3.6 SOME APPLICATIONS OF THE CHAIN RULE

The Chain Rule will help us determine the derivatives of logarithms and exponential functions $a^x$. We will also use it to answer some applied questions and to find slopes of graphs given by parametric equations.

DERIVATIVES OF LOGARITHMS

\[
D(\ln(x)) = \frac{1}{x} \quad \text{and} \quad D(\ln(g(x))) = \frac{g'(x)}{g(x)}.
\]

Proof: We know that the natural logarithm $\ln(x)$ is the logarithm with base $e$, and $e^{\ln(x)} = x$ for $x > 0$.

We also know that $D(e^x) = e^x$, so using the Chain Rule we have $D(e^{f(x)}) = e^{f(x)} f'(x)$. Differentiating each side of the equation $e^{\ln(x)} = x$, we get that

\[
D(e^{\ln(x)}) = D(x) \quad \text{use} \quad D(e^{f(x)}) = e^{f(x)} f'(x) \quad \text{with} \quad f(x) = \ln(x)
\]

\[
e^{\ln(x)} \cdot D(\ln(x)) = 1 \quad \text{replace} \quad e^{\ln(x)} \quad \text{with} \quad x
\]

\[
x \cdot D(\ln(x)) = 1 \quad \text{and solve for} \quad D(\ln(x)) \quad \text{to get} \quad D(\ln(x)) = \frac{1}{x}.
\]

The function $\ln(g(x))$ is the composition of $f(x) = \ln(x)$ with $g(x)$, so by the Chain Rule,

\[
D(\ln(g(x))) = D(f(g(x))) = f'(g(x)) \cdot g'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}.
\]

Example 1: Find $D(\ln(\sin(x)))$ and $D(\ln(x^2 + 3))$.

Solution: (a) Using the pattern $D(\ln(g(x))) = \frac{g'(x)}{g(x)}$ with $g(x) = \sin(x)$, then

\[
D(\ln(\sin(x))) = \frac{\cos(x)}{\sin(x)} = \cot(x).
\]

(b) Using the pattern with $g(x) = x^2 + 3$, we have $D(\ln(x^2 + 3)) = \frac{2x}{x^2 + 3}$.

We can use the Change of Base Formula from algebra to rewrite any logarithm as a natural logarithm, and then we can differentiate the resulting natural logarithm.

Change of Base Formula for logarithms: $\log_a x = \frac{\log_b x}{\log_b a}$ for all positive $a$, $b$ and $x$. 

Source URL: http://scidiv.bellevuecollege.edu/dh/Calculus_all/Calculus_all.html
Saylor URL: http://www.saylor.org/courses/ma005/

Attributed to: Dale Hoffman

Saylor.org
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Example 2: Use the Change of Base formula and your calculator to find \( \log_{\pi} 7 \) and \( \log_2 8 \).

Solution: \( \log_{\pi} 7 = \frac{\ln 7}{\ln \pi} \approx \frac{1.946}{1.145} \approx 1.700 \). (Check that \( \pi^{1.7} \approx 7 \)) \( \log_2 8 = \frac{\ln 8}{\ln 2} = 3 \).

Practice 1: Find the values of \( \log_9 20 \), \( \log_3 20 \) and \( \log_{\pi} e \).

Putting \( b = e \) in the Change of Base Formula, \( \log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a} \), so any logarithm can be written as a natural logarithm divided by a constant. Then any logarithm is easy to differentiate.

\[
\begin{align*}
D(\log_a(x)) &= \frac{1}{x \ln(a)} \\
D(\log_a(f(x))) &= \frac{f'(x)}{f(x)} \cdot \frac{1}{\ln(a)}
\end{align*}
\]

Proof: \( D(\log_a(x)) = D\left( \frac{\ln x}{\ln a} \right) = \frac{1}{\ln(a)} \cdot D(\ln x) = \frac{1}{\ln(a)} \cdot \frac{1}{x} = \frac{1}{x \ln(a)} \).

The second differentiation formula follows from the Chain Rule.

Practice 2: Calculate \( D(\log_{10}(\sin(x))) \) and \( D(\log_{\pi}(e^x)) \).

The number \( e \) might seem like an "unnatural" base for a natural logarithm, but of all the logarithms to different bases, the logarithm with base \( e \) has the nicest and easiest derivative. The natural logarithm is even related to the distribution of prime numbers. In 1896, the mathematicians Hadamard and Valle–Poussin proved the following conjecture of Gauss: (The Prime Number Theorem) For large values of \( x \), \( \{ \text{number of primes less than } x \} \approx \frac{x}{\ln(x)} \).

DERIVATIVE OF \( a^x \)

Once we know the derivative of \( e^x \) and the Chain Rule, it is relatively easy to determine the derivative of \( a^x \) for any \( a > 0 \).

\[
D(a^x) = a^x \cdot \ln a \quad \text{for} \quad a > 0.
\]

Proof: If \( a > 0 \), then \( a^x > 0 \) and \( a^x = e^{\ln(a^x)} = e^{x \cdot \ln a} \).

\[
D(a^x) = D(e^{\ln(a^x)}) = D(e^{x \cdot \ln a}) = e^{x \cdot \ln a} \cdot D(x \cdot \ln a) = a^x \cdot \ln a .
\]
Example 3: Calculate $D(7^x)$ and $\frac{d}{dt}(2 \sin(t))$

Solution: (a) $D(7^x) = 7^x \ln 7 \approx (1.95) 7^x$.

(b) We can write $y = 2 \sin(t)$ as $y = 2^u$ with $u = \sin(t)$. Using the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = 2^u \cdot \ln(2) \cdot \cos(t) = 2^{\sin(t)} \cdot \ln(2) \cdot \cos(t).$$

Practice 3: Calculate $D(\sin(2^x))$ and $\frac{d}{dt}(3t^2)$.

SOME APPLIED PROBLEMS

Now we can examine applications which involve more complicated functions.

Example 4: A ball at the end of a rubber band (Fig. 1) is oscillating up and down, and its height

(in feet) above the floor at time $t$ seconds is $h(t) = 5 + 2 \sin(\frac{t}{2})$. ($t$ is in radians)

(a) How fast is the ball travelling after 2 seconds? after 4 seconds? after 60 seconds?

(b) Is the ball moving up or down after 2 seconds? after 4 seconds? after 60 seconds?

(c) Is the vertical velocity of the ball ever 0?

Solution: (a) $v(t) = D(h(t)) = D(5 + 2 \sin(\frac{t}{2}))$

$$= 2 \cos(\frac{t}{2}) \cdot D(\frac{t}{2}) = \cos(\frac{t}{2}) \text{ feet/second so}
\quad\quad v(2) = \cos(2) \approx 0.540 \text{ ft/s}, \quad v(4) = \cos(4) \approx -0.416 \text{ ft/s}, \text{ and}
\quad\quad v(60) = \cos(60) \approx 0.154 \text{ ft/s}.$$

(b) The ball is moving upward when $t = 2$ and 60 seconds, downward when $t = 4$.

(c) $v(t) = \cos(\frac{t}{2})$ and $\cos(\frac{t}{2}) = 0$ when $t = \pi \pm n \cdot 2\pi$ ($n = 1, 2, \ldots$).

Example 5: If 2400 people now have a disease, and the number of people with the disease appears to
double every 3 years, then the number of people expected to have the disease in $t$ years is $y = 2400 \cdot 2^{t/3}$.

(a) How many people are expected to have the disease in 2 years?

(b) When are 50,000 people expected to have the disease?

(c) How fast is the number of people with the disease expected to grow now and 2 years from now?
Solution: (a) In 2 years, \( y = 2400 \cdot 2^{2/3} \approx 3,810 \) people.

(b) We know \( y = 50,000 \), and we need to solve \( 50,000 = 2400 \cdot 2^{t/3} \) for \( t \). Taking logarithms of each side of the equation, \( \ln(50,000) = \ln(2400 \cdot 2^{t/3}) = \ln(2400) + (t/3) \cdot \ln(2) \) so \( 10.819 = 7.783 + .231t \) and \( t \approx 13.14 \) years. We expect 50,000 people to have the disease about 13.14 years from now.

(c) This is asking for \( \frac{dy}{dt} \) when \( t = 0 \) and 2 years. \( \frac{dy}{dt} = \frac{dt}{dt} \cdot \frac{2400 \cdot 2^{t/3}}{d(2400 \cdot 2^{t/3})/dt} = 2400 \cdot 2^{t/3} \cdot \ln(2) \cdot (1/3) \approx 554.5 \cdot 2^{t/3} \). Now, at \( t = 0 \), the rate of growth of the disease is approximately \( 554.5 \cdot 2^0 \approx 554.5 \) people/year. In 2 years the rate of growth will be approximately \( 554.5 \cdot 2^{2/3} \approx 880 \) people/year.

Example 6: You are riding in a balloon, and at time \( t \) (in minutes) you are \( h(t) = t + \sin(t) \) feet high. If the temperature at an elevation \( h \) is \( T(h) = \frac{72}{1 + h} \) degrees Fahrenheit, then how fast is your temperature changing when \( t = 5 \) minutes? (Fig. 2)

Solution: As \( t \) changes, your elevation will change, and, as your elevation changes, so will your temperature. It is not difficult to write the temperature as a function of time, and then we could calculate \( \frac{d}{dt} T(t) = T'(t) \) and evaluate \( T'(5) \), or we could use the Chain Rule:

\[
\frac{dT(t)}{dt} = \frac{d}{dh} T(h) \cdot \frac{dh(t)}{dt} = \frac{d}{dh} T(h) \cdot h'(t) = \frac{72}{(1 + h)^2} \cdot (1 + \cos(t)).
\]

When \( t = 5 \), then \( h(t) = 5 + \sin(5) \approx 4.04 \) so \( T'(5) \approx \frac{-72}{(1 + 4.04)^2} \cdot (1 + .284) \approx -3.64^\circ/minute.\)

Practice 4: Write the temperature \( T \) in the previous example as a function of the variable \( t \) alone and then differentiate \( T \) to determine the value of \( \frac{dT}{dt} \) when \( t = 5 \) minutes.

Example 7: A scientist has determined that, under optimum conditions, an initial population of 40 bacteria will grow "exponentially" to \( f(t) = 40 \cdot e^{t/5} \) bacteria after \( t \) hours.

(a) Graph \( y = f(t) \) for \( 0 \leq t \leq 15 \) . Calculate \( f(0) \), \( f(5) \), \( f(10) \).

(b) How fast is the population increasing at time \( t \)? (Find \( f'(t) \).)

(c) Show that the rate of population increase, \( f'(t) \), is proportional to the population, \( f(t) \), at any time \( t \). (Show \( f'(t) = K \cdot f(t) \) for some constant \( K \).)
Solution: (a) The graph of \( y = f(t) \) is given in Fig. 3.

\[
f(0) = 40 \cdot e^{0/5} = 40 \text{ bacteria. } f(5) = 40 \cdot e^{5/5} \approx 109 \text{ bacteria,}
\]

and \( f(10) = 40 \cdot e^{10/5} \approx 296 \text{ bacteria.} \)

\[
(b) f'(t) = \frac{d}{dt} ( f(t) ) = \frac{d}{dt} ( 40 \cdot e^{t/5} ) = 40 \cdot e^{t/5} \frac{d}{dt} ( t/5 )
\]

\[
= 40 \cdot e^{t/5} \left( \frac{1}{5} \right) = 8 \cdot e^{t/5} \text{ bacteria/hour.}
\]

(c) \( f'(t) = 8 \cdot e^{t/5} = \frac{1}{5} \left( 40 \cdot e^{t/5} \right) = \frac{1}{5} f(t) \) so \( f'(t) = K \cdot f(t) \) with \( K = 1/5 \).

PARAMETRIC EQUATIONS

Suppose a robot has been programmed to move in the xy-plane so at time \( t \) its \( x \) coordinate will be \( \sin(t) \) and its \( y \) coordinate will be \( t^2 \). Both \( x \) and \( y \) are functions of the independent parameter \( t \), \( x(t) = \sin(t) \) and \( y(t) = t^2 \), and the path of the robot (Fig. 4) can be found by plotting \( (x,y) = (x(t), y(t)) \) for lots of values of \( t \).

<table>
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<th>( t )</th>
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<th>( y(t) = t^2 )</th>
<th>plot point</th>
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<tr>
<td>2.0</td>
<td>2.91</td>
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Typically we know \( x(t) \) and \( y(t) \) and need to find \( \frac{dy}{dx} \), the slope of the tangent line to the graph of \( (x(t), y(t)) \). The Chain Rule says that

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}, \text{ so, algebraically solving for } \frac{dy}{dx}, \text{ we get } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.
\]

If we can calculate \( dy/dt \) and \( dx/dt \), the derivatives of \( y \) and \( x \) with respect to the parameter \( t \), then we can determine \( dy/dx \), the rate of change of \( y \) with respect to \( x \).
If \( x = x(t) \) and \( y = y(t) \) are differentiable with respect to \( t \), and \( \frac{dx}{dt} \neq 0 \), then
\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.
\]

**Example 8:** Find the slope of the tangent line to the graph of \((x, y) = (\sin(t), t^2)\) when \( t = 2 \)?

Solution: \( \frac{dx}{dt} = \cos(t) \) and \( \frac{dy}{dt} = 2t \). When \( t = 2 \), the object is at the point \((\sin(2), 2^2) \approx (0.91, 4)\)
and the slope of the tangent line to the graph is
\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\cos(t)} = \frac{2 \cdot 2}{\cos(2)} \approx \frac{4}{-0.42} \approx -9.61.
\]

**Practice 5:** Graph \((x, y) = (3\cos(t), 2\sin(t))\) and find the slope of the tangent line when \( t = \pi/2 \).

When we calculated \( \frac{dy}{dx} \), the slope of the tangent line to the graph of \((x(t), y(t))\), we used the derivatives \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \), and each of these derivatives also has a geometric meaning:
- \( \frac{dx}{dt} \) measures the rate of change of \( x(t) \) with respect to \( t \) -- it tells us whether the x-coordinate is increasing or decreasing as the t-variable increases.
- \( \frac{dy}{dt} \) measures the rate of change of \( y(t) \) with respect to \( t \).

**Example 9:** For the parametric graph in Fig. 5, tell whether \( \frac{dx}{dt} \), \( \frac{dy}{dt} \) and \( \frac{dy}{dx} \) is positive or negative when \( t = 2 \).

Solution: As we move through the point B (where \( t = 2 \)) in the direction of increasing values of \( t \), we are moving to the left so \( x(t) \) is decreasing and \( \frac{dx}{dt} \) is negative.

Similarly, the values of \( y(t) \) are increasing so \( \frac{dy}{dt} \) is positive. Finally, the slope of the tangent line, \( \frac{dy}{dx} \), is negative.

(As check on the sign of \( \frac{dy}{dx} \) we can also use the result \( \frac{dy}{dx} = \frac{dy / dt}{dx / dt} = \frac{\text{positive}}{\text{negative}} = \text{negative.} \)

**Practice 6:** For the parametric graph in the previous example, tell whether \( \frac{dx}{dt} \), \( \frac{dy}{dt} \) and \( \frac{dy}{dx} \) is positive or negative when \( t = 1 \) and when \( t = 3 \).
Speed

If we know the position of an object at every time, then we can determine its speed. The formula for speed comes from the distance formula and looks a lot like it, but with derivatives.

If \( x = x(t) \) and \( y = y(t) \) give the location of an object at time \( t \) and are differentiable functions of \( t \), then the speed of the object is

\[
\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2.
\]

Proof: The speed of an object is the limit, as \( \Delta t \to 0 \), of \( \frac{\text{change in position}}{\text{change in time}} \). (Fig. 6)

\[
\lim_{\Delta t \to 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} = \sqrt{\left( \frac{\Delta x}{\Delta t} \right)^2 + \left( \frac{\Delta y}{\Delta t} \right)^2} 
\to \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \quad \text{as} \quad \Delta t \to 0.
\]

Exercise 10: Find the speed of the object whose location at time \( t \) is \((x,y) = (\sin(t), t^2)\) when \( t = 0 \) and \( t = 1 \).

Solution: \( \frac{dx}{dt} = \cos(t) \) and \( \frac{dy}{dt} = 2t \) so speed = \( \sqrt{(\cos(t))^2 + (2t)^2} = \sqrt{\cos^2(t) + 4t^2} \).

When \( t = 0 \), speed = \( \sqrt{\cos^2(0) + 4(0)^2} = \sqrt{1 + 0} = 1 \). When \( t = 1 \), speed = \( \sqrt{\cos^2(1) + 4(1)^2} \approx 2.07 \).

Practice 7: Show that an object whose location at time \( t \) is \((x,y) = (3\sin(t), 3\cos(t))\) has a constant speed. (This object is moving on a circular path.)

Practice 8: Is the object whose location at time \( t \) is \((x,y) = (3\cos(t), 2\sin(t))\) travelling faster at the top of the ellipse (at \( t = \pi/2 \)) or at the right edge of the ellipse (at \( t = 0 \))?
PROBLEMS FOR SOLUTION

Differentiate the functions in problems 1 – 19.

1. \( \ln(5x) \)
2. \( \ln(x^2) \)
3. \( \ln(x^k) \)
4. \( \ln(x^x) = x^x \ln(x) \)
5. \( \ln(\cos(x)) \)
6. \( \cos(\ln(x)) \)
7. \( \log_2 5x \)
8. \( \log_2 kx \)
9. \( \ln(\sin(x)) \)
10. \( \ln(kx) \)
11. \( \log_2(\sin(x)) \)
12. \( \ln(e^x) \)
13. \( \log_5 5^x \)
14. \( \ln(e^{rx}) \)
15. \( x \ln(3x) \)
16. \( e^x \cdot \ln(x) \)
17. \( \frac{\ln(x)}{x} \)
18. \( \sqrt{x + \ln(3x)} \)
19. \( \ln(\sqrt{5x - 3}) \)
20. \( \frac{d}{dt} \ln(\cos(t)) \)
21. \( \frac{d}{dw} \cos(\ln(w)) \)
22. \( \frac{d}{dx} \ln(ax + b) \)
23. \( \frac{d}{dt} \ln(\sqrt{t + 1}) \)
24. \( D(3^x) \)
25. \( D(5^{\sin(x)}) \)
26. \( D(x \ln(x) - x) \)
27. \( \frac{d}{dx} \ln(\sec(x) + \tan(x)) \)

28. Find the slope of the line tangent to \( f(x) = \ln(x) \) at the point \( (e, 1) \). Find the slope of the line tangent to \( g(x) = e^x \) at the point \( (1, e) \). How are the slopes of \( f \) and \( g \) at these points related?

29. Find a point \( P \) on the graph of \( f(x) = \ln(x) \) so the tangent line to \( f \) at \( P \) goes through the origin.

30. You are moving from left to right along the graph of \( y = \ln(x) \) (Fig. 7).
   (a) If the x-coordinate of your location at time \( t \) seconds is \( x(t) = 3t + 2 \), then how fast is your elevation increasing?
   (b) If the x-coordinate of your location at time \( t \) seconds is \( x(t) = e^t \), then how fast is your elevation increasing?

31. Rumor. The percent of a population, \( p(t) \), who have heard a rumor by time \( t \) is often modeled as
   \[ p(t) = \frac{100}{1 + Ae^{-t}} = 100 (1 + Ae^{-t})^{-1} \]
   for some positive constant \( A \). Calculate how fast the rumor is spreading, \( \frac{dp(t)}{dt} \).
32. Radioactive decay. If we start with \( A \) atoms of a radioactive material which has a "half-life" (the amount of time for half of the material to decay) of 500 years, then the number of radioactive atoms left after \( t \) years is \( r(t) = A e^{-Kt} \) where \( K = \frac{\ln(2)}{500} \). Calculate \( r'(t) \) and show that \( r'(t) \) is proportional to \( r(t) \) (\( r'(t) = b r(t) \) for some constant \( b \)).

In problems 33 – 41, find a function with the given derivative.

33. \( f'(x) = \frac{8}{x} \)  
34. \( h'(x) = \frac{3}{3x + 5} \)  
35. \( f'(x) = \frac{\cos(x)}{3 + \sin(x)} \)

36. \( g'(x) = \frac{x}{1 + x^2} \)  
37. \( g'(x) = 3e^{5x} \)  
38. \( h'(x) = e^2 \)

39. \( f'(x) = 2xe^{(x^2)} \)  
40. \( g'(x) = \cos(x)e^{\sin(x)} \)  
41. \( h'(x) = \frac{\cos(x)}{\sin(x)} \)

42. Define \( A(x) \) to be the area bounded between the \( x \)-axis, the graph of \( f(x) \), and a vertical line at \( x \) (Fig. 8). The area under each "hump" of \( f \) is 2 square inches.

(a) Graph \( A(x) \) for \( 0 \leq x \leq 9 \).

(b) Graph \( A'(x) \) for \( 0 \leq x \leq 9 \).

Problems 43 – 48 involve parametric equations.

43. At time \( t \) minutes, robot A is at \( (t, 2t + 1) \) and robot B is at \( (t^2, 2t^2 + 1) \).

(a) Where is each robot when \( t=0 \) and \( t=1 \)?

(b) Sketch the path each robot follows during the first minute.

(c) Find the slope of the tangent line, \( dy/dx \), to the path of each robot at \( t = 1 \) minute.

(d) Find the speed of each robot at \( t = 1 \) minute.

(e) Discuss the motion of a robot which follows the path \(( \sin(t), 2\sin(t) + 1 )\) for 20 minutes.

44. \( x(t) = t + 1 \) \( , y(t) = t^2 \).

(a) Graph \(( x(t), y(t) ) \) for \(-1 \leq t \leq 4 \).

(b) Find \( dx/dt \), \( dy/dt \), the tangent slope \( dy/dx \), and speed when \( t = 1 \) and \( t = 4 \).
45. For the parametric graph in Fig. 9, determine whether \( \frac{dx}{dt} \), \( \frac{dy}{dt} \) and \( \frac{dy}{dx} \) are positive, negative or zero when \( t = 1 \) and \( t = 3 \).

46. For the parametric graph in Fig. 10, determine whether \( \frac{dx}{dt} \), \( \frac{dy}{dt} \) and \( \frac{dy}{dx} \) are positive, negative or zero when \( t = 1 \) and \( t = 3 \).

47. \( x(t) = R(t - \sin(t)) \), \( y(t) = R(1 - \cos(t)) \).  
   (a) Graph \( (x(t), y(t)) \) for \( 0 \leq t \leq 4\pi \).  
   (b) Find \( \frac{dx}{dt} \), \( \frac{dy}{dt} \), the tangent slope \( \frac{dy}{dx} \), and speed when \( t = \pi/2 \) and \( \pi \). 
   (The graph of \( (x(t), y(t)) \) is called a cycloid and is the path of a light attached to the edge of a rolling wheel with radius \( R \).)

48. Describe the motion of two particles whose locations at time \( t \) are \( (\cos(t), \sin(t)) \) and \( (\cos(t), -\sin(t)) \).

49. Describe the path of a robot whose location at time \( t \) is 
   (a) \( (3\cos(t), 5\sin(t)) \)  
   (b) \( (A\cos(t), B\sin(t)) \)  
   (c) Give the parametric equations so the robot will move along the same path as in part (a) but in the opposite direction.
50. After \( t \) seconds, a projectile hurled with initial velocity \( v \) and angle \( \theta \) will be at \( x(t) = v \cos(\theta)t \) feet and \( y(t) = v \sin(\theta)t - 16t^2 \) feet. (Fig. 11) (This formula neglects air resistance.)

(a) For an initial velocity of 80 feet/second and an angle of \( \pi/4 \), find \( t > 0 \) so \( y(t) = 0 \). What does this value for \( t \) represent physically? Evaluate \( x(t) \).

(b) For \( v \) and \( \theta \) in part (a), calculate \( dy/dx \). Find \( t \) so \( dy/dx = 0 \) at \( t \), and evaluate \( x(t) \). What does \( x(t) \) represent physically?

(c) What initial velocity is needed so a ball hit at an angle of \( \pi/4 \approx 0.7854 \) will go over a 40 foot high fence 350 feet away?

(d) What initial velocity is needed so a ball hit at an angle of \( 0.7 \) will go over a 40 foot high fence 350 feet away?
Section 3.6 PRACTICE Answers

Practice 1:  \[ \log_9 20 = \frac{\log(20)}{\log(9)} \approx 1.3634165 \] \[ \log_3 20 = \frac{\log(20)}{\log(3)} \approx 2.726833 \]

\[ \log_\pi e = \frac{\log(e)}{\log(\pi)} \approx \frac{\ln(e)}{\ln(\pi)} = \frac{1}{\ln(\pi)} \]

Practice 2:  \[ D(\log_{10}(\sin(x))) = \frac{1}{\sin(x)\ln(10)} \]

\[ D(\log_{\pi}(\pi^x)) = \frac{1}{\pi^x\ln(\pi)} \]

\[ D(\ln(\pi t^2)) = 3(t^2) \ln(3) \]

Practice 3:  \[ D(\sin(2^x)) = \cos(2^x)D(2^x) = \cos(2^x)2^x\ln(2) \]

Practice 4:  \[ T = \frac{72}{1+h} = \frac{72}{1+t+\sin(t)} \]

\[ \frac{dT}{dt} = \frac{(1+t+\sin(t))D(72) - 72D(1+t+\sin(t))}{(1+t+\sin(t))^2} = \frac{-72(1+\cos(t))}{(1+t+\sin(t))^2} \]

When \( t = 5 \), \[ \frac{dT}{dt} = \frac{-72(1+\cos(5))}{(1+t+\sin(t))^2} \approx -3.63695 \]

Practice 5:  \[ x(t) = 3\cos(t) \quad \text{so} \quad \frac{dx}{dt} = -3\sin(t) \]

\[ y(t) = 2\sin(t) \quad \text{so} \quad \frac{dy}{dt} = 2\cos(t) \]

When \( t = \pi/2 \), \[ \frac{dy}{dx} = \frac{2\cos(\pi/2)}{-3\sin(\pi/2)} = \frac{2-0}{-3-1} = 0 \] (See Fig. 12)

Practice 6:  When \( x=1 \): pos., pos., pos.  When \( x=3 \): pos., neg., neg.

Practice 7:  \[ x(t) = 3\sin(t) \quad \text{and} \quad y(t) = 3\cos(t) \quad \text{so} \quad \frac{dx}{dt} = 3\cos(t) \quad \text{and} \quad \frac{dy}{dt} = -3\sin(t) \]

Then \[ \text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3\cos(t))^2 + (-3\sin(t))^2} \]

\[ = \sqrt{9\cos^2(t) + 9\sin^2(t)} = \sqrt{9} = 3, \text{ a constant.} \]
**Practice 8:** \( x(t) = 3\cos(t) \) and \( y(t) = 2\sin(t) \) so \( \frac{dx}{dt} = -3\sin(t) \) and \( \frac{dy}{dt} = 2\cos(t) \). Then

\[
\text{speed} = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} = \sqrt{(-3\sin(t))^2 + (2\cos(t))^2} = \sqrt{9\sin^2(t) + 4\cos^2(t)} .
\]

When \( t = 0 \), the speed is \( \sqrt{9(0)^2 + 4(1)^2} = 2 \).

When \( t = \pi/2 \), the speed is \( \sqrt{9(1)^2 + 4(0)^2} = 3 \) (faster).