2.1 Tangent Lines and Rates of change

1. a. Let $y = f(x)$. For $x_0 = 3,\ f(x) = \frac{1}{2}x^2 = \frac{1}{2}(3)^2 = \frac{9}{2}$ and for $x_0 = 4,\ f(4) = \frac{1}{2}x^2 = \frac{1}{2}(4)^2 = \frac{16}{2} = 8$.

The average rate of change of y with respect to x over $[3,4]$ is $m_{\text{sec}} = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(3)-f(1)}{3-1} = \frac{8-\frac{9}{2}}{4-3} = \frac{\frac{7}{2}}{1} = \frac{7}{2}$.

b. First, find $f'(x)$.

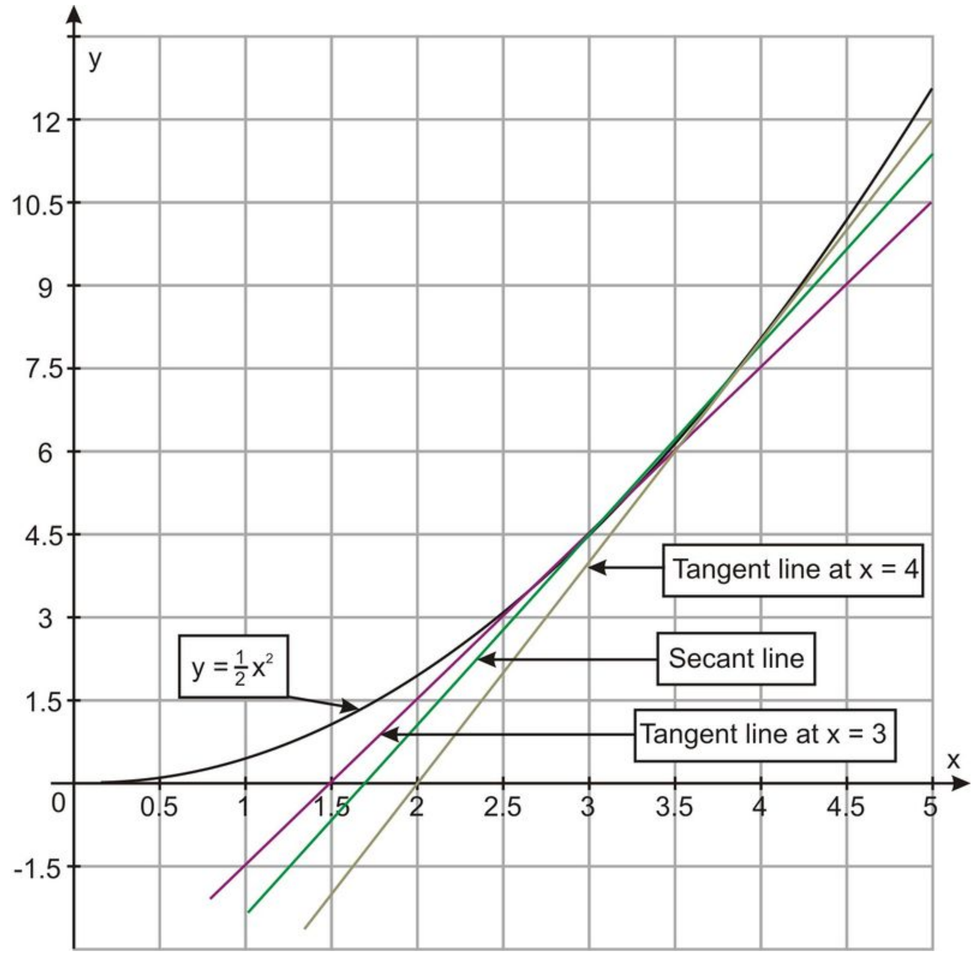
$$\begin{aligned}
f'(x_0) &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{2}(x_0+h)^2-\frac{1}{2}x_0}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{2}x_0^2+2hx_0+h^2-\frac{1}{2}x_0}{h}\\
&= \lim_{h\to 0}\frac{2hx_0+h^2}{h}\\
&= \lim_{h\to 0}\frac{h(2x_0+h)}{h}\\
&= \lim_{h\to 0}(2x_0+h)\\
&= 2x_0
\end{aligned}$$

The instantaneous rate of change of $y = f(x)$ with respect to x at $x_0 = 3$ is $f'(3) = 2x_0 = 2(3) = 6$.

c. The slope of the tangent line at $x_1 = 4$ is $f'(4) = 2x_0 = 2(4) = 8$.

d. The slope of the secant line between $x_0 = 3$ and $x_1 = 4$ is the same as the average rate of change of y with respect to x over the interval $[3,4]$. Then $m_{\text{sec}} = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{8-\frac{9}{2}}{4-3} = \frac{\frac{7}{2}}{1} = \frac{7}{2}$.

e.



2. a. Let $y = f(x)$. For $x_0 = 2,\ f(2) = \frac{1}{2}$ and for $x_0 = 3,\ f(3) = \frac{1}{3}$.

The average rate of change of y with respect to x over $[2,3]$ is $m_{\sec} = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{\frac{1}{3}-\frac{1}{2}}{3-2} = \frac{\frac{2}{6}-\frac{3}{6}}{1} = -\frac{1}{6}$.

b. First, find $f'(x)$.

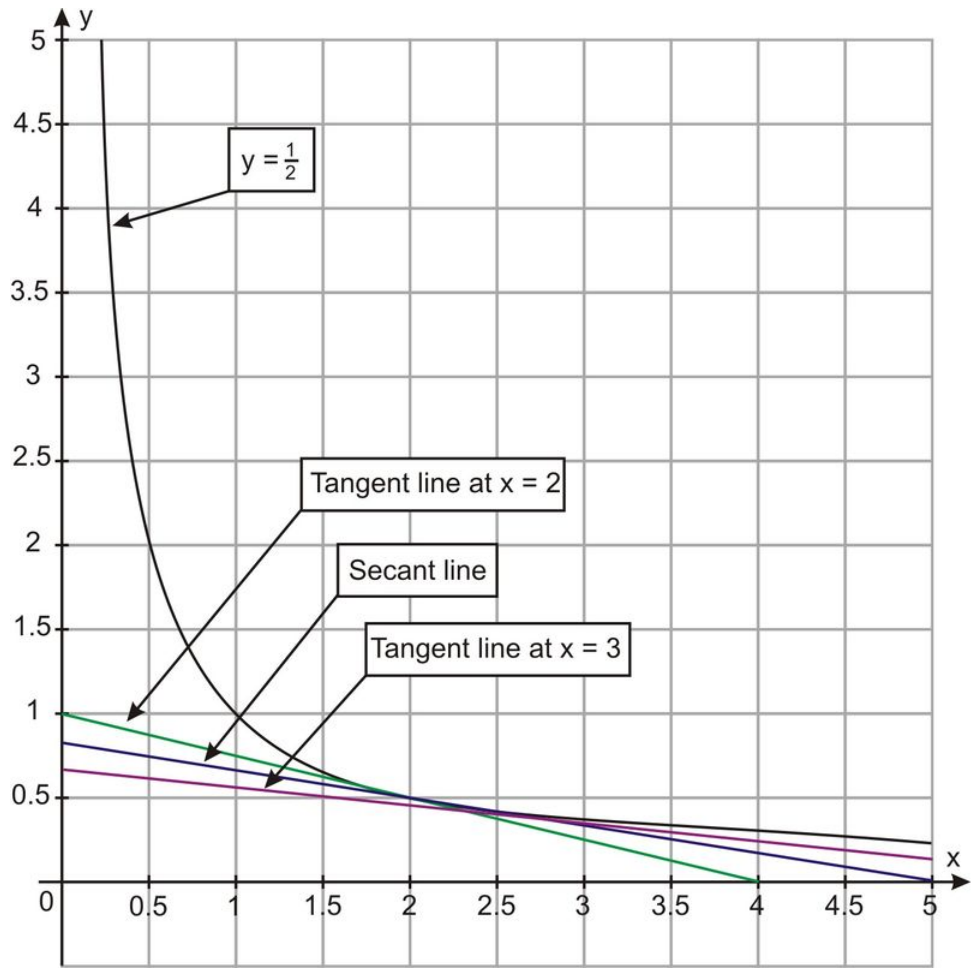
$$\begin{aligned}
f'(x_0) &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{x_0+h}-\frac{1}{x_0}}{h}\\
&= \lim_{h\to 0}\frac{\frac{x_0}{(x_0+h)x_0}-\frac{x_0+h}{(x_0+h)x_0}}{h}\\
&= \lim_{h\to 0}\frac{\frac{x_0-(x_0+h)}{(x_0+h)x_0}}{h}\\
&= \lim_{h\to 0}\frac{\frac{x_0-x_0-h}{(x_0+h)x_0}}{h}\\
&= \lim_{h\to 0}\frac{\frac{-h}{(x_0+h)x_0}}{h}\\
&= \lim_{h\to 0}\left(\frac{-h}{(x_0+h)x_0}\times\frac{1}{h}\right)\\
&= \lim_{h\to 0}-\frac{1}{(x_0+h)x_0}\\
&= \lim_{h\to 0}-\frac{1}{x_0^2+hx_0}\\
&= -\frac{1}{x_0^2}
\end{aligned}$$

The instantaneous rate of change of $y=f(x)$ with respect to x at $x_0=2$ is $f'(2)=-\frac{1}{x_0^2}=-\frac{1}{2^2}=-\frac{1}{4}$.

c. The slope of the tangent line at $x_1=3$ is $f'(3)=-\frac{1}{x_0^2}=-\frac{1}{3^2}=-\frac{1}{9}$.

d. The slope of the secant line between $x_0=2$ and $x_1=3$ is the same as the average rate of change of y with respect to x over the interval $[2,3]$. Then $m_{\text{sec}}=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{\frac{1}{3}-\frac{1}{2}}{3-2}=\frac{\frac{2}{6}-\frac{3}{6}}{1}=-\frac{1}{6}$.

e.



3. The slope of the tangent line at a point x is $f'(x)$.

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \\ &= \lim_{h\to 0} \frac{(x_0+h)^2+1-\left(x_0^2+1\right)}{h} \\ &= \lim_{h\to 0} \frac{x_0^2+2x_0^2h+h^2+1-x_0^2-1}{h} \\ &= \lim_{h\to 0} \frac{2x_0^2h+h^2}{h} \\ &= \lim_{h\to 0} \frac{h\left(2x_0^2+h\right)}{h} \\ &= \lim_{h\to 0} \left(2x_0^2+h\right) \\ &= 2x_0^2 \end{aligned}$$

The slope of the tangent line at $x_0 = 6$ is $f'(6) = 2x_0^2 = 2(6) = 12$.

4. a. Let $y = f(x)$. For $x_0 = 1,\ f(1) = \frac{1}{\sqrt{1}} = 1$ and for $x_0 = 4,\ f(3) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

The average rate of change of y with respect to x over $[1,3]$ is $m_{\text{sec}} = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(3)-f(1)}{3-1} = \frac{\frac{\sqrt{3}}{3}-1}{2} = \left(\frac{\sqrt{3}}{3}-1\right)\left(\frac{1}{2}\right) =$

$\frac{\sqrt{3}}{6} - \frac{1}{2}$.

b.

$$\begin{aligned}
f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x_0+h}} - \frac{1}{\sqrt{x_0}}}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{1}{\sqrt{x_0+h}} - \frac{1}{\sqrt{x_0}}\right)}{h} \times \frac{\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)}{\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)} \\
&= \lim_{h\to 0} \frac{\frac{\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)}{\left(\sqrt{x_0+h}\right)} - \frac{\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)}{\sqrt{x_0}}}{h\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)} \\
&= \lim_{h\to 0} \frac{\sqrt{x_0} - \sqrt{x_0+h}}{h\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{x_0} - \sqrt{x_0+h}\right)}{h\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)} \times \frac{\left(\sqrt{x_0} + \sqrt{x_0+h}\right)}{\left(\sqrt{x_0} + \sqrt{x_0+h}\right)} \\
&= \lim_{h\to 0} \frac{x_0 - (x_0+h)}{h\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)\left(\sqrt{x_0} + \sqrt{x_0+h}\right)} \\
&= \lim_{h\to 0} \frac{-h}{h\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)\left(\sqrt{x_0} + \sqrt{x_0+h}\right)} \\
&= \lim_{h\to 0} \frac{1}{\left(\sqrt{x_0+h}\right)\left(\sqrt{x_0}\right)\left(\sqrt{x_0} + \sqrt{x_0+h}\right)} \\
&= -\frac{1}{\left(\sqrt{x_0+0}\right)\left(\sqrt{x_0}\right)\left(\sqrt{x_0} + \sqrt{x_0+0}\right)} \\
&= \frac{1}{\left(\sqrt{x_0}\right)\left(\sqrt{x_0}\right)\left(\sqrt{x_0} + \sqrt{x_0}\right)} \\
&= -\frac{1}{\left(\sqrt{x_0}\right)\left(\sqrt{x_0}\right)\left(2\sqrt{x_0}\right)} \\
&= -\frac{1}{2\left(\sqrt{x_0}\right)^3}
\end{aligned}$$

c. $f'(1) = -\frac{1}{2\left(\sqrt{1}\right)^3} = -\frac{1}{2}$

5. a. $h(35) = 4.9(3.5)^2 = 6002.5$ meters

b.

$$\begin{aligned}
v &= \frac{h(35) - h(0)}{35 - 0} \\
&= \frac{4.9(35)^2 - 4.9(0)^2}{35} \\
&= \frac{4.9(35)^2}{35} \\
&= 4.9(35) \\
&= 171.5 \text{ m/sec}
\end{aligned}$$

c. Set $h(t) = 200$ and solve for the numbers of minutes t that pass for the rocket to travel 200 meters.

$$\begin{aligned}
200 &= 4.9t^2 \\
\frac{200}{4.9} &= t^2 \\
40.82 &= t^2 \\
6.39 &= t^2
\end{aligned}$$

Next, find the average velocity:

$$\begin{aligned}
v &= \frac{200 - h(0)}{6.39 - 0} \\
&= \frac{200 - 4.9(0)^2}{6.39} \\
&= \frac{200}{6.39} \\
&= 31.3 \text{ m/sec}
\end{aligned}$$

d. Write $h(t) = f(t)$ do avoid confusing $h(t)$ with h in the formula for the instantaneous rate of change.

$$\begin{aligned}
f'(t_0) &= \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} \\
&= \lim_{h \to 0} \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h} \\
&= \lim_{h \to 0} \frac{4.9\left(t_0^2 + 2ht_0 + h^2\right) - 4.9t_0^2}{h} \\
&= \lim_{h \to 0} \frac{4.9t_0^2 + 9.8hx_0 + 4.9h^2 - 4.9t_0^2}{h} \\
&= \lim_{h \to 0} \frac{9.8ht_0 + 4.9h}{h} \\
&= \lim_{h \to 0} \frac{4.9h(2x_0 + 1)}{h} \\
&= \lim_{h \to 0} 4.9(2t_0 + 1) \\
&= 9.8t_0
\end{aligned}$$

Rewrite the formula with $h'(t)$: $h'(t) = 9.8t_0$. Then $h'(35) = 9.8(35) = 343$ m/sec.

6. a.

$$\begin{aligned}
v &= \frac{\varkappa(2) - \varkappa(0)}{2 - 0} \\
&= \frac{9.9(2)^3 - 9.9(0)^3}{2 - 0} \\
&= \frac{79.2}{2} \\
&= 39.6 \text{ nanometers/nanosecond}
\end{aligned}$$

b.

$$\begin{aligned}
\varkappa'(t_0) &= \lim_{h\to 0}\frac{\varkappa(t_0+h)-\varkappa(t_0)}{h}\\
&= \lim_{h\to 0}\frac{9.9\,(t_0+h)^3-9.9t_0^3}{h}\\
&= \lim_{h\to 0}\frac{9.9\left(t_0^3+3t_0^2h+3t_0h^2+h^3\right)-9.9t_0^3}{h}\\
&= \lim_{h\to 0}\frac{9.9t_0^3+29.7t_0^2h+29.7t_0h^2+9.9h^3-9.9t_0^3}{h}\\
&= \lim_{h\to 0}\frac{29.7t_0^2h+29.7t_0h^2+9.9h^3}{h}\\
&= \lim_{h\to 0}\frac{9.9h\left(3t_0^2+3t_0h+h^2\right)}{h}\\
&= \lim_{h\to 0}9.9\left(3t_0^2+3t_0h+h\right)\\
&= 29.7t_0^2
\end{aligned}$$

The instantaneous velocity at $t_0 = 2$ is $\varkappa'(2) = 29.7\,(2)^2 = 118.8$ nanometers/nanosecond.