3.6 Analyzing the Graph of a Function

1. $f(x) = x^3 + 3x^2 - x - 3$

Domain and Range

The function is a polynomial. The domain D is the set of all real numbers. The range R is the set of all real numbers.

Intercepts and Zeros

Find the zeros or $x-$intercepts by solving $x^3 + 3x^2 - x - 3 = 0$.

The possible integer roots of the polynomial are $\pm 1, \pm 3$.

We find that $f(1) = 0$. Thus 1 is a zero of the polynomial. Then we divide $f(x)$ by the factor $x-1$. The quotient is a 2nd-degree polynomial. Factoring that polynomial, the two roots there are $-1, -3$. Thus, the zeros are $1, -1$, and 3.

The $y-$intercept is $f(0) = -3$. The $y-$intercept is at $(0, -3)$.

Asymptotes and limits at infinity

The function does not have any asymptotes.

$$\lim_{x\to\infty}\left(x^3 + 3x^2 - x - 3\right) = +\infty$$
$$\lim_{x\to-\infty}\left(x^3 + 3x^2 - x - 3\right) = -\infty$$

Differentiability

The function is a polynomial and hence is differentiable everywhere.

Intervals where f is increasing and decreasing

$$f(x) = x^3 + 3x^2 - x - 3$$
$$f'(x) = 3x^2 + 6x - 1$$

Find critical values:

Set $3x^2 + 6x - 1 = 0$ and solve for x.

The critical values are

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{36+12}}{6} \\ &= \frac{-6 \pm \sqrt{48}}{6} \\ &= \frac{-6 \pm 4\sqrt{3}}{6} \\ &= \frac{-3 \pm 2\sqrt{3}}{3} \end{aligned}$$

Function values of the two critical values:

$$f\left(\frac{-3-2\sqrt{3}}{3}\right)=3.07$$
$$f\left(\frac{-3+2\sqrt{3}}{3}\right)=3.07$$

Interval	$\left(-\infty,\frac{-3-2\sqrt{3}}{3}\right)$	$\left(\frac{-3-2\sqrt{3}}{3},\frac{-3+2\sqrt{3}}{3}\right)$	$\left(\frac{-3+2\sqrt{3}}{3},\infty\right)$
Test point $x=c$	$c=-3$	$c=0$	$c=1$
evaluating	$3(-3)^2+6(-3)-1$	$3(0)^2+6(0)-1$	$3(2)^2+6(2)-1$
$f'(x)=3x^2+6x-1$	$=27-18-1>0$	$=-1<0$	$=12+12-1>0$
at $x=c$			
sign of $f'(x)$	$f'(x)>0$	$f'(x)<0$	$f'(x)>0$
Increasing/Decreasing	Increasing	Decreasing	Increasing

Relative Extrema

There is a relative maximum at $x=\frac{-3-2\sqrt{3}}{3}$ located at $(-2.15,3.07)$. There is a relative minimum at $x=\frac{-3+2\sqrt{3}}{3}$ located at $(0.15,3.07)$.

Concavity

Find $f''(x)$ and set it equal to 0. Solve for x to find possible inflection points.

$$f''(x)=6x+6$$
$$6x+6=0$$
$$6x=-6$$
$$x=-1$$

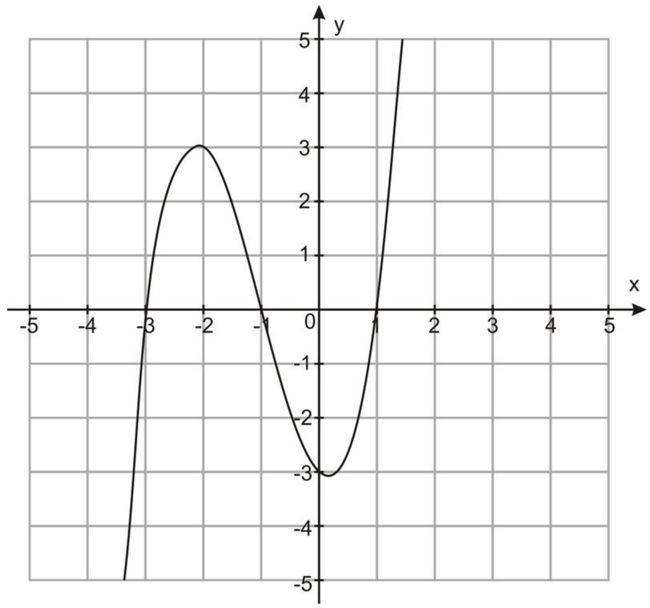
Function value of $x=-1:\ f(-1)=0$

Interval	$(-\infty,-1)$	$(-1,\infty)$
Test point $x=c$	$c=-2$	$c=0$
$f''(x)=6x+6$	$6(-2)+6$	$6(0)+6$
$f''(c)$	$=-12+6<0$	$=6>0$
Sign of $f''(x)$	$f''(x)<0$	$f''(x)>0$
Concavity	Concave down	Concave up

Inflection Points

$$f(-1)=-1+3+1-3=0$$

There is an inflection point at $(-1,0)$.



2. $f(x) = -x^4 + 4x^3 - 4x^2$

Domain and Range

The domain D is the set of all real numbers. The range R is the set of all real numbers.

Intercepts and Zeros

Find the zeros or $x-$intercepts by solving $-x^4 + 4x^3 - 4x^2 = 0$.

$$\begin{aligned}
-x^4 + 4x^3 - 4x^2 &= 0 \\
-x^2\left(x^2 - 4x - 4\right) &= 0 \\
-x^2(x-2)(x-2) &= 0 \\
x &= 0 \text{ or } x = 2
\end{aligned}$$

The zeros are $x = 0$ and $x = 2$.

The $y-$intercept is $f(0) = 0$. The $y-$intercept is at $(0,0)$.

Asymptotes and limits at infinity

The function does not have any asymptotes.

$$\begin{aligned}
\lim_{x\to\infty}\left(-x^4 + 4x^3 - 4x^2\right) &= -\infty \\
\lim_{x\to-\infty}\left(-x^4 + 4x^3 - 4x^2\right) &= -\infty
\end{aligned}$$

Differentiability

The function is a polynomial and hence is differentiable everywhere.

Intervals where f is increasing and decreasing

$$\begin{aligned}
f(x) &= -x^4 + 4x^3 - 4x^2 \\
f'(x) &= -4x^3 + 12x^2 - 8x
\end{aligned}$$

Find critical values:

Set $-4x^3+12x-8x=0$ and solve for x.

The critical values are

$$\begin{aligned}
-4x^3+12x-8x&=0\\
-4x\left(x^2-3x^2+2\right)&=0\\
x\left(x^2-3x+2\right)&=0\\
x(x-2)(x-1)&=0\\
&x=0 \text{ or } x=2 \text{ or } x=1
\end{aligned}$$

Function values of these points:

$$\begin{aligned}
f(0)&=0\\
f(1)&=-1\\
f(2)&=0
\end{aligned}$$

Interval	$(-\infty,0)$	$(0,1)$	$(1,2)$
Test point $x=c$	$c=-1$	$c=\frac{1}{2}$	$c=\frac{3}{2}$
evaluating	$-4(-1)^3+12(-1)^2-8(-1)$	$-4\left(\frac{1}{4}\right)^3+12\left(\frac{1}{2}\right)^2-8\left(\frac{1}{2}\right)$	$-4\left(\frac{3}{2}\right)^3+12\left(\frac{3}{2}\right)^2-8\left(\frac{3}{2}\right)$
$f'(x)=-4x^3+12x-8$ at $x=c$	$=4+12+8>0$	$=-\frac{4}{8}+\frac{12}{4}-4<0$	$=-13.5+27-12>0$
sign of $f'(x)$	$f'(x)>0$	$f'(x)<0$	$f'(x)>0$
Increasing/Decreasing	Increasing	Decreasing	Increasing

Relative Extrema

There is a relative maximum at $x=0$ located at $(0,0)$. There is a relative minimum at $x=1$ located at $(1,-1)$. There is a relative maximum at $x=2$ located at the point $(1,-1)$.

Concavity

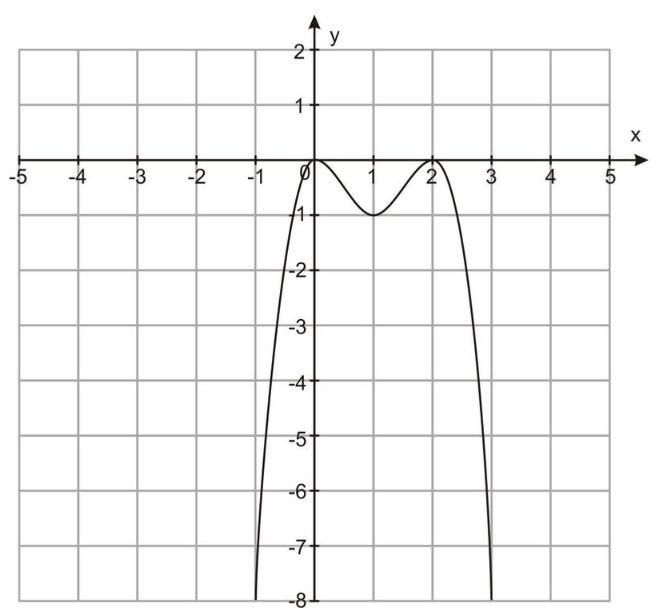
Find $f''(x)$ and set it equal to 0. Solve for x to find possible inflection points.

$$f''(x) = -12x^2 + 12x - 8$$
$$12x^2 + 24x - 8 = 0$$
$$-4\left(3x^2 - 6x + 2\right) = 0$$
$$3x^2 - 6x + 2 = 0$$
$$x = \frac{3 \pm \sqrt{3}}{3}$$
$$x = 0.42, 1.58$$
$$f\left(\frac{3-\sqrt{3}}{3}\right) = -0.44$$
$$f\left(\frac{3+\sqrt{3}}{3}\right) = -0.44$$

Interval	$\left(-\infty, \frac{3-\sqrt{3}}{3}\right)$	$\left(\frac{3-\sqrt{3}}{3}, \frac{3+\sqrt{3}}{3}\right)$	$\left(\frac{3+\sqrt{3}}{3}, \infty\right)$
Test point $x = c$	$c = 0$	$c = 1$	$c = 2$
$f''(x) = -12x^2 + 24x - 8$	$-12(0)^2 + 24(0) - 8$	$-12(1)^2 + 24(1) - 8$	$-12(2)^2 + 24(2) - 8$
$f''(c)$	$= -8 < 0$	$= 4 > 0$	$= -8 < 0$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Concavity	Concave down	Concave up	Concave down

Inflection Points

$$\left(\frac{3-\sqrt{3}}{3}, -0.44\right), \left(\frac{3+\sqrt{3}}{3}, 0.44\right)$$



3. $f(x) = \frac{2x-2}{x^2}$

Domain and Range

The function is undefined when $x^2 = 0$, or $x = 0$. The domain D is $(-\infty, 0) \cup (0, \infty)$. The range R is the set of all real numbers.

Intercepts and Zeros

zeros:

$$\begin{aligned} 2x - 2 &= 0 \\ x &= 2 \end{aligned}$$

$(2, 0)$ is a zero.

The $y-$intercept is $f(0)$, which is undefined. There is no $y-$intercept.

Asymptotes and Limits at Infinity

$$\begin{aligned} &\lim_{x \to \infty} \frac{2x-2}{x^2} = \frac{\frac{2x}{x^2} - \frac{2}{x^2}}{\frac{x^2}{x^2}} = \frac{\frac{2}{x} - \frac{2}{x^2}}{1} = 0 \\ &\lim_{x \to \infty} \frac{2x-2}{x^2} = 0 \end{aligned}$$

There is a horizontal asymptote: $y = 0$.

There is a vertical asymptote: $x = 0$.

Differentiability

The function is differentiable everywhere, except at $x = 0$.

Intervals where f is increasing and decreasing

$$\begin{aligned} f'(x) &= \frac{x^2(2) - (2x-2)(2x)}{x^4} \\ &= \frac{2x^2 - 4x^2 + 4x}{x^4} \\ &= \frac{-2x^2 + 4x}{x^4} \\ &= \frac{-2x + 4}{x^3} \end{aligned}$$

Critical values: $x = 2,\ x = 0$

$$f(2) = \frac{1}{2} = 0.5$$

Interval	$(-\infty,0)$	$(0,2)$	$(2,\infty)$
Test point $x=c$	$c=-1$	$c=1$	$c=3$
evaluating $\frac{-2x+4}{x^3}$ at $x=c$	$\frac{-2(-1)+4}{(-1)^3}<0$	$\frac{-2(1)+4}{(1)^3}>0$	$\frac{-2(3)+4}{(3)^3}<0$
Sign of $f'(x)$	$f'(x)<0$	$f'(x)>0$	$f'(x)<0$
Increasing/Decreasing	Decreasing	Increasing	Decreasing

Relative extrema

There is a relative maximum at $x=2$ located at $(2,0.5)$.

Concavity

$$\begin{aligned}
f''(x) &= \frac{x^3(-2)-(-2x+4)\left(3x^2\right)}{x^6}\\
&= \frac{4x^3-12x^2}{x^6}\\
&= \frac{4x-12}{x^4}\\
4x-12 &= 0\\
4x &= 12\\
x &= 3
\end{aligned}$$

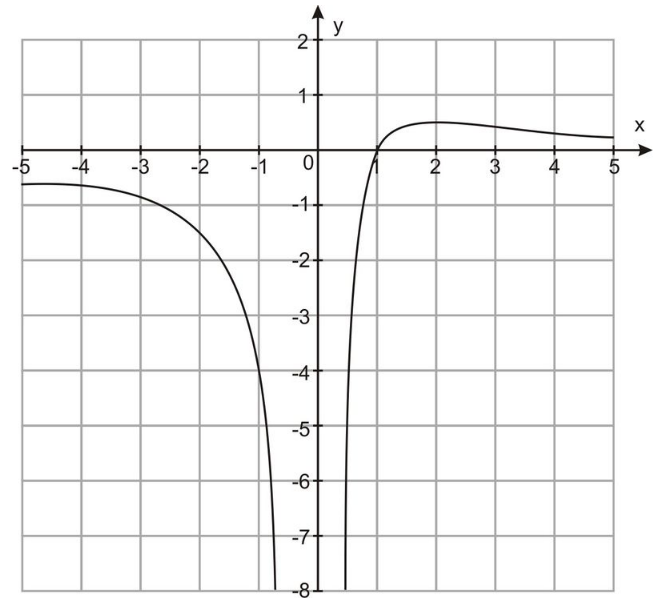
$f''(x)$ is undefined at $x=0$.

$$f(3)=\frac{4}{9}$$

Interval	$(-\infty,0)$	$(0,3)$	$(3,\infty)$
Test point $x=c$	$c=-1$	$c=1$	$c=4$
$f''(x)=\frac{4x-12}{x^4}$	$\frac{4(-1)-12}{(-1)^4}$	$\frac{4(1)-12}{(1)^4}$	$\frac{4(4)-12}{4^4}$
$f''(c)$	$=-8<0$	$=-8<0$	$=\frac{1}{4^3}>0$
Sign of $f''(x)$	$f''(x)<0$	$f''(x)<0$	$f''(x)>0$
Concavity	Concave down	Concave down	Concave up

Inflection Points

There is an inflection point at $x=3$ located at $\left(3,\frac{4}{9}\right)$.



4. $f(x) = x - x^{\frac{1}{3}}$

Domain and Range

The domain of each individual function is the set of real numbers. The domain D of the function is $(-\infty, \infty)$. The range R is the set of all real numbers.

Intercepts and Zeros

$$x - x^{\frac{1}{3}} = 0$$
$$x^{\frac{1}{3}}\left(x^{\frac{2}{3}} - 1\right) = 0$$

$$\begin{aligned} x = 0 \text{ or } x^{\frac{2}{3}} - 1 &= 0 \\ x^{\frac{2}{3}} &= 1 \\ x^2 &= 1^3 \\ x &= \pm 1 \end{aligned}$$

The $x-$intercepts are $x = 0,\ x = -1$, and $x = 1$.

The $y-$intercept is $(0,0)$.

Asymptotes and Limits at Infinity

There are no asymptotes.

$$\lim_{x \to \infty}\left(x - x^{\frac{1}{3}}\right) = \infty$$
$$\lim_{x \to -\infty}\left(x - x^{\frac{1}{3}}\right) = -\infty$$

Differentiability

$f'(x) = 1 - \frac{1}{3}x^{-\frac{2}{3}} = 1 - \frac{1}{3\sqrt[3]{x^2}}$. The function is differentiable on $(-\infty, 0) \cup (0, \infty)$.

Intervals where f is increasing and decreasing

$$\begin{aligned}
1-\frac{1}{3\sqrt[3]{x^2}} &= 0\\
-\frac{1}{3\sqrt[3]{x^2}} &= -1\\
1 &= 3\sqrt[3]{x^2}\\
\frac{1}{3} &= \sqrt[3]{x^2}\\
\frac{1}{27} &= x^2\\
\pm\sqrt{\frac{1}{27}} &= x\\
\pm\frac{1}{3\sqrt{3}} &= x\\
\pm\frac{\sqrt{3}}{9} &= x
\end{aligned}$$

$f'(x)$ is undefined at $x=0$.

Critical values: $x=\pm\frac{\sqrt{3}}{9}, x=0$

$$\begin{aligned}
f\left(-\frac{\sqrt{3}}{9}\right) &= 0.384\\
f\left(\frac{\sqrt{3}}{9}\right) &= 0.384
\end{aligned}$$

Intervals Where f is Increasing or decreasing

Interval	$\left(-\infty,-\frac{\sqrt{3}}{9}\right)$	$\left(-\frac{\sqrt{3}}{9},0\right)$	$\left(0,\frac{\sqrt{3}}{9}\right)$	$\left(\frac{\sqrt{3}}{9},\infty\right)$
Test point $x=c$	$c=-1$	$c=-0.1$	$c=0.1$	$c=1$
evaluating $1-\frac{1}{3\sqrt[3]{(x)^2}}$ at $x=c$	$1-\frac{1}{3\sqrt[3]{(-1)^2}}>0$	$1-\frac{1}{3\sqrt[3]{(-0.1)^2}}<0$	$1-\frac{1}{3\sqrt[3]{(0.1)^2}}<0$	$1-\frac{1}{3\sqrt[3]{1^2}}>0$
Sign of $f'(x)$	$f'(x)>0$	$f'(x)<0$	$f'(x)<0$	$f'(x)>0$
Increasing/Decreasing	Increasing	Decreasing	Decreasing	Increasing

Relative Extrema

There is a relative maximum at $x=-\frac{\sqrt{3}}{9}$ located at $\left(-\frac{\sqrt{3}}{9},0.384\right)$. There is a relative minimum at $x=-\frac{\sqrt{3}}{9}$ located at $\left(\frac{\sqrt{3}}{9},0.384\right)$.

Concavity

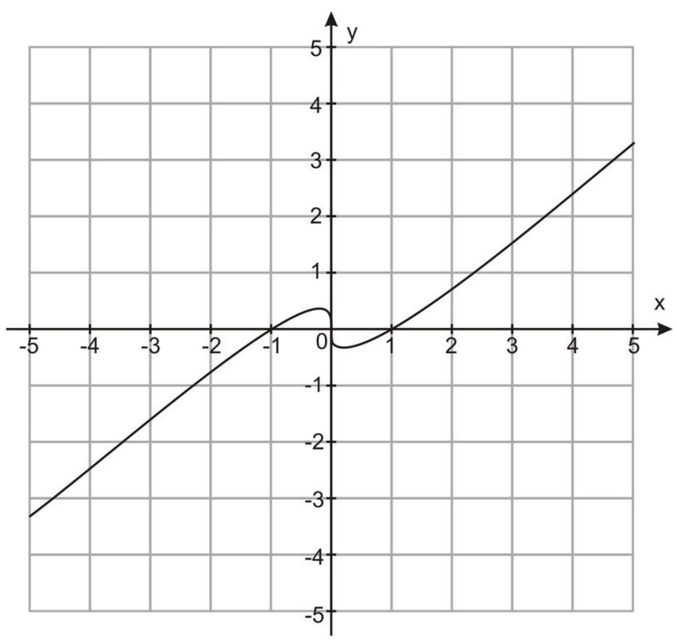
$$f''(x) = 0 + x^{-\frac{5}{3}} = \frac{1}{\sqrt[3]{x^5}}$$

$f''(x)$ is undefined at 0.

Interval	$(-\infty, 0)$	$(0, \infty)$
Test point $x = c$	$c = -1$	$c = 1$
evaluating $\frac{1}{\sqrt[3]{x^5}}$ at $x = c$	$\frac{1}{\sqrt[3]{(-1)^5}} < 0$	$\frac{1}{\sqrt[3]{(1)^5}} < 0$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Concavity	Concave down	Concave up

Inflection Points

There is an inflection point at $x = 0$ located at $(0, 0)$.



5. $f(x) = -\sqrt{2x-6} + 3$

Domain and Range

$2x - 6$ needs to be greater than or equal to 0.

$$\begin{aligned} 2x - 6 &\geq 0 \\ 2x &\geq 6 \\ x &\geq 3 \end{aligned}$$

Domain $D = [3, \infty)$

The greatest that f can be is 3. The range is $(\infty, 3]$.

Intercepts and Zeros

Since 0 is not in the domain, there is no $y-$intercept.

Find the zeros:

$$\begin{aligned}
-\sqrt{2x-6}+3&=0\\
-\sqrt{2x-6}&=-3\\
2x-6&=9\\
2x&=15\\
x&=\frac{15}{2}
\end{aligned}$$

There is a zero at $x=\frac{15}{2}$.

Asymptotes and Limits at Infinity

$$\lim_{x\to\infty}\left(-\sqrt{2x-6}+3\right)=-\infty$$

There are no asymptotes.

Differentiability

$$\begin{aligned}
f'(x)&=-\frac{1}{2}(2x-6)^{-\frac{1}{2}}(2)\\
&=\frac{-1}{(2x-6)^{\frac{1}{2}}}
\end{aligned}$$

The derivative is undefined at $x=3$.

$$f(3)=3$$

The function is differentiable on $(3,\infty)$.

Intervals Where f is increasing/decreasing

Check the sign of $f'(x)$ on $(3,\infty)$. One test point is $x=4$. $f'(4)=-\frac{1}{\sqrt{2(4)-6}}<0$. The function is decreasing on $(3,\infty)$.

Relative Extrema

There is an absolute maximum at $x=3$ located at $(3,3)$.

Concavity

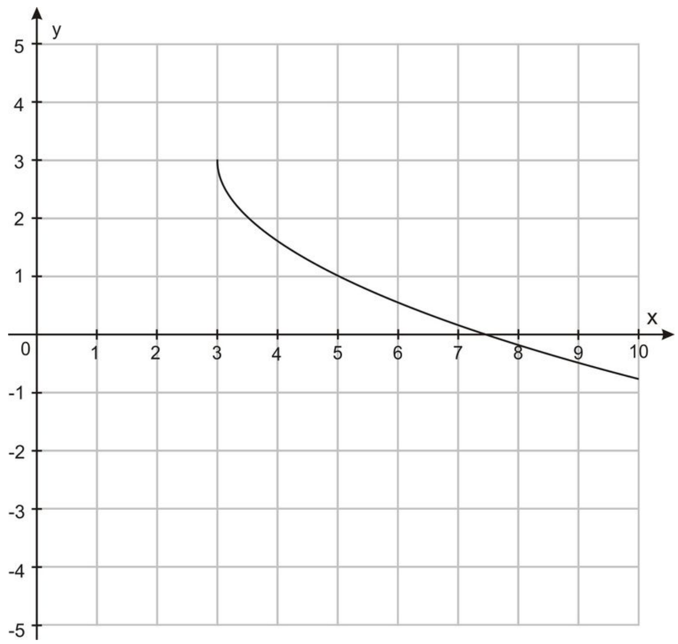
$$\begin{aligned}
f''(x)&=\frac{1}{2}(2x-6)^{-\frac{3}{2}}(2)\\
&=(2x-6)^{-\frac{3}{2}}\\
&=\frac{1}{\sqrt{(2x-6)^3}}
\end{aligned}$$

$f''(x)$ is undefined at $x = 0$.

On the interval $(3, \infty)$, $f''(x) > 0$ for all values in the interval. The function is concave up in the interval.

Inflection Points

There are no inflection points.



6. $f(x) = x^2 - 2\sqrt{x}$

Domain and Range

Because of the square root function, the domain D is $[0, \infty)$.

From using the first derivative, the minimum value of f is -1.19. The range is $[-1.19, \infty)$.

Intercepts and Zeros

zeros:

$$\begin{aligned} x^2 - 2\sqrt{x} &= 0 \\ x^{\frac{1}{2}}\left(x^{\frac{3}{2}} - 2\right) &= 0 \\ x = 0 \text{ or } x^{\frac{3}{2}} &= 2 \\ x &= \sqrt[3]{4} \end{aligned}$$

There are two zeros: $x = \sqrt[3]{4}$ and $x = 0$.

$$f(0) = 0$$

Asymptote and Limits at Infinity

There are no asymptotes.

$$\lim_{x\to\infty} x^2 - 2\sqrt{x} = \infty$$

Differentiability

$$f'(x) = 2x - 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = 2x - \frac{1}{\sqrt{x}}$$

Find critical values:

$$\begin{aligned}
2x - \frac{1}{\sqrt{x}} &= 0\\
2x^{\frac{3}{2}} - 1 &= 0\\
2x^{\frac{3}{2}} &= 1\\
x^{\frac{3}{2}} &= \frac{1}{2}\\
x^3 &= \left(\frac{1}{2}\right)^2\\
x &= \left(\frac{1}{4}\right)^{\frac{1}{3}} = \frac{1}{\sqrt[3]{4}} = \frac{\sqrt[3]{16}}{4}\\
f\left(\frac{\sqrt[3]{16}}{4}\right) &= -1.19
\end{aligned}$$

$f'(x)$ is undefined at $x = 0$. The function is differentiable in $(0, \infty)$.

Intervals where f is increasing/decreasing

Interval	$\left(0, \frac{\sqrt[3]{16}}{4}\right)$	$\left(\frac{\sqrt[3]{16}}{4}, \infty\right)$
Test point $x = c$	$c = 0.5$	$c = 1$
evaluating $2x - \frac{1}{\sqrt{x}}$ at $x = c$	$2(.5) - \frac{1}{\sqrt{0.5}} < 0$	$2(1) - \frac{1}{\sqrt{1}} < 0$
Sign of $f'(x)$	$f'(x) < 0$	$f'(x) > 0$
Increasing/Decreasing	Decreasing	Increasing

Relative Extrema

There is a relative minimum at $x = \frac{3\sqrt{16}}{4}$ located at the point $\left(\frac{\sqrt[3]{16}}{4}, -1.19\right)$

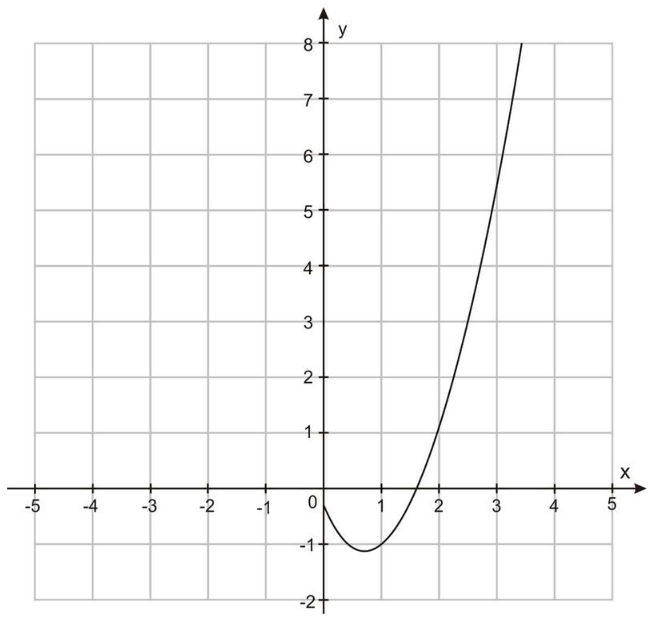
Concavity

$$f''(x) = 2 - \left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = 2 + \frac{1}{\sqrt[3]{x^2}}$$

The second derivative is undefined at $x=0$. It is greater than 0 for all values in $(0,\infty)$. The function is concave up on $(0,\infty)$.

Inflection Points

There are no inflection points.



7. $f(x)=1+\cos x$ on $[-\pi,\pi]$

Domain and Range

The domain D is $[-\pi,\pi]$. Since $-1\le\cos x\le 1$, the range of $1+\cos x$ is $[0,2]$.

Intercepts and Zeros

zeros:

$$\begin{aligned}1+\cos x&=0\\ \cos x&=-1\\ x=-\pi &\text{ or } x=\pi\end{aligned}$$

$y-$intercept

$$f(0)=1+\cos 0=2$$

There is a $y-$intercept at $(0,2)$.

Asymptotes and limits at infinity

There are no asymptotes. The limits at infinity are not relevant because we are looking at a function on a finite, closed interval.

Differentiability

The function is differentiable at every point of its domain.

Intervals where f is Increasing/Decreasing

$$\begin{aligned} f'(x) &= -\sin x \\ -\sin x &= 0 \\ x &= 0 \\ f(0) &= 1+\cos 0 = 2 \end{aligned}$$

Interval	$(-\pi, 0)$	$(0, \pi)$
Test point $x = c$	$c = -\frac{\pi}{2}$	$c = \frac{\pi}{2}$
evaluating $\sin x$ at $x = c$	$-\sin\left(-\frac{\pi}{2}\right) > 0$	$-\sin\left(\frac{\pi}{2}\right) < 0$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$
Increasing/Decreasing	Increasing	Decreasing

Because we are analyzing a continuous function on a closed interval, we check the endpoints for extrema.

$$\begin{aligned} f(-\pi) &= 0 \\ f(\pi) &= 0 \end{aligned}$$

There is an absolute maximum at $x = 0$ located at $(0, 2)$. There are absolute minimums at $x = -\pi$ and $x = \pi$ located at $(-\pi, 0)$ and $(\pi, 0)$.

Concavity

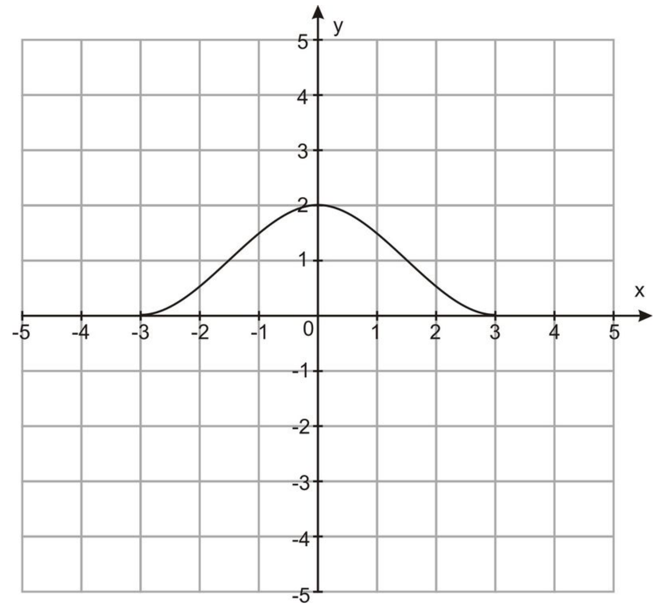
$$\begin{aligned} f''(x) &= -\cos x \\ -\cos x &= 0 \\ x = -\frac{\pi}{2} \text{ or } x &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} f\left(-\frac{\pi}{2}\right) &= 1 \\ f\left(\frac{\pi}{2}\right) &= 1 \end{aligned}$$

Interval	$\left(-\pi, -\frac{\pi}{2}\right)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \pi\right)$
Test point $x = c$	$c = -\frac{3\pi}{2}$	$c = 0$	$c = \frac{3\pi}{4}$
evaluating $-\sin x$ at $x = c$	$-\cos\left(-\frac{3\pi}{4}\right) > 0$	$-\cos 0 < 0$	$-\cos\left(\frac{3\pi}{4}\right) > 0$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Concavity	Concave up	Concave down	Concave up

Inflection Points

The points of inflection are at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ located at $\left(-\frac{\pi}{2}, 1\right)$ and $\left(\frac{\pi}{2}, 1\right)$.


y
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