. Together, these two inequalities say that $0 \leq x \leq 4$, so the endpoints are $x = 0$ and $x = 4$. (Note that the value $x = 6$ is not in this interval, so $x = 6$ does not maximize the volume and we do not consider it further.)

The maximum volume must occur at one of the critical points $x = 0, 5/3$, or 4: $V(0) = 0$,
$V(5/3) = 2450/27 \approx 90.74$ cubic inches, and $V(4) = 0$. The maximum volume of the box occurs when a 5/3 inch by 5/3 inch square is removed from each corner, and resulting box is 5/3 inches high,
$8 - 2(5/3) = 14/3$ inches wide, and $15 - 2(5/3) = 35/3$ inches long.

Practice 1: If you start with 7 inch by 15 inch pieces of tin, what size square should you remove from each corner so the box will have as large a volume as possible?
(Hint: $12x^2 - 88x + 105 = (2x - 3)(6x - 35)$)

We were fortunate in the previous example and practice problem because the functions we created to describe the volume were functions of only one variable. In some problems, the function we get will have more than one variable, and we will need to use additional information to change our function into a function of one variable. Typically the constraints will contain the additional information we need.

Example 2: We want to fence a rectangular area in our backyard for a garden. One side of the garden is along the edge of the yard which is already fenced, so we only need to build a new fence along the other 3 sides of the rectangle (Fig. 2). If we have 80 feet of fencing available, what dimensions should the garden have in order to enclose the largest possible area?

Fig. 2

Solution: The first step is to understand the problem, and a **diagram** or picture of the situation often helps. Next, we need to **identify the variables**: in this case, the length, call it x, and width, call it y, of the garden. Fig. 3 shows the **labeled diagram** so now we can write a formula for the function which we want to maximize:

Maximize A = area of the rectangle = (length)•(width) = $x \cdot y$.

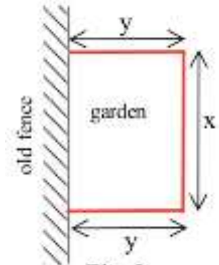

Fig. 3

Unfortunately, our function A has two variables, x and y , so we need to find a relationship between them (an equation containing both x and y) which we can solve for one of x or y. The constraint in this problem says that "we have 80 feet of fencing available" so $x + 2y = 80$ and $y = 40 - (x/2)$. Then $A = x \cdot y = x \cdot (40 - (x/2)) = 40x - \frac{x^2}{2}$, a function of one variable. We want to maximize A.

$A' = 40 - x$. The only time $A' = 0$ is when $x = 40$, so $x = 40$ so there is only one critical point of type (i). A is differentiable for all x so there are no critical numbers of the type (ii). Finally, $0 \le x \le 80$ (why?) so the only critical points of type (iii) are when $x = 0$ and $x = 80$. The only critical points of A are when $x = 0, 40$, and 80 , and the maximum area occurs at one of them:

at the critical number $x = 0$, $A = 40(0) - \frac{(0)^2}{2} = 0$ square feet

at the critical number $x = 40$, $A = 40(40) - \frac{(40)^2}{2} = 800 \text{ ft}^2$

at the critical number $x = 80$, $A = 40(80) - \frac{(80)^2}{2} = 0 \text{ ft}^2$

so the largest rectangular garden has an area of 800 square feet and dimensions $x = 40$ feet by $y = 40 - (x/2) = 40 - (40/2) = 20$ feet.

Practice 2: Suppose you decide to fence the rectangular garden in the corner of your yard. Then two sides of the garden are bounded by the yard fence which is already there, so you only need to use the 80 feet of fencing to enclose the other two sides. What are the dimensions of the new garden of largest area? What are the dimensions of the rectangular garden of largest area in the corner of the yard if you have F feet of new fencing available?

Example 3: You need to reach home as quickly as possible, but you are in a rowboat 4 miles from shore and your home is 2 miles up the coast (Fig. 4). If you can row at 3 miles per hour and walk at 5 miles per hour, toward which point on the shore should you row? Toward which point should you row if your home is 7 miles up the coast?

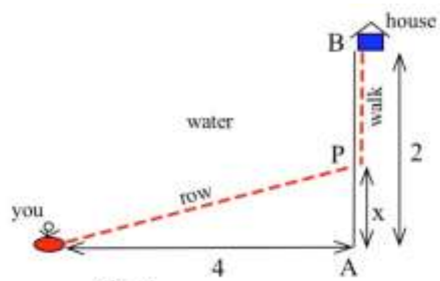

Fig. 4

Solution: Fig. 4 shows a labeled diagram with the variable x representing the distance from point A, the nearest shore point, to point P, the point you row toward. Then the total time, rowing and walking, is

T = total time = (rowing time from boat to P) + (walking time from P to B)

= (distance from boat to P)/(rate from boat to P) + (distance from P to B)/(rate from P to B)

$$= \sqrt{x^2 + 4^2}/3 + (2 - x)/5 = \frac{\sqrt{x^2 + 16}}{3} + \frac{2 - x}{5} .$$

It is not reasonable to row to a point below A and then walk home, so $x \ge 0$. Similarly, we can conclude that $x \le 2$, so our interval is $0 \le x \le 2$ and the endpoints are $x = 0$ and $x = 2$.

To find the other critical numbers of T between $x = 0$ and $x = 2$, we need the derivative of T.

$$T'(x) = \frac{1}{3} \cdot \frac{1}{2}(x^2 + 16)^{-1/2}(2x) - \frac{1}{5} = \frac{x}{3\sqrt{x^2 + 16}} - \frac{1}{5} .$$

To find where T '(x) is zero, set T '(x) = 0 and solve:

$$T'(x) = \frac{x}{3\sqrt{x^2 + 16}} - \frac{1}{5} = 0 \text{ so } \frac{x}{3\sqrt{x^2 + 16}} = \frac{1}{5} \text{ and}$$

$5x = 3\sqrt{x^2 + 16}$ so $25x^2 = 9x^2 + 144$ and $x = \pm 3$. Neither of these numbers, however, is in our interval $0 \le x \le 2$ so neither of them gives a minimum time.

T is differentiable for all values of x, so there are no critical numbers of type (ii).

The only critical numbers for T on this interval are $x = \mathbf{0}$ and $x = \mathbf{2}$: $T(\mathbf{0}) = \frac{\sqrt{\mathbf{0} + 16}}{3} + \frac{2 - \mathbf{0}}{5} = \frac{4}{3} + \frac{2}{5} \approx 1.73$ hours and $T(\mathbf{2}) = \frac{\sqrt{\mathbf{2}^2 + 16}}{3} + \frac{2 - \mathbf{2}}{5} = \frac{\sqrt{20}}{3} + 0 \approx 1.49$ hours. The quickest route is when P is 2 miles down the coast. You should row directly toward home.

If your home is 7 miles down the coast, then the interval for x is $0 \le x \le 7$ which has the endpoints $x = 0$ and $x = 7$. Our function for the travel time is $T(x) = \frac{\sqrt{x^2 + 16}}{3} + \frac{\mathbf{7} - x}{5}$ and

$T'(x) = \frac{x}{3\sqrt{x^2 + 16}} - \frac{1}{5}$ so the only point in our interval where $T(x)' = 0$ is at $x = 3$.

The only critical numbers for T in the interval are x = **0**, x = **3**, and x = **7**:

$$T(\mathbf{0}) = \frac{\sqrt{\mathbf{0}^2 + 16}}{3} + \frac{7 - \mathbf{0}}{5} = \frac{4}{3} + \frac{7}{5} \approx 2.73 \text{ hours}$$

$$T(\mathbf{7}) = \frac{\sqrt{\mathbf{7}^2 + 16}}{3} + \frac{7 - \mathbf{7}}{5} = \frac{\sqrt{65}}{3} + 0 \approx 2.68 \text{ hours}$$

$$T(\mathbf{3}) = \frac{\sqrt{\mathbf{3}^2 + 16}}{3} + \frac{7 - \mathbf{3}}{5} = \frac{5}{3} + \frac{4}{5} \approx 2.47 \text{ hours.}$$

The quickest way home is to aim for a point P which is 3 miles down the coast, row directly to P, and then walk along the coast to home.

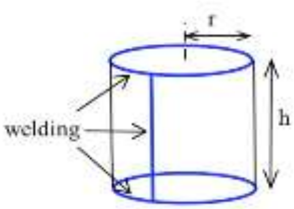

Fig. 5

One challenge of max/min problems is that they may require geometry or trigonometry or other mathematical facts and relationships.

Example 4: Find the height and radius of the least expensive closed cylinder which has a volume of 1000 cubic inches. Assume that the materials are free, but that it costs 80¢ per inch to weld the top and bottom onto the cylinder and to weld the seam up the side of the cylinder (Fig. 5).

Solution: If we let r be the radius of the cylinder and h be its height, then the volume $V = \pi r^2 h = 1000$. The function we want to minimize is cost, and

C = total welding cost = (top seam cost) + (bottom seam cost) + (side seam cost)

= (top seam length)·(80¢/inch) + (bottom seam length)·(80¢/in) + (side seam length)·(80¢/in)

= (2πr)·(80) + (2πr)·(80) + (h)·(80) = 320πr + 80 h.

Unfortunately, our function C is a function of two variables, r and h, but we can use the information in the constraint, $V = \pi r^2 h = 1000$, to solve for h and then substitute this h into the formula for C: $1000 = \pi r^2 h$ so $h = \frac{1000}{\pi r^2}$ and then $C = 320\pi r + 80\,\mathbf{h} = 320\pi r + 80(\frac{\mathbf{1000}}{\boldsymbol{\pi} \mathbf{r^2}})$, a function of one variable. $C' = 320\pi - \frac{160000}{\pi r^3}$, and C is a minimum when $C' = 0$: at

$$r = \sqrt[3]{\frac{500}{\pi^2}} \approx 3.7 \text{ inches and } h = \frac{1000}{\pi r^2} \approx \frac{1000}{\pi (3.7)^2} \approx 23.3 \text{ inches.}$$

Practice 3: Find the height and radius of the least expensive closed cylinder which has a volume of 1000 cubic inches. Assume that the only cost for this cylinder is the cost of the materials: the material

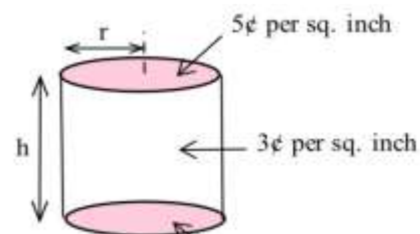

Fig. 6

for the top and bottom costs 5¢ per square inch, and the material for the sides costs 3¢ per square inch (Fig. 6).

Example 5: Find the dimensions of the least expensive rectangular box which is three times as long as it is wide and which holds 100 cubic centimeters of water. The material for the bottom costs 7¢ per cm^2, the sides cost 5¢ per cm^2 and the top costs 2¢ per cm^2.

Solution: Label the box so w = width, l = length, and h = height. Then our cost function C is

$$\begin{aligned} C &= (\text{bottom cost}) + (\text{cost of front and back}) + (\text{cost of ends}) + (\text{top cost}) \\ &= (\text{bottom area})\cdot(7¢) + (\text{front and back area})\cdot(5¢) + (\text{ends area})\cdot(5¢) + (\text{top area})\cdot(2¢) \\ &= (wl)\cdot(7) + (2lh)\cdot(5) + (2wh)\cdot(5) + (wl)\cdot(2) = 7wl + 10lh + 10wh + 2wl = 9(wl) + 10(lh) + 10(wh). \end{aligned}$$

Unfortunately, C is a function of 3 variables, w, l, and h, but we can use the other information in the constraints to eliminate some of the variables:

the box is "three times as long as it is wide" so $l = 3w$ and

$C = 9w\mathbf{l} + 10\mathbf{l}h + 10wh = 9w(\mathbf{3w}) + 10(\mathbf{3w})h + 10wh = 27w^2 + 40wh$.

We also know that the volume V is 100 in^3 and $V = lwh = 3w^2h$ (since $l = 3w$), so $h = \dfrac{100}{3w^2}$.

Then $C = 27w^2 + 40w\mathbf{h} = 27w^2 + 40w\left(\dfrac{100}{3w^2}\right) = 27w^2 + \dfrac{4000}{3w}$, a function of one variable.

$C' = 54w - \dfrac{4000}{3w^2}$, and C is minimized when $w = \sqrt[3]{\dfrac{4000}{162}} \approx 2.91$ inches ($l = 3w \approx 8.73$ inches, and $h = \dfrac{100}{3w^2} \approx 3.94$ inches). The minimum cost is approximately \$6.87 .

Problems described in words are usually more difficult to solve because we first need to understand and "translate" the problem into a mathematical problem, and, unfortunately, those skills only seem to come with practice. With practice, however, you will start to recognize patterns for understanding, translating, and solving these problems, and you will develop the skills you need. So read carefully, draw pictures, think hard, and do the best you can.

Problems

1. (a) You have 200 feet of fencing to enclose a rectangular vegetable garden. What should the dimensions of your garden be in order to enclose the largest area?
   (b) Show that if you have P feet of fencing available, the garden of greatest area is a square.
   (c) What are the dimensions of the largest rectangular garden you can enclose with P feet of fencing if one edge of the garden borders a straight river and does not need to be fenced?

(d) Just thinking — calculus will not help with this one: What do you think is the shape of the largest garden which can be enclosed with P feet of fencing if we do not require the garden to be rectangular? What do you think is the shape of the largest garden which can be enclosed with P feet of fencing if one edge of the garden borders a river and does not need to be fenced?

2. (a) You have 200 feet of fencing available to construct a rectangular pen with a fence divider down the middle (see Fig. 7). What dimensions of the pen enclose the largest total area?

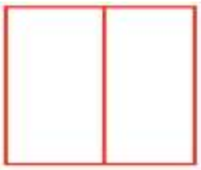
Fig. 7

(b) If you need 2 dividers, what dimensions of the pen enclose the largest area?

(c) What are the dimensions in parts (a) and (b) if one edge of the pen borders on a river and does not require any fencing?

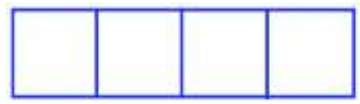
Fig. 8

3. You have 120 feet of fencing to construct a pen with 4 equal sized stalls.

(a) If the pen is rectangular and shaped like the Fig. 8, what are the dimensions of the pen of largest area and what is that area?

(b) The square pen in Fig. 9 uses 120 feet of fencing and encloses a larger area (400 square feet) than the best design in part (a). Design a pen which uses only 120 feet of fencing and has 4 equal sized stalls but which encloses even more than 400 square feet. (Suggestion: don't use rectangles and squares.)

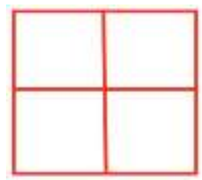
Fig. 9

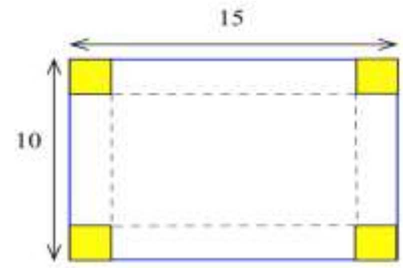

Fig. 10

4. (a) You have a 10 inch by 15 inch piece of tin which you plan to form into a box (without a top) by cutting a square from each corner and folding up the sides (see Fig. 10). How much should you cut from each corner so the resulting box has the greatest volume?

   (b) If the piece of tin is A inches by B inches, how much should you cut from each corner so the resulting box has the greatest volume?

5. You have a 10 inch by 10 inch piece of cardboard which you plan to cut and fold as shown in Fig. 11 to form a box with a top. Find the dimensions of the box which has the largest volume.

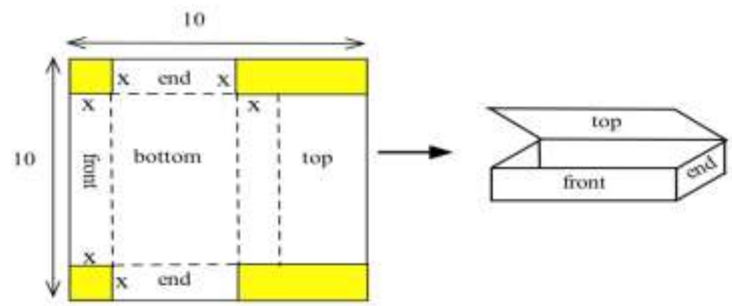

Fig. 11

6. (a) You have been asked to bid on the construction of a square-bottomed box with no top which will hold 100 cubic inches of water. If the bottom and sides are made from the same material, what are the dimensions of the box which uses the least material? (Assume that no material is wasted.)

   (b) Suppose the box in part (a) uses different materials for the bottom and the sides. If the bottom material costs 5¢ per square inch and the side material costs 3¢ per square inch, what are the dimensions of the least expensive box which will hold 100 cubic inches of water?

   (This is a "classic" problem which has many variations. We could require that the box be twice as long as it is wide, or that the box have a top, or that the ends cost a different amount than the front and back, or even that it costs some amount of money to weld each inch of edge. You should be able to set up the cost equations for these variations.)

7. (a) Determine the dimensions of the least expensive cylindrical can which will hold 100 cubic inches if the materials cost 2¢, 5¢ and 3¢ respectively for the top, bottom and sides.

   (b) How do the dimensions of the least expensive can change if the bottom material costs more than 5¢ per square inch?

8. You have 100 feet of fencing to build a pen in the shape of a circular sector, the "pie slice" in Fig. 12. The area of such a sector is rs/2.
   (a) What value of r maximizes the enclosed area?
   (b) What is the central angle when the area is maximized?

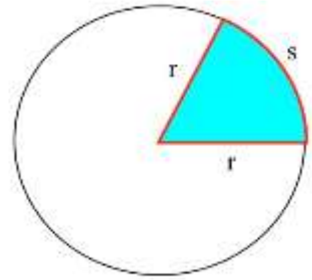

Fig. 12

9. You are a lifeguard standing at the edge of the water when you notice a swimmer in trouble (Fig. 13). Assuming you can run about 8 meters per second and swim about 2 m/s, how far along the shore should you run before diving into the water in order to reach the swimmer as quickly as possible?

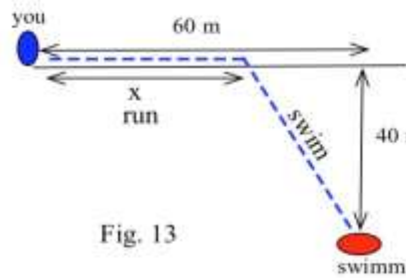

Fig. 13

10. (a) You have been asked to determine the least expensive route for a telephone cable which connects Andersonville with Beantown (see Fig. 14). If it costs \$5000 per mile to lay the cable on land and \$8000 per mile to lay the cable across the river and the cost of the cable is negligible, find the least expensive route.
    (b) What is the least expensive route if the cable costs \$7000 per mile plus the cost to lay it.

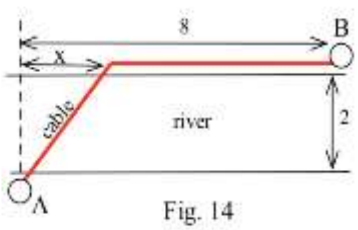

Fig. 14

11. You have been asked to determine where a water works should be built along a river between Chesterville and Denton (see Fig. 15) to minimize the total cost of the pipe to the towns.
    (a) Assume that the same size (and cost) pipe is used to each town. (This part can be done quickly without using calculus.)
    (b) Assume that the pipe to Chesterville costs \$3000 per mile and to Denton it costs \$7000 per mile.

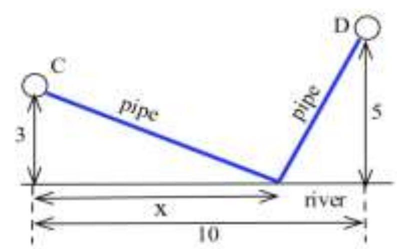

Fig. 15

12. Light from a bulb at A is reflected off a flat mirror to your eye at point B (see Fig. 16). If the time (and length of the path) from A to the mirror and then to your eye is a minimum, show that the angle of incidence equals the angle of reflection. (Hint: This is similar to the previous problem.)

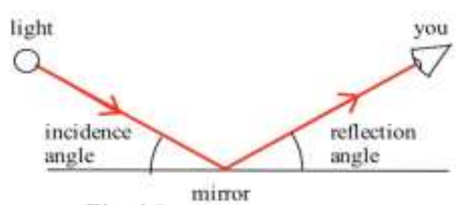

Fig. 16

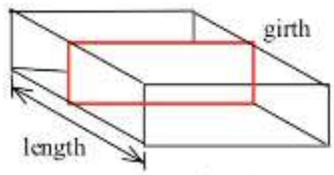

Fig. 17

13. U.S. postal regulations state that the sum of the length and girth (distance around) of a package must be no more than 108 inches. (Fig. 17)

(a) Find the dimensions of the acceptable box with a square end which has the largest volume.

(b) Find the dimensions of the acceptable box which has the largest volume if its end is a rectangle twice as long as it is wide.

(c) Find the dimensions of the acceptable box with a circular end which has the largest volume.

14. Just thinking — you don't need calculus for this problem: A spider and a fly are located on opposite corners of a cube (see Fig. 18). What is the shortest path along the surface of the cube from the spider to the fly?

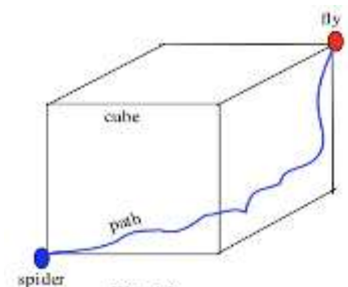

Fig. 18

15. Two sides of a triangle are 7 and 10 inches respectively. What is the length of the third side so the area of the triangle will be greatest? (This problem can be done without using calculus. How? If you do use calculus, consider the angle θ between the two sides.)

16. Find the shortest distance from the point (2,0) to the curve

(a) $y = 3x - 1$ (b) $y = x^2$ (c) $x^2 + y^2 = 1$ (d) $y = \sin(x)$

17. Find the dimensions of the rectangle with the largest area if the base must be on the x-axis and its other two corners are on the graph of

(a) $y = 16 - x^2$ on $[-4, 4]$ (b) $x^2 + y^2 = 1$ on $[-1, 1]$

(c) $|x| + |y| = 1$ on $[-1, 1]$ (d) $y = \cos(x)$ on $[-\pi/2, \pi/2]$

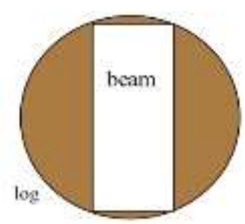

Fig. 19

18. The strength of a wooden beam is proportional to the product of its width and the square of its height (Fig. 19).

(a) What are the dimensions of the strongest log which can be cut from a log with diameter 12 inches?

(b) What are the dimensions of the strongest log which can be cut from a log with diameter d inches?

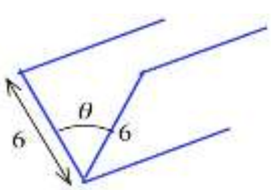

Fig. 20

19. You have a long piece of 12 inch wide metal which you are going to fold along the center line to form a V–shaped gutter (Fig. 20). What angle θ will give the gutter which holds the most water (the largest cross–sectional area)?

20. You have a long piece of 8 inch wide metal which you are going to make into a gutter by bending up 3 inches on each side (Fig. 21). What angle θ will give the gutter which holds the most water (with the largest cross–sectional area)?

Fig. 21

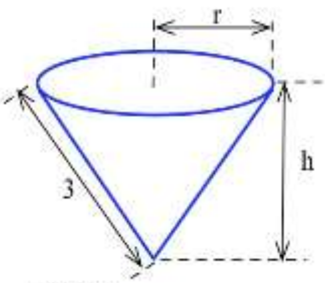

Fig. 22

21. You have a 6 inch diameter circle of paper which you want to form into a drinking cup by removing a pie–shaped wedge and forming the remaining paper into a cone (Fig. 22). Find the height and top radius of so the volume of the cup is as large as possible.

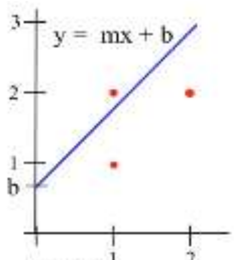

Fig. 23

22. (a) What value of b minimizes the sum of the squares of the vertical distances of the line y=2x+b from the points (1, 1), (1, 2) and (2, 2)? (Fig. 23)
(b) What slope m minimizes the sum of the squares of the vertical distances of the line y=mx from the points (1, 1), (1, 2) and (2, 2)?
(c) What slope m minimizes the sum of the squares of the vertical distances of the line y = mx from the points (2, 1), (4, 3), (–2, –2), and (–4, –2)?

23. You own a small airplane which holds a maximum of 20 passengers. It costs you $100 per flight from St. Thomas to St. Croix for gas and wages plus an additional $6 per passenger for the extra gas required by the extra weight. The charge per passenger is $30 each if 10 people charter your plane (10 is the minimum number you will fly), and this charge is reduced by $1 per passenger for each passenger over 10 who goes (that is, if 11 go they each pay $29, if 12 go they each pay $28, etc.). What number of passengers on a flight will maximize your profits?

24. Prove: If f and g are differentiable functions and if the vertical distance between f and g is greatest at x = c, then f '(c) = g '(c) and the tangent lines to f and g are parallel when x = c. (Fig. 24)

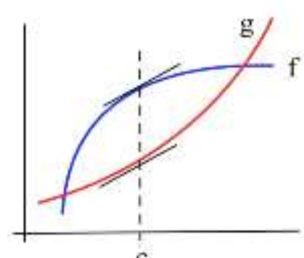

Fig. 24

25. Profit is revenue minus expenses. Assume that revenue and expenses are differentiable functions and show that when profit is maximized, then marginal revenue (dR/dx) equals marginal expense (dE/dx).

26. D. Simonton claims that the "productivity levels" of people in different fields can be described as a function of their "career age" t by $p(t) = e^{-at} - e^{-bt}$ where a and b are constants which depend on the field of work, and career age is approximately 20 less than the actual age of the individual.
    (a) Based on this model, at what ages do mathematicians (a=.03, b=.05), geologists (a=.02, b=.04), and historians (a=.02, b=.03) reach their maximum productivity?
    (b) Simonton says "With a little calculus we can show that the curve (p(t)) maximizes at $t = \frac{1}{b-a} \ln\left(\frac{b}{a}\right)$." Use calculus to show that Simonton is correct.

    Note: Models of this type have uses for describing the behavior of groups, but it is dangerous and usually invalid to apply group descriptions or comparisons to **individuals** in the group.
    (Scientific Genius, by Dean Simonton, Cambridge University Press, 1988, pp. 69 – 73)

27. After the table was wiped and the potato chips dried off, the question remained: "Just how far could a can of cola be tipped before it fell over?"

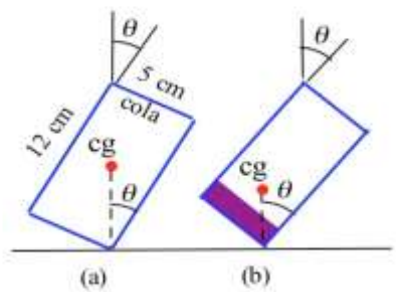

Fig. 25

(i) For a full can or an empty can the answer was easy: the center of gravity (cg) of the can is at the middle of the can, half as high as the height of the can, and we can tilt the can until the cg is directly above the bottom rim. (Fig. 25a) Find θ.

(ii) For a partly filled can more thinking was needed. Some ideas you will see in chapter 5 let us calculate that the cg of a can containing x cm of cola is $C(x) = \frac{360 + 9.6x^2}{60 + 19.2x}$ cm above the bottom of the can. Find the height **x** of cola in the can which will make the cg as low as possible.

(iii) Assuming that the cola is frozen solid (so the top of the cola stays parallel to the bottom of the can), how far can we tilt a can containing **x** cm of cola. (Fig. 25b)

(iv) If the can contained **x** cm of liquid cola, could we tilt it more or less far than the frozen cola before it would fall over?

28. Just thinking — calculus will not help with this one:
    (a) Four towns are located at the corners of a square (see Fig. 26). What is the shortest length of road we can construct so that it is possible to travel along the road from any town to any other town?
    (b) What is the shortest connecting path for 5 towns located on the corners of a pentagon?

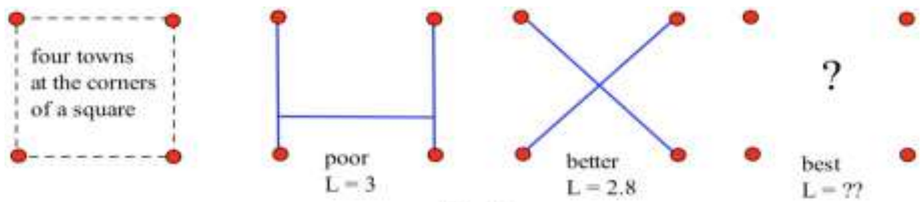

Fig. 26

(The problem of finding the shortest path connecting several points in the plane is called the "Steiner problem." It is important for designing computer chips and telephone networks to be as efficient as possible.)

Generalized Max/Min Problems

The previous max/min problems were all numerical problems: the amount of fencing in problem 2 was 200 feet, the sides of the piece of tin in problem 4 were 10 and 15, and the parabola in problem 17a was $y = 16 - x^2$. In doing those problems you might have noticed some patterns among the numbers in the problem and the numbers in your answers, and you might have wondered if the pattern was an accident of the numbers or if it there really was a pattern at work. Rather than trying several numerical examples to see if the "pattern" holds, mathematicians, engineers, scientists and others sometimes resort to generalizing the problem. We free the problem from the particular numbers by replacing the numbers with letters, and then we solve the generalized problem. In this way, relationships between the values in the problem and those in the solution can become more obvious. Solutions to these generalized problems are also useful if you want to program a computer or calculator to quickly provide numerical answers.

29. (a) Find the dimensions of the rectangle with the greatest area that can be built so the base of the rectangle is on the x–axis between 0 and 1 ($0 \le x \le 1$) and one corner of the rectangle is on the curve $y = x^2$ (Fig. 27a). What is the area of this rectangle?

    (b) Generalize the problem in part (a) for the parabola $y = Cx^2$ with $C > 0$ and $0 \le x \le 1$ (Fig. 27b).

    (c) Generalize for the parabola $y = Cx^2$ with $C > 0$ and $0 \le x \le B$ (Fig. 27c).

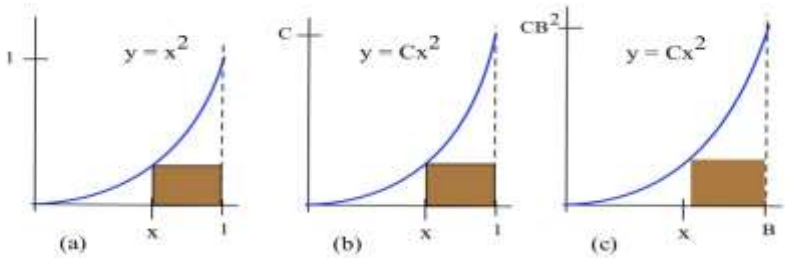

Fig. 27

30. (a) Find the dimensions of the rectangle with the greatest area that can be built so the base of the rectangle is on the x–axis between 0 and 1 ($0 \le x \le 1$) and one corner of the rectangle is on the curve $y = x^3$. What is the area of this rectangle?

    (b) Generalize the problem in part (a) for the curve $y = Cx^3$ with $C > 0$ and $0 \le x \le 1$.

    (c) Generalize for the curve $y = Cx^3$ with $C > 0$ and $0 \le x \le B$.

    (d) Generalize for the curve $y = Cx^n$ with $C > 0$, n a positive integer, and $0 \le x \le B$.

31. (a) The base of a right triangle is 50 and the height is 20 (Fig. 28a). Find the dimensions and area of the rectangle with the greatest area that can be enclosed in the triangle if the base of the rectangle must lie on the base of the triangle.

    (b) The base of a right triangle is B and the height is H (Fig. 28b). Find the dimensions and area of the rectangle with the greatest area that can be enclosed in the triangle if the base of the rectangle must lie on the base of the triangle.

    (c) State your general conclusion from part (b) in words.

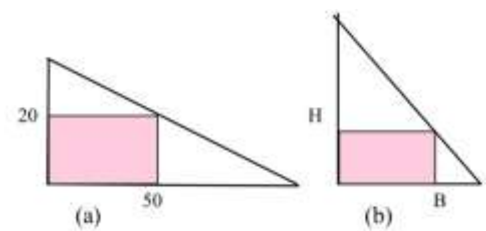

Fig. 28

32. (a) You have T dollars to buy fence to enclose a rectangular plot of land (Fig. 29). The fence for the top and bottom costs \$5 per foot and for the sides it costs \$3 per foot. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom, and for the sides?

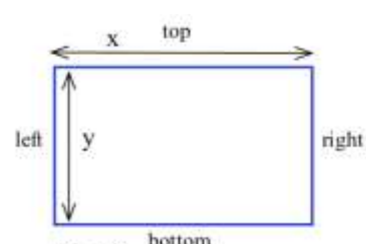

Fig. 29

    (b) You have T dollars to buy fence to enclose a rectangular plot of land.

The fence for the top and bottom costs \$A per foot and for the sides it costs \$B per foot. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom (together), and for the sides (together)?

(c) You have T dollars to buy fence to enclose a rectangular plot of land. The fence costs \$A per foot for the top, \$B/foot for the bottom, \$C/ft for the left side and \$D/ft for the right side. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom (together), and for the sides (together)?

33. Determine the dimensions of the least expensive cylindrical can which will hold V cubic inches if the top material costs \$A per square inch, the bottom material costs \$B/in^2, and the side material costs \$C/in^2.

34. Find the location of C in Fig. 30 so the sum of the distances from A to C and from C to B is a minimum.

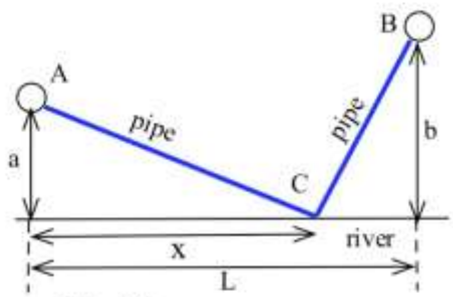

Fig. 30

Section 4.5 **PRACTICE Answers**

Practice 1: $V(x) = x(15 - 2x)(7 - 2x) = 4x^3 - 44x^2 + 105x$.

$V'(x) = 12x^2 - 88x + 105 = (2x - 3)(6x - 35)$ which is defined for all x so the only critical numbers are the endpoints $x = 0$ and $x = 7/2$ and the places where V ' equals 0, at $x = 3/2$ and $x = 35/6$ (but $35/6$ is not in the interval $[0, 7/2]$ so it is not practical for this applied problem).

The maximum volume must occur when $x = 0$, $x = 3/2$, or $x = 7/2$):

$V(0) = 0\cdot(15 - 2\cdot 0)\cdot(7 - 2\cdot 0) = 0$

$V(\frac{3}{2}) = \frac{3}{2}\cdot(15 - 2\cdot\frac{3}{2})\cdot(7 - 2\cdot\frac{3}{2})$
$= \frac{3}{2}(12)(4) = 72$ max.

$V(\frac{7}{2}) = \frac{7}{2}\cdot(15 - 2\cdot\frac{7}{2})\cdot(7 - 2\cdot\frac{7}{2})$
$= \frac{7}{2}(8)(0) = 0$

Fig. 31 shows the graph of V(x).

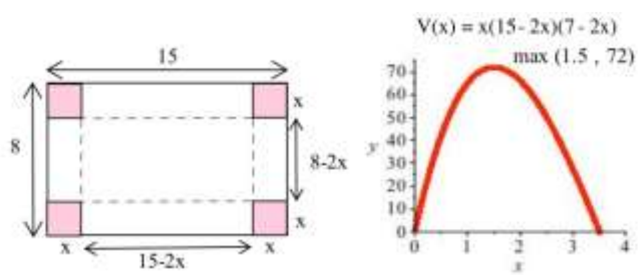

Fig. 31

Practice 2: (a) We have 80 feet of fencing. (See Fig. 32). Our assignment is to maximize the area of the garden: $A = x \cdot y$ (two variables). Fortunately we have the constraint that $x + y = 80$ so $y = 80 - x$, and our assignment reduces to maximizing a function of one variable:

maximize $A = x \cdot y = x \cdot (80 - x) = 80x - x^2$.

$A' = 80 - 2x$ so $A' = 0$ when $x = 40$.

$A'' = -2$ so A is concave down, and A has a maximum at $x = 40$.

The maximum area is $A = 40 \cdot 40 = 1600$ square feet when $x = 40$ ft. and $y = 40$ ft. The maximum area garden is a square.

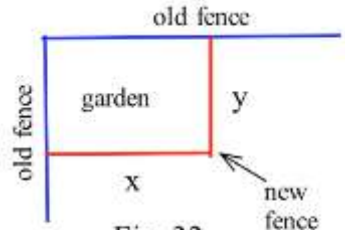

Fig. 32

(b) This is very similar to part (a) except we have F feet of fencing instead of 80 feet.

$x + y = F$ so $y = F - x$, and we want to maximize $A = xy = x(F - x) = Fx - x^2$.

$A' = F - 2x$ so $A' = 0$ when $x = F/2$ and $y = F/2$. The maximum area is $A = F^2/4$ square feet and that occurs when the garden is a square and half of the new fence is used on each of the two new sides.

Practice 3: Cost C = 5(area of top) + 3(area of sides) + 5(area of bottom) $= 5(\pi r^2) + 3(2\pi rh) + 5(\pi r^2)$ so our assignment is to minimize $C = 10\pi r^2 + 6\pi rh$, a function of two variables r and h.

Fortunately we also have the constraint that volume $= 1000 \text{ in}^3 = \pi r^2 h$ so $h = \dfrac{1000}{\pi r^2}$. Then

$C = 10\pi r^2 + 6\pi r(\dfrac{1000}{\pi r^2}) = 10\pi r^2 + \dfrac{6000}{r}$ so $C' = 20\pi r - \dfrac{6000}{r^2}$. $C' = 0$ if $20\pi r - \dfrac{6000}{r^2} = 0$ so $20\pi r^3 = 6000$ and $r = (\dfrac{6000}{20\pi})^{1/3} \approx 4.57$ in. Then $h = \dfrac{1000}{\pi r^2} \approx \dfrac{1000}{\pi(4.57)^2} \approx 15.24$ in.

($C'' = 20\pi + \dfrac{12000}{r^3} > 0$ for all $r > 0$ so C is concave up and we have found a minimum of C.)