WORK, FLUID PRESSURES AND FORCES

Elizabeth Wood

WORK

When we push a chair a distance along a straight-line with constant force, we can determine the work done by the force on the chair by using the formula **W = Fd.** Unfortunately, not all forces are constant, so we have to take the variation of F into consideration. To do this, we must integrate the force. Here is the formula for calculating work.

$$W = \int_a^b F(x)\,dx$$

There are three types of work problems. The first is hauling up a weight. The next type is pumping liquids, and the last type is compressing or stretching a spring. We will start with the first type of problem.

EXAMPLE 1: A mountain climber is about to haul up a 50-m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?

SOLUTION:

First, let us determine the function for the force.

Force = (weight) * (length of rope that is still hanging) = 0.624(50 - x)

The limits of integration will be $0 \le x \le 50$. The reason for this is that the mountain climber starts with 50-m of rope. When he is done hauling in the rope, then there will not be any rope hanging and force acting upon it.

Now, we will set up the work integral.

$$Work = \int_0^{50} 0.624(50 - x)\,dx$$

$$= 0.624\left(50x - \frac{x^2}{2}\right)\Bigg|_0^{50} = 780 N \cdot m = 780 J$$

EXAMPLE 2: You drove an 800-gal truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-ft elevation change in 50 minutes. Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8-lb./U.S. gal.

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SOLUTION:

The force required to lift the water is equal to the water's weight which varies 8*800 lbs. to 8*400 lbs. over the 4750 ft change in elevation. Since it looses half of the water when the truck reaches its destination, it would loose all of the water if it went twice the distance. When the truck is x feet of the base of Mt. Washington, the water's weight is the following proportion.

$$F(x) = 8 * 800\left(\frac{2 * 4750 - x}{2 * 4750}\right) = 6400\left(1 - \frac{x}{9500}\right)$$

Notice that the formula satisfies the initial conditions. When the truck is at the base of the mountain (x = 0ft), the force is 6400 lbs. When the truck has reached the summit (x = 4750 ft), there force is 3200 lbs. Now, let us calculate the work done by moving this force.

$$Work = \int_0^{4750} 6400\left(1 - \frac{x}{9500}\right)dx$$

$$= 6400\left(x - \frac{x^2}{2 * 9500}\right)\Bigg|_0^{4750} = 22{,}800{,}000 \ ft - lbs$$

Now, let us do an example of calculating the work done in pumping a liquid.

EXAMPLE 3: Pumping water from a lake 15-ft below the bottom of the tank can fill the cylindrical tank shown here. (See **figure 1**) There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reason for answer.

SOLUTION:

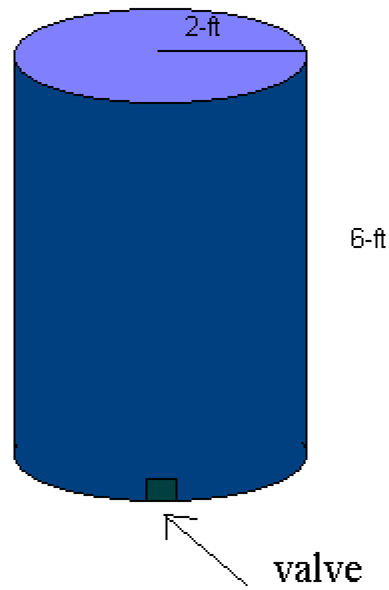


figure 1

PUMPING WATER TO THE BOTTOM OF THE TANK

In the statement of the problem, it states that the water is being pumped from a lake that is 15-ft below the tank. That is the distance it takes to get the water from the lake to the valve, but this is not the total distance that the water is moved. We are forcing the water up into the tank, so the water travels a distance of y in the tank.

The total distance the water travels is 15 + y.

Water weight 62.4 lbs./ft ^{3}.

The volume of a cross-section of the tank would be area of the circle times the height of the cross-section.

Volume of the cross-section = $\pi (2)^2 dy$

The limits for integration are $0 \le y \le 6$. These are the values that y can take on. (The first 15-ft are a must.) Now, let us calculate the work done in pumping water to the bottom of the tank.

$$Work_1 = \int_0^6 62.4(4\pi)(15+y)\,dy$$

$$= 784.14\left(15y - \frac{y^2}{2}\right)\Bigg|_0^6 = 84{,}687.3 \ ft-lbs.$$

PUMPING WATER TO THE TOP OF THE TANK

Now we are pumping the water to the top of the tank and letting it pour in. Therefore, the distance that the water is pumped is 15 ft plus 6 ft for a total of 21 ft. The cross-section volume is the same as above. It is 4π dy. The limits of integration are $0 \le y \le 6$. These are the values that y can take on. Now, let us calculate the work done in pumping the water to the top of the tank.

$$Work_2 = \int_0^6 62.4(4\pi)\,dy$$

$$= 16466.97y\Big|_0^6 = 98{,}801.82 \ ft-lbs.$$

Which way will be faster?

If we assume that the pump produces a constant amount of work per hour, then it will take more time to pump the water to the top of the tank. Therefore, pumping the water to the bottom of the tank is the fastest way.

Finally, we will talk about the third type of work problems - springs. When calculating the work done to either compress or stretch a spring, we must use **Hooke's Law** to determine the force. Hooke's law states that the force it takes to stretch or compress a spring x length units from its natural length is proportional to x. The formula for force is **F = kx**, where **k** is the spring constant.

EXAMPLE 4: A spring has a natural length of 10 inches. An 800-lb force stretches the spring 14-inches. **(a)** Find the force constant. **(b)** How much work is done in stretching the spring from 10 inches to 12 inches? **(c)** How far beyond it natural length will a 1600-lb. force stretch the spring?

SOLUTION:

PART (a)

To find the force constant (spring constant), we must solve for k. Substitute 800 in for F and (14 - 10) in for x. F = kx → 800 = k(14 - 10) → k = 800/4 = 200 lbs./ft

PART (b)

To calculate the work done in stretching the spring from 10 inches to 12 inches, we must integrate the force function found in part **a**.

$$Work = \int_0^2 200x\,dx = 100x^2\Big|_0^2 = 400\,in - lbs.$$

PART (c)

To determine how far 1600-lb. force will stretch the spring, we do not need to integrate. We will set the new force equal to the old force and solve for x.

1600 = 200x → x = 8 inches

EXAMPLE 5: A force of 200 N will stretch a garage door spring 0.8-m beyond its unstressed length. How far will a 300-N-force stretch the spring? How much work does it take to stretch the spring this far?

SOLUTION:

To determine how far a 300-N-force will stretch the spring, we must first of all determine the force constant.

200 = 0.8k → k = 250 N/m

Now set the new force equal to the old force and solve for x.

300 = 250x → x = 1.2 m

To determine the work done to stretch the spring this far, we will integrate the force with our limits of integration being 0 and 1.2.

$$Work = \int_0^{1.2} 250x\,dx = 125x^2 \Big|_0^{1.2} = 180 \ N-m = 180 \ J$$

FLUID PRESSURES AND FORCES

Consider the construction of the Hoover Dam, the bottom of this dam is wider than it is at the top. The reason for this is that the pressure is greater at the bottom of the dam than it is at the top. To determine the pressure and forces acting on a dam, we must use the pressure-depth equation to derive the formula we will use to determine the force of the fluid against the surface.

THE PRESSURE-DEPTH EQUATION

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h**:** **p = wh**. If the fluid is pressing against a horizontal base of a vat, then the total force exerted by the fluid against the base is **F = pressure * area = whA**.

If we would like to determine the force exerted by a fluid against one side of a vertical plate submerged in the fluid, then we would have to use the variable-depth formula.

THE INTEGRAL FOR FLUID FORCE

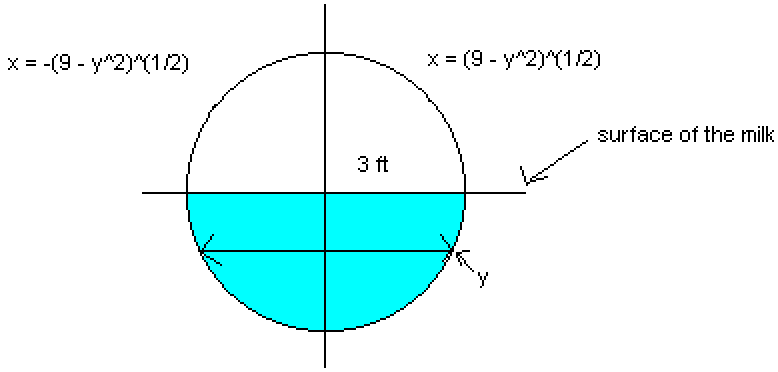
$$F = \int_a^b w * (strip\ depth) * L(y)\,dy$$

L (y) is the length of a horizontal strip from left to right at level y, and w is the weight density of the fluid.

EXAMPLE 6: A tank truck hauls milk in a 6-ft diameter horizontal right circular cylindrical tank. How must force does the milk exert on each end of the tank when the tank is half full?

SOLUTION:

We will use symmetry when determining L (y).



$$L(y) = 2x = 2\sqrt{9 - y^2}$$

The depth of the horizontal strip is -y. (I am assuming that the surface of the milk is the x-axis, so the horizontal strip is below the x-axis.)

The weight density of milk is 64.5 lbs/ft ^{3}.

$$F = \int_{-3}^{0} 64.5(-y)(2)\sqrt{9 - y^2}\, dy$$

$$u = 9 - y^2 \rightarrow du = -2y\,dy$$

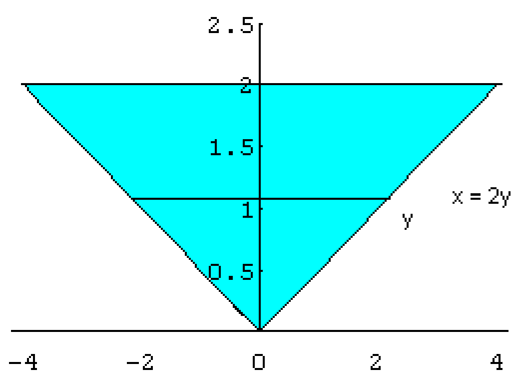
When y = -3, then u = 0, and when y = 0, then u = 9.

$$= 64.5\int_0^9 u^{\frac{1}{2}}\, du = 64.5\left(\frac{2}{3}\right)u^{\frac{3}{2}}\Big|_0^9 = 1161\, lbs.$$

EXAMPLE 7: The vertical triangular plate shown here is the end plate of a trough full of water (w = 62.4 lbs./ft ^{3}). What is the fluid force against the plate?

SOLUTION:

Using symmetry, L (y) = 2x = 4y. The horizontal strip's depth is 2 - y, so now let us calculate the force.



$$F = \int_0^2 62.4(2 - y)(4y)\,dy$$

$$= 62.4\int_0^2 (8y - 4y^2)\,dy$$

$$= 62.4\left(4y^2 - \frac{4}{3}y^3\right)\Bigg|_0^2 = 332.81\,lbs.$$

Work through the examples of calculating work and fluid forces. Remember to always sketch a diagram for fluid force problems.