# AREAS OF SURFACES OF REVOLUTION

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Suppose that we are given a function $y = f(x) \geq 0$ on the interval [a, b]. Revolving this function about the x-axis generates a solid. (See **figure 1**) From supplemental notes 26 and 27, we learned how to calculate the volume of this solid. Now we will learn how to find it's surface area.

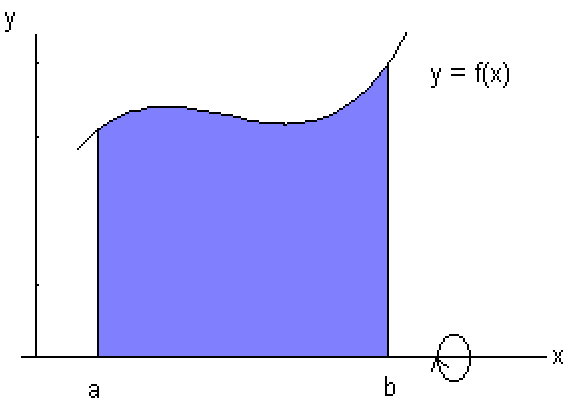



**figure 1**

To determine the surface area, we will cut the surface along the curve, and then flatten it out. (E.g. Consider a metal can. Remove the top and the

down the side of it. When you flatten it, you will discover that the flatten cylinder is in the bottom, then cut shape of a rectangle.) The length of this flatten cylinder is the circumference of the surface. Recall that the circumference of a circle is $C = 2\pi r$. Therefore, the circumference of this surface is $C = 2\pi f(x)$. The width of this rectangle is the length of the plane curve. The formula for the length of a plane curve was discussed in the supplemental notes 28. Therefore, the surface area formula for the revolution about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

The surface area formula for revolution for $x = g(y)$ on [c, d] about the y-axis is

$$S = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\,dy.$$

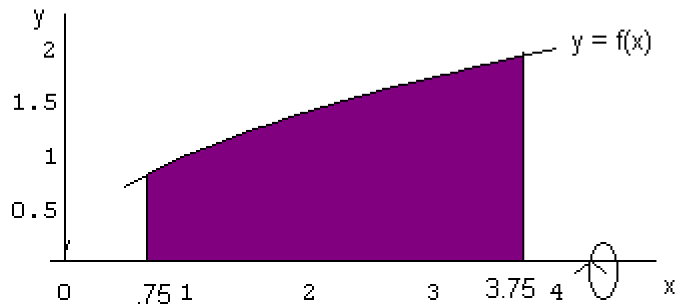
**EXAMPLE 1:** Find the area of the surface generated by revolving

$$y = \sqrt{x}$$

on the interval [3/4, 15/4] about the x-axis.

**SOLUTION:**

Here is the graph of this function.




$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$1 + (f'(x))^2 = 1 + \frac{1}{4x} = \frac{4x+1}{4x}$$

$$S = \int_{.75}^{3.75} 2\pi\sqrt{x}\,\frac{\sqrt{4x+1}}{2\sqrt{x}}\,dx \quad = \int_{.75}^{3.75} \pi\sqrt{4x+1}\,dx \quad = \int_{4}^{16} \pi\frac{1}{4}u^{\frac{1}{2}}\,du$$

Let $u = 4x + 1$, then $\frac{du}{4} = dx$.

When $x = 0.75$, then $u = 4$, and when $x = 3.75$, then $u = 16$.

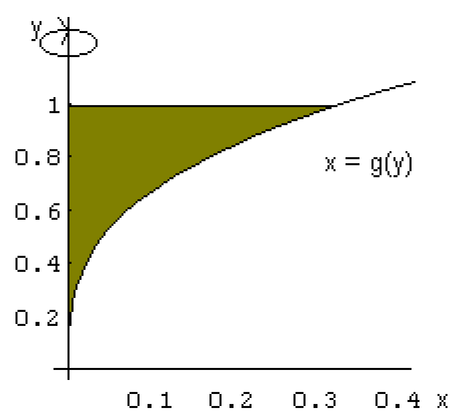
$$= \frac{1}{6}u^{\frac{3}{2}}\Big|_{4}^{16} = \frac{28}{3}$$

If integration by substitution is giving you problems, refer back to the supplemental notes 20.

**EXAMPLE 2:** Find the area of the surface generated by revolving $x = y^3/3$ on the interval [0, 1] about the y-axis.

**SOLUTION:**

Here is the graph of this function.




$$g'(y) = y^2$$

$$1 + (g'(y))^2 = 1 + y^4$$

$$S = \int_0^1 2\pi\frac{y^3}{3}\sqrt{1+y^4}\,dy \quad = \int_1^2 \frac{\pi}{6}u^{\frac{3}{2}}\,du = \frac{28\pi}{8}$$

Let $u = 1 + y^4$, then $\frac{du}{4} = y^3\,dy$.

When $y = 0$, then $u = 1$, and when $y = 1$, then $u = 2$.

$$= \frac{\pi}{9} u^{\frac{3}{2}} \Big|_1^2 = \frac{\pi}{9}\left(2\sqrt{2} - 1\right)$$

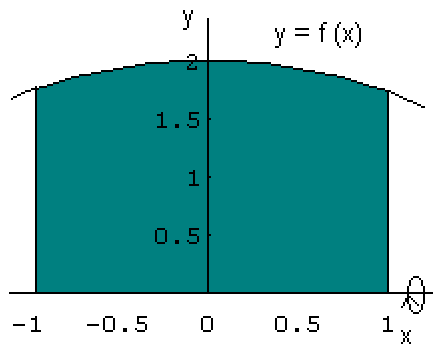
**EXAMPLE 3:** Find the area of the surface generated by revolving

$$y = \sqrt{4 - x^2}$$

on the interval [-1, 1] about the x-axis.

**SOLUTION:**

Here is the graph of this function.




$$f'(x) = \frac{1}{2}\left(4 - x^2\right)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

$$1 + (f'(x))^2 = 1 + \frac{x^2}{4 - x^2} = \frac{4 - x^2 + x^2}{4 - x^2} = \frac{4}{4 - x^2}$$

$$S = \int_{-1}^{1} 2\pi\sqrt{4 - x^2}\sqrt{\frac{4}{4 - x^2}}\,dx = \int_{-1}^{1} 2\pi\sqrt{4 - x^2}\,\frac{2}{\sqrt{4 - x^2}}\,dx = \int_{-1}^{1} 4\pi\,dx$$

$$= 4\pi x\Big|_{-1}^{1} = 8\pi$$

As you can see, the integral for finding the surface area of a solid of revolution is based on the integral to find the length of a plane curve. You start off the same way. Work through these examples.