2.5 DEFINITION OF LIMIT

It may seem strange that we have been using and calculating the values of limits for awhile without having a precise definition of limit, but the history of mathematics shows that many concepts, including limits, were successfully used before they were precisely defined or even fully understood. We have chosen to follow the historical sequence in this chapter and to emphasize the intuitive and graphical meaning of limit because most students find these ideas and calculations easier than the definition. Also, this intuitive and graphical understanding of limit was sufficient for the first hundred years of the development of calculus (from Newton and Leibniz in the late 1600's to Cauchy in the early 1800's), and it is sufficient for using and understanding the results in beginning calculus.

Mathematics, however, is more than a collection of useful tools, and part of its power and beauty comes from the fact that in mathematics terms are precisely defined and results are rigorously proved. Mathematical tastes (what is mathematically beautiful, interesting, useful) change over time, but because of these careful definitions and proofs, the results remain true, everywhere and forever. Textbooks seldom give all of the definitions and proofs, but it is important to mathematics that such definitions and proofs exist.

The goal of this section is to provide a precise definition of the limit of a function. The definition will not help you calculate the values of limits, but it provides a precise statement of what a limit is. The definition of limit is then used to verify the limits of some functions, and some general results are proved.

The Intuitive Approach

The precise ("formal") definition of limit carefully defines the ideas that we have already been using graphically and intuitively. The following side–by–side columns show some of the phrases we have been using to describe limits, and those phrases, particularly the last ones, provide the basis to building the definition of limit.

A Particular Limit	General Limit
$\lim_{x \to 3} 2x - 1 = 5$	$\lim_{x \to a} f(x) = L$
"as the values of x approach 3, the values of 2x–1 approach (are arbitrarily close to) 5"	"as the values of x approach a, the values of f(x) approach (are arbitrarily close to) L"
"when x is close to 3 (but not equal to 3), the value of 2x–1 is close to 5"	"when x is close to a (but not equal to a), the value of f(x) is close to L"
"we can guarantee that the values of f(x) = 2x–1 are as close to 5 as we want by starting with values of x sufficiently close to 3 (but not equal to 3)"	"we can guarantee that the values of f(x) are as close to L as we want by starting with values of x sufficiently close to a (but not equal to a)"

Let's examine what the last phrase ("we can ...") means for the Particular Limit.

Example 1: We know $\lim_{x \to 3} 2x - 1 = 5$. Show that we can guarantee that the values of $f(x) = 2x - 1$ are as close to 5 as we want by starting with values of x sufficiently close to 3.

(a) What values of x guarantee that $f(x) = 2x - 1$ is within 1 unit of 5 ? (Fig. 1a)

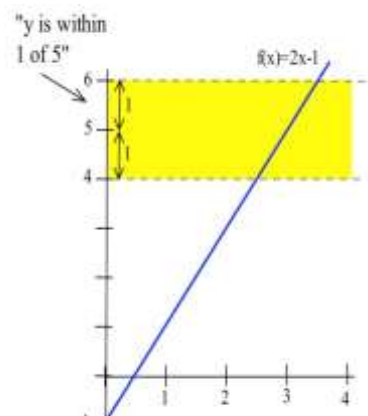


Fig. 1a

Solution: "within 1 unit of 5" means between $5–1 = 4$ and $5+1 = 6$, so the question can be rephrased as "for what values of x is $y = 2x – 1$ between 4 and 6: $4 < 2x – 1 < 6$?" We want to know which values of x put the values of $y = 2x –1$ into the shaded band in Fig. 1a. The algebraic process is straightforward: solve $4 < 2x – 1 < 6$ for x to get $5 < 2x < 7$ and $2.5 < x < 3.5$. We can restate this result as follows: "If x is within **0.5** units of 3, then $y = 2x–1$ is within 1 unit of 5." (Fig. 1b)

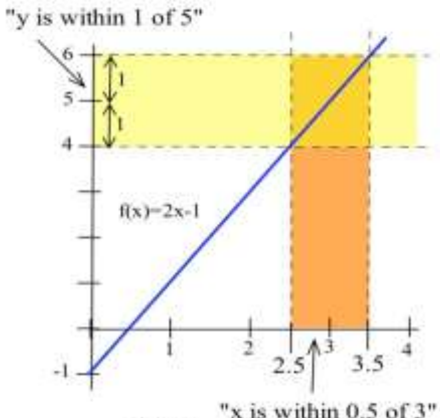


Fig. 1b

Any smaller distance also satisfies the guarantee: e.g., "If x is within **0.4** units of 3, then $y = 2x–1$ is within 1 unit of 5." (Fig. 1c)

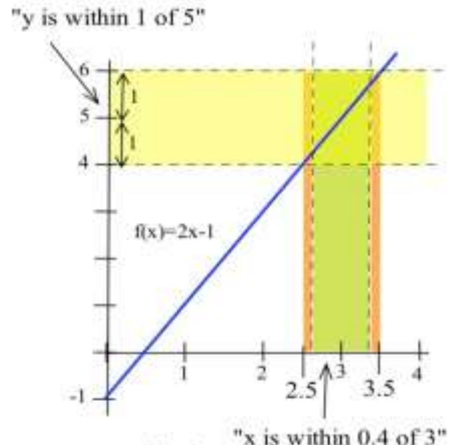


Fig. 1c

(b) What values of x guarantee the $f(x) = 2x – 1$ is within 0.2 units of 5? (Fig. 2a)

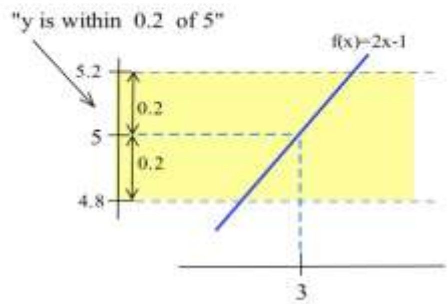
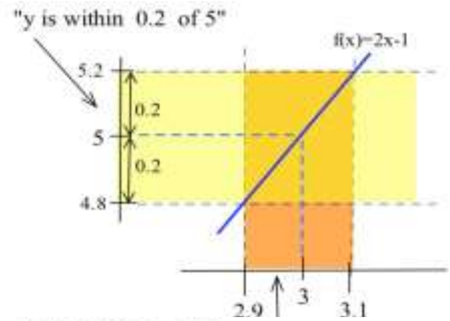


Fig. 2a

Solution: "within 0.2 units of 5" means between $5–0.2 = 4.8$ and $5+0.2 = 5.2$, so the question can be rephrased as "for what values of x is $y = 2x – 1$ between 4.8 and 5.2: $4.8 < 2x – 1 < 5.2$?" Solving for x, we get $5.8 < 2x < 6.2$ and $2.9 < x < 3.1$. "If x is within 0.1 units of 3, then $y = 2x–1$ is within 0.2 units of 5." (Fig. 2b) Any smaller distance also satisfies the guarantee.



Rather than redoing these calculations for every possible distance from 5, we can do the work once, generally:

(c) What values of x guarantee that $f(x) = 2x – 1$ is within E units of 5? (Fig. 3a)

Solution: "within E unit of 5" means between 5–E and 5+E , so the question is "for what values of x is $y = 2x – 1$ between 5–E and 5+ε: $5–E < 2x – 1 < 5+E$?" Solving $5–E < 2x – 1 < 5+E$ for x, we get $6–E < 2x < 6+E$ and $3 – E/2 < x < 3 + E/2$. "If x is within E/2 units of 3, then $y = 2x–1$ is within E units of 5." (Fig. 3b) Any smaller distance also satisfies the guarantee.

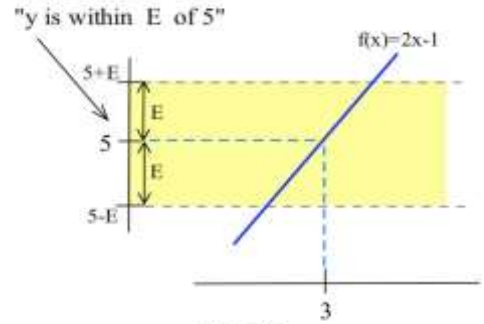


Fig. 3a

Part (c) of Example 1 illustrates a little of the power of general solutions in mathematics. Rather than doing a new set of similar calculations every time someone demands that $f(x) = 2x – 1$ be within some given distance of 5, we did the calculations once. And then we can respond for any given distance. For the question "What values of x guarantee that $f(x) = 2x – 1$ is within 0.4, 0.1 and 0.006 units of 5?", we can answer "If x is within 0.2 (= 0.4/2), 0.05 (=0.1/2) and 0.003 (=0.006/2) units of 3."

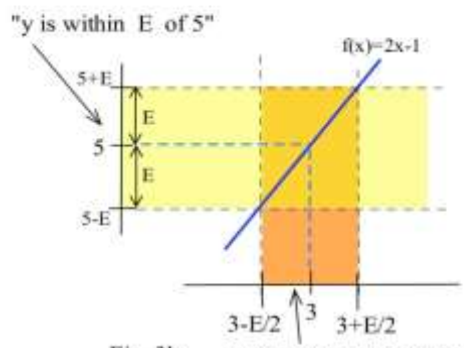


Fig. 3b

Practice 1: $\lim_{x \to 2} 4x - 5 = 3$. What values of x guarantee that

$f(x) = 4x - 5$ is within

(a) 1 unit of 3?

(b) 0.08 units of 3?

(c) E units of 3? (Fig. 4)

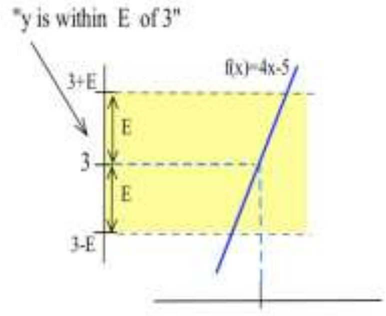


Fig. 4

The same ideas work even if the graphs of the functions are not straight lines, but the calculations are more complicated.

Example 2: $\lim_{x \to 2} x^2 = 4$. (a) What values of x guarantee that $f(x) = x^2$ is within 1 unit of 4? (b) Within 0.2 units of 4? (Fig. 5a) State each answer in the form "If x is within ______ units of 2, then f(x) is within 1 (or 0.2) unit of 4."

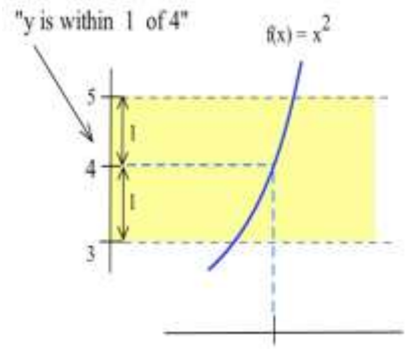


Fig. 5a

Solution: (a) If x^2 is within 1 unit of 4, then $3 < x^2 < 5$ so $\sqrt{3} < x < \sqrt{5}$

or $1.732 < x < 2.236$. The interval containing these x values extends from $2 - \sqrt{3} \approx 0.268$ units to the left of 2 to $\sqrt{5} - 2 \approx 0.236$ units to the right of 2. Since we want to specify a single distance on each side of 2, we can pick the **smaller** of the two distances, 0.236 . (Fig. 5b)

"If x is within _0.236_ units of 2,
then f(x) is within 1 unit of 4."

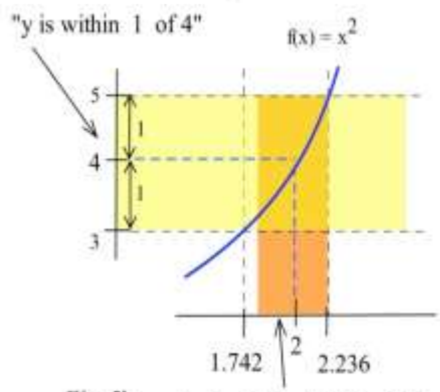


Fig. 5b **"x is within 0.236 of 3"**

(b) Similarly, if x^2 is within 0.2 units of 4, then $3.8 < x^2 < 4.2$ so $\sqrt{3.8} < x < \sqrt{4.2}$ or $1.949 < x < 2.049$. The interval containing these x values extends from $2 - \sqrt{3.8} \approx 0.051$ units to the left of 2 to $\sqrt{4.2} - 2 \approx 0.049$ units to the right of 2. Again picking the smaller of the two distances, "If x is within _0.049_ units of 2, then f(x) is within 1 unit of 4."

The situation in Example 2 of different distances on the left and right sides is very common, and we **always** pick our single distance to be the **smaller** of the distances to the left and right. By using the smaller distance, we can be certain that if x is within that smaller distance on either side, then the value of f(x) is within the specified distance of the value of the limit.

Practice 2: $\lim_{x \to 9} \sqrt{x} = 3$. What values of x guarantee that $f(x) = \sqrt{x}$ is within 1 unit of 3? Within 0.2 units of 3? (Fig. 6) State each answer in the form

"If x is within ______ units of 2, then f(x) is within 1 (or 0.2) unit of 4."

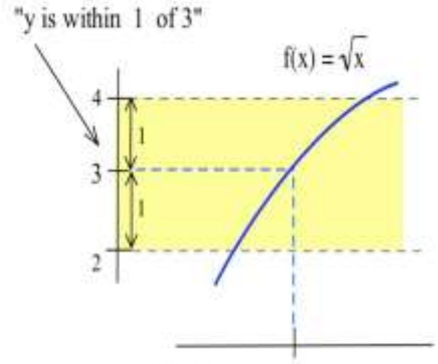


Fig. 6

The same ideas can also be used when the function and the specified distance are given graphically, and in that case we can give the answer graphically.

Example 3: In Fig. 7, $\lim_{x \to 2} f(x) = 3$. What values of x guarantee that y = f(x) is within E units (given graphically) of 3? State your answer in the form "If x is within ______ (show a distance D graphically) of 2, then f(x) is within E units of 3."

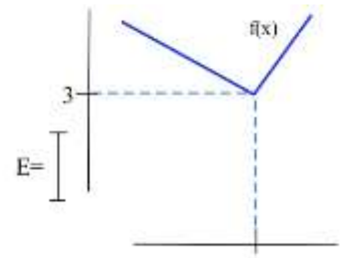


Fig. 7

Solution: The solution process requires several steps as illustrated in Fig. 8:

i. Use the given distance E to find the values 3 – E and 3 + E on the y–axis.

ii. Sketch the horizontal band which has its lower edge at y = 3 – E and its upper edge at y = 3 + E.

iii. Find the first locations to the right and left of x = 2 where the graph of y = f(x) crosses the lines y = 3 – E and y = 3 + E, and at these locations draw vertical lines to the x–axis.

iv. On the x–axis, graphically determine the distance from 2 to the vertical line on the left (labeled D_L) and from 2 to the vertical line on the right (labeled D_R).

v. Let the length D be the smaller of the lengths D_L and D_R.

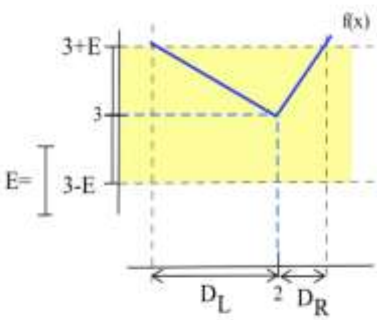


Fig. 8a: steps i and ii

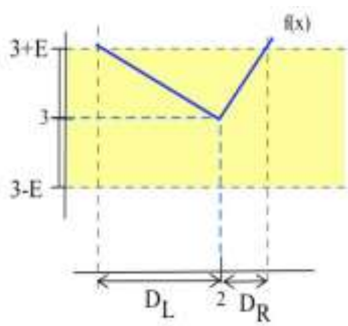


Fig. 8b: steps iii and iv

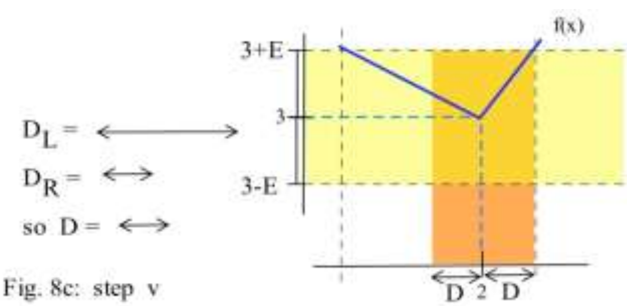


Fig. 8c: step v

Practice 3: In Fig. 9, $\lim_{x \to 3} f(x) = 1.8$. What values of x guarantee that $y = f(x)$ is within E units of 1.8?

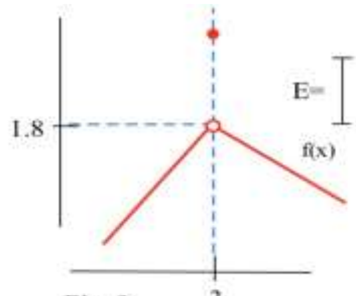


Fig. 9

The Formal Definition of Limit

The ideas of the previous examples and practice problems can be stated for general functions and limits, and they provide the basis for the definition of limit which is given in the box. The use of the lower case Greek letters ε (epsilon) and δ (delta) in the definition is standard, and this definition is sometimes called the "epsilon–delta" definition of limit.

Definition of $\lim_{x \to a} f(x) = L$**:**

$\lim_{x \to a} f(x) = L$ means

for every given $\varepsilon > 0$ there is a $\delta > 0$ so that (Fig. 10)

if x is within δ units of a (and $x \neq a$)

then f(x) is within ε units of L.

(Equivalently: $| f(x) - L | < \varepsilon$ whenever $0 < | x - a | < \delta$.)

In this definition, ε represents the given distance on either side of the limiting value y = L, and δ is the distance on each side of the point x = a on the x–axis that we have been finding in the previous examples. This definition has the form of a a "challenge and reponse:" for any positive challenge ε (make f(c) within ε of L), there is a positive response δ (start with x within δ of a and x ≠ a).

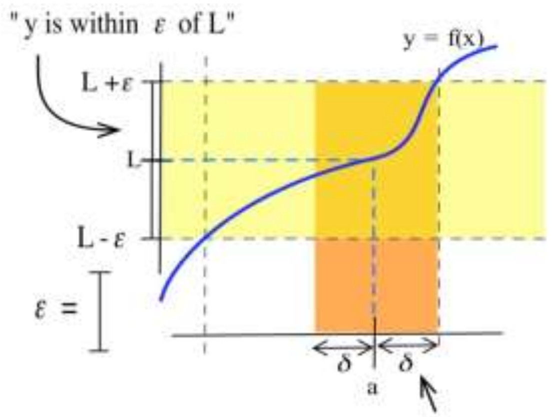


Fig. 10

Example 4: In Fig. 11a, $\lim_{x \to a} f(x) = L$, and a value for ε is given graphically as a length. Find a length for δ that satisfies the definition of limit (so "if x is within δ of a (and x ≠ a), then f(x) is within ε of L").

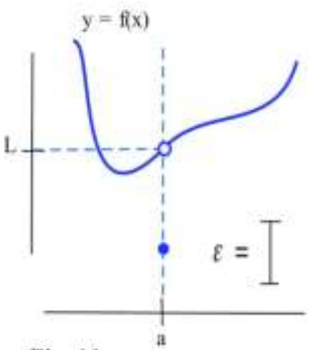
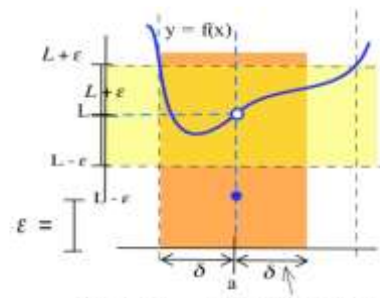


Fig. 11a

Solution: Follow the steps outlined in Example 3. The length for δ is shown in Fig. 11b, and any shorter length for δ also satisfies the definition.



Practice 4: In Fig. 12, $\lim_{x \to a} f(x) = L$, and a value for ε is given graphically as a length. Find a length for δ that satisfies the definition of limit

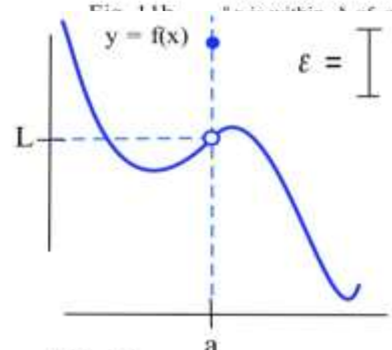


Fig. 12

Example 5: Prove that $\lim_{x \to 3} 4x - 5 = 7$.

Solution: We need to show that

"for every given ε > 0 there is a δ > 0 so that

if x is within δ units of 3 (and x ≠ 3)

then 4x – 5 is within ε units of 7."

Actually there are two things we need to do. First, we need to find a value for δ (typically depending on ε), and, second, we need to show that our δ really does satisfy the "if – then" part of the definition.

i. Finding δ is similar to part (c) in Example 1 and Practice 1: assume $4x - 5$ is within ε units of 7 and solve for x. If $7 - \varepsilon < 4x - 5 < 7 + \varepsilon$, then $12 - \varepsilon < 4x < 12 + \varepsilon$ and

$3 - \varepsilon/4 < x < 3 + \varepsilon/4$, so x is within $\varepsilon/4$ units of 3. Put $\delta = \varepsilon/4$.

ii. To show that $\delta = \varepsilon/4$ satisfies the definition, we merely reverse the order of the steps in part i. Assume that x is within δ units of 3. Then $3 - \delta < x < 3 + \delta$ so

$3 - \varepsilon/4 < x < 3 + \varepsilon/4$ (replacing δ with $\varepsilon/4$),

$12 - \varepsilon < 4x < 12 + \varepsilon$ (multiplying by 4), and

$7 - \varepsilon < 4x - 5 < 7 + \varepsilon$ (subtracting 5), so

we can conclude that $f(x) = 4x - 5$ is within ε units of 7. This formally verifies that $\lim_{x\to 3} 4x - 5 = 7$.

Practice 5: Prove that $\lim_{x\to 4} 5x + 3 = 23$.

The method used to prove the values of the limits for these particular linear functions can also be used to prove the following general result about the limits of linear functions.

Theorem: $\lim_{x\to a} mx + b = ma + b$

Proof: Let $f(x) = mx + b$.

Case 1: $m = 0$. Then $f(x) = 0x + b = b$ is simply a constant function, and any value for $\delta > 0$ satisfies the definition. Given any value of $\varepsilon > 0$, let $\delta = 1$ (any positive value for δ works). If x is is within 1 unit of a, then $f(x) - f(a) = b - b = 0 < e$, so we have shown that for any

$\varepsilon > 0$, there is a $\delta > 0$ which satisfies the definition.

Case 2: $m \neq 0$. Then $f(x) = mx + b$. For any $\varepsilon > 0$, put $\delta = \dfrac{\varepsilon}{|m|} > 0$. If x is within $\delta = \dfrac{\varepsilon}{|m|}$ of a, then $a - \dfrac{\varepsilon}{|m|} < x < a + \dfrac{\varepsilon}{|m|}$ so $-\dfrac{\varepsilon}{|m|} < x - a < \dfrac{\varepsilon}{|m|}$ and $|\,x - a\,| < \dfrac{\varepsilon}{|m|}$.

Then the distance between f(x) and $L = ma + b$ is

$|\,f(x) - L\,| = |\,(mx + b) - (ma + b)\,| = |m|\cdot|\,x - a\,| < |m|\dfrac{\varepsilon}{|m|}$

$= \varepsilon$

so f(x) is within ε

of $L = ma + b$. (Fig. 13)

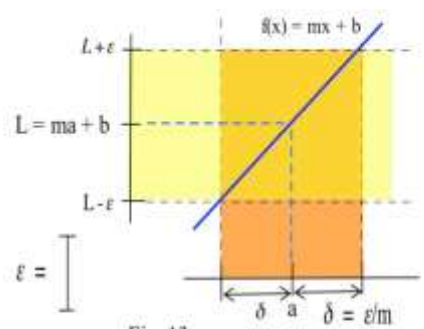


Fig. 13

In each case, we have shown that "given any $\varepsilon > 0$, there is a

$\delta > 0$" that satisfies the rest of the definition is satisfied.

If there is even a single value of ε for which there is no δ, then the function does not satisfy the difinition, and we say that the limit "does not exist.:

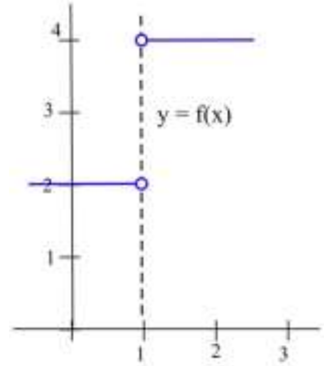


Fig. 14

Example 6: Let $f(x) = \begin{cases} 2 & \text{if } x < 1 \\ 4 & \text{if } x > 1 \end{cases}$ as is shown in Fig. 14 .

Use the definition to prove that $\lim\limits_{x \to 1} f(x)$ does not exist.

Solution: One common proof technique in mathematics is called "proof by contradiction," and that is the method we use here. Using that method in this case, (i) we assume that the limit does exist and equals some number L, (ii) we show that this assumption leads to a contradiction, and (iii) we conclude that the assumption must have been false. Therefore, we conclude that the limit does not exist.

(i) Assume that the limit exists: $\lim\limits_{x \to 1} f(x) = L$ for some value for L. Let $\varepsilon = \frac{1}{2}$. (The definition says "for every ε" so we can pick this value. Why we chose this value for ε shows up later in the proof.) Then, since we are assuming that the limit exists, there is a $\delta > 0$ so that if x is within δ of 1 then $f(x)$ is within ε of L.

(ii) Let x_1 be between 1 and $1 + \delta$. Then $x_1 > 1$ so $f(x_1) = 4$. Also, x_1 is within δ of 1 so $f(x_1) = 4$ is within $\frac{1}{2}$ of L, and L is between 3.5 and 4.5: $\mathbf{3.5 < L < 4.5}$.

Let x_2 be between 1 and $1 - \delta$. Then $x_2 < 1$ so $f(x_2) = 2$. Also, x_2 is within δ of 1 so $f(x_2) = 2$ is within $\frac{1}{2}$ of L, and L is between 1.5 and 2.5: $\mathbf{1.5 < L < 2.5}$.

(iii) The two inequalities in bold print provide the contradiction we were hoping to find. There is no value L that simultaneously satisfies $3.5 < L < 4.5$ **and** $1.5 < L < 2.5$, so we can conclude that our assumption was false and that $f(x)$ does not have a limit as $x \to 1$.

Practice 6: Use the definition to prove that $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist (Fig. 15).

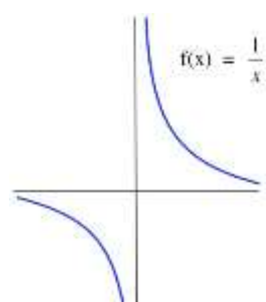


Fig. 15

Two Limit Theorems

The theorems and their proofs are included here so you can see how such proofs proceed — you have already used these theorems to evaluate limits of functions.. There are rigorous proofs of all of the other limit properties, but they are somewhat more complicated than the proofs given here.

Theorem: If $\lim\limits_{x \to a} f(x) = L$, then $\lim\limits_{x \to a} k \cdot f(x) = k \cdot L$.

Proof: Case $k = 0$: The Theorem is true but not very interesting: $\lim\limits_{x \to a} 0 \cdot f(x) = \lim\limits_{x \to a} 0 = 0 \cdot L$.

Case $k \neq 0$: Since $\lim\limits_{x \to a} f(x) = L$, then, by the definition, for every $\varepsilon > 0$ there is a $\delta > 0$ so that $| f(x) - L | < \varepsilon$ whenever $| x - a | < \delta$. For any $\varepsilon > 0$, we know $\frac{\varepsilon}{|k|} > 0$ and pick a value of δ that satisfies $| f(x) - L | < \frac{\varepsilon}{|k|}$ whenever $| x - a | < \delta$. When

$| x - a | < \delta$ ("x is within δ of a") then

$| f(x) - L | < \frac{\varepsilon}{|k|}$ ("f(x) is within $\frac{\varepsilon}{|k|}$ of L") so

$|k| \cdot | f(x) - L | < \varepsilon$ (multiplying each side by $|k| > 0$) and

$| k \cdot f(x) - k \cdot L | < \varepsilon$ ($k \cdot f(x)$ is within ε of $k \cdot L$).

Theorem: If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} f(x) + g(x) = L + M$.

Proof: Assume that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Then, given any $\varepsilon > 0$, we know $\varepsilon/2 > 0$ and that there are deltas for f and g, δ_f and δ_g, so that

if $| x - a | < \delta_f$, then $| f(x) - L | < \varepsilon/2$ ("if x is within δ_f of a, then f(x) is within $\varepsilon/2$ of L", and

if $| x - a | < \delta_g$, then $| g(x) - M | < \varepsilon/2$ ("if x is within δ_g of a, then g(x) is within $\varepsilon/2$ of M").

Let δ be the smaller of δ_f and δ_g. If $| x - a | < \delta$, then $| f(x) - L | < \varepsilon/2$ and $| g(x) - M | < \varepsilon/2$ so

$| (f(x) + g(x)) - (L + M)) | = | (f(x) - L) + (g(x) - M) |$ (rearranging the terms)

$\leq | f(x) - L | + | g(x) - M |$ (by the Triangle Inequality for absolute values)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$ (by the definition of the limits for f and g).

Problems for Solution

In problems 1–4, state each answer in the form "If x is within _____ units of . . . "

1. $\lim_{x \to 3} 2x + 1 = 7$. What values of x guarantee that $f(x) = 2x + 1$ is (a) within 1 unit of 7?

   (b) within 0.6 units of 7? (c) within 0.04 units of 7? (d) within ε units of 7?

2. $\lim_{x \to 1} 3x + 2 = 5$. What values of x guarantee that $f(x) = 3x + 2$ is within 1 unit of 5 ?

   (b) within 0.6 units of 5? (c) within 0.09 units of 5? (d) within ε units of 5?

3. $\lim_{x \to 2} 4x - 3 = 5$. What values of x guarantee that $f(x) = 4x - 3$ is within 1 unit of 5 ?

   (b) within 0.4 units of 5? (c) within 0.08 units of 5? (d) within ε units of 5?

4. $\lim_{x \to 1} 5x - 3 = 2$. What values of x guarantee that $f(x) = 5x - 3$ is within 1 unit of 2 ?

   (b) within 0.5 units of 5? (c) within 0.01 units of 5? (d) within ε units of 5?

5. For problems 1 – 4, list the slope of each function f and the δ (as a function of ε). For these linear functions f, how is δ related to the slope?

6. You have been asked to cut two boards (exactly the same length after the cut) and place them end to end. If the combined length must be within 0.06 inches of 30 inches, then each board must be within how many inches of 15?

7. You have been asked to cut three boards (exactly the same length after the cut) and place them end to end. If the combined length must be within 0.06 inches of 30 inches, then each board must be within how many inches of 10?

8. $\lim_{x \to 3} x^2 = 9$. What values of x guarantee that $f(x) = x^2$ is within 1 unit of 9? within 0.2 units?

9. $\lim_{x \to 2} x^3 = 8$. What values of x guarantee that $f(x) = x^3$ is within 0.5 unit of 8? within 0.05 units?

10. $\lim_{x \to 16} \sqrt{x} = 4$. What values of x guarantee that $f(x) = \sqrt{x}$ is within 1 unit of 4? Within 0.1 units?

11. $\lim_{x \to 3} \sqrt{1+x} = 2$. What values of x guarantee that $f(x) = \sqrt{1+x}$ is within 1 unit of 2? Within 0.0002 units?

12. You have been asked to cut four pieces of wire (exactly the same length after the cut) and form them into a square. If the area of the square must be within 0.06 inches of 100 inches, then each piece of wire must be within how many inches of 10?

13. You have been asked to cut four pieces of wire (exactly the same length after the cut) and form them into a square. If the area of the square must be within 0.06 inches of 25 inches, then each piece of wire must be within how many inches of 5?

In problems 14 – 17, $\lim_{x \to a} f(x) = L$ and the function f and a value for ε are given graphically. Find a length for δ that satisfies the definition of limit for the given function and value of ε.

14. f and ε as shown in Fig. 16.

15. f and ε as shown in Fig. 17.

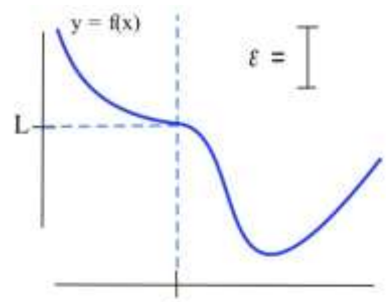


Fig. 16

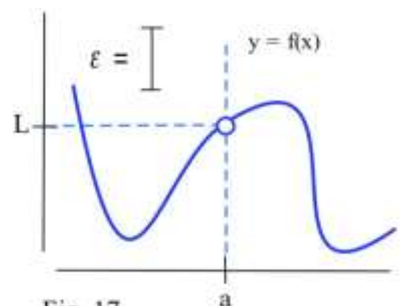


Fig. 17

16. f and ε as shown in Fig. 18.

17. f and ε as shown in Fig. 19.

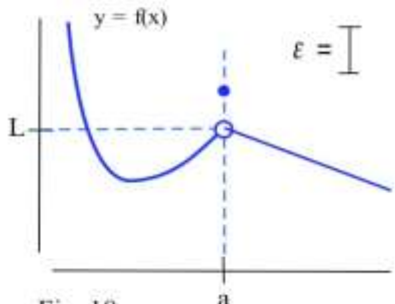


Fig. 18

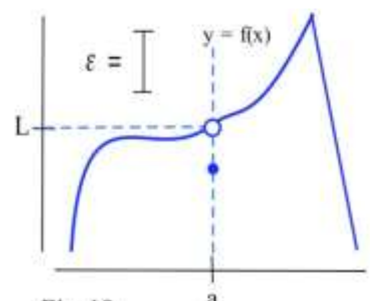


Fig. 19

18. Redo each of problems 14 – 17 taking a new value of ε to be half the value of ε given in the problem.

In problems 19–22, use the definition to prove that the given limit does not exist. (Find a value for $\varepsilon > 0$ for which there is no δ that satisfies the definition.)

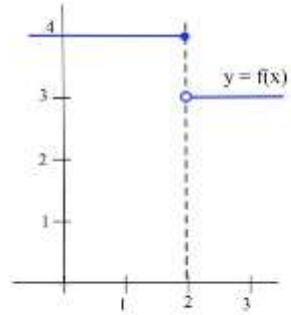


Fig. 20

19. $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 3 & \text{if } x > 2 \end{cases}$ as is shown in Fig. 20.

Show $\lim\limits_{x\to 2} f(x)$ does not exist.

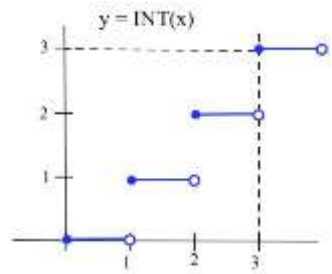


Fig. 21

20. $f(x) = \text{INT}(x)$ as is shown in Fig. 21.

Show $\lim\limits_{x\to 3} f(x)$ does not exist.

21. $f(x) = \begin{cases} x & \text{if } x < 2 \\ 6-x & \text{if } x > 2 \end{cases}$. Show $\lim\limits_{x\to 2} f(x)$ does not exist.

22. $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 & \text{if } x > 1 \end{cases}$. Show $\lim\limits_{x\to 2} f(x)$ does not exist.

23. Prove: If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then $\lim\limits_{x\to a} f(x) - g(x) = L - M$.

Section 2.5 **PRACTICE Answers**

Practice 1: (a) $3 - 1 < 4x - 5 < 3 + 1$ so $7 < 4x < 9$ and $1.75 < x < 2.25$: "x within 1/4 unit of 2."

(b) $3 - 0.08 < 4x - 5 < 3 + 0.08$ so $7.92 < 4x < 8.08$ and $1.98 < x < 2.02$: " x within 0.02 units of 2."

(c) $3 - E < 4x - 5 < 3 + E$ so $8 - E < 4x < 8 + E$ and $2 - \frac{E}{4} < x < 2 + \frac{E}{4}$: "x within E/4 units of 2."

Practice 2: "within 1 unit of 3": If $2 < \sqrt{x} < 4$, then $4 < x < 16$ which extends from 5 units to the left of 9 to 7 units to right of 9. Using the **smaller** of these two distances from 9, "If x is within 5 units of 9, then $\sqrt{x}$ is within 1 unit of 3."
"within 0.2 units of 3": If $2.8 < \sqrt{x} < 3.2$, then $7.84 < x < 10.24$ which extends from 1.16 units to the left of 9 to 1.24 units to the right of 9. "If x is within 1.16 units of 9, then $\sqrt{x}$ is wqithin 0.2 units of 3.

Practice 3: See Fig. 22. **Practice 4:** See Fig. 23

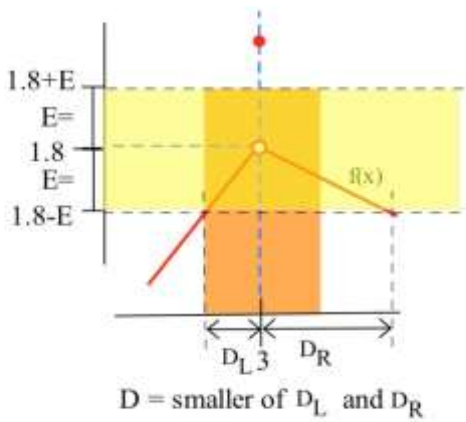


Fig. 22 (based on Fig. 9)

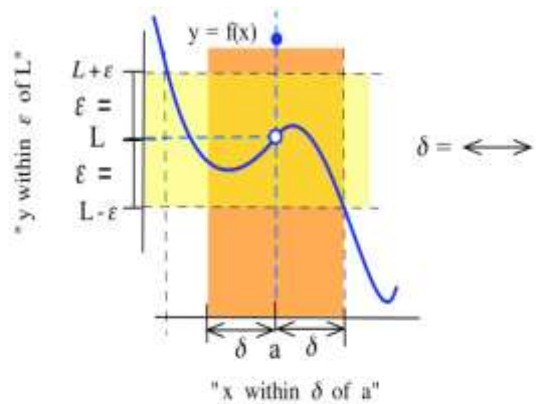


Fig. 23 (based on Fig. 12)

Practice 5: Given any $\varepsilon > 0$, take $\delta = \varepsilon/5$.

If x is within $\delta = \varepsilon/5$ of 4, then

$4 - \varepsilon/5 < x < 4 + \varepsilon/5$ so

$-\varepsilon/5 < x - 4 < \varepsilon/5$ (subtracting 4)

$-\varepsilon < 5x - 20 < \varepsilon$ (multiplying by 5)

$-\varepsilon < (5x + 3) - 23 < \varepsilon$ (rearranging to get the form we want)

so, finally, $f(x) = 5x + 3$ is within ε of $L = 23$.

We have shown that "for any $\varepsilon > 0$, there is a $\delta > 0$ (namely $\delta = \varepsilon/5$)" so that the rest of the definition is satisfied.

Practice 6: This is a much more sophisticated (= harder) problem.

Using "proof by contradiction" as outlined in the solution to Example 6.

(i) Assume that the limit exists: $\lim_{x \to 0} \frac{1}{x} = L$ for some value for L. Let $\varepsilon = 1$. (The definition says "for every ε" so we can pick this value. For this limit, the definition fails for every $\varepsilon > 0$.) Then, since we are assuming that the limit exists, there is a $\delta > 0$ so that if x is within δ of 0 then

$f(x) = \frac{1}{x}$ is within $\varepsilon = 1$ of L.

(ii) (See Fig. 24) Let x_1 be between 0 and $0 + \delta$ and also require that $x_1 < \frac{1}{2}$. Then $0 < x_1 < \frac{1}{2}$ so $f(x_1) = \frac{1}{x_1} > 2$. Since x_1 is within δ of 0, $f(x_1) > 2$ is within $\varepsilon = 1$ of L, so L is greater than $2 - \varepsilon = 1$: $\mathbf{1 < L}$.

Let x_2 be between 0 and $0 - \delta$ and also require that $x_2 > -\frac{1}{2}$. Then $0 > x_2 > \frac{1}{2}$ so $f(x_2) = \frac{1}{x_2} < -2$. Since x_2 is within δ of 0, $f(x_2) < -2$ is within $\varepsilon = 1$ of L, so L is less than $-2 + \varepsilon = -1$: $\mathbf{-1 > L}$.

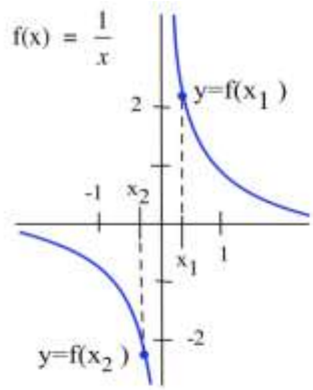


Fig. 23 (based on Fig. 15)

(iii) The two inequalities in bold print provide the contradiction we were hoping to find. There is no value L that satisfies

BOTH $\mathbf{1 < L}$ **and** $\mathbf{L < -1}$,

so we can conclude that our assumption was false and that $f(x) = \frac{1}{x}$ does not have a limit as $x \to 0$.