3.3 DERIVATIVES: PROPERTIES AND FORMULAS

This section begins with a look at which functions have derivatives. Then we'll examine how to calculate derivatives of elementary combinations of basic functions. By knowing the derivatives of some basic functions and just a few differentiation patterns, you will be able to calculate the derivatives of a tremendous variety of functions. This section contains most, but not quite all, of the general differentiation patterns you will ever need.

WHICH FUNCTIONS HAVE DERIVATIVES?

Theorem:	If	a function is **differentiable** at a point,
	then	it is **continuous** at that point.

The contrapositive form of this theorem tells about some functions which do not have derivatives:

Contrapositive Form of the Theorem:

If	f is **not continuous** at a point,
then	f is **not differentiable** at that point.

Proof of the Theorem: We assume that the hypothesis , f is differentiable at the point c , is true so

$\lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$ exists and equals f '(c). We want to show that f must necessarily be continuous at c : $\lim\limits_{h\to 0} f(c+h) = f(c)$.

Since f(c + h) can be written as $f(c+h) = f(c) + \left\{\dfrac{f(c+h)-f(c)}{h}\right\}\cdot h$, we have

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0}\left(f(c) + \left\{\frac{f(c+h)-f(c)}{h}\right\}\cdot h\right)$$

$$= \lim_{h\to 0} f(c) + \lim_{h\to 0}\left\{\frac{f(c+h)-f(c)}{h}\right\}\cdot\lim_{h\to 0}(h) = f(c) + f'(c)\cdot 0 = f(c) .$$

Therefore f is continuous at c .

It is important to clearly understand what is meant by this theorem and what is not meant: If the function is differentiable at a point, then the function is automatically continuous at that point. If the function is continuous at a point, then the function **may** or **may not** have a derivative at that point.

If the function is not continuous at a point, then the function **is not** differentiable at that point.

Example 1: Show that $f(x) = [\,x\,] = INT(x)$ is not continuous and not differentiable at 2 (Fig. 1).

Solution: The one–sided limits, $\lim_{x \to 2^+} INT(x) = 2$ and $\lim_{x \to 2^-} INT(x) = 1$, have different values so $\lim_{x \to 2} INT(x)$ does not exist, and $INT(x)$ is not continuous at 2. Since $f(x) = INT(x)$ is not continuous at 2, it is not differentiable there.

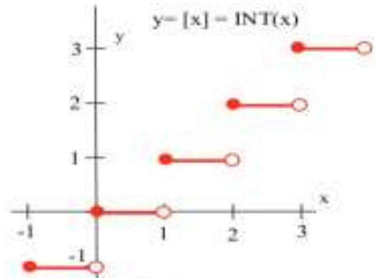

Fig. 1

Lack of continuity is enough to imply lack of differentiability, but the next two examples show that continuity is **not** enough to guarantee differentiability.

Example 2: Show that $f(x) = |\,x\,|$ is continuous but **not** differentiable at $x = 0$ (Fig. 2)

Solution: $\lim_{x \to 0} |x| = 0 = |0|$ so f is continuous at 0, but we showed in Section 2.1 that the absolute value function was not differentiable at $x = 0$.

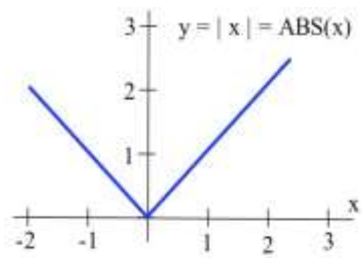

Fig. 2

A function is not differentiable at a cusp or a"corner."

Example 3: Show that $f(x) = \sqrt[3]{x} = x^{1/3}$ is continuous but **not** differentiable at $x = 0$ (Fig. 3)

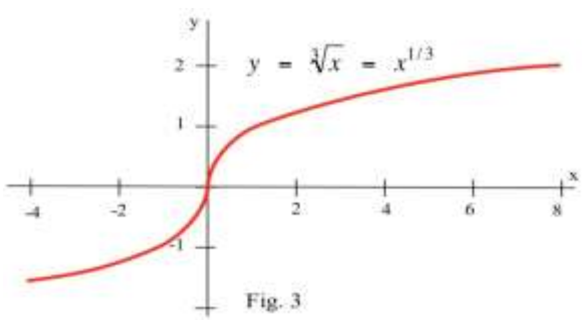

Fig. 3

Solution: $\lim_{x \to 0^-} \sqrt[3]{x} = \lim_{x \to 0^+} \sqrt[3]{x} = 0$ so

$\lim_{x \to 0} \sqrt[3]{x} = 0 = \sqrt{0}$, and f is continuous at 0.

$f'(x) = \frac{1}{3} x^{-(2/3)} = \frac{1}{3x^{2/3}}$ which is undefined at $x = 0$ so f is not differentiable at 0.

A function is not differentiable where its tangent line is vertical.

Practice 1: At which integer values of x is the graph of f in Fig. 4 continuous? differentiable?

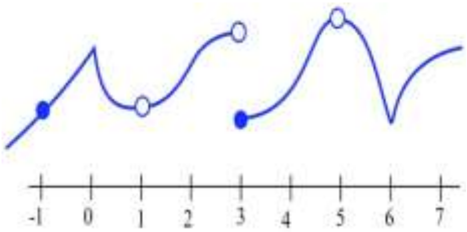

Fig. 4

Graphically, a function is **continuous** if and only if its graph is **connected** and does not have any holes or breaks.

Graphically, a function is **differentiable** if and only if it is continuous and its graph is **smooth** with no corners or vertical tangent lines.

DERIVATIVES OF ELEMENTARY COMBINATIONS OF FUNCTIONS

Example 4: The derivative of f(x) = x is **D**f(x) = 1, and the derivative of g(x) = 5 is **D**g(x) = 0.

What are the derivatives of their elementary combinations: 3f, f + g, f – g, f·g and f/g ?

Solution: **D(** 3 f(x) **)** = **D(** 3x **)** = 3 = 3 **D(** f(x) **).**

D(f(x) + g(x) **)** = **D(** x + 5 **)** = 1 = **D(** f(x) **)** + **D(** g(x) **)** .

D(f(x) – g(x) **)** = **D(** x – 5 **)** = 1 = **D(** f(x) **)** – **D(** g(x) **)** .

Unfortunately, the derivatives of f·g and f/g don't follow the same easy patterns:

D(f(x) · g(x) **)** = **D(** 5x **)** = 5 but **D(** f(x) **)** · **D(** g(x) **)** = (1)·(0) = 0 , and

D(f(x) / g(x) **)** = **D(** x / 5 **)** = 1/5 but **D(** f(x) **)** / **D(** g(x) **)** is undefined.

These two very simple functions show that, in general, **D(** f· g **)** ≠ **D(** f **)** · **D(** g **)** and **D(** f/g **)** ≠ **D(** f **)** / **D(** g **)**.

The Main Differentiation Theorem below states the correct patterns for differentiating products and quotients.

Practice 2: For f(x) = 6x + 8 and g(x) = 2 , what are the derivatives of 3f, f+g, f–g, f· g and f/g ?

The following theorem says that the simple patterns in the example for constant multiples of functions and sums and differences of functions are true for all differentiable functions. It also includes the correct patterns for derivatives of products and quotients of differentiable functions.

Main Differentiation Theorem: If f and g are differentiable at x, then

(a) **Constant Multiple Rule:** $\mathbf{D}(\, kf(x)\,) = k\mathbf{D}(\, f(x)\,) = k\mathbf{Df}$

or $(\, kf(x)\,)' = k\, f'(x)$

(b) **Sum Rule:** $\mathbf{D}(\, f(x) + g(x)\,) = \mathbf{D}(\, f(x)\,) + \mathbf{D}(\, g(x)\,) = \mathbf{Df} + \mathbf{Dg}$

or $(\, f(x) + g(x)\,)' = f'(x) + g'(x)$

(c) **Difference Rule:** $\mathbf{D}(\, f(x) - g(x)\,) = \mathbf{D}(\, f(x)\,) - \mathbf{D}(\, g(x)\,) = \mathbf{Df} - \mathbf{Dg}$

or $(\, f(x) - g(x)\,)' = f'(x) - g'(x)$

(d) **Product Rule:** $\mathbf{D}(\, f(x)\cdot g(x)\,) = f(x)\cdot \mathbf{D}(\, g(x)\,) + g(x)\cdot \mathbf{D}(\, f(x)\,) = f\,\mathbf{Dg} + g\,\mathbf{Df}$

or $(\, f(x)\cdot g(x)\,)' = f(x)\cdot \mathbf{g'(x)} + g(x)\cdot \mathbf{f'(x)}$

(e) **Quotient Rule:**

$$\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot\mathbf{D}(\, f(x)\,) - f(x)\cdot\mathbf{D}(\, g(x)\,)}{(g(x))^2}$$

$$= \frac{g\mathbf{Df} - f\mathbf{Dg}}{g^2} \quad \text{or} \quad \frac{g(x)\cdot\mathbf{f'(x)} - f(x)\cdot\mathbf{g'(x)}}{g^2(x)}$$

(provided $g(x) \neq 0$)

The proofs of parts (a), (b), and (c) of this theorem are straightforward, but parts (d) and (e) require some clever algebraic manipulations. Lets look at an example first.

Example 5: Recall that $\mathbf{D}(\, x^2\,) = 2x$ and $\mathbf{D}(\, \sin(x)\,) = \cos(x)$. Find $\mathbf{D}(\, 3\sin(x)\,)$ and $\frac{\mathbf{d}}{\mathbf{dx}}(\, 5x^2 - 7\sin(x)\,)$.

Solution: $\mathbf{D}(\, 3\sin(x)\,)$ is an application of part (a) of the theorem with $k = 3$ and $f(x) = \sin(x)$ so

$\mathbf{D}(\, 3\sin(x)\,) = 3\,\mathbf{D}(\, \sin(x)\,) = 3\cos(x)$.

$\frac{\mathbf{d}}{\mathbf{dx}}(\, 5x^2 - 7\sin(x)\,)$ uses part (c) of the theorem with $f(x) = 5x^2$ and $g(x) = 7\sin(x)$ so

$$\frac{\mathbf{d}}{\mathbf{dx}}(\, 5x^2 - 7\sin(x)\,) = \frac{\mathbf{d}}{\mathbf{dx}}(\, 5x^2\,) - \frac{\mathbf{d}}{\mathbf{dx}}(\, 7\sin(x)\,) = 5\frac{\mathbf{d}}{\mathbf{dx}}(\, x^2\,) - 7\frac{\mathbf{d}}{\mathbf{dx}}(\, \sin(x)\,)$$

$$= 5(\, \mathbf{2x}\,) - 7(\, \mathbf{\cos(x)}\,) = 10x - 7\cos(x).$$

Practice 3: Find $\mathbf{D}(\, x^3 - 5\sin(x)\,)$ and $\frac{\mathbf{d}}{\mathbf{dx}}(\, \sin(x) - 4x^3\,)$.

Practice 4: Fill in the values in the table for **D(** 3f(x) **)**, **D(** 2f(x)+g(x) **)**, and **D(** 3g(x) – f(x) **)**

x	f(x)	f '(x)	g(x)	g '(x)	**D(** 3f(x) **)**	**D(** 2f(x) + g(x) **)**	**D(** 3g(x) – f(x) **)**
0	3	–2	–4	3			
1	2	–1	1	0			
2	4	2	3	1			

Proof of the Main Derivative Theorem (a) and (c): The only general fact we have about derivatives is the definition as a limit, so our proofs here will have to recast derivatives as limits and then use some results about limits. The proofs are applications of the definition of the derivative and results about limits.

(a) **D(** kf(x) **)** $\equiv \lim_{h\to 0} \frac{k\cdot f(x+h)-k\cdot f(x)}{h} = \lim_{h\to 0} k\cdot \frac{f(x+h)-f(x)}{h} = k\cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$ k·**D(**f(x)**).**

(c) **D(** f(x) – g(x) **)** $= \lim_{h\to 0} \frac{\{f(x+h)-g(x+h)\}-\{f(x)-g(x)\}}{h} =$

$\lim_{h\to 0} \frac{\{f(x+h)-f(x)\}-\{g(x+h)-g(x)\}}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$

= **D(** f(x) **)** – **D(** g(x) **) .**

The proof of part (b) is very similar to these two proofs, and is left for you as the next Practice Problem.

The proof for the Product Rule and Quotient Rules will be given later.

Practice 5: Prove part (b) of the theorem, the Sum Rule: **D(** f(x) + g(x) **)** = **D(** f(x) **)** + **D(** g(x) **)** .

Practice 6: Use the Main Differentiation Theorem and the values in the table to fill in the rest of the table.

x	f(x)	f '(x)	g(x)	g '(x)	**D(** f(x)·g(x) **)**	**D(** f(x)/g(x) **)**	**D(** g(x)/f(x) **)**
0	3	–2	–4	3			
1	2	–1	1	0			
2	4	2	3	1			

Example 6: Determine **D(** x^2 ·sin(x) **)** and $\frac{\mathbf{d}}{\mathbf{dx}}\left(\frac{x^3}{\sin(x)}\right)$.

Solution: (a) We can use the Product Rule with $f(x) = x^2$ and g(x) = sin(x) :

D(x^2·sin(x) **)** = **D(** f(x)·g(x) **)** = f(x)·**D(** g(x) **)** + g(x)·**D(** f(x) **)**

= (x^2)·**D(** sin(x) **)** + sin(x)·**D(** x^2 **)**

= (x^2)·**(cos(x))** + sin(x)·**(2x)** = x^2·cos(x) + 2x·sin(x)

(b) We can use the Quotient Rule with $f(x) = x^3$ and $g(x) = \sin(x)$:

$$\frac{\mathbf{d}}{\mathbf{dx}}\left(\frac{x^3}{\sin(x)}\right) = \frac{g(x)\cdot\mathbf{D}(f(x)) - f(x)\cdot\mathbf{D}(g(x))}{g^2(x)} = \frac{\sin(x)\cdot\mathbf{D}(x^3) - x^3\cdot\mathbf{D}(\sin(x))}{(\sin(x))^2}$$

$$= \frac{\sin(x)\cdot\mathbf{(3x^2)} - x^3\cdot\mathbf{\cos(x)}}{\sin^2(x)} = \frac{3x^2\sin(x) - x^3\cos(x)}{\sin^2(x)}$$

Practice 7: Determine $\mathbf{D}((x^2+1)(7x-3))$, $\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{3t-2}{5t+1}\right)$ and $\mathbf{D}\left(\frac{\cos(x)}{x}\right)$.

Proof of the Product Rule: The proofs of parts (d) and (e) of the theorem are complicated but only involve elementary techniques, used in just the right way. Sometimes we will omit such computational proofs, but the Product and Quotient Rules are fundamental techniques you will need hundreds of times.

By the hypothesis, f and g are differentiable so $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$ and $\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x)$.
Also, both f and g are continuous (why?) so $\lim_{h\to 0} f(x+h) = f(x)$ and $\lim_{h\to 0} g(x+h) = g(x)$.

(d) Product Rule: Let $P(x) = f(x)\cdot g(x)$. Then $P(x+h) = f(x+h)\cdot g(x+h)$.

$$\mathbf{D}(f(x)\cdot g(x)) = \mathbf{D}(P(x)) = \lim_{h\to 0}\frac{P(x+h)-P(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)+\{-f(x)g(x+h)+f(x)g(x+h)\}-f(x)g(x)}{h}$$ adding and subtracting f(x)g(x+h)

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h}$$ regrouping the terms

$$= \lim_{h\to 0}\left(g(x+h)\right)\cdot\left(\frac{f(x+h)-f(x)}{h}\right) + \left(\mathbf{f(x)}\right)\cdot\left(\frac{g(x+h)-g(x)}{h}\right)$$ finding common factors

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$ taking limit as $h \to 0$

$$\left(g(x)\right) \cdot \left(f'(x)\right) + \left(f(x)\right) \cdot \left(g'(x)\right) = g\,\mathbf{Df} + f\,\mathbf{Dg}.$$

(e) The steps for a proof of the Quotient Rule are shown in Problem 55.

USING THE DIFFERENTIATION RULES

You definitely need to memorize the differentiation rules, but it is vitally important that you also know how to use them. Sometimes it is clear that the function we want to differentiate is a sum or product of two obvious functions, but we commonly need to differentiate functions which involve several operations and functions. Memorizing the differentiation rules is only the first step in learning to use them.

Example 7: Calculate $\mathbf{D}(\, x^5 + x\cdot\sin(x)\,)$ **.**

Solution: This function is more difficult because it involves both an addition and a multiplication. Which rule(s) should we use, or, more importantly, which rule should we use first?

$\mathbf{D}(\, x^5 + x\cdot\sin(x)\,) = \mathbf{D}(\, x^5\,) + \mathbf{D}(\, x\cdot\sin(x)\,)$ applying the Sum Rule and trading one derivative for two easier ones

$= 5x^4 + \{\, x\cdot\mathbf{D}(\sin(x)) + \sin(x)\cdot\mathbf{D}(\, x\,)\,\}$ applying the product rule to $\mathbf{D}(\, x\cdot\sin(x)\,)$

$= 5x^4 + x\cdot\cos(x) + \sin(x)$ this expression has no more derivatives so we are done.

If you were evaluating the function $x^5 + x\sin(x)$ for some particular value of x, you would (1) raise x to the 5th power, (2) calculate $\sin(x)$, (3) multiply $\sin(x)$ by x, and (4) your FINAL evaluation step, SUM the values of x^5 and $x\sin(x)$.

The FINAL step of your evaluation of f indicates the FIRST rule to use to calculate the derivative of f.

Practice 8: Which differentiation rule should you apply FIRST for each of the following:

(a) $x\cdot\cos(x) - x^3\cdot\sin(x)$ (b) $(2x-3)\cdot\cos(x)$ (c) $2\cos(x) - 7x^2$ (d) $\dfrac{\cos(x)+3x}{\sqrt{x}}$

Practice 9: Calculate $\mathbf{D}\left(\dfrac{x^2-5}{\sin(x)}\right)$ and $\dfrac{\mathbf{d}}{\mathbf{dt}}\left(\dfrac{t^2-5}{t\cdot\sin(t)}\right)$.

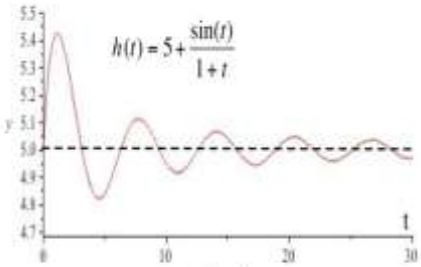

Fig. 5

Example 8: A weight attached to a spring is oscillating up and down. Over a period of time, the motion becomes "damped" because of friction and air resistance (Fig. 5), and its height at time t seconds is $h(t) = 5 + \frac{\sin(t)}{1 + t}$ feet.

What are the height and velocity of the weight after 2 seconds?

Solution: The height is

$$h(2) = 5 + \frac{\sin(2)}{1 + 2} = 5 + \frac{.909}{3} = 5.303 \text{ feet above the ground.}$$

The velocity is h '(2) .

$$h'(t) = 0 + \frac{(1+t)\cdot\mathbf{D}(\sin(t)) - \sin(t)\cdot\mathbf{D}(1+t)}{(1 + t)^2} = \frac{(1 + t)\cdot\cos(t) - \sin(t)}{(1 + t)^2}$$

$$\text{so } h'(2) = \frac{3\cos(2) - \sin(2)}{9} = \frac{-2.158}{9} \approx -0.24 \text{ feet per second .}$$

Practice 10: What are the height and velocity of the weight in the previous example after 5 seconds? What are the height and velocity of the weight be after a "long time" ?

Example 9: Calculate $\mathbf{D}(x\cdot\sin(x)\cdot\cos(x))$.

Solution: Clearly we need to use the Product Rule since the only operation in this function is multiplication, but the Product Rule deals with a product of <u>two</u> functions and we have the product of <u>three</u> ; x and sin(x) and cos(x) . However, if we think of our two functions as $f(x) = \mathbf{x\cdot\sin(x)}$ and $g(x) = \mathbf{\cos(x)}$, then we do have the product of two functions and

$$\begin{aligned}\mathbf{D}(x\cdot\sin(x)\cdot\cos(x)) &= \mathbf{D}(f(x)\cdot g(x)) = f(x)\cdot\mathbf{D}(g(x)) + g(x)\cdot\mathbf{D}(f(x)) \\ &= x\sin(x)\cdot\mathbf{D}(\cos(x)) + \cos(x)\cdot\mathbf{D}(x\sin(x))\end{aligned}$$

We are not done, but we have traded one hard derivative for two easier ones. We know that $\mathbf{D}(\cos(x)) = -\sin(x)$, and we can use the Product Rule (again) to calculate $\mathbf{D}(x\sin(x))$. Then the last line of our calculation becomes

$$= x\sin(x)\cdot(-\sin(x)) + \cos(x)\cdot\{x\,\mathbf{D}(\sin(x)) + \sin(x)\,\mathbf{D}(x)\}$$

$$= -x\sin^2(x) + \cos(x)\{x\,\mathbf{\cos(x)} + \sin(x)\,(\mathbf{1})\} = -x\sin^2(x) + x\cos^2(x) + \cos(x)\sin(x) .$$

EVALUATING A DERIVATIVE AT A POINT

The derivative of a function f is a new **function f '(x)** which gives the slope of the line tangent to the graph of f at each point x. To find the slope of the tangent line at a particular point (c, f(c)) on the graph of f, we should first calculate the derivative f '(x) and then evaluate the **function f '(x)** at the point x = c to get the **number f '(c)**. If you mistakenly evaluate f first, you get a number f(c), and the derivative of a constant is always equal to 0.

Example 10: Determine the slope of the line tangent to $f(x) = 3x + \sin(x)$ at (0, f(0)) and (1, f(1)):

Solution: $f'(x) = \mathbf{D}(3x + \sin(x)) = \mathbf{D}(3x) + \mathbf{D}(\sin(x)) = 3 + \cos(x)$. When $x = 0$, the graph of $y = 3x + \sin(x)$ goes through the point $(0, 3(0)+\sin(0))$ with slope $f'(0) = 3 + \cos(0) = 4$. When $x = 1$, the graph goes through the point $(1, 3(1)+\sin(1)) = (1, 3.84)$ with slope $f'(1) = 3 + \cos(1) \approx 3.54$.

Practice 11: Where do $f(x) = x^2 - 10x + 3$ and $g(x) = x^3 - 12x$ have a horizontal tangent lines ?

IMPORTANT RESULTS OF THIS SECTION

Differentiability and Continuity: If a function is differentiable then it must be continuous.
If a function is not continuous then it cannot be differentiable.
A function may be continuous at a point and not differentiable there.

Graphically: CONTINUOUS means **connected.**
DIFFERENTIABLE means continuous, **smooth** and not vertical.

Differentiation Patterns:

$$\mathbf{D}(kf(x)) = k\cdot\mathbf{D}(f(x))$$
$$\mathbf{D}(f+g) = \mathbf{D}f + \mathbf{D}g$$
$$\mathbf{D}(f-g) = \mathbf{D}f - \mathbf{D}g$$
$$\mathbf{D}(f\cdot g) = f\cdot\mathbf{D}g + g\cdot\mathbf{D}f$$
$$\mathbf{D}(f/g) = \frac{g\cdot\mathbf{D}f - f\cdot\mathbf{D}g}{g^2}$$

The FINAL STEP used to evaluate f indicates the FIRST RULE to use to differentiate f.

To evaluate a derivative at a point, first differentiate and then evaluate.

PROBLEMS FOR SOLUTION

1. The graph of $y = f(x)$ is given in Fig. 6.
   (a) At which integers is f continuous?
   (b) At which integers is f differentiable?

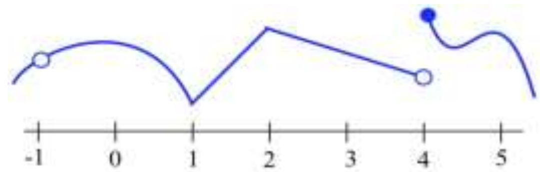

Fig. 6

2. The graph of $y = g(x)$ is given in Fig. 7.
   (a) At which integers is g continuous?
   (b) At which integers is g differentiable?

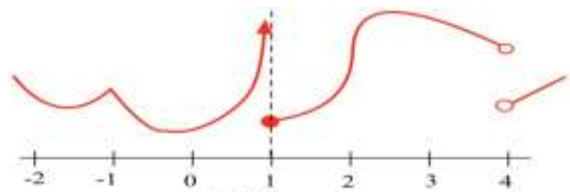

Fig. 7

3. Use the values given in the table to determine the values of $f \cdot g$, **D(** $f \cdot g$ **)**, f/g and **D(** f/g **)**.

x	f(x)	f '(x)	g(x)	g '(x)	f(x)•g(x)	**D(** f(x)•g(x) **)**	f(x)/g(x)	**D(** f(x)/g(x) **)**
0	2	3	1	5				
1	–3	2	5	–2				
2	0	–3	2	4				
3	1	–1	0	3				

4. Use the values given in the table to determine the values of $f \cdot g$, **D(** $f \cdot g$ **)**, f/g and **D(** f/g **)**.

x	f(x)	f '(x)	g(x)	g '(x)	f(x)•g(x)	**D(** f(x)•g(x) **)**	f(x)/g(x)	**D(** f(x)/g(x) **)**
0	4	2	3	–3				
1	0	3	2	1				
2	–2	5	0	–1				
3	–1	–2	–3	4				

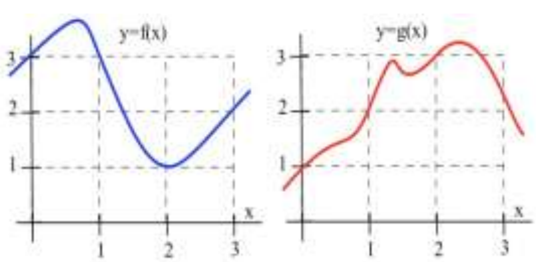

Fig. 8

5. Use the information in Fig. 8 to plot the values of the functions $f + g$, $f \cdot g$ and f/g and their derivatives at $x = 1, 2$ and 3.

6. Use the information in Fig. 8 to plot the values of the functions $2f$, $f - g$ and g/f and their derivatives at $x = 1, 2$ and 3.

7. Calculate **D(** $(x - 5)(3x + 7)$ **)** by (a) using the product rule and (b) expanding the product and then differentiating. Verify that both methods give the same result.

Source URL: http://scidiv.bellevuecollege.edu/dh/Calculus_all/Calculus_all.html
Saylor URL: http://www.saylor.org/courses/ma005/

8. Calculate $\mathbf{D}(\sqrt{x} \cdot \sin(x))$.

9. Calculate $\frac{\mathbf{d}}{\mathbf{dx}}\left(\frac{\cos(x)}{x^2}\right)$.

10. Calculate $\mathbf{D}(\sin(x) + \cos(x))$.

11. Calculate $\mathbf{D}(\sin^2(x))$ and $\mathbf{D}(\cos^2(x))$.

12. Calculate $\mathbf{D}(\sin(x))$, $\frac{\mathbf{d}}{\mathbf{dx}}(\sin(x) + 7)$, $\mathbf{D}(\sin(x) - 8000)$ and $\mathbf{D}(\sin(x) + k)$.

13. Find values for the constants a, b and c so that the parabola $f(x) = ax^2 + bx + c$ has $f(0) = 0$, $f'(0) = 0$ and $f'(10) = 30$.

14. If f is a differentiable function,
    (a) how are the graphs of $y = f(x)$ and $y = f(x) + k$ related?
    (b) how are the derivatives of $f(x)$ and $f(x) + k$ related?

15. If f and g are differentiable functions which always differ by a constant ($f(x) - g(x) = k$ for all x), then what can you conclude about their graphs and their derivatives?

16. If f and g are differentiable functions whose sum is a constant ($f(x) + g(x) = k$ for all x), then what can you conclude about (a) their graphs? (b) their derivatives?

17. If the product of f and g is a constant ($f(x)g(x) = k$ for all x), then how are $\frac{\mathbf{D}(f(x))}{f(x)}$ and $\frac{\mathbf{D}(g(x))}{g(x)}$ related?

18. If the quotient of f and g is a constant ($f(x)/g(x) = k$ for all x), then how are $g \cdot f'$ and $f \cdot g'$ related?

In problems 19 – 28, (a) calculate $f'(1)$ and (b) determine when $f'(x) = 0$.

19. $f(x) = x^2 - 5x + 13$

20. $f(x) = 5x^2 - 40x + 73$

21. $f(x) = 3x - 2\cos(x)$

22. $f(x) = | x + 2 |$

23. $f(x) = x^3 + 9x^2 + 6$

24. $f(x) = x^3 + 3x^2 + 3x - 1$

25. $f(x) = x^3 + 2x^2 + 2x - 1$

26. $f(x) = \frac{7x}{x^2 + 4}$

27. $f(x) = x \cdot \sin(x)$ and $0 \le x \le 5$. (You may need to use the Bisection Algorithm or the "trace" option on a calculator to approximate where $f'(x) = 0$.)

28. $f(x) = A x^2 + B x + C$ A, B and C are constants and $A \ne 0$.

29. $f(x) = x^3 + Ax^2 + Bx + C$ with constants A, B and C. Can you find conditions on the constants A, B and C which will guarantee that the graph of $y = f(x)$ has two distinct "vertices"? (Here a "vertex" means a place where the curve changes from increasing to decreasing or from decreasing to increasing.)

Where are the functions in problems 30 – 37 differentiable?

30. $f(x) = |x| \cos(x)$

31. $f(x) = \dfrac{x-5}{x+3}$

32. $f(x) = \tan(x)$

33. $f(x) = \dfrac{x^2 + x}{x^2 - 3x}$

34. $f(x) = | x^2 - 4 |$

35. $f(x) = | x^3 - 1 |$

36. $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sin(x) & \text{if } x \geq 0 \end{cases}$

37. $f(x) = \begin{cases} x & \text{if } x < 0 \\ \sin(x) & \text{if } x \geq 0 \end{cases}$

38. For what value(s) of A is $f(x) = \begin{cases} Ax - 4 & \text{if } x < 2 \\ x^2 + x & \text{if } x \geq 2 \end{cases}$ differentiable at $x = 2$?

39. For what values of A and B is $f(x) = \begin{cases} Ax + B & \text{if } x < 1 \\ x^2 + x & \text{if } x \geq 1 \end{cases}$ differentiable at $x = 1$?

40. An arrow shot straight up from ground level with an initial velocity of 128 feet per second will be at height $h(x) = -16x^2 + 128x$ feet at x seconds. (Fig.9)

    (a) Determine the velocity of the arrow when x = 0, 1 and 2 seconds.
    (b) What is the velocity of the arrow, v(x), at any time x?
    (c) At what time x will the velocity of the arrow be 0?
    (d) What is the greatest height the arrow reaches?
    (e) How long will the arrow be aloft?
    (f) Use the answer for the velocity in part (b) to determine the acceleration, $a(x) = v'(x)$, at any time x.

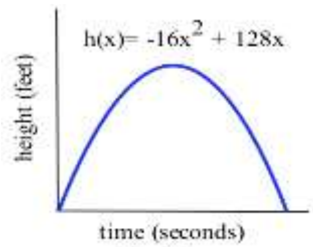

Fig. 9

41. If an arrow is shot straight up from ground level on the moon with an initial velocity of 128 feet per second, its height will be $h(x) = -2.65x^2 + 128x$ feet at x seconds. Do parts (a) – (e) of problem 40 using this new equation for h.

42. In general, if an arrow is shot straight upward with an initial velocity of 128 feet per second from ground level on a planet with a constant gravitational acceleration of g feet per second^2, then its height will be $h(x) = -\frac{g}{2} x^2 + 128x$ feet at x seconds. Answer the questions in problem 40 for arrows shot on Mars and Jupiter (Use the values in Fig. 10).

43. If an object on Earth is propelled upward from ground level with an initial velocity of v_0 feet per second, then its height at x seconds will be $h(x) = -16x^2 + v_0 x$.
   (a) What will be the object's velocity after x seconds?
   (b) What is the greatest height the object will reach?
   (c) How long will the object remain aloft?

Object	g (ft/sec^2)	g (cm/sec^2)
Mercury	11.8	358
Venus	20.1	887
Earth	32.2	981
moon	5.3	162
Mars	12.3	374
Jupiter	85.3	2601
Saturn	36.6	1117
Uranus	34.4	1049
Neptune	43.5	1325
Pluto	7.3	221

Source: CRC Handbook of Chemistry and Physics

Fig. 10: Values of g

44. In order for a 6 foot tall basketball player to dunk the ball, the player must achieve a vertical jump of about 3 feet. Use the information in the previous problems to answer the following questions.
   (a) What is the smallest initial vertical velocity the player can have to dunk the ball?
   (b) With the initial velocity achieved in part (a), how high would the player jump on the moon?

45. The best high jumpers in the world manage to lift their centers of mass approximately 6.5 feet.
   (a) What is the initial vertical velocity these high jumpers attain?
   (b) How long are these high jumpers in the air?
   (c) With the initial velocity in part (a), how high would they lift their centers of mass on the moon?

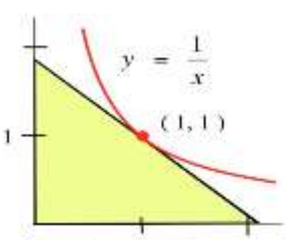

Fig. 11

46. (a) Find the equation of the line L which is tangent to the curve $y = \frac{1}{x}$ at the point (1,1) .
   (b) Determine where L intersects the x–axis and the y–axis.
   (c) Determine the area of the region in the first quadrant bounded by L, the x–axis and the y–axis. (Fig. 11)

47. (a) Find the equation of the line L which is tangent to the curve $y = \frac{1}{x}$ at the point $(2, \frac{1}{2})$.
    (b) Graph $y = 1/x$ and L and determine where L intersects the x–axis and the y–axis.
    (c) Determine the area of the region in the first quadrant bounded by L, the x–axis and the y–axis.
48. (a) Find the equation of the line L which is tangent to the curve $y = \frac{1}{x}$ at the point $(p, \frac{1}{p})$, $p \geq 0$.
    (b) Determine where L intersects the x–axis and the y–axis.
    (c) Determine the area of the region in the first quadrant bounded by L, the x–axis and the y–axis.
    (d) How does the area of the triangle in part (c) depend on the initial point $(p, \frac{1}{p})$?

49. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes through the point (1,4) and is tangent to the line $y = 9x - 13$ at the point (3, 14).

50. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes through the point (0,1) and is tangent to the line $y = 3x - 2$ at the point (2,4).

51. (a) Find a function f so that **D(** f(x) **)** $= 3x^2$.
    (b) Find another function g so that **D(** g(x) **)** $= 3x^2$.
    (c) Can you find more functions whose derivatives are $3x^2$?

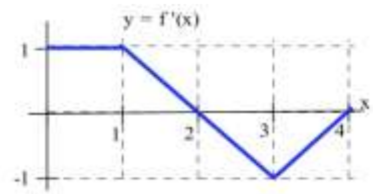

Fig. 12

52. (a) Find a function f so that **f '(** x **)** = 6x + cos(x) .

    (b) Find another function g so that **g '(** x **)** = 6x + cos(x) .

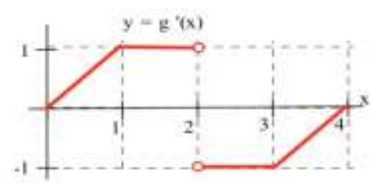

Fig. 13

53. The graph of y = f '(x) is given in Fig. 12.
    (a) Assume f(0) = 0 and sketch the graph of y = f(x) .
    (b) Assume f(0) = 1 and graph y = f(x) .

54. The graph of y = g '(x) is given in Fig. 13. Assume that g is continuous.
    (a) Assume g(0) = 0 and sketch the graph of y = g(x) .
    (b) Assume g(0) = 1 and graph y = g(x) .

55. Assume that f and g are differentiable functions and that $g(x) \neq 0$. State why each step in the following proof of the Quotient Rule is valid.

$$\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \lim_{h\to 0} \frac{1}{h}\left\{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right\} = \lim_{h\to 0} \frac{1}{h}\left\{\frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)}\right\}$$

$$= \lim_{h\to 0} \frac{1}{g(x+h)g(x)}\left\{\frac{f(x+h)g(x) + \left(-f(x)g(x) + f(x)g(x)\right) - g(x+h)f(x)}{h}\right\}$$

$$= \lim_{h\to 0} \frac{1}{g(x+h)g(x)}\left\{g(x)\frac{f(x+h) - f(x)}{h} - f(x)\frac{g(x+h) - g(x)}{h}\right\}$$

$$= \frac{1}{g^2(x)}\left\{g(x)\cdot f'(x) - f(x)\cdot g'(x)\right\} = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g^2(x)}.$$

Section 3.3 **PRACTICE Answers**

Practice 1: f is continuous at $x = -1, 0, 2, 4, 6$, and 7. f is differentiable at $x = -1, 2, 4$, and 7.

Practice 2: $f(x) = 6x + 8$ and $g(x) = 2$ so $\mathbf{D}(f(x)) = 6$ and $\mathbf{D}(g(x)) = 0$.

$\mathbf{D}(3f(x)) = 3\mathbf{D}(f(x)) = 3(6) = \mathbf{18}$, $\mathbf{D}(f(x) + g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x)) = 6 + 0 = \mathbf{6}$

$\mathbf{D}(f(x) - g(x)) = \mathbf{D}(f(x)) - \mathbf{D}(g(x)) = 6 - 0 = \mathbf{6}$

$\mathbf{D}(f(x)g(x)) = f(x)\mathbf{D}(g(x)) + g(x)\mathbf{D}(f(x)) = (6x + 8)(0) + (2)(6) = \mathbf{12}$

$$\mathbf{D}(f(x)/g(x)) = \frac{g(x)\mathbf{D}(f(x)) - f(x)\mathbf{D}(g(x))}{(g(x))^2} = \frac{(2)(6) - (6x+8)(0)}{2^2} = \frac{12}{4} = \mathbf{3}$$

Practice 3: $\mathbf{D}(x^3 - 5\sin(x)) = \mathbf{D}(x^3) - 5\mathbf{D}(\sin(x)) = \mathbf{3x^2 - 5cos(x)}$

$$\frac{\mathbf{d}}{\mathbf{dx}}(\sin(x) - 4x^3) = \frac{\mathbf{d}}{\mathbf{dx}}\sin(x) - 4\frac{\mathbf{d}}{\mathbf{dx}}x^3 = \mathbf{cos(x) - 12x^2}$$

Practice 4:

x	f(x)	f '(x)	g(x)	g '(x)	**D(** 3f(x) **)**	**D(** 2f(x) + g(x) **)**	**D(** 3g(x) – f(x) **)**
0	3	–2	–4	3	**–6**	**–1**	**11**
1	2	–1	1	0	**–3**	**–2**	**1**
2	4	2	3	1	**6**	**5**	**1**

Practice 5:

$$\mathbf{D}(f(x) + g(x)) = \lim_{h\to 0}\frac{\{f(x+h)+g(x+h)\}-\{f(x)+g(x)\}}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$$

$$= \left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)+\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right) = \mathbf{D}(f(x)) + \mathbf{D}(g(x)).$$

Practice 6:

x	f(x)	f '(x)	g(x)	g '(x)	**D(** f(x)·g(x) **)**	**D(** f(x)/g(x) **)**	**D(** g(x)/f(x) **)**
0	3	–2	–4	3	3·3+(–4)(–2)=**17**	$\frac{-4(-2)-(3)(3)}{(-4)^2} = \mathbf{\frac{-1}{16}}$	$\frac{(3)(3)-(-4)(-2)}{3^2} = \mathbf{\frac{1}{9}}$
1	2	–1	1	0	2·0+1(–1) = **–1**	$\frac{1(-1)-(2)(0)}{1^2} = \mathbf{-1}$	$\frac{2(0)-1(-1)}{2^2} = \mathbf{\frac{1}{4}}$
2	4	2	3	1	4·1 + 3·2 = **10**	$\frac{3(2)-(4)(1)}{3^2} = \mathbf{\frac{2}{9}}$	$\frac{4(1)-3(2)}{4^2} = \mathbf{\frac{-2}{16}}$

Practice 7:

$\mathbf{D}((x^2+1)(7x-3)) = (x^2+1)\mathbf{D}(\ 7x-3\ ) + (7x-3)\mathbf{D}(\ x^2+1\ ) = (x^2+1)(\mathbf{7}) + (7x-3)(\mathbf{2x}) = \mathbf{21x^2 - 6x + 7}$

or $\mathbf{D}((x^2+1)(7x-3)) = \mathbf{D}(\ 7x^3 - 3x^2 + 7x\ ) = \mathbf{21x^2 - 6x + 7}$

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{3t-2}{5t+1}\right) = \frac{(5t+1)\mathbf{D}(\ 3t-2\ ) - (3t-2)\mathbf{D}(\ 5t+1\ )}{(5t+1)^2} = \frac{(5t+1)(\mathbf{3}) - (3t-2)(\mathbf{5})}{(5t+1)^2} = \frac{13}{(5t+1)^2}$$

$$\mathbf{D}\left(\frac{\cos(x)}{x}\right) = \frac{x\mathbf{D}(\ \cos(x)\ ) - \cos(x)\mathbf{D}(\ x\ )}{(x)^2} = \frac{x(\mathbf{-\sin(x)}\ ) - \cos(x)(\mathbf{1})}{x^2} = \frac{-x\cdot\sin(x) - \cos(x)}{x^2}$$

Practice 8: (a) difference rule (b) product rule (c) difference rule (d) quotient rule

Practice 9:

$$\mathbf{D}\left(\frac{x^2-5}{\sin(x)}\right) = \frac{\sin(x)\mathbf{D}(\ x^2-5\ ) - (x^2-5)\mathbf{D}(\ \sin(x)\ )}{(\ \sin(x)\ )^2} = \frac{\sin(x)(\mathbf{2x}) - (x^2-5)\mathbf{\cos(x)}}{\sin^2(x)}$$

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{t^2-5}{t\cdot\sin(t)}\right) = \frac{t\cdot\sin(t)\mathbf{D}(t^2-5) - (t^2-5)\mathbf{D}(t\cdot\sin(t))}{(\ t\cdot\sin(t)\ )^2} = \frac{t\cdot\sin(t)(\mathbf{2t}) - (t^2-5)\{\ \mathbf{t\cdot\cos(t) + \sin(t)}\ \}}{t^2\cdot\sin^2(t)}$$

Practice 10: (a) $h(5) = 5 + \dfrac{\sin(5)}{1+5} \approx \mathbf{4.84\ \ ft.}$ $v(5) = h\,'(5) = \dfrac{(1+5)\cos(5) - \sin(5)}{(1+5)^2} \approx \mathbf{0.074\ \ ft/sec.}$

"long time": $\displaystyle\lim_{t\to\infty} h(t) = \lim_{t\to\infty}\ 5 + \frac{\sin(t)}{1+t} = \mathbf{5\ feet}.$

$$\lim_{t\to\infty} h\,'(t) = \lim_{t\to\infty}\frac{(1+t)\cos(t) - \sin(t)}{(1+t)^2} = \lim_{t\to\infty}\left\{\frac{\cos(t)}{1+t} - \frac{\sin(t)}{(1+t)^2}\right\} = \mathbf{0\ \ ft/sec}.$$

Practice 11: $f\,'(x) = 2x - 10$. $f\,'(x) = 0$ when $2x - 10 = 0$ so when $\mathbf{x = 5}$.

$g\,'(x) = 3x^2 - 12$. $g\,'(x) = 0$ when $3x^2 - 12 = 0$ so $x^2 = 4$ and $\mathbf{x = -2, +2}$.