f' is $-x/\sqrt{r^2-x^2}$, so the surface area is given by

$$\begin{aligned} A &= 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx \\ &= 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}\,dx \\ &= 2\pi\int_{-r}^{r} r\,dx = 2\pi r\int_{-r}^{r} 1\,dx = 4\pi r^2 \end{aligned}$$

□

If the curve is rotated around the y axis, the formula is nearly identical, because the length of the line segment we use to approximate a portion of the curve doesn't change. Instead of the radius $f(x_i^*)$, we use the new radius $\bar{x}_i = (x_i + x_{i+1})/2$, and the surface area integral becomes

$$\int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx.$$

EXAMPLE 9.32 Compute the area of the surface formed when $f(x) = x^2$ between 0 and 2 is rotated around the y-axis.

We compute $f'(x) = 2x$, and then

$$2\pi\int_0^2 x\sqrt{1+4x^2}\,dx = \frac{\pi}{6}(17^{3/2}-1),$$

by a simple substitution. □

Exercises 9.10.

1. Compute the area of the surface formed when $f(x) = 2\sqrt{1-x}$ between -1 and 0 is rotated around the x-axis. ⇒
2. Compute the surface area of example 9.32 by rotating $f(x) = \sqrt{x}$ around the x-axis.
3. Compute the area of the surface formed when $f(x) = x^3$ between 1 and 3 is rotated around the x-axis. ⇒
4. Compute the area of the surface formed when $f(x) = 2 + \cosh(x)$ between 0 and 1 is rotated around the x-axis. ⇒
5. Consider the surface obtained by rotating the graph of $f(x) = 1/x$, $x \ge 1$, around the x-axis. This surface is called **Gabriel's horn** or **Toricelli's trumpet**. In exercise 13 in section 9.7 we saw that Gabriel's horn has finite volume. Show that Gabriel's horn has infinite surface area.
6. Consider the circle $(x-2)^2 + y^2 = 1$. Sketch the surface obtained by rotating this circle about the y-axis. (The surface is called a **torus**.) What is the surface area? ⇒

7. Consider the ellipse with equation $x^2/4 + y^2 = 1$. If the ellipse is rotated around the x-axis it forms an **ellipsoid**. Compute the surface area. ⇒

8. Generalize the preceding result: rotate the ellipse given by $x^2/a^2 + y^2/b^2 = 1$ about the x-axis and find the surface area of the resulting ellipsoid. You should consider two cases, when $a > b$ and when $a < b$. Compare to the area of a sphere. ⇒