MOMENTS AND CENTERS OF MASS

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Suppose that we have a circular plate of uniform density. If we tried to balance this plate on the tip of our finger, we would have to find the point in which the plate would balance. This point is what we call the center of mass. Another example of the use of the center of mass would be the use of a lever to lift an object. Think of a seesaw. Recall that when one person weighed more than the other, the smaller person would have to move closer to the fulcrum. When the person did this, the seesaw would balance, and the fulcrum would be at the center of mass.

In this set of supplemental notes, we will work two types of center of mass problems. The first type will be Wires/Thin Rods, and the other Plane Regions.

WIRES AND THIN RODS

MOMENT, MASS, AND CENTER OF MASS OF A THIN ROD ALONG THE X-AXIS WITH DENSITY FUNCTION δ (x)

MOMENT ABOUT THE ORIGIN:

$$M_o = \int_a^b x\, \delta(x)\, dx$$

MASS:

$$M = \int_a^b \delta(x)\, dx$$

CENTER OF MASS:

$$\bar{x} = \frac{M_o}{M}$$

EXAMPLE 1: Find the rod's moment about the origin, mass, and center of mass if its density function δ (x) = 1 + x/3 on [0, 3].

SOLUTION:

$$Mass = \int_0^3 \left(1+\frac{x}{3}\right)dx = x + \left.\frac{x^2}{6}\right|_0^3 = \frac{9}{2}$$

$$M_o = \int_0^3 x\left(1+\frac{x}{3}\right)dx = \int_0^3 \left(x+\frac{x^2}{3}\right)dx = \left.\frac{x^2}{2}+\frac{x^3}{9}\right|_0^3 = \frac{15}{2}$$

$$\bar{x} = \frac{M_o}{M} = \frac{\frac{15}{2}}{\frac{9}{2}} = \frac{5}{3}$$

EXAMPLE 2: Find the rod's moment about the origin, mass, and center of mass if it density function is the following.

$$\delta(x) = \begin{cases} 2 - x & 0 \le x \le 1 \\ x & 1 \le x \le 2 \end{cases}$$

SOLUTION:

$$Mass = \int_0^1 (2-x)\,dx + \int_1^2 x\,dx = 2x - \frac{x^2}{2}\Big|_0^1 + \frac{x^2}{2}\Big|_1^2 = 3$$

$$M_o = \int_0^1 x(2-x)\,dx + \int_1^2 x^2\,dx = \int_0^1 (2x - x^2)dx + \int_1^2 x^2\,dx$$

$$= x^2 - \frac{x^3}{3}\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 = 3$$

$$\bar{x} = \frac{M_o}{M} = \frac{3}{3} = 1$$

THIN, FLAT PLATES

MOMENTS, MASS, AND CENTER OF MASS OF THIN, FLAT PLATES WITH DENSITY FUNCTION δ (x)

Here are the formulas for the moments, mass, and center of mass of a thin, flat plate with density δ (x). **In these formulas, dm = δ (x) dA.**

$$\text{Moment about the } x-\text{axis} \quad M_x = \int \tilde{y}\,dm$$

$$\text{Moment about the } y-\text{axis} \quad M_y = \int \tilde{x}\,dm$$

$$\text{Mass} \quad M = \int dm$$

$$\text{Center of mass} \quad \bar{x} = \frac{M_y}{M} \quad \bar{y} = \frac{M_x}{M}$$

If the density is constant, then the center of mass is called a centroid.

EXAMPLE 3: Find the center of mass of a thin plate of constant density δ covering the region bounded by the parabola $y = x^2$ and the line y = 4.

SOLUTION:

The first step in finding the center of mass for this problem is to sketch the region. Here is the graph of this region. (See **figure 1**)

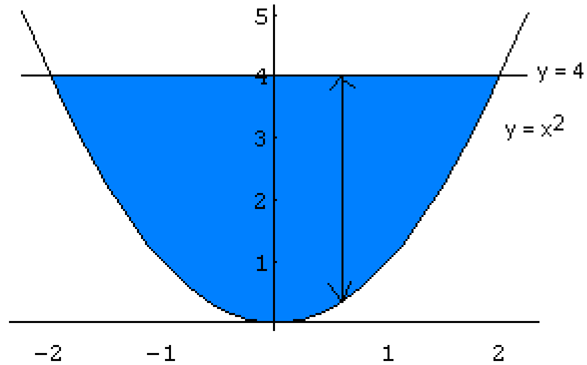

figure 1

$$(\tilde{x},\tilde{y}) = \left(x, \frac{4+x^2}{2}\right)$$

Length = upper curve - lower curve = 4 - x^2

Width = dx and **dm** = δ (4 - x^2)dx Now we have to find the bounds of integration.

$x^2 = 4 \rightarrow x = \rightarrow 2$

$$M = \int_{-2}^{2} \delta(4 - x^2)\,dx = \delta\left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2} = \frac{32\delta}{3}$$

$$M_x = \int_{-2}^{2} \delta\left(\frac{4+x^2}{2}\right)(4 - x^2)\,dx$$

$$= \delta\int_{-2}^{2} \frac{16 - x^4}{2}\,dx = \frac{\delta}{2}\left(16x - \frac{x^5}{5}\right)\Bigg|_{-2}^{2} = \frac{128\delta}{5}$$

$$M_y = \int_{-2}^{2} \delta x(4 - x^2)\,dx$$

$$= \delta\int_{-2}^{2} (4x - x^3)\,dx = \delta\left(2x^2 - \frac{x^4}{4}\right)\Bigg|_{-2}^{2} = 0$$

Why is this answer zero? Look at the graph of the region. The region is symmetric with respect to the y-axis, and since the plate is of constant density, the moment about the y-axis is zero. We could have used symmetry in finding the mass and the moment about the x-axis too. As an example, let us determine the mass using symmetry.

$$Mass = 2\,\delta\int_0^2 \left(4 - x^2\right)dx$$

$$= 2\,\delta\left(4x - \frac{x^3}{3}\right)\Bigg|_0^2 = \frac{32\,\delta}{3}$$

Do not use symmetry when finding the moment about the axis the graph is symmetric about. Also, do not use symmetry when the density is not a constant function. The reason for this is that the plate does not have the same density throughout the region. The center of mass for this problem is the following.

$$\bar{x} = 0 \quad \bar{y} = \frac{\frac{128\,\delta}{5}}{\frac{32\,\delta}{3}} = \frac{12}{5}$$

EXAMPLE 4: Find the center of mass of a thin plate of constant density δ covering the region bounded by the curves $x = y^2 - 2y$ and $y = x$.

SOLUTION:

First, let us graph this region. (See **figure 2**)

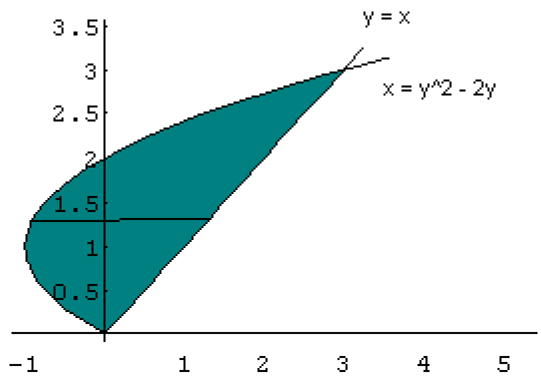

figure 2

$$(\tilde{x}, \tilde{y}) = \left(\frac{y^2 - 2y + y}{2}, y\right)$$

$$= \left(\frac{y^2 - 2y}{2}, y\right)$$

Length $= y - (y^2 - 2y) = 3y - y^2$

Width = dy and **dm** $= \delta\,(3y - y^2)\,dy$. Now we have to find the limits of integration.

$y^2 - 2y = y \rightarrow y^2 - 3y = 0 \rightarrow y = 0$ and $y = 3$

$$M = \delta\int_0^3 \left(3y - y^2\right)dy$$

$$= \delta\left(\frac{3y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^3 = \frac{9\,\delta}{2}$$

$$M_x = \delta\int_0^3 y\left(3y - y^2\right)dy \quad = \delta\int_0^3 \left(3y^2 - y^3\right)dy$$

$$= \delta\left(y^3 - \frac{y^4}{4}\right)\Bigg|_0^3 = \frac{27\ \delta}{4}$$

$$M_y = \delta\int_0^3 \left(\frac{y^2 - y}{2}\right)\left(3y - y^2\right)dy$$

$$= \frac{\delta}{2}\int_0^3 \left(-y^4 + 4y^3 - 3y^2\right)dy$$

$$= \frac{\delta}{2}\left(-\frac{y^5}{5} + y^4 - y^3\right)\Bigg|_0^3 = \frac{27\ \delta}{10}$$

$$\bar{x} = \frac{\frac{27\ \delta}{10}}{\frac{9\ \delta}{2}} = \frac{3}{5} \qquad \bar{y} = \frac{\frac{27\ \delta}{4}}{\frac{9\ \delta}{2}} = \frac{3}{2}$$

EXAMPLE 4: Find the center of mass of a thin plate covering the region between the x-axis and the curve $y = 2/x^2$, $1 \le x \le 2$, if the plate's density at the point (x, y) is $\delta(x) = x^2$.

SOLUTION:

Let us look at the graph of this region. (See **figure 3**)

$$(\tilde{x}, \tilde{y}) = \left(x, \frac{\frac{2}{x^2} + 0}{2}\right)$$

$$= \left(x, \frac{1}{x^2}\right)$$

$$Length = \frac{2}{x^2}$$

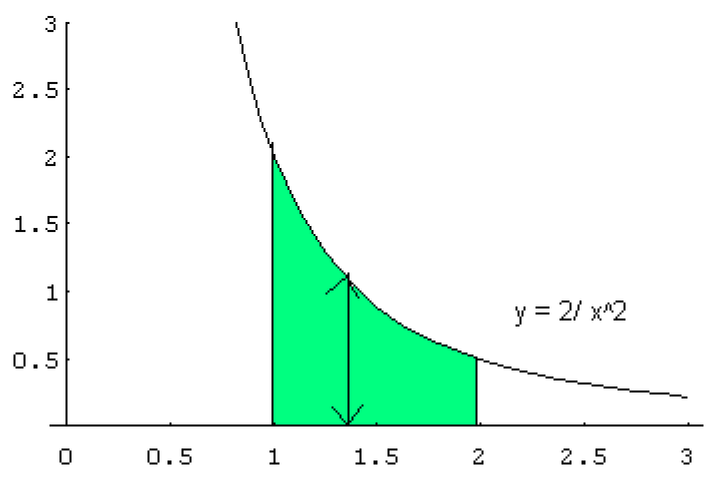

figure 3

Width = dx

$$dm = x^2\left(\frac{2}{x^2}\right) = 2$$

$$M = \int_1^2 2\,dx = 2x\Big|_1^2 = 2$$

$$M_x = \int_1^2 \frac{2}{x^2}\,dx = -\frac{2}{x}\Big|_1^2 = 1$$

$$M_y = \int_1^2 2x\,dx = x^2\Big|_1^2 = 3$$

$$\bar{x} = \frac{3}{2} \qquad \bar{y} = \frac{1}{2}$$

Work through these examples, and then try to do a few more from the book. If you follow the steps that I have used in these examples, a daunting task will be made straightforward.

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