# 3.1 INTRODUCTION TO DERIVATIVES

## PREVIEW OF CHAPTER 3

The two previous chapters have laid the foundation for the study of calculus. They provided a review of some material you will need, and they started to emphasize the various ways we will need to view and use functions: functions given by graphs, equations, and tables of values.

Chapter 2 will focus on the idea of tangent lines. We will get a definition for the derivative of a function and calculate the derivatives of some functions using this definition. Then we will examine some of the properties of derivatives, see some relatively easy ways to calculate the derivatives, and begin to look at some ways we can use derivatives. Chapter 2 will emphasize what derivatives are, how to calculate them, and some of their applications.

This section begins with a very graphical approach to slopes of tangent lines. It then examines the problem of finding the slopes of the tangent lines for a single function, $y = x^2$, in some detail, and illustrates how these slopes can help us solve fairly sophisticated problems.

### Slopes of Tangent Lines: Graphically

Fig. 1 is the graph of a function $y = f(x)$. We can use the information in the graph to fill in the table:

| x | y = f(x) | m(x) = the **estimated** SLOPE of the tangent line to y=f(x) at the point (x,y) |
|---|---|---|
| 0 | **0** | **1** |
| 1 | **1** | **0** |
| 2 | **0** | **– 1** |
| 3 | **– 1** | **0** |
| 4 | **1** | **1** |
| 5 | **2** | **1/2** |

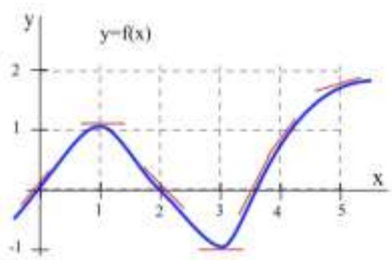

Fig. 1

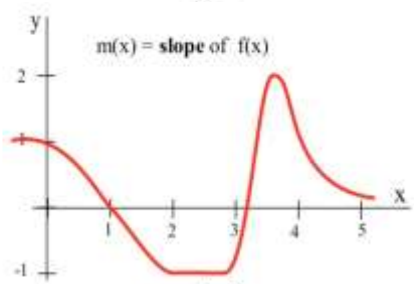

Fig. 2

We can estimate the values of m(x) at some non-integer values of x, $m(.5) \approx 0.5$ and $m(1.3) \approx -0.3$, and even over entire intervals, if $0 < x < 1$, then m(x) is positive.

The values of m(x) definitely depend on the values of x, and m(x) is a function of x. We can use the results in the table to help sketch the graph of m(x) in Fig. 2.

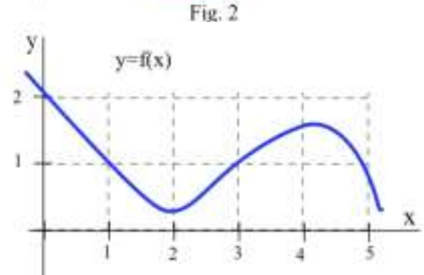

Fig. 3

**Practice 1:** The graph of $y = f(x)$ is given in Fig. 3. Set up a table of values for $x$ and $m(x)$ (the slope of the line tangent to the graph of $y=f(x)$ at the point $(x,y)$ ) and then graph the function $m(x)$.

In some applications, we need to know where the graph of a function $f(x)$ has horizontal tangent lines (slopes = 0). In Fig. 3, the slopes of the tangent lines to graph of $y = f(x)$ are 0 when $x = 2$ or $x \approx 4.5$ .

**Practice 2:** At what values of $x$ does the graph of $y = g(x)$ in Fig. 4 have horizontal tangent lines?

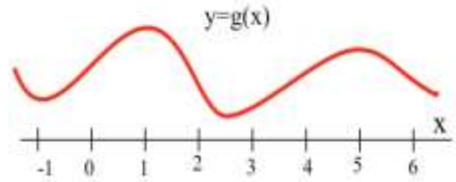

Fig. 4

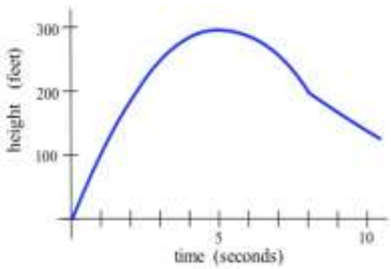

Fig. 5

**Example 1:** Fig. 5 is the graph of the height of a rocket at time t. Sketch the graph of the **velocity** of the rocket at time t. (Velocity is the **slope of the tangent** to the graph of position or height.)

Solution: The lower graph in Fig. 6 shows the velocity of the rocket.

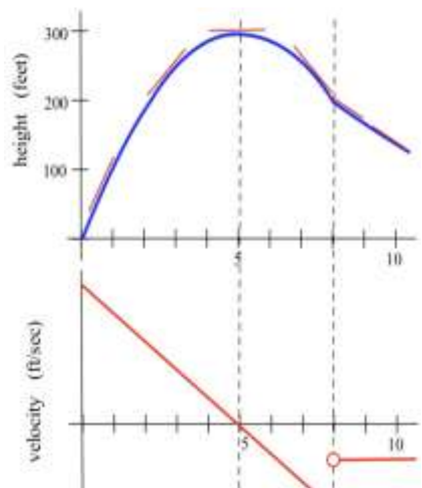

Fig. 6

**Practice 3:** Fig. 7 shows the temperature during a summer day in Chicago. Sketch the graph of the **rate** at which the temperature is changing. (This is just the graph of the **slopes** of the lines which are tangent to the temperature graph.)

The function $m(x)$ , the slope of the line tangent to the graph of $f(x)$, is called the **derivative of f(x)** . We have used the idea of the slope of the tangent line throughout Chapter 1. In the Section 2.1, we will formally define the derivative of a function and begin to examine some of the properties of the derivative function, but first lets see what we can do when we have a formula for $f(x)$.

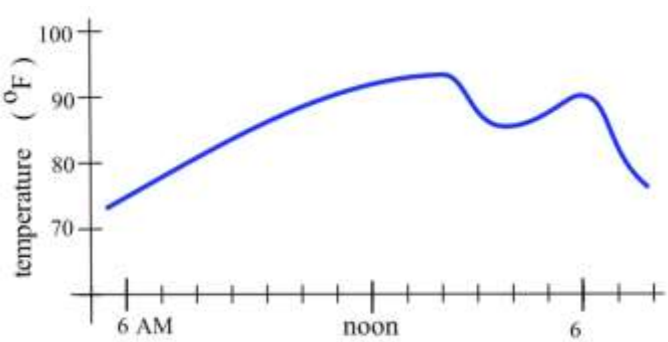

Fig. 7

## Tangents to $y = x^2$

When we have a formula for a function, we can determine the slope of the tangent line at a point ( x, f(x) ) by calculating the slope of the secant line through the points ( x, f(x) ) and ( x+h, f(x+h) ),

$m_{sec} = \frac{f(x+h) - f(x)}{(x+h) - (x)}$ , and then taking the limit of $m_{sec}$ as h approaches 0 (Fig. 8) :

$$m_{tan} = \lim_{h\to 0} m_{sec} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{(x+h)-(x)} .$$

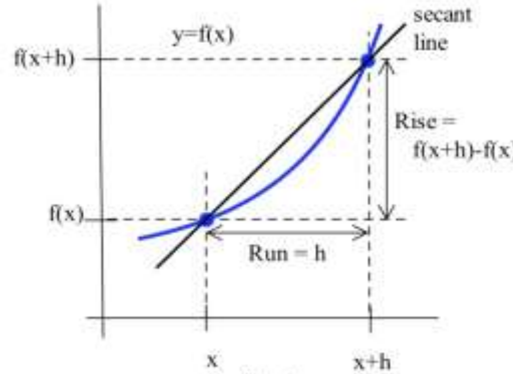

Fig. 8

**Example 2:** Find the **slope** of the line tangent to the graph of $y = f(x) = x^2$ at the point (2,4). (Fig. 9).

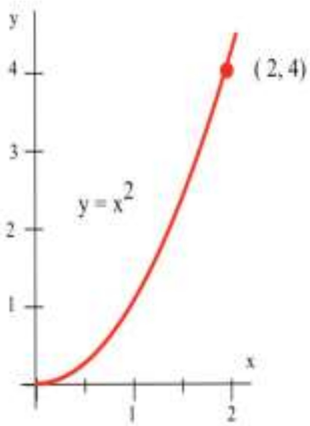

Fig. 9

Solution: In this example x = 2 , so x + h = 2 + h and $f(x + h) = f(2+h) = (2+h)^2$ .

The slope of the tangent line at (2,4) is

$$m_{tan} = \lim_{h\to 0} m_{sec} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{(2+h)-(2)}$$

$$= \lim_{h\to 0} \frac{(2+h)^2 - (2)^2}{(2+h)-(2)} = \lim_{h\to 0} \frac{4+4h+h^2-4}{h}$$

$$= \lim_{h\to 0} \frac{4h+h^2}{h} = \lim_{h\to 0} \frac{h(4+h)}{h} = \lim_{h\to 0} (4+h) = 4 .$$

The tangent line to the graph of $y = x^2$ at the point (2,4) has slope 4 .

We can use the point–slope formula for a line to find the equation of the tangent line:

$y - y_o = m(x - x_o)$ so $y - 4 = 4(x - 2)$ and $y = 4x - 4$.

**Practice 4:** Use the method of Example 2 to show that the **slope** of the line tangent to the graph of $y = f(x) = x^2$ at the point (1,1) is $m_{tan} = 2$. Also find the values of $m_{tan}$ at (0,0) and (–1,1).

It is possible to find the slopes of the tangent lines one point at a time, but that is not very efficient. You should have noticed in the Practice 4 that the algebra for each point was very similar, so let's do all the work once for an arbitrary point $( x, f(x) ) = ( x, x^2 )$ and then use the general result for our particular problems. The slope of the line tangent to the graph of $y = f(x) = x^2$ at the arbitrary point $( x, x^2 )$ is

$$m_{tan} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-(x)} = \lim_{h\to 0}\frac{(x+h)^2-(x)^2}{(x+h)-(x)} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = \mathbf{2x}\,.$$

The **slope** of the line tangent to the graph of $y = f(x) = x^2$ at the point $(x, x^2)$ is $m_{tan} = \mathbf{2x}$. We can use this general result at any value of x without going through all of the calculations again. The slope of the line tangent to $y = f(x) = x^2$ at the point $(4, 16)$ is $m_{tan} = 2(4) = 8$ and the slope at $(\pi, \pi^2)$ is $m_{tan} = 2(\pi) = 2\pi$. The value of x determines where we are on the curve ( at $y = x^2$ ) as well as the slope of the tangent line, $m_{tan} = 2x$ , at that point. The slope $m_{tan} = 2x$ is a **function** of x and is called the **derivative of y =** $\mathbf{x^2}$ .

Simply knowing that the slope of the line tangent to the graph of $y = x^2$ is $m_{tan} = 2x$ at a point $(x,y)$ can help us quickly find the equation of the line tangent to the graph of $y = x^2$ at any point and answer a number of difficult–sounding questions.

**Example 3:** Find the equations of the lines tangent to $y = x^2$ at $(3, 9)$ and $(p, p^2)$.

Solution: At $(3, 9)$, the slope of the tangent line is $2x = 2(3) = 6$, and the equation of the line is
$y - y_o = m(x - x_o)$ so $y - 9 = 6(x - 3)$ and $y = 6x - 9$.

At $(p, p^2)$, the slope of the tangent line is $2x = 2(p) = 2p$, and the equation of the line is
$y - y_o = m(x - x_o)$ so $y - p^2 = 2p(x - p)$ and $y = 2px - p^2$.

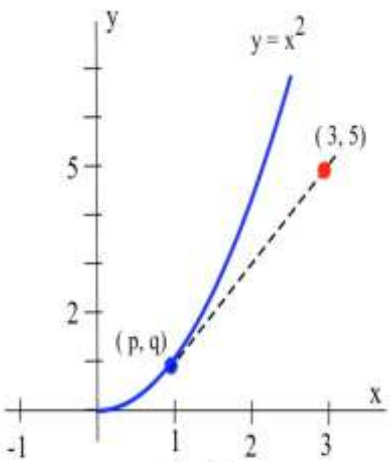

Fig. 10

**Example 4:** A rocket has been programmed to follow the path $y = x^2$ in space (from left to right along the curve), but an emergency has arisen and the crew must return to their base which is located at coordinates (3,5). At what point on the path $y = x^2$ should the captain turn off the engines so the ship will coast along the tangent to the curve to return to the base? (Fig. 10)

Solution: You might spend a few minutes trying to solve this problem without using the relation $m_{tan} = 2x$, but the problem is much easier if we do use that result.

Lets assume that the captain turns off the engine at the point (p,q) on the curve $y = x^2$ , and then try to determine what values p and q must have so that the resulting tangent line to the curve will go through the point (3,5). The point (p,q) is on the curve $y = x^2$ , so $q = p^2$ , and the equation of the tangent line, found in Example 3, is $y = 2px - p^2$ .

To find the value of p so that the tangent line will go through the point (3,5), we can substitute the values x = 3 and y = 5 into
the equation of the tangent line and solve for p :

$$y = 2px - p^2 \quad \text{so} \quad 5 = 2p(3) - p^2 \quad \text{and} \quad p^2 - 6p + 5 = 0.$$

The only solutions of $p^2 - 6p + 5 = (p - 1)(p - 5) = 0$ are $p = 1$ and $p = 5$, so the only possible points are (1,1) and (5,25). You can verify that the tangent lines to $y = x^2$ at (1,1) and (5,25) go through the base at the point (3,5) (Fig. 11). Since the ship is moving from left to right along the curve, the captain should turn off the engines at the point (1,1). Why not at (5,25)?

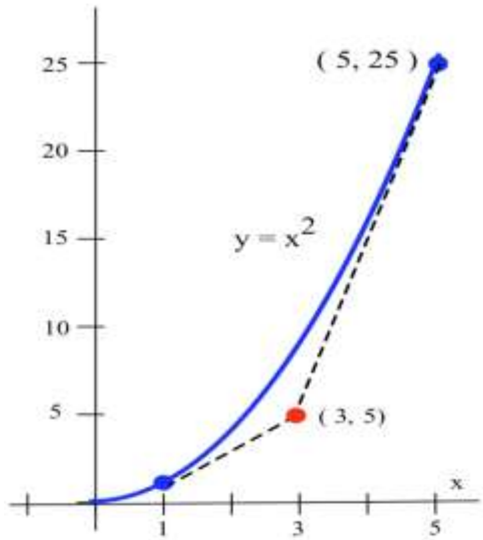

Fig. 11

**Practice 5:** Verify that if the rocket engines in Example 4 are shut off at (2,4), then the rocket will go through the point (3,8).

## Problems for Solution

1. Use the function in Fig. 12 to fill in the table and then graph m(x).

| x | y = f(x) | m(x) = the estimated slope of the tangent line to y=f(x) at the point (x,y) |
|---|---|---|
| 0 | | |
| 0.5 | | |
| 1.0 | | |
| 1.5 | | |
| 2.0 | | |
| 2.5 | | |
| 3.0 | | |
| 3.5 | | |
| 4.0 | | |

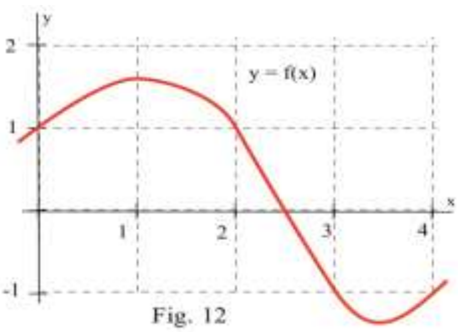

Fig. 12

2. Use the function in Fig. 13 to fill in the table and then graph m(x).

| x | y = g(x) | m(x) = the estimated slope of the tangent line to y=g(x) at the point (x,y) |
|---|---|---|
| 0 | | |
| 0.5 | | |
| 1.0 | | |
| 1.5 | | |
| 2.0 | | |
| 2.5 | | |
| 3.0 | | |
| 3.5 | | |
| 4.0 | | |

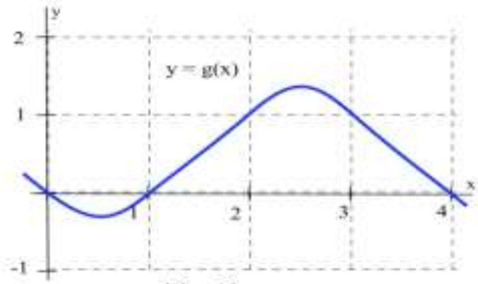

Fig. 13

3. (a) At what values of x does the graph of f in Fig. 14 have a horizontal tangent line?
   (b) At what value(s) of x is the value of f the largest? smallest?
   (c) Sketch the graph of m(x) = the slope of the line tangent to the graph of f at the point (x,y).

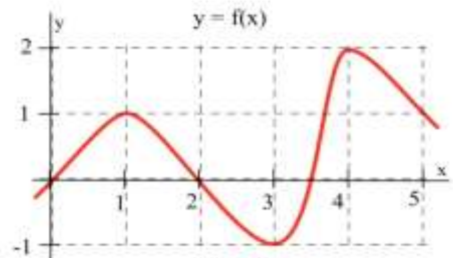

Fig. 14

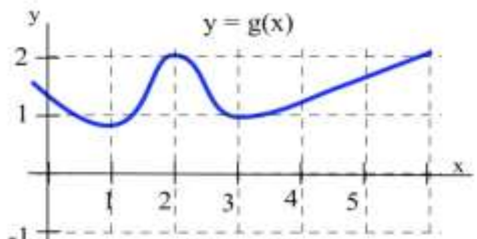

Fig. 15

4. (a) At what values of x does the graph of g in Fig. 15 have a horizontal tangent line?
   (b) At what value(s) of x is the value of g the largest? smallest?
   (c) Sketch the graph of m(x) = the slope of the line tangent to the graph of g at the point (x,y).

5. (a) Sketch the graph of f(x) = sin(x) for $-3 \le x \le 10$.
   (b) Sketch the graph of m(x) = slope of the line tangent to the graph of sin(x) at the point (x, sin(x)).
   (c) Your graph in part (b) should look familiar. What function is it?

6. Match the situation descriptions with the corresponding **time–velocity** graphs in Fig. 16.
   (a) A car quickly leaving from a stop sign.
   (b) A car sedately leaving from a stop sign.
   (c) A student bouncing on a trampoline.
   (d) A ball thrown straight up.
   (e) A student confidently striding across campus to take a calculus test.
   (f) An unprepared student walking across campus to take a calculus test.

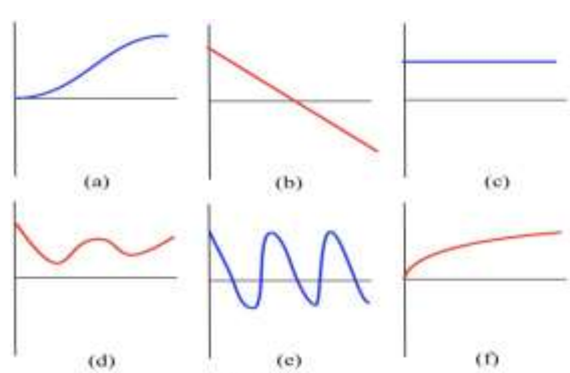

Fig. 16

Problems 7 – 10 assume that a rocket is following the path $y = x^2$, from left to right.

7. At what point should the engine be turned off in order to coast along the tangent line to a base at (5,16)?

8. At what point should the engine be turned off in order to coast along the tangent line to a base at (3,–7)?

9. At what point should the engine be turned off in order to coast along the tangent line to a base at (1,3)?

10. Which points in the plane can not be reached by the rocket? Why not?

For each function f(x) in problems 11 – 16, perform steps (a) – (d):

(a) calculate $m_{sec} = \dfrac{f(x+h) - f(x)}{(x+h) - (x)}$ and simplify (b) determine $m_{tan} = \lim_{h \to 0} m_{sec}$

(c) evaluate $m_{tan}$ at $x = 2$, (d) find the equation of the line tangent to the graph of f at (2, f(2) )

11. $f(x) = 3x - 7$ 12. $f(x) = 2 - 7x$ 13. $f(x) = ax + b$ where a and b are constants

14. $f(x) = x^2 + 3x$ 15. $f(x) = 8 - 3x^2$ 16. $f(x) = ax^2 + bx + c$ where a, b and c are constants

In problems 17 and 18, use the result that if $f(x) = ax^2 + bx + c$ then $m_{tan} = 2ax + b$.

17. $f(x) = x^2 + 2x$. At which point(s) (p, f(p) ) does the line tangent to the graph at that point also go through the point (3, 6) ?

18. (a) If $a \neq 0$, then what is the shape of the graph of $y = f(x) = ax^2 + bx + c$ ?
(b) At what value(s) of x is the line tangent to the graph of f(x) horizontal?

**Section 3.1** **PRACTICE Answers**

**Practice 1:** Approximate values of m(x) are in the table below. Fig. 17 is a g

| x | y = f(x) | m(x) = the **estimated** SLOPE of the tangent line to y=f(x) at the point (x,y) |
|---|---|---|
| 0 | **2** | **–1** |
| 1 | **1** | **–1** |
| 2 | **1/3** | **0** |
| 3 | **1** | **1** |
| 4 | **3/2** | **1/2** |
| 5 | **1** | **–2** |

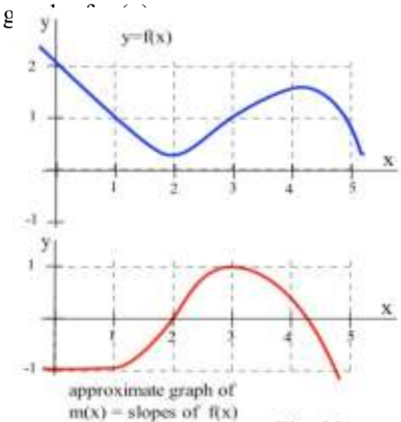


Fig. 17

**Practice 2:** The tangent lines to the graph of g are horizontal (slope = 0) when x ≈ –1, 1, 2.5, and 5.

**Practice 3**: Fig. 18 is a graph of the approximate **rate** of temperature change (slope).

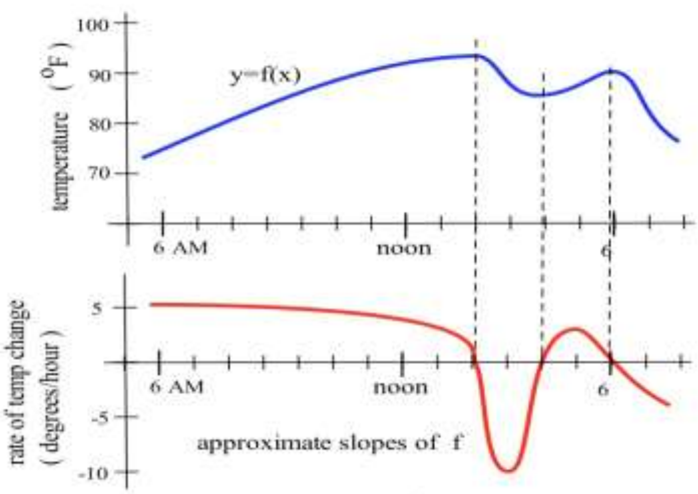


Fig. 18

*Source URL: http://scidiv.bellevuecollege.edu/dh/Calculus_all/Calculus_all.html*
*Saylor URL: http://www.saylor.org/courses/ma005/*

**Practice 4**: $y = x^2$.

At (1,1), $m_{tan} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{(1+h)-(1)} = \lim_{h\to 0} \frac{(1+h)^2-(1)^2}{h} = \lim_{h\to 0} \frac{\{1+2h+h^2\}-1}{h}$

$= \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0} \frac{h(1+h)}{h} = \lim_{h\to 0} (2+h) = 2$

At (0,0), $m_{tan} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{(0+h)-(0)} = \lim_{h\to 0} \frac{(0+h)^2-(0)^2}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = \mathbf{0}$.

At (–1,1), $m_{tan} = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{(-1+h)-(-1)} = \lim_{h\to 0} \frac{\{1-2h+h^2\}-1}{h} = \lim_{h\to 0} \frac{-2h+h^2}{h} = \mathbf{-2}$.

**Practice 5**: From Example 4 we know the slope of the tangent line is $m_{tan} = 2x$ so the slope of the tangent line at (2,4) is $m_{tan} = 2x = 2(2) = 4$. The tangent line has slope 4 and goes through the point (2,4) so the equation of the tangent line (using $y - y_o = m(x - x_o)$ ) is $y - 4 = 4(x - 2)$ or $y = 4x - 4$. The point (3,8) satisfies the equation $y = 4x - 4$ so the point (3,8) lies on the tangent line.