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Elements of the Hull-White Model: An Implementation Guide

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Abstract

This is a self-contained implementation guide to the Hull-White model. We derive from scratch formulas that are essential to coding up the model. Some numerical issues are also handled with care.

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Chapter 1

One-factor Hull-White model

1.1 Dynamics of short rate r_t and state variable X_t under risk-neutral measure

Under the assumption of one-factor Hull-White model, the short rate process under the risk-neutral measure Q (the martingale measure associated with money market account numeraire) follows the dynamics

$$dr_t = (b_t - \kappa r_t)dt + \sigma_t dW_t,$$

where κ is a constant, b_t and σ_t are deterministic functions of t , and W is a standard Brownian motion.

Solving the SDE gives

$$r_t = e^{-\kappa t} r_0 + e^{-\kappa t} \int_0^t e^{\kappa s} b_s ds + e^{-\kappa t} \int_0^t e^{\kappa s} \sigma_s dW_s.$$

Setting $\theta_t = e^{-\kappa t} r_0 + e^{-\kappa t} \int_0^t e^{\kappa s} b_s ds$ and $X_t = e^{-\kappa t} \int_0^t e^{\kappa s} \sigma_s dW_s$. Then θ_t is a deterministic function of t and X_t is Gaussian process with mean 0 and variance $e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma_s^2 ds$. In summary, we have

$$r_t = \theta_t + X_t, \quad dX_t = -\kappa X_t dt + \sigma_t dW_t, \quad X_0 = 0, \quad E^Q[X_t] = 0, \quad E^Q[X_t^2] = e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma_s^2 ds \quad (1.1)$$

1.2 Pricing formula of zero coupon bond

1.2.1 Formula

Denote by $P(t, T)$ the time- t price of a zero-coupon bond with maturity T , we have (note $P(t, T)$ is a function of the state variable X_t)

$$\begin{cases} P(t, T) = P(t, T; X_t) = \frac{P(0, T)}{P(0, t)} \exp \left\{ -H(T-t) \left[X_t + \nu^h(t) + \frac{1}{2} \nu(t) H(T-t) \right] \right\} \\ P(0, t) = \exp \left\{ - \int_0^t \theta_s ds + \nu_t^H \right\} \end{cases} \quad (1.2)$$

where

$$\begin{cases} h(t) = e^{-\kappa t} \\ H(t) = \int_0^t h(s) ds \\ \nu(t) = e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma_s^2 ds \\ \nu^h(t) = h * \nu(t) = \int_0^t e^{-\kappa(t-s)} \nu(s) ds \\ \nu^H(t) = H * \nu(t) = \int_0^t H(t-s) \nu(s) ds. \end{cases}$$

We also note that $\frac{d}{dt} \nu^H(t) = \nu^h(t)$.

1.2.2 Derivation

For a derivation of formula (1.2), define $\alpha_t = \sigma_t e^{\kappa t}$. It's easy to see $X_t = h(t) \int_0^t \alpha_s dW_s$. For the convenience of later computation, we note for $u > t$,

$$X_u = e^{-\kappa u} \int_0^t \alpha_s dW_s + e^{-\kappa u} \int_t^u \alpha_s dW_s = h(u-t)X_t + h(u) \int_t^u \alpha_s dW_s.$$

Therefore, risk neutral pricing formula yields

$$P(t, T; X_t = x) = E^Q \left[e^{-\int_t^T \theta_u du} \middle| \mathcal{F}_t, X_t = x \right] = e^{-\int_t^T \theta_u du - H(T-t)x} E^Q \left[e^{-\int_t^T h(u)(\int_t^u \alpha_s dW_s) du} \right]$$

Define $\eta = \int_t^T h(u)(\int_t^u \alpha_s dW_s) du$. Then η is a Gaussian random variable with 0 mean and

$$\begin{aligned} \eta^2 &= \left[H(u) \int_t^u \alpha_s dW_s \Big|_{u=t}^u - \int_t^T H(u) \alpha_u dW_u \right]^2 \\ &= H^2(T) \left(\int_t^T \alpha_s dW_s \right)^2 - 2H(T) \int_t^T \alpha_s dW_s \int_t^T H(u) \alpha_u dW_u + \left(\int_t^T H(u) \alpha_u dW_u \right)^2. \end{aligned}$$

Therefore the variance of η is equal to

$$E^Q[\eta^2] = H^2(T) \int_t^T \alpha_s^2 ds - 2H(T) \int_t^T H(s) \alpha_s^2 ds + \int_t^T H^2(s) \alpha_s^2 ds$$

and

$$\begin{aligned} P(t, T; X_t = x) &= e^{-\int_t^T \theta_u du - H(T-t)x} E^Q[e^{-\eta}] = e^{-\int_t^T \theta_u du - H(T-t)x} \exp \left\{ \frac{1}{2} \text{Var}(\eta) \right\} \\ &= \exp \left\{ -\int_t^T \theta_u du - H(T-t)x + \frac{1}{2} \left[H^2(T) \int_t^T \alpha_s^2 ds + \int_t^T H^2(s) \alpha_s^2 ds \right] - H(T) \int_t^T H(s) \alpha_s^2 ds \right\}. \end{aligned}$$

As particular cases, we have

$$\begin{cases} P(0, T) = \exp \left\{ -\int_0^T \theta_u du + \frac{1}{2} \left[H^2(T) \int_0^T \alpha_s^2 ds + \int_0^T H^2(s) \alpha_s^2 ds \right] - H(T) \int_0^T H(s) \alpha_s^2 ds \right\} \\ P(0, t) = \exp \left\{ -\int_0^t \theta_u du + \frac{1}{2} \left[H^2(t) \int_0^t \alpha_s^2 ds + \int_0^t H^2(s) \alpha_s^2 ds \right] - H(t) \int_0^t H(s) \alpha_s^2 ds \right\}. \end{cases}$$

Therefore, we have

$$\begin{aligned} &\frac{P(0, T)}{P(0, t)P(t, T; X_t = x)} \\ &= \exp \left\{ H(T-t)x + \frac{1}{2} \left[H^2(T) \int_0^t \alpha_s^2 ds + \int_0^t H^2(s) \alpha_s^2 ds \right] - H(T) \int_0^t H(s) \alpha_s^2 ds \right\} \\ &\quad \cdot \exp \left\{ -\frac{1}{2} \left[H^2(t) \int_0^t \alpha_s^2 ds + \int_0^t H^2(s) \alpha_s^2 ds \right] + H(t) \int_0^t H(s) \alpha_s^2 ds \right\} \\ &= \exp \left\{ H(T-t)x + \frac{1}{2} [H^2(T) - H^2(t)] \int_0^t \alpha_s^2 ds - [H(T) - H(t)] \int_0^t H(s) \alpha_s^2 ds \right\}. \end{aligned}$$

That is,

$$\begin{aligned} &P(t, T; X_t = x) \\ &= \frac{P(0, T)}{P(0, t)} \exp \left\{ -H(T-t)x - \frac{1}{2} [H^2(T) - H^2(t)] \int_0^t \alpha_s^2 ds + [H(T) - H(t)] \int_0^t H(s) \alpha_s^2 ds \right\} \end{aligned}$$

Note $[H(T) - H(t)]e^{\kappa t} = \frac{e^{-\kappa t} - e^{-\kappa T}}{\kappa}e^{\kappa t} = H(T - t)$, we get

$$\begin{aligned}
& P(t, T; X_t = x) \\
&= \frac{P(0, T)}{P(0, t)} \exp \left\{ -H(T - t) \left[x + \frac{H(T) + H(t)}{2} e^{-\kappa t} \int_0^t \alpha_s^2 ds - e^{-\kappa t} \int_0^t H(s) \alpha_s^2 ds \right] \right\} \\
&= \frac{P(0, T)}{P(0, t)} \exp \left\{ -H(T - t) \left[x + H(t) e^{-\kappa t} \int_0^t \alpha_s^2 ds + \frac{H(T) - H(t)}{2} e^{-\kappa t} \int_0^t \alpha_s^2 ds \right. \right. \\
&\quad \left. \left. - e^{-\kappa t} \int_0^t H(s) \alpha_s^2 ds \right] \right\} \\
&= \frac{P(0, T)}{P(0, t)} \exp \left\{ -H(T - t) \left[x + H(t) e^{-\kappa t} \int_0^t \alpha_s^2 ds + \frac{1}{2} H(T - t) \nu(t) - e^{-\kappa t} \int_0^t H(s) \alpha_s^2 ds \right] \right\}.
\end{aligned}$$

Note

$$\begin{aligned}
& H(t) e^{-\kappa t} \int_0^t \alpha_s^2 ds - e^{-\kappa t} \int_0^t H(s) \alpha_s^2 ds \\
&= h(t) \left[H(t) \int_0^t \alpha_s^2 ds - H(s) \int_0^s \alpha_u^2 du \Big|_0^t + \int_0^t h(s) \left(\int_0^s \alpha_u^2 du \right) ds \right] \\
&= h(t) \int_0^t e^{\kappa s} \nu(s) ds = \nu^h(t),
\end{aligned}$$

We have obtained

$$P(t, T; X_t = x) = \frac{P(0, T)}{P(0, t)} \exp \left\{ -H(T - t) \left[x + \nu^h(t) + \frac{1}{2} \nu(t) H(T - t) \right] \right\},$$

which gives formula (1.2).

1.3 Joint density of $(\int_0^t X_s ds, X_t)$ and value of $E^Q \left[e^{-\int_0^t X_s ds} \Big| X_t \right]$ under risk-neutral measure

To price a European contingent claim with payoff $f(X_T)$ at terminal time T , we typically need to evaluate

$$\begin{aligned}
V_0 &= E^Q \left[e^{-\int_0^T r_s ds} f(X_T) \right] = e^{-\int_0^T \theta_s ds} E^Q \left[e^{-\int_0^T X_s ds} f(X_T) \right] \\
&= e^{-\int_0^T \theta_s ds} E^Q \left[E^Q \left[e^{-\int_0^T X_s ds} \Big| X_T \right] f(X_T) \right].
\end{aligned}$$

This demands the knowledge of the joint density of $(\int_0^t X_s ds, X_t)$ or the value of $E^Q \left[e^{-\int_0^t X_s ds} \Big| X_t \right]$. We derive these two quantities in this section.

It's easy to see $Z_t := \int_0^t X_s ds$ and X_t are jointly Gaussian. In order to know their joint density, it's sufficient to know their respective mean and variance, as well as their covariance. In this regard, we note $E[X_t] = E[Z_t] = 0$. Define

$$v_X(t) = \sqrt{E[X_t^2]}, \quad v_Z(t) = \sqrt{E[Z_t^2]}, \quad \rho_{XZ}(t) = \frac{E[X_t Z_t]}{v_X(t) v_Z(t)}, \quad c_{XZ}(t) = E[X_t Z_t].$$

Then $v_X^2(t) = E[X_t^2] = e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma_s^2 ds = \nu(t)$, and by integration-by-parts formula, we have

$$X_t Z_t = \int_0^t Z_s dX_s + \int_0^t X_s^2 ds = -\kappa \int_0^t X_s Z_s ds + \int_0^t X_s^2 ds + \text{mart. part.}$$

Taking expectation on both sides gives

$$c_{XZ}(t) = -\kappa \int_0^t c_{XZ}(s)ds + \int_0^t \nu(s)ds.$$

Solving this integral equation gives

$$c_{XZ}(t) = e^{-\kappa t} \int_0^t e^{\kappa s} \nu(s)ds = \nu^h(t).$$

Finally, note $\frac{d}{dt}E[Z_t^2] = 2E[Z_t X_t] = 2c_{XZ}(t) = 2\nu^h(t)$, we have $v_Z^2(t) = E[Z_t^2] = 2 \int_0^t \nu^h(s)ds = 2\nu^H(t)$. In summary, we have

$$\begin{cases} v_X^2(t) = E[X_t^2] = \nu(t) \\ v_Z^2(t) = E[Z_t^2] = 2\nu^H(t) \\ c_{XZ}(t) = E[X_t Z_t] = \nu^h(t) \\ \rho_{XZ}^2(t) = \frac{(E[X_t Z_t])^2}{v_X^2(t)v_Z^2(t)} = \frac{(\nu^h(t))^2}{2\nu(t)\nu^H(t)} \end{cases}$$

Therefore, the covariance matrix Σ_t of the pair (Z_t, X_t) is

$$\Sigma_t = \begin{pmatrix} E[Z_t^2] & E[X_t Z_t] \\ E[X_t Z_t] & E[X_t^2] \end{pmatrix} = \begin{pmatrix} 2\nu^H(t) & \nu^h(t) \\ \nu^h(t) & \nu(t) \end{pmatrix}$$

and its inverse is

$$\Sigma_t^{-1} = \frac{1}{1 - \rho_{XZ}^2(t)} \begin{pmatrix} \frac{1}{v_Z^2(t)} & -\frac{\rho_{XZ}(t)}{v_X(t)v_Z(t)} \\ -\frac{\rho_{XZ}(t)}{v_X(t)v_Z(t)} & \frac{1}{v_X^2(t)} \end{pmatrix}$$

The joint density function of the pair (X_t, Z_t) is therefore

$$\begin{aligned} g(x, z; t) &= \frac{|\Sigma_t|^{-\frac{1}{2}}}{2\pi} e^{-\frac{1}{2}(x, z)\Sigma_t^{-1}(x, z)'} \\ &= \boxed{\frac{1}{2\pi v_X(t)v_Z(t)\sqrt{1 - \rho_{XZ}^2(t)}} \exp \left\{ -\frac{1}{2(1 - \rho_{XZ}^2(t))} \left[\frac{x^2}{v_X^2(t)} - 2\rho_{XZ}(t)\frac{xz}{v_X(t)v_Z(t)} + \frac{z^2}{v_Z^2(t)} \right] \right\}} \quad (1.3) \end{aligned}$$

To compute the other quantity, note when conditioning on $X_t = x$,

$$Z_t \sim N \left(\frac{c_{XZ}(t)}{v_X^2(t)} x, v_Z^2(t) - \frac{c_{XZ}^2(t)}{v_X^2(t)} \right) = N \left(x \frac{\nu^h(t)}{\nu(t)}, 2\nu^H(t) - \frac{(\nu^h(t))^2}{\nu(t)} \right).$$

So

$$\boxed{E^Q [e^{-Z_t} | X_t] = \exp \left\{ -X_t \frac{\nu^h(t)}{\nu(t)} + \nu^H(t) - \frac{(\nu^h(t))^2}{2\nu(t)} \right\}} \quad (1.4)$$

1.4 Pricing formula of European contingent claim

To price a European contingent claim with payoff $f(X_T)$ at terminal time T , we note by formula (1.4)

$$\begin{aligned} V_0 &= E^Q \left[e^{-\int_0^T r_s ds} f(X_T) \right] = e^{-\int_0^T \theta_s ds} E^Q \left[e^{-\int_0^T X_s ds} f(X_T) \right] \\ &= e^{-\int_0^T \theta_s ds} E^Q \left[E^Q \left[e^{-\int_0^T X_s ds} | X_T \right] f(X_T) \right] \\ &= \exp \left\{ - \int_0^T \theta_s ds + \nu^H(T) - \frac{(\nu^h(T))^2}{2\nu(T)} \right\} E^Q \left[f(X_T) \exp \left\{ -X_T \frac{\nu^h(T)}{\nu(T)} \right\} \right] \\ &= \boxed{P(0, T) \exp \left\{ -\frac{(\nu^h(T))^2}{2\nu(T)} \right\} E^Q \left[f(X_T) \exp \left\{ -X_T \frac{\nu^h(T)}{\nu(T)} \right\} \right]} \quad (1.5) \end{aligned}$$

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