Elements of the LGM Model: An Implementation Guide

Yan Zeng

Version 1.0, last revised on 2012-07-26.

This book is for sale at https://leanpub.com/sqf8-lgm-impl

This version was published on 2023-11-16

This is a Leanpub book. Leanpub empowers authors and publishers with the Lean Publishing process. Lean Publishing is the act of publishing an in-progress ebook using lightweight tools and many iterations to get reader feedback, pivot until you have the right book and build traction once you do.

Abstract

In this note, we document elements of the Linear Gaussian Markov (LGM) model and its calibration to swaptions.

Contents

Chapter 1

Elements of one-factor LGM model

In this section, we review the elements of one-factor LGM model and its calibration to swaptions, as presented in Hagan [1] and Piza [6].

1.1 HJM framework

We assume that we have a family of zero-coupon bonds traded in the market. The price at time t of a zero-coupon bond with maturity T $(0 \leq t \leq T)$ will be denoted by $P(t,T)$. We assume the bond price satisfies the following SDE:

$$dP(t,T) = P(t,T)\big[A(t,T)dt + B(t,T)dW_t\big],\ P(T,T) = 1,\ A(T,T) = B(T,T) = 0,$$

where W is a 1-dimensional standard Brownian motion. We assume there is also a strictly positive process N, which will be chosen as the numéraire, that satisfies the following SDE:

$$dN_t = N_t\big(\mu_t^N dt + \sigma_t^N dW_t\big),\ N_0 = 1.$$

By the Fundamental Theorem of Asset Pricing, a necessary and sufficient conditoin for the no arbitrage property (more precisely, no-free-lunch-with-vanishing-risk, NFLVR, for allowable strategies) is that we can find a probability measure Q such that the discounted bond price process

$$\bar{P}(t,T) := \frac{P(t,T)}{N_t}$$

is a Q-local martingale. Itô calculus yields

$$\frac{d\bar{P}(t,T)}{\bar{P}(t,T)} = \big[B(t,T) - \sigma_t^N\big]\left[\frac{A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)}{B(t,T) - \sigma_t^N}dt + dW_t\right]$$

provided $B(t,T) - \sigma_t^N \neq 0$, $0 \leq t \leq T$.

If the probability measure Q is defined by (P denotes the original probability measure)

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = D_t = \exp\left\{\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right\},$$

we necessarily have

$$\frac{A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)}{B(t,T) - \sigma_t^N} = -\theta_t,$$

which must be independent of T. We are already in the risk-neutral measure (i.e. $P = Q$) if and only if

$$A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T) = 0.$$

1.2 Forward rate model

The results in HJM model can be translated into those in forward rate model. Denote by $f(t,T)$ the forward rate such that $P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$. Assume $f(t,T)$ follows the SDE

$$df(t,T) = a(t,T)dt + b(t,T)dW_t.$$

We then have the following relations

$$A(t,T) = f(t,t) - \int_t^T a(t,s)ds + \frac{1}{2}\left(\int_t^T b(t,s)ds\right)^2,\ B(t,T) = -\int_t^T b(t,s)ds$$

and

$$a(t,T) = \frac{\partial B(t,T)}{\partial T}B(t,T) - \frac{\partial A(t,T)}{\partial T},\ b(t,T) = -\frac{\partial B(t,T)}{\partial T}$$

Then the condition $A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T) = 0$ translates into

$$a(t,T) = \int_t^T b(t,s)ds \cdot b(t,T) + \sigma_t^N b(t,T).$$

1.3 The LGM model

To get the LGM model, we assume that we are already under the risk-neutral measure associated with the numeraire N, where N is specified by the following parameter specification

$$\begin{cases} b(t,T) = H'(T)\alpha_t \\ \sigma_t^N = H(t)\alpha_t \end{cases}$$

Here H and α are two deterministic functions with $H(0) = 0$. This specification gives

$$a(t,T) = H(T)H'(T)\alpha_t^2,\ B(t,T) = -[H(T) - H(t)]\alpha_t.$$

Define $\zeta_t = \int_0^t \alpha_s^2 ds$ and $X_t = \int_0^t \alpha_s dW_s$, we have $f(t,T) = f(0,T) + H'(T)H(T)\zeta_t + H'(T)X_t$. This gives

$$A(t,T) = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t - [H(T) - H(t)]H(t)\alpha_t^2$$

and

$$\mu_t^N = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t + H^2(t)\alpha_t^2.$$

In summary, the HJM parameter specifications of LGM model are

$$\boxed{\begin{cases} A(t,T) = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t - [H(T) - H(t)]H(t)\alpha_t^2 \\ B(t,T) = -[H(T) - H(t)]\alpha_t \\ a(t,T) = H(T)H'(T)\alpha_t^2 \\ b(t,T) = H'(T)\alpha_t \\ \mu_t^N = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t + H^2(t)\alpha_t^2 \\ \sigma_t^N = H(t)\alpha_t \end{cases}} \tag{1.1}$$

where H and α are two deterministic functions with $H(0) = 0$, $\zeta_t = \int_0^t \alpha_s^2 ds$, $X_t = \int_0^t \alpha_s dW_s$, and $f(0,t)$ is given by market quoted yield curve.

Consequently, we have $r_t := f(t,t) = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t$,

$$P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\} = \frac{P(0,T)}{P(0,t)}\exp\left\{-[H(T) - H(t)]X_t - \frac{1}{2}[H^2(T) - H^2(t)]\zeta_t\right\}.$$

and

$$\frac{d\bar{P}(t,T)}{\bar{P}(t,T)} = [B(t,T) - \sigma_t^N]dW_t.$$

The last SDE gives

$$\bar{P}(t,T) = P(0,T)\exp\left\{-H(T)X_t - \frac{1}{2}H^2(T)\zeta_t\right\}.$$

Therefore

$$N_t = \frac{P(t,T)}{\bar{P}(t,T)} = \frac{1}{P(0,t)}\exp\left\{H(t)X_t + \frac{1}{2}H^2(t)\zeta_t\right\}.$$

In summary, we have

$$\boxed{\begin{cases} f(t,T) = f(0,T) + H'(T)H(T)\zeta_t + H'(T)X_t \\ r_t = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t \\ P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left\{-[H(T)-H(t)]X_t - \frac{1}{2}[H^2(T) - H^2(t)]\zeta_t\right\} \\ \bar{P}(t,T) = P(0,T)\exp\left\{-H(T)X_t - \frac{1}{2}H^2(T)\zeta_t\right\} \\ N_t = \frac{1}{P(0,t)}\exp\left\{H(t)X_t + \frac{1}{2}H^2(t)\zeta_t\right\} \end{cases}} \tag{1.2}$$

1.4 Connection with one-factor Hull-White model

Denote by Q the martingale measure associated with money market account numeraire. The one-factor Hull-White model assumes the short rate process r_t follows the following dynamics under Q

$$dr_t = (b_t - \kappa r_t)dt + \sigma_t dW_t^Q,$$

where κ is a constant, b_t and σ_t are deterministic functions of t, and W^Q is a standard Brownian motion under Q.

Define $\theta_t = e^{-\kappa t}r_0 + e^{-\kappa t}\int_0^t e^{\kappa s}b_s ds$ and $X_t^Q = e^{-\kappa t}\int_0^t e^{\kappa s}\sigma_s dW_s^Q$. Then θ_t is a deterministic function of t and X_t^Q is Gaussian process with mean 0 and variance $e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds$. In summary, we have

$$\boxed{r_t = \theta_t + X_t^Q,\ dX_t^Q = -\kappa X_t^Q dt + \sigma_t dW_t^Q,\ X_0^Q = 0,\ E[X_t^Q] = 0,\ E[(X_t^Q)^2] = e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds.}$$

It's easy to verify that

$$\boxed{\begin{cases} P(t,T) = P(t,T;X_t^Q) = \frac{P(0,T)}{P(0,t)}\exp\left\{-H^Q(T-t)\left[X_t^Q + \nu^h(t) + \frac{1}{2}\nu(t)H^Q(T-t)\right]\right\} \\ P(0,t) = \exp\left\{-\int_0^t \theta_s ds + \nu_t^{H^Q}\right\} \end{cases}} \tag{1.3}$$

where

$$\begin{cases} h(t) = e^{-\kappa t} \\ H^Q(t) = \int_0^t h(s)ds \\ \nu(t) = e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds \\ \nu^h(t) = h * v(t) = \int_0^t e^{-\kappa(t-s)}\nu(s)ds \\ \nu^{H^Q}(t) = H^Q * \nu(t) = \int_0^t H^Q(t-s)\nu(s)ds. \end{cases}$$

We also note that $\frac{d}{dt}\nu^{H^Q}(t) = \nu^h(t)$. The one-to-one correspondence between one-factor LGM model and one-factor Hull-White model is therefore

$$\boxed{\begin{cases} \alpha_t = e^{\kappa t}\sigma_t \\ \zeta_t = e^{2\kappa t}\nu(t) = \int_0^t \alpha_s^2 ds = \int_0^t e^{2\kappa s}\sigma_s^2 ds \\ H(t) = H^Q(t) = \int_0^t e^{-\kappa s}ds. \end{cases}}$$

To verify this relationship, we note

$$\begin{aligned}
& -H^Q(T-t)[X_t^Q + \nu^h(t) + \frac{1}{2}\nu(t)H^Q(T-t)] \\
= & -H(T-t)\left[e^{-\kappa t}\int_0^t e^{\kappa s}\sigma_s dW_s^Q + e^{-\kappa t}\int_0^t e^{\kappa s}e^{-2\kappa s}\zeta_s ds + \frac{1}{2}e^{-2\kappa t}\zeta_t H(T-t)\right] \\
= & -[H(T)-H(t)]\left[\int_0^t e^{\kappa s}\sigma_s dW_s^Q + \int_0^t e^{-\kappa s}\zeta_s ds\right] - \frac{1}{2}[H(T)-H(t)]^2\zeta_t \\
= & -[H(T)-H(t)]\left[\int_0^t e^{\kappa s}\sigma_s dW_s^Q - \int_0^t H(s)e^{2\kappa s}\sigma_s^2 ds\right] - \frac{1}{2}[H^2(T)-H^2(t)]\zeta_t.
\end{aligned}$$

We shall show $\int_0^t e^{\kappa s}\sigma_s dW_s^Q - \int_0^t H(s)e^{2\kappa s}\sigma_s^2 ds = \int_0^t e^{\kappa s}\sigma_s(dW_s^Q - H(s)e^{\kappa s}\sigma_s ds) = \int_0^t e^{\kappa s}\sigma_s dW_s = X_t$, and thus prove that formula (1.3) agrees with the zero coupon bond price formula in (1.2). Indeed, the Radon-Nikodym derivative of Q^N w.r.t. Q is

$$D_t = \frac{N_t}{e^{\int_0^t r_u du}}.$$

So $d\ln D_t = \frac{dN_t}{N_t} + (...)dt$. Since D_t is a martingale under Q, we conclude

$$dD_t = D_t\sigma_t^N dW_t^Q = D_t H(t)\alpha_t dW_t^Q.$$

Girsanov's Theorem (see Appendix 3.1) implies $W_t^Q - \int_0^t H(s)\alpha_s$ is a martingale under Q^N. This proves our claim.

1.5 Pricing formula of swap

Consider a swap with start date t_0, fixed leg pay dates t_1, t_2, $\cdots$, t_n, and fixed rate K. Then the fixed leg makes the payments (assuming notional is one unit of currency)

$$\begin{cases} \tau_i K & \text{paid at } t_i, \text{ for } i = 1, 2, \cdots, n-1 \\ 1 + \tau_n K & \text{paid at } t_n, \end{cases}$$

where τ_i is the day count of $[t_{i-1}, t_i]$ in year fraction. For any $t \le t_0$, these payments have the value

$$V_{fix}(t) = K\sum_{i=1}^n \tau_i P(t, t_i) + P(t, t_n).$$

The swap's floating leg usually has a different frequency than the fixed leg, so let this leg's start and pay dates be

$$t_0 = u_0 < u_1 < \cdots < u_m = t_n.$$

The floating leg pays

$$\begin{cases} \tilde{\tau}_j L_j & \text{paid at } u_j, \text{ for } j = 1, 2, \cdots, m-1 \\ 1 + \tilde{\tau}_m L_m & \text{paid at } u_m = t_n \end{cases}$$

where $\tilde{\tau}_j$ is the day count of $[u_{j-1}, u_j]$ in year fraction and L_j is the Libor or Euribor floating rate for the interval $[u_{j-1}, u_j]$. The rate L_j is set on the fixing date, which is generally two London business days before the interval starts on u_{j-1}. In formula,

$$L_j = \frac{1}{\tilde{\tau}_j}\left[\frac{P(u_{j-1}^{fix}, u_{j-1})}{P(u_{j-1}^{fix}, u_j)} - 1\right] + s_j,$$

where the first part of the formula stands for risk-free floating rate, and the second part s_j stands for a spread for credit risk. The payment of $\tilde{\tau}_j L_j$ at time u_j is equal to a payment of

$$[P(u_{j-1}^{fix}, u_{j-1}) - P(u_{j-1}^{fix}, u_j)] + \tilde{\tau}_j s_j P(u_{j-1}^{fix}, u_j)$$

at time u_j^{fix}, which is further equal to a payment of

$$[P(t, u_{j-1}) - P(t, u_j)] + \tilde{\tau}_j s_j P(t, u_j)$$

at time t. The value of the floating leg is therefore

$$V_{flt}(t) = P(t, t_0) + \sum_{j=1}^{m} \tilde{\tau}_j s_j P(t, u_j).$$

The value of the receiver swap (receiving the fixed leg, paying the floating leg) is

$$\boxed{V_{rec}(t) = K \sum_{i=1}^{n} \tau_i P(t, t_i) + P(t, t_n) - P(t, t_0) - \sum_{j=1}^{m} \tilde{\tau}_j s_j P(t, u_j)} \tag{1.4}$$

For $t = 0$, we can write the formula in a nicer form

$$V_{rec}(0) = K^{adj} \sum_{i=1}^{n} \tau_i P(0, t_i) + P(0, t_n) - P(0, t_0)$$

where $K^{adj} = K - \frac{\sum_{j=1}^{m} \tilde{\tau}_j s_j P(0,u_j)}{\sum_{i=1}^{n} \tau_i P(0,t_i)}$. This leads to the following pragmatic approximation

$$\boxed{V_{rec}(t) \approx K^{adj} \sum_{i=1}^{n} \tau_i P(t, t_i) + P(t, t_n) - P(t, t_0)} \tag{1.5}$$

1.6 Pricing formula of swaption

The value of a receiver swaption at time zero is ($t_{ex} \leq t_0$ is the option exercise time)

$$V_{rec}^{opt}(0) = N_0 E^{Q_N} \left[\frac{\max\{V_{rec}(t_{ex}), 0\}}{N_{t_{ex}}} \right] \approx E^{Q_N} \left[\left(K^{adj} \sum_{i=1}^{n} \tau_i \bar{P}(t_{ex}, t_i; X_{t_{ex}}) + \bar{P}(t_{ex}, t_n; X_{t_{ex}}) - \bar{P}(t_{ex}, t_0; X_{t_{ex}}) \right)^+ \right]$$

where $X_{t_{ex}} \sim N(0, \zeta_{t_{ex}})$ under the martingale measure Q_N associated with numeraire N. By change of variable $y = x + H(t_0)\zeta_{t_{ex}}$, we have

$$\begin{aligned} V_{rec}^{opt}(0) &\approx \frac{1}{\sqrt{2\pi\zeta_{t_{ex}}}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\zeta_{t_{ex}}}} \left(K^{adj} \sum_{i=1}^{n} \tau_i P(0, t_i) \exp\left\{ -H(t_i)x - \frac{1}{2} H^2(t_i)\zeta_{t_{ex}} \right\} \right. \\ &\quad \left. + P(0, t_n) \exp\left\{ -H(t_n)x - \frac{1}{2} H^2(t_n)\zeta_{t_{ex}} \right\} - P(0, t_0) \exp\left\{ -H(t_0)x - \frac{1}{2} H^2(t_0)\zeta_{t_{ex}} \right\} \right)^+ dx \\ &= \frac{1}{\sqrt{2\pi\zeta_{t_{ex}}}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\zeta_{t_{ex}}}} \left(K^{adj} \sum_{i=1}^{n} \tau_i D_i \exp\left\{ -(H_i - H_0)y - \frac{1}{2} (H_i - H_0)^2 \zeta_{t_{ex}} \right\} \right. \\ &\quad \left. + D_n \exp\left\{ -(H_n - H_0)y - \frac{1}{2} (H_n - H_0)^2 \zeta_{t_{ex}} \right\} - D_0 \right)^+ dx \end{aligned}$$

where $H_i = H(t_i)$, $D_i = P(0, t_i)$ for $i = 0, 1, \cdots, n$.

We now assume without loss of generality that H is a strictly increasing function so that $H' > 0$. Then

$$\exp\left\{-[H(T)-H(t)]y - \frac{1}{2}[H(T)-H(t)]^2\zeta_{t_{ex}}\right\},\ t_{ex} \le t \le T$$

is a monotone decreasing function of y, with limit 0 as $y \to \infty$ and limit ∞ as $y \to -\infty$. So there exists a unique break-even point y^* such that the term inside $(\cdots)^+$ is

$$\begin{cases} < 0 & \text{if } y > y^* \\ = 0 & \text{if } y = y^* \\ > 0 & \text{if } y < y^* \end{cases}$$

Then

$$\begin{aligned} & V_{rec}^{opt}(0) \\ \approx\ & \frac{1}{\sqrt{2\pi\zeta_{t_{ex}}}} \int_{-\infty}^{y^*} e^{-\frac{y^2}{2\zeta_{t_{ex}}}} \left(K^{adj} \sum_{i=1}^{n} \tau_i D_i e^{-(H_i - H_0)y - \frac{1}{2}(H_i-H_0)^2\zeta_{t_{ex}}} + D_n e^{-(H_n-H_0)y - \frac{1}{2}(H_n-H_0)^2\zeta_{t_{ex}}} - D_0 \right) dx \\ =\ & \boxed{K^{adj} \sum_{i=1}^{n} \tau_i D_i \Phi\left(\frac{y^* + (H_i - H_0)\zeta_{t_{ex}}}{\sqrt{\zeta_{t_{ex}}}}\right) + D_n \Phi\left(\frac{y^* + (H_n - H_0)\zeta_{t_{ex}}}{\sqrt{\zeta_{t_{ex}}}}\right) - D_0 \Phi\left(\frac{y^*}{\sqrt{\zeta_{t_{ex}}}}\right)} \end{aligned} \tag{1.6}$$

where $\Phi(\cdot)$ is the c.d.f. of a standard normal distribution and y^* is the unique solution of

$$K^{adj} \sum_{i=1}^{n} \tau_i D_i e^{-[H(t_i)-H(t_0)]y^* - \frac{1}{2}[H(t_i)-H(t_0)]^2\zeta_{t_{ex}}} + D_n e^{-[H(t_n)-H(t_0)]y^* - \frac{1}{2}[H(t_n)-H(t_0)]^2\zeta_{t_{ex}}} = D_0.$$

1.7 Calibration to swaption market

We define the forward swap rate S as

$$S(t) = \frac{P(t,t_0) - P(t,t_n)}{\sum_{i=1}^{n} \tau_i P(t,t_i)},\ t \le t_0$$

and the annuity numeraire as

$$L(t) = \sum_{i=1}^{n} \tau_i P(t,t_i),\ t \le t_0.$$

Then

$$V_{rec}(t) \approx K^{adj} \sum_{i=1}^{n} \tau_i P(t,t_i) + P(t,t_n) - P(t,t_0) = (K^{adj} - S(t))L(t)$$

and the rule of change-of-numeraire gives us

$$V_{rec}^{opt}(0) = N_0 E^{Q_N}\left[\frac{\max\{V_{rec}(t_{ex}),0\}}{N_{t_{ex}}}\right] = L_0 E^{Q_L}\left[\frac{\max\{V_{rec}(t_{ex}),0\}}{L_{t_{ex}}}\right] \approx L_0 E^{Q_L}[(K^{adj} - S(t_{ex}))^+].$$

By the pricing formula of zero coupon bond, $S(t)$ is a function of t and X_t. So Ito's formula yields

$$dS(t) = dS(t, X_t) = \left.\frac{\partial S(t,x)}{\partial x}\right|_{x=X_t} \alpha_t dW_t + (...)dt.$$

Since $S(t)$ has the form of $\frac{\text{tradable}}{\text{numeraire}}$, it is a martingale under the martingale measure Q_L associated with the annuity numeraire L. Therefore

$$dS(t) = \left.\frac{\partial S(t,x)}{\partial x}\right|_{x=X_t} \alpha_t dW_t^L,$$