3.4 MORE DIFFERENTIATION PROBLEMS

Polynomials are very useful, but they are not the only functions we need. This section uses the ideas of the two previous sections to develop techniques for differentiating **powers of functions**, and to determine the derivatives of some particular functions which occur often in applications, the **trigonometric** and **exponential** functions.

As you focus on learning how to differentiate different types and combinations of functions, it is important to remember what derivatives are and what they measure. Calculators and personal computers are available to calculate derivatives. Part of your job as a professional will be to decide which functions need to be differentiated and how to use the resulting derivatives. You can succeed at that only if you understand what a derivative is and what it measures.

A POWER RULE FOR FUNCTIONS: $\mathbf{D}(f^n(x))$

If we apply the Product Rule to the product of a function with itself, a familiar pattern emerges.

$$\mathbf{D}(f^2) = \mathbf{D}(f \cdot f) = f \cdot \mathbf{D}(f) + f \cdot \mathbf{D}(f) = 2f \cdot \mathbf{D}(f).$$

$$\mathbf{D}(f^3) = \mathbf{D}(f^2 \cdot f) = f^2 \cdot \mathbf{D}(f) + f \cdot \mathbf{D}(f^2) = f^2 \cdot \mathbf{D}(f) + f\,\{ 2f \cdot \mathbf{D}(f) \} = f^2 \cdot \mathbf{D}(f) + 2f^2 \cdot \mathbf{D}(f) = 3 f^2 \cdot \mathbf{D}(f).$$

$$\mathbf{D}(f^4) = \mathbf{D}(f^3 \cdot f) = f^3 \cdot \mathbf{D}(f) + f \cdot \mathbf{D}(f^3) = f^3 \cdot \mathbf{D}(f) + f\{ 3f^2 \cdot \mathbf{D}(f)\} = f^3 \cdot \mathbf{D}(f) + 3f^3 \cdot \mathbf{D}(f) = 4 f^3 \cdot \mathbf{D}(f).$$

Practice 1: What is the pattern here? What do you think the results will be for $\mathbf{D}(f^5)$ and $\mathbf{D}(f^{13})$?

We could keep differentiating higher and higher powers of f(x) by writing them as products of lower powers of f(x) and using the Product Rule, but the Power Rule For Functions guarantees that the pattern we just saw for the small integer powers also works for all constant powers of functions.

Power Rule For Functions: If n is any constant
then $\mathbf{D}(f^n(x)) = n\, f^{n-1}(x) \cdot \mathbf{D}(f(x))$

The Power Rule for Functions is a special case of a more general theorem, the Chain Rule, which we will examine in Section 2.4. The Power Rule For Functions will be proved after the Chain Rule.

Example 1: Use the Power Rule for Functions to find

(a) $\mathbf{D}(\,(x^3-5)^2\,)$ (b) $\frac{\mathbf{d}}{\mathbf{dx}}(\,\sqrt{2x+3x^5}\,)$ (c) $\mathbf{D}(\,\sin^2(x)\,) = \mathbf{D}(\,(\sin(x))^2\,)$.

Solution: (a) To match the pattern of the Power Rule for $\mathbf{D}(\,(x^3-5)^2\,)$, let $f(x) = x^3 - 5$ and $n = 2$.

Then $\mathbf{D}(\,(x^3-5)^2\,) = \mathbf{D}(\,f^n(x)\,) \quad = n\,f^{n-1}(x)\cdot\mathbf{D}(\,f(x)\,)$

$$= 2(x^3-5)^1\mathbf{D}(\,x^3-5\,) = 2(\,x^3-5\,)(3x^2) \quad = \mathbf{6x^2(x^3-5)}.$$

(b) To match the pattern for $\frac{\mathbf{d}}{\mathbf{dx}}(\,\sqrt{2x+3x^5}\,) = \frac{\mathbf{d}}{\mathbf{dx}}(\,(2x+3x^5)^{1/2}\,)$, we can let $f(x) = 2x + 3x^5$ and take $n = 1/2$. Then

$$\frac{\mathbf{d}}{\mathbf{dx}}(\,(2x+3x^5)^{1/2}\,) = \frac{\mathbf{d}}{\mathbf{dx}}(\,f^n(x)\,) = n\,f^{n-1}(x)\cdot\frac{\mathbf{d}}{\mathbf{dx}}(\,f(x)\,) = \frac{1}{2}(2x+3x^5)^{-1/2}\frac{\mathbf{d}}{\mathbf{dx}}(\,2x+3x^5\,)$$

$$= \mathbf{\frac{1}{2}(2x+3x^5)^{-1/2}(2+15x^4)} = \frac{2+15x^4}{2\sqrt{2x+3x^5}}.$$

(c) To match the pattern for $\mathbf{D}(\,\sin^2(x)\,)$, Let $f(x) = \sin(x)$ and $n = 2$. Then

$$\mathbf{D}(\,\sin^2(x)\,) = \mathbf{D}(\,f^n(x)\,) = n\,f^{n-1}(x)\cdot\mathbf{D}(\,f(x)\,) = 2\sin^1(x)\,\mathbf{D}(\,\sin(x)\,) = \mathbf{2\,\sin(x)\,\cos(x)}.$$

Practice 2: Use the Power Rule for Functions to find

(a) $\frac{\mathbf{d}}{\mathbf{dx}}(\,(2x^5-\pi)^2\,)$, (b) $\mathbf{D}(\,\sqrt{x+7x^2}\,)$, (c) $\mathbf{D}(\,\cos^4(x)\,) = \mathbf{D}(\,(\cos(x))^4\,)$.

Example 2: Use calculus to show that the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3,4)$ has slope $-3/4$.

Solution: The top half of the circle is the graph of $y = f(x) = \sqrt{25-x^2}$ so $f'(x) = \mathbf{D}(\,(25-x^2)^{1/2}\,)$

$$= \frac{1}{2}(25-x^2)^{-1/2}\,\mathbf{D}(\,25-x^2\,) = \frac{-x}{\sqrt{25-x^2}} \quad \text{and} \quad f'(3) = \frac{-3}{\sqrt{25-3^2}} = \frac{-3}{4}.$$

As a check, you can verify that the slope of the radial line through the center of the circle (0,0) and the point (3,4) has slope 4/3 and is perpendicular to the tangent line which has a slope of –3/4.

DERIVATIVES OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

We have some general rules which apply to any elementary combination of differentiable functions, but in order to use the rules we still need to know the derivatives of each of the particular functions. Here we will add to the list of functions whose derivatives we know.

Derivatives of the Trigonometric Functions

We know the derivatives of the sine and cosine functions, and each of the other four trigonometric functions is just a ratio involving sines or cosines. Using the Quotient Rule, we can differentiate the rest of the trigonometric functions.

Theorem: $\mathbf{D}(\tan(x)) = \sec^2(x)$ $\qquad$ $\mathbf{D}(\sec(x)) = \sec(x)\tan(x)$

$\mathbf{D}(\cot(x)) = -\csc^2(x)$ $\qquad$ $\mathbf{D}(\csc(x)) = -\csc(x)\cot(x)$.

Proof: From trigonometry we know $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$, $\sec(x) = \dfrac{1}{\cos(x)}$, and $\csc(x) = \dfrac{1}{\sin(x)}$,

and we know $\mathbf{D}(\sin(x)) = \cos(x)$ and $\mathbf{D}(\cos(x)) = -\sin(x)$. Using the Quotient Rule,

$$\mathbf{D}(\tan(x)) = \mathbf{D}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cdot\mathbf{D}(\sin(x)) - \sin(x)\cdot\mathbf{D}(\cos(x))}{(\cos(x))^2}$$

$$= \frac{\cos(x)\,\mathbf{\cos(x)} - \sin(x)\{\mathbf{-\sin(x)}\}}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \mathbf{\sec^2(x)} .$$

$$\mathbf{D}(\sec(x)) = \mathbf{D}\left(\frac{1}{\cos(x)}\right) = \frac{\cos(x)\,\mathbf{D}(1) - 1\,\mathbf{D}(\cos(x))}{\cos^2(x)}$$

$$= \frac{\cos(x)\,(\mathbf{0}) - 1\,\{\mathbf{-\sin(x)}\}}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)}\,\frac{1}{\cos(x)} = \mathbf{\tan(x)\cdot\sec(x)} .$$

Instead of the Quotient Rule, we could have used the Power Rule to calculate $\mathbf{D}(\sec(x)) = \mathbf{D}((\cos(x))^{-1})$.

Practice 3: Use the Quotient Rule on $f(x) = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$ to prove that $f'(x) = -\csc^2(x)$.

Practice 4: Prove that $\mathbf{D}(\csc(x)) = -\csc(x)\cdot\cot(x)$. The justification of this result is very similar to the justification for $\mathbf{D}(\sec(x))$.

Practice 5: Find (a) $\mathbf{D}(x^5 \cdot \tan(x))$, (b) $\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\sec(t)}{t} \right)$ and (c) $\mathbf{D}(\sqrt{\cot(x) - x})$.

Derivative of e^x

We can use graphs of exponential functions to estimate the slopes of their tangent lines or we can numerically approximate the slopes.

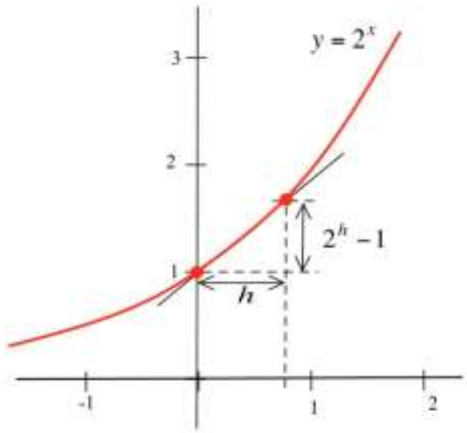

Fig. 1

Example 3: Estimate the derivative of $f(x) = 2^x$ at the point $(0, 2^0) = (0, 1)$ by approximating the slope of the line tangent to $f(x) = 2^x$ at that point.

Solution: We can get estimates from the graph of $f(x) = 2^x$ by carefully graphing $f(x) = 2^x$ for small values of x, sketching secant lines, and then measuring the slopes of the secant lines (Fig. 1).

We can also find the slope numerically by using the definition of the derivative,

$$f'(0) \equiv \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{2^{0+h}-2^0}{h} = \lim_{h\to 0} \frac{2^h-1}{h}$$, and evaluating $\frac{2^h - 1}{h}$

for some very small values of h.

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$	$\frac{e^h - 1}{h}$
0.1	0.717734625		
–0.1	0.669670084		
0.01	0.69555		
–0.01	0.690750451		
0.001	0.6933874		
–0.001	0.69290695		
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	≈ 0.693	≈ 1.099	1

From the table we can see that $f'(0) \approx .693$.

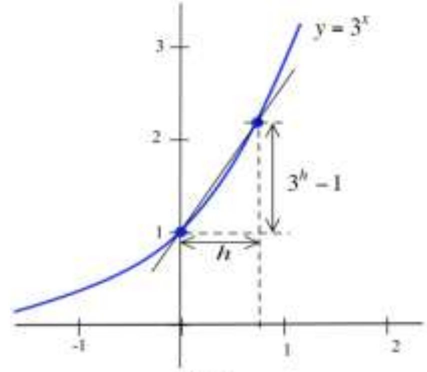

Fig. 2

Practice 6: Fill in the table for $\dfrac{3^h - 1}{h}$, and show that the slope of the line tangent to $g(x) = 3^x$ at (0,1) is approximately 1.099 . (Fig. 2)

At (0,1), the slope of the tangent to $y = 2^x$ is less than 1 , and the slope of the tangent to $y = 3^x$ is slightly greater than 1. (Fig. 3) There is a number, denoted **e**, between 2 and 3 so that the slope of the tangent to $y = e^x$ is exactly 1: $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h} = 1$. The number $e \approx 2.71828182845904$.

e is irrational and is very important and common in calculus and applications.

Once we grant that $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h} = 1$, it is relatively straightforward to calculate **D(** e^x **)**.

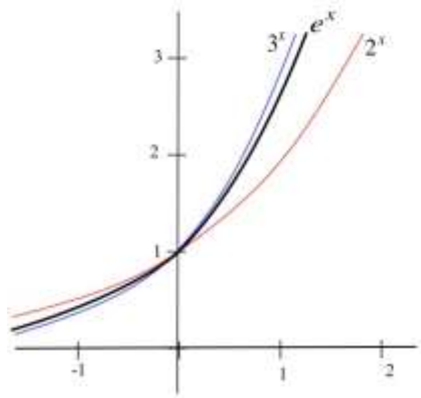

Fig. 3

Theorem: **D(** e^x **)** = $\mathbf{e^x}$.

Proof: **D(** e^x **)** $\displaystyle\equiv \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h}$

$$= \lim_{h\to 0} \left(e^x\right) \cdot \left(\frac{e^h - 1}{h}\right)$$

$$= \lim_{h\to 0} \left(e^x\right) \cdot \lim_{h\to 0} \left(\frac{e^h - 1}{h}\right) = (e^x)(1) = \mathbf{e^x}.$$

The function $f(x) = e^x$ is its own derivative: $f'(x) = f(x)$. The **height** of $f(x) = e^x$ at any point and the **slope** of the tangent to $f(x) = e^x$ at that point are the same: as the graph gets higher, its slope gets steeper.

Example 4: Find (a) $\dfrac{\mathbf{d}}{\mathbf{dt}}$ **(** $t \cdot e^t$ **)** , (b) **D(** $e^x / \sin(x)$ **)** and (c) **D(** e^{5x} **)** = **D(** $(e^x)^5$ **)**

Solution: (a) Using the Product Rule with $f(t) = t$ and $g(t) = e^t$,

$$\frac{\mathbf{d}}{\mathbf{dt}}(t \cdot e^t) = t \cdot \mathbf{D}(e^t) + e^t \cdot \mathbf{D}(t) = t \cdot \mathbf{e^t} + e^t \cdot \mathbf{(1)} = t \cdot e^t + e^t = (t+1)e^t .$$

(b) Using the Quotient Rule with $f(x) = e^x$ and $g(x) = \sin(x)$,

$$\mathbf{D}\left(\frac{e^x}{\sin(x)}\right) = \frac{\sin(x)\,\mathbf{D}(e^x) - e^x\,\mathbf{D}(\sin(x))}{\sin^2(x)} = \frac{\sin(x)\,\mathbf{e^x} - e^x\,\mathbf{\cos(x)}}{\sin^2(x)} .$$

(c) Using the Power Rule for Functions with $f(x) = e^x$ and $n = 5$,

$$\mathbf{D}((e^x)^5) = 5(e^x)^4 \cdot \mathbf{D}(e^x) = 5(e^x)^4 \cdot \mathbf{e^x} = 5e^{4x}e^x = 5e^{5x} .$$

Practice 7: Find (a) $\mathbf{D}(x^3 e^x)$ and (b) $\mathbf{D}((e^x)^3)$.

Higher Derivatives: Derivatives of Derivatives

The derivative of a function f is a new function **f '** , and we can calculate the derivative of this new function to get the derivative of the derivative of f, denoted by f '' and called the second derivative of f. For example, if $f(x) = x^5$ then $f'(x) = 5x^4$ and $\mathbf{f''(x)} = (f'(x))' = (5x^4)' = \mathbf{20x^3}$.

Definitions: The first derivative of f is $f'(x)$, the rate of change of f.

The second derivative of f is $f''(x) = (f'(x))'$, the rate of change of f '.

The third derivative of f is $f'''(x) = (f''(x))'$, the rate of change of f ''.

For $y = f(x)$, $f'(x) = \frac{dy}{dx}$, $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$, $f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$, and so on.

Practice 8: Find f ', f '', and f ''' for $f(x) = 3x^7$, $f(x) = \sin(x)$, and $f(x) = x\cos(x)$.

If f(x) represents the position of a particle at time x, then $v(x) = f'(x)$ will represent the velocity (rate of change of the position) of the particle and $a(x) = v'(x) = f''(x)$ will represent the acceleration (the rate of change of the velocity) of the particle.

Example 5: The height (feet) of a particle at time t seconds is $t^3 - 4t^2 + 8t$. Find the height, velocity and acceleration of the particle when $t = 0$, 1, and 2 seconds.

Solution: $f(t) = t^3 - 4t^2 + 8t$ so $f(0) = 0$ feet, $f(1) = 5$ feet, and $f(2) = 8$ feet.

The velocity is $v(t) = f'(t) = 3t^2 - 8t + 8$ so $v(0) = 8$ ft/s, $v(1) = 3$ ft/s, and $v(2) = 4$ ft/s. At each of these times the velocity is positive and the particle is moving upward, increasing in height.

The acceleration is $a(t) = 6t - 8$ so $a(0) = -8$ ft/s^2, $a(1) = -2$ ft/s^2 and $a(2) = 4$ ft/s^2.

We will examine the geometric meaning of the second derivative later.

Bent and Twisted Functions

In Section 1.2 we saw that the "holey" function $h(x) = \begin{cases} 2 & \text{if } x \text{ is a \textbf{rational} number} \\ 1 & \text{if } x \text{ is an \textbf{irrational} number} \end{cases}$

is discontinuous at every value of x, so at every x h(x) is not differentiable. We can create graphs of continuous functions that are not differentiable at several places just by putting corners at those places, but how many corners can a continuous function have? How badly can a continuous function fail to be differentiable?

In the mid–1800s, the German mathematician Karl Weierstrass surprised and even shocked the mathematical world by creating a function which was **continuous everywhere but differentiable nowhere** — a function whose graph was everywhere connected and everywhere bent! He used techniques we have not investigated yet, but we can start to see how such a function could be built.

Start with a function f_1 (Fig. 4) which zigzags between the values $+1/2$ and $-1/2$ and has a "corner" at each integer. This starting function f_1 is continuous everywhere and is differentiable everywhere except at the integers.

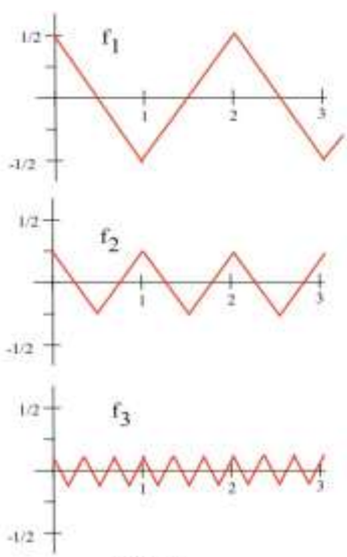

Fig. 4

Next create a list of functions f_2, f_3, f_4, . . . , each of which is a lot shorter but with many more "corners" than the previous ones. For example, we might make f_2 zigzag between the values $+1/4$ and $-1/4$ and have "corners" at $\pm 1/2, \pm 3/2, \pm 5/2$, etc., and f_3 zigzag between $+1/9$ and $-1/9$ and have "corners" at $\pm 1/3, \pm 2/3, \pm 4/3$, etc. If we add f_1 and f_2, we get a continuous function (since the sum of two continuous functions is continuous) which will have corners at $0, \pm 1/2, \pm 1, \pm 3/2, \ldots$ If we then add f_3 to the previous sum, we get a new continuous function with even more corners. If we continue adding the functions in our list "indefinitely", the final result will be a continuous function which is differentiable nowhere.

We haven't developed enough mathematics here to precisely describe what it means to add an infinite number of functions together or to verify that the resulting function is nowhere differentiable, but we will. You can at least start to imagine what a strange, totally "bent" function it must be.

Until Weierstrass created his "everywhere continuous, nowhere differentiable" function, most mathematicians thought a continuous function could only be "bad" in a few places, and Weierstrass' function was (and is) considered "pathological", a great example of how bad something can be. The mathematician Hermite expressed a reaction shared by many when they first encounter Weierstrass' function:

"I turn away with fright and horror from this lamentable evil of functions which do not have derivatives."

IMPORTANT RESULTS

Power Rule For Functions: $\mathbf{D}(f^n(x)) = n \cdot f^{n-1}(x) \cdot \mathbf{D}(\mathbf{f(x)})$

Derivatives of the Trigonometric Functions:

$\mathbf{D}(\sin(x)) = \cos(x)$ $\quad$ $\mathbf{D}(\tan(x)) = \sec^2(x)$ $\quad$ $\mathbf{D}(\sec(x)) = \sec(x)\tan(x)$

$\mathbf{D}(\cos(x)) = -\sin(x)$ $\quad$ $\mathbf{D}(\cot(x)) = -\csc^2(x)$ $\quad$ $\mathbf{D}(\csc(x)) = -\csc(x)\cot(x)$

Derivatives of the Exponential Function: $\mathbf{D}(e^x) = \mathbf{e^x}$

PROBLEMS FOR SOLUTION

1. Let f(1) = 2 and f '(1) = 3. Find the values of $\mathbf{D}(f^2(x))$, $\mathbf{D}(f^5(x))$, and $\mathbf{D}(\sqrt{f(x)})$ at x=1.

2. Let f(2) = –2 and f '(2) = 5. Find the values of $\mathbf{D}(f^2(x))$, $\mathbf{D}(f^{-3}(x))$, and $\frac{\mathbf{d}}{\mathbf{dx}}(\sqrt{f(x)})$ at x=2.

3. Estimate the values of f(x) and f '(x) in Fig. 5 and determine

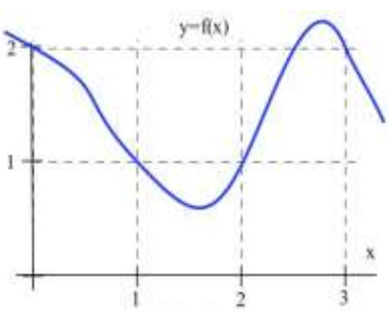

Fig. 5

(a) $\frac{\mathbf{d}}{\mathbf{dx}}(f^2(x))$ at x = 1 and 3 (b) $\mathbf{D}(f^3(x))$ at x = 1 and 3

(c) $\mathbf{D}(f^5(x))$ at x = 1 and 3 .

4. Estimate the values of f(x) and f '(x) in Fig. 5 and determine

(a) $\mathbf{D}(f^2(x))$ at x = 0 and 2 (b) $\frac{\mathbf{d}}{\mathbf{dx}}(f^3(x))$ at x = 0 and 2

(c) $\frac{\mathbf{d}}{\mathbf{dx}}(f^5(x))$ at x = 0 and 2.

In problems 5 – 10 , find the derivative of each function.

5. $f(x) = (2x - 8)^5$
6. $f(x) = (6x - x^2)^{10}$
7. $f(x) = x \cdot (3x + 7)^5$
8. $f(x) = (2x + 3)^6 \cdot (x - 2)^4$
9. $f(x) = \sqrt{x^2 + 6x - 1}$
10. $f(x) = \dfrac{x - 5}{(x + 3)^4}$

11. A weight attached to a spring is at a height of $h(t) = 3 - 2\sin(t)$ feet above the floor t seconds after it is released. (a) Graph $h(t)$ (b) At what height is the weight when it is released?
(c) How high does the weight ever get above the floor and how close to the floor does it ever get?
(d) Determine the height, velocity and acceleration at time t. (Be sure to include the correct units.)
(e) Why is this an unrealistic model of the motion of a weight on a real spring?

12. A weight attached to a spring is at a height of $h(t) = 3 - \frac{2\sin(t)}{1 + 0.1t^2}$ feet above the floor t seconds after it is released. (a) Graph $h(t)$ (b) At what height is the weight when it is released?
(c) Determine the height and velocity at time t.
(d) What happens to the height and the velocity of the weight after a "long time?"

13. The kinetic energy K of an object of mass m and velocity v is $\frac{1}{2} m \cdot v^2$.
(a) Find the kinetic energy of an object with mass m and height $h(t) = 5t$ feet at $t = 1$ and 2 seconds.
(b) Find the kinetic energy of an object with mass m and height $h(t) = t^2$ feet at $t = 1$ and 2 seconds.

14. An object of mass m is attached to a spring and has height $h(t) = 3 + \sin(t)$ feet at time t seconds.
(a) Find the height and kinetic energy of the object when $t = 1$, 2, and 3 seconds.
(b) Find the rate of change in the kinetic energy of the object when $t = 1$, 2, and 3 seconds.
(c) Can K ever be negative? Can dK/dt ever be negative? Why?

In problems 15 – 20, find the derivatives **df/dx**.

15. $f(x) = x \cdot \sin(x)$
16. $f(x) = \sin^5(x)$
17. $f(x) = e^x - \sec(x)$
18. $f(x) = \sqrt{\cos(x) + 1}$
19. $f(x) = e^{-x} + \sin(x)$
20. $f(x) = \sqrt{x^2 - 4x + 3}$

In problems 21 – 26, find the equation of the line tangent to the graph of the function at the given point.

21. $f(x) = (x - 5)^7$ at $(4, -1)$
22. $f(x) = e^x$ at $(0,1)$
23. $f(x) = \sqrt{25 - x^2}$ at $(3,4)$
24. $f(x) = \sin^3(x)$ at $(\pi,0)$
25. $f(x) = (x - a)^5$ at $(a,0)$
26. $f(x) = x \cdot \cos^5(x)$ at $(0, 0)$

27. Find the equation of the line tangent to $f(x) = e^x$ at the point $(3, e^3)$. Where will this tangent line intersect the x–axis? Where will the tangent line to $f(x) = e^x$ at the point (p, e^p) intersect the x–axis?

In problems 28 – 33, calculate **f '** and **f ''**.

28. $f(x) = 7x^2 + 5x - 3$
29. $f(x) = \cos(x)$
30. $f(x) = \sin(x)$
31. $f(x) = x^2 \cdot \sin(x)$
32. $f(x) = x \cdot \sin(x)$
33. $f(x) = e^x \cdot \cos(x)$

34. Calculate the first 8 derivatives of $f(x) = \sin(x)$. What is the pattern?
    What is the 208th derivative of $\sin(x)$?

35. What will the 2^{nd} derivative of a quadratic polynomial be? The 3^{rd} derivative? The 4^{th} derivative?

36. What will the 3^{rd} derivative of a cubic polynomial be? The 4^{th} derivative?

37. What can you say about the n^{th} and $(n+1)^{st}$ derivatives of a polynomial of degree n?

In problems 38 – 42, you are given f'. Find a function f with the given derivative.

38. $f'(x) = 4x + 2$

39. $f'(x) = 5e^x$

40. $f'(x) = 3 \cdot \sin^2(x) \cdot \cos(x)$

41. $f'(x) = 5(1 + e^x)^4 \cdot e^x$

42. $f'(x) = e^x + \sin(x)$

43. The function $f(x) = \begin{cases} x \cdot \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ in Fig. 6 is continuous at 0 since $\lim_{h \to 0} f(x) = 0 = f(0)$. Is f differentiable at 0? (Use the definition of $f'(0)$ and consider $\lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$.)

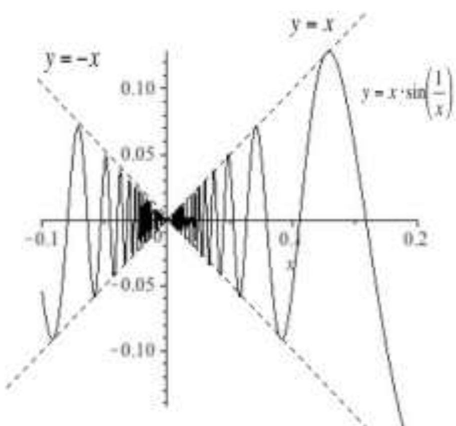

Fig. 6

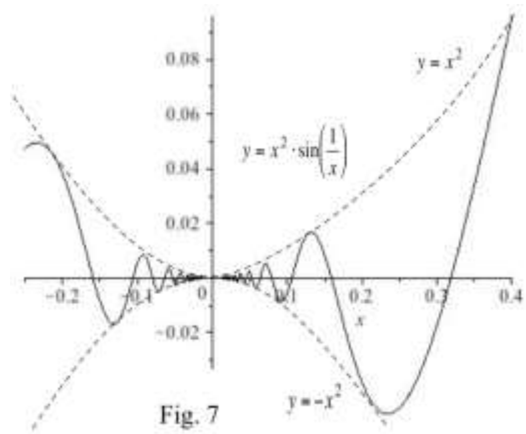

Fig. 7

44.

The number e appears in a variety of unusual situations. Problems 45 – 48 illustrate a few of them.

45. Use your calculator to examine the values of $\left(1 + \frac{1}{x}\right)^x$ when x is relatively large, for example, $x = 100$, 1000, and 10000. Try some other large values for x. If x is large, the value of $\left(1 + \frac{1}{x}\right)^x$ is close to what number?

46. If you put \$1 into a bank which pays 1% interest per year and compounds the interest x times a year, then after one year you will have earned $\left(1 + \frac{.01}{x}\right)^x$ dollars in the bank.

    (a) How much money will you have after 1 year if the bank calculates the interest once a year?

    (b) How much money will you have after 1 year if the bank calculates the interest twice a year?

    (c) How much money will you have after 1 year if the bank calculates the interest 365 times a year?

    (d) How does your answer in part (c) compare with $e^{.01}$?

47. (a) Calculate the value of the sums $s_1 = 1 + \frac{1}{1!}$, $s_2 = 1 + \frac{1}{1!} + \frac{1}{2!}$, $s_3 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$, $s_4 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$, $s_5 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$, and $s_6 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$.

    (b) What value do the sums in part (a) seem to be approaching? Calculate s_7 and s_8 .

    (n! = product of all positive integers from 1 to n. For example, 2! = 1•2 = 2, 3! = 1•2•3 = 6, 4! = 24.)

48. If it is late at night and you are tired of studying calculus, try the following experiment with a friend. Take the 2 through 10 of hearts from a regular deck of cards and shuffle these 9 cards well. Have your friend do the same with the 2 through 10 of spades. Now compare your cards one at a time. If there is a match, for example you both play a 5, then the game is over and you win. If you make it through the entire 9 cards with no match, then your friend wins. If you play the game **many times**, then the ratio $\frac{\text{total number of games played}}{\text{number of times your friend wins}}$ will be approximately equal to e.

Section 3.4 **PRACTICE Answers**

Practice 1: The pattern is $\mathbf{D}(\, f^n(x)\,) = n\, f^{n-1}(x)\cdot\mathbf{D}(\, \mathbf{f(x)}\,)$. $\mathbf{D}(\, f^5\,) = 5f^4\, \mathbf{D(f)}$ and $\mathbf{D}(\, f^{13}\,) = 13f^{12}\mathbf{D(f)}$.

Practice 2: $\frac{d}{dx}(2x^5 - \pi)^2 = 2(2x^5 - \pi)^1\, \mathbf{D}(\, 2x^5 - \pi\,) = 2(2x^5 - \pi)^1\, (\mathbf{10x^4}) = 40x^9 - 20\pi x^4$.

$$\mathbf{D}(\, (x + 7x^2)^{1/2}\,) = \frac{1}{2}(x + 7x^2)^{-1/2}\, \mathbf{D}(\, x + 7x^2\,) = \frac{\mathbf{1 + 14x}}{2\sqrt{x + 7x^2}} .$$

$$\mathbf{D}(\, (\cos(x)\,)^4\,) = 4(\,\cos(x)\,)^3\mathbf{D}(\,\cos(x)\,) = 4(\,\cos(x)\,)^3\, (\,\mathbf{-\sin(x)}\,) = \mathbf{-4\cos^3(x)\sin(x)} .$$

Practice 3: $\mathbf{D}\left(\frac{\cos(x)}{\sin(x)}\right) = \frac{\sin(x)\mathbf{D}(\,\cos(x)\,) - \cos(x)\mathbf{D}(\,\sin(x)\,)}{(\,\sin(x)\,)^2}$

$$= \frac{\sin(x)(\,\mathbf{-\sin(x)}\,) - \cos(x)(\,\mathbf{\cos(x)}\,)}{\sin^2(x)} = \frac{-(\,\sin^2(x) + \cos^2(x)\,)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = \mathbf{-\csc^2(x)} .$$

Practice 4: $\mathbf{D}(\csc(x)) = \mathbf{D}\left(\frac{1}{\sin(x)}\right) = \frac{\sin(x)\mathbf{D}(1) - 1\mathbf{D}(\sin(x))}{\sin^2(x)}$

$$= \frac{\sin(x)\mathbf{(0)} - \mathbf{\cos(x)}}{\sin^2(x)} = -\frac{\cos(x)}{\sin(x)}\frac{1}{\sin(x)} = \mathbf{-\cot(x)\csc(x)}\ .$$

Practice 5: $\mathbf{D}(x^5 \cdot \tan(x)) = x^5\mathbf{D}(\tan(x)) + \tan(x)\mathbf{D}(x^5) = x^5\,\mathbf{\sec^2(x)} + \tan(x)\mathbf{(5x^4)}$.

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\sec(t)}{t}\right) = \frac{t\mathbf{D}(\sec(t)) - \sec(t)\mathbf{D}(t)}{t^2} = \frac{t\cdot\sec(t)\cdot\tan(t) - \sec(t)}{t^2}\ .$$

$$\mathbf{D}\left((\cot(x) - x)^{1/2}\right) = \frac{1}{2}(\cot(x) - x)^{-1/2}\,\mathbf{D}(\cot(x) - x)$$

$$= \frac{1}{2}(\cot(x) - x)^{-1/2}\,\mathbf{(-\csc^2(x) - 1)} = \frac{-\csc^2(x) - 1}{2\sqrt{\cot(x) - x}}\ .$$

Practice 6:

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$	$\frac{e^h - 1}{h}$
0.1	0.717734625	**1.16123174**	1.051709181
–0.1	0.669670084	**1.040415402**	0.9516258196
0.01	0.69555	**1.104669194**	1.005016708
–0.01	0.690750451	**1.092599583**	0.9950166251
0.001	0.6933874	**1.099215984**	1.000500167
–0.001	0.69290695	**1.098009035**	0.9995001666
↓	↓	↓	↓
0	≈ 0.693	≈ 1.099	1

Practice 7: $\mathbf{D}(x^3e^x) = x^3\,\mathbf{D}(e^x) + e^x\,\mathbf{D}(x^3) = x^3\,\mathbf{(e^x)} + e^x\,\mathbf{(3x^2)} = \mathbf{x^2 \cdot e^x \cdot (x + 3)}$.

$\mathbf{D}((e^x)^3) = 3(e^x)^2\,\mathbf{D}(e^x) = 3(e^x)^2\,\mathbf{(e^x)} = 3e^{2x}\cdot e^x = \mathbf{3\,e^{3x}}$ or

$\mathbf{D}((e^x)^3) = \mathbf{D}(e^{3x}) = e^{3x}\,\mathbf{D}(3x) = \mathbf{3\,e^{3x}}$.

Practice 8:

$f(x) = 3x^7$	$f(x) = \sin(x)$	$f(x) = x \cdot \cos(x)$
$f'(x) = 21x^6$	$f'(x) = \cos(x)$	$f'(x) = -x \cdot \sin(x) + \cos(x)$
$f''(x) = 126x^5$	$f''(x) = -\sin(x)$	$f''(x) = -x \cdot \cos(x) - 2\sin(x)$
$f'''(x) = 630x^4$	$f'''(x) = -\cos(x)$	$f'''(x) = x \cdot \sin(x) - 3\cos(x)$