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Elements of the Hull-White Model: An Implementation Guide

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Abstract

This is a self-contained implementation guide to the Hull-White model. We derive from scratch formulas that are essential to coding up the model. Some numerical issues are also handled with care.

Contents

Chapter 1

One-factor Hull-White model

1.1 Dynamics of short rate r_t and state variable X_t under risk-neutral measure

Under the assumption of one-factor Hull-White model, the short rate process under the risk-neutral measure Q (the martingale measure associated with money market account numeraire) follows the dynamics

$$dr_t = (b_t - \kappa r_t)dt + \sigma_t dW_t,$$

where κ is a constant, b_t and σ_t are deterministic functions of t, and W is a standard Brownian motion.

Solving the SDE gives

$$r_t = e^{-\kappa t} r_0 + e^{-\kappa t} \int_0^t e^{\kappa s} b_s ds + e^{-\kappa t} \int_0^t e^{\kappa s} \sigma_s dW_s.$$

Setting $\theta_t = e^{-\kappa t} r_0 + e^{-\kappa t} \int_0^t e^{\kappa s} b_s ds$ and $X_t = e^{-\kappa t} \int_0^t e^{\kappa s} \sigma_s dW_s$. Then θ_t is a deterministic function of t and X_t is Gaussian process with mean 0 and variance $e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma_s^2 ds$. In summary, we have

$$\boxed{r_t = \theta_t + X_t,\ dX_t = -\kappa X_t dt + \sigma_t dW_t,\ X_0 = 0,\ E^Q[X_t] = 0,\ E^Q[X_t^2] = e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma_s^2 ds} \tag{1.1}$$

1.2 Pricing formula of zero coupon bond

1.2.1 Formula

Denote by $P(t,T)$ the time-t price of a zero-coupon bond with maturity T, we have (note $P(t,T)$ is a function of the state variable X_t)

$$\boxed{\begin{cases} P(t,T) = P(t,T;X_t) = \frac{P(0,T)}{P(0,t)} \exp\left\{-H(T-t)\left[X_t + \nu^h(t) + \frac{1}{2}\nu(t)H(T-t)\right]\right\} \\ P(0,t) = \exp\left\{-\int_0^t \theta_s ds + \nu_t^H\right\} \end{cases}} \tag{1.2}$$

where

$$\begin{cases} h(t) = e^{-\kappa t} \\ H(t) = \int_0^t h(s)ds \\ \nu(t) = e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma_s^2 ds \\ \nu^h(t) = h * v(t) = \int_0^t e^{-\kappa(t-s)} \nu(s) ds \\ \nu^H(t) = H * \nu(t) = \int_0^t H(t-s)\nu(s)ds. \end{cases}$$

We also note that $\frac{d}{dt}\nu^H(t) = \nu^h(t)$.

1.2.2 Derivation

For a derivation of formula (1.2), define $\alpha_t = \sigma_t e^{\kappa t}$. It's easy to see $X_t = h(t)\int_0^t \alpha_s dW_s$. For the convenience of later computation, we note for $u > t$,

$$X_u = e^{-\kappa u}\int_0^t \alpha_s dW_s + e^{-\kappa u}\int_t^u \alpha_s dW_s = h(u-t)X_t + h(u)\int_t^u \alpha_s dW_s.$$

Therefore, risk neutral pricing formula yields

$$P(t,T;X_t=x) = E^Q\left[e^{-\int_t^T r_u du}\middle|\mathcal{F}_t, X_t = x\right] = e^{-\int_t^T \theta_u du - H(T-t)x}E^Q\left[e^{-\int_t^T h(u)(\int_t^u \alpha_s dW_s)du}\right]$$

Define $\eta = \int_t^T h(u)(\int_t^u \alpha_s dW_s)du$. Then η is a Gaussian random variable with 0 mean and

$$\begin{aligned}\eta^2 &= \left[H(u)\int_t^u \alpha_s dW_s|_{u=t}^{u=T} - \int_t^T H(u)\alpha_u dW_u\right]^2 \\ &= H^2(T)\left(\int_t^T \alpha_s dW_s\right)^2 - 2H(T)\int_t^T \alpha_s dW_s \int_t^T H(u)\alpha_u dW_u + \left(\int_t^T H(u)\alpha_u dW_u\right)^2.\end{aligned}$$

Therefore the variance of η is equal to

$$E^Q[\eta^2] = H^2(T)\int_t^T \alpha_s^2 ds - 2H(T)\int_t^T H(s)\alpha_s^2 ds + \int_t^T H^2(s)\alpha_s^2 ds$$

and

$$\begin{aligned}&P(t,T;X_t=x) = e^{-\int_t^T \theta_u du - H(T-t)x}E^Q[e^{-\eta}] = e^{-\int_t^T \theta_u du - H(T-t)x}\exp\left\{\frac{1}{2}Var(\eta)\right\} \\ &= \exp\left\{-\int_t^T \theta_u du - H(T-t)x + \frac{1}{2}\left[H^2(T)\int_t^T \alpha_s^2 ds + \int_t^T H^2(s)\alpha_s^2 ds\right] - H(T)\int_t^T H(s)\alpha_s^2 ds\right\}.\end{aligned}$$

As particular cases, we have

$$\begin{cases} P(0,T) = \exp\left\{-\int_0^T \theta_u du + \frac{1}{2}\left[H^2(T)\int_0^T \alpha_s^2 ds + \int_0^T H^2(s)\alpha_s^2 ds\right] - H(T)\int_0^T H(s)\alpha_s^2 ds\right\} \\ P(0,t) = \exp\left\{-\int_0^t \theta_u du + \frac{1}{2}\left[H^2(t)\int_0^t \alpha_s^2 ds + \int_0^t H^2(s)\alpha_s^2 ds\right] - H(t)\int_0^t H(s)\alpha_s^2 ds\right\}.\end{cases}$$

Therefore, we have

$$\begin{aligned}&\frac{P(0,T)}{P(0,t)P(t,T;X_t=x)} \\ &= \exp\left\{H(T-t)x + \frac{1}{2}\left[H^2(T)\int_0^t \alpha_s^2 ds + \int_0^t H^2(s)\alpha_s^2 ds\right] - H(T)\int_0^t H(s)\alpha_s^2 ds\right\} \\ &\quad \cdot \exp\left\{-\frac{1}{2}\left[H^2(t)\int_0^t \alpha_s^2 ds + \int_0^t H^2(s)\alpha_s^2 ds\right] + H(t)\int_0^t H(s)\alpha_s^2 ds\right\} \\ &= \exp\left\{H(T-t)x + \frac{1}{2}[H^2(T) - H^2(t)]\int_0^t \alpha_s^2 ds - [H(T) - H(t)]\int_0^t H(s)\alpha_s^2 ds\right\}.\end{aligned}$$

That is,

$$\begin{aligned}&P(t,T;X_t=x) \\ &= \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)x - \frac{1}{2}[H^2(T) - H^2(t)]\int_0^t \alpha_s^2 ds + [H(T) - H(t)]\int_0^t H(s)\alpha_s^2 ds\right\}\end{aligned}$$

Note $[H(T)-H(t)]e^{\kappa t} = \frac{e^{-\kappa t}-e^{-\kappa T}}{\kappa}e^{\kappa t} = H(T-t)$, we get

$$
\begin{aligned}
& P(t,T;X_t=x) \\
= \; & \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)\left[x+\frac{H(T)+H(t)}{2}e^{-\kappa t}\int_0^t \alpha_s^2 ds - e^{-\kappa t}\int_0^t H(s)\alpha_s^2 ds\right]\right\} \\
= \; & \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)\left[x+H(t)e^{-\kappa t}\int_0^t \alpha_s^2 ds + \frac{H(T)-H(t)}{2}e^{-\kappa t}\int_0^t \alpha_s^2 ds\right.\right. \\
& \left.\left.-e^{-\kappa t}\int_0^t H(s)\alpha_s^2 ds\right]\right\} \\
= \; & \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)\left[x+H(t)e^{-\kappa t}\int_0^t \alpha_s^2 ds + \frac{1}{2}H(T-t)\nu(t) - e^{-\kappa t}\int_0^t H(s)\alpha_s^2 ds\right]\right\}.
\end{aligned}
$$

Note

$$
\begin{aligned}
& H(t)e^{-\kappa t}\int_0^t \alpha_s^2 ds - e^{-\kappa t}\int_0^t H(s)\alpha_s^2 ds \\
= \; & h(t)\left[H(t)\int_0^t \alpha_s^2 ds - H(s)\int_0^s \alpha_u^2 du\Big|_0^t + \int_0^t h(s)\left(\int_0^s \alpha_u^2 du\right)ds\right] \\
= \; & h(t)\int_0^t e^{\kappa s}\nu(s)ds = \nu^h(t),
\end{aligned}
$$

We have obtained

$$
P(t,T;X_t=x) = \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)\left[x+\nu^h(t)+\frac{1}{2}\nu(t)H(T-t)\right]\right\},
$$

which gives formula (1.2).

1.3 Joint density of $(\int_0^t X_s ds, X_t)$ and value of $E^Q\left[e^{-\int_0^t X_s ds}\Big| X_t\right]$ under risk-neutral measure

To price a European contingent claim with payoff $f(X_T)$ at terminal time T, we typically need to evaluate

$$
\begin{aligned}
V_0 &= E^Q\left[e^{-\int_0^T r_s ds}f(X_T)\right] = e^{-\int_0^T \theta_s ds}E^Q\left[e^{-\int_0^T X_s ds}f(X_T)\right] \\
&= e^{-\int_0^T \theta_s ds}E^Q\left[E^Q\left[e^{-\int_0^T X_s ds}\Big| X_T\right]f(X_T)\right].
\end{aligned}
$$

This demands the knowledge of the joint density of $(\int_0^t X_s ds, X_t)$ or the value of $E^Q\left[e^{-\int_0^t X_s ds}\Big| X_t\right]$. We derive these two quantities in this section.

It's easy to see $Z_t := \int_0^t X_s ds$ and X_t are jointly Gaussian. In order to know their joint density, it's sufficient to know their respective mean and variance, as well as their covariance. In this regard, we note $E[X_t] = E[Z_t] = 0$. Define

$$
v_X(t) = \sqrt{E[X_t^2]},\ v_Z(t) = \sqrt{E[Z_t^2]},\ \rho_{XZ}(t) = \frac{E[X_t Z_t]}{v_X(t)v_Z(t)},\ c_{XZ}(t) = E[X_t Z_t].
$$

Then $v_X^2(t) = E[X_t^2] = e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds = \nu(t)$, and by integration-by-parts formula, we have

$$
X_t Z_t = \int_0^t Z_s dX_s + \int_0^t X_s^2 ds = -\kappa\int_0^t X_s Z_s ds + \int_0^t X_s^2 ds + \text{mart. part.}
$$

Taking expectation on both sides gives

$$c_{XZ}(t) = -\kappa \int_0^t c_{XZ}(s)ds + \int_0^t \nu(s)ds.$$

Solving this integral equation gives

$$c_{XZ}(t) = e^{-\kappa t} \int_0^t e^{\kappa s}\nu(s)ds = \nu^h(t).$$

Finally, note $\frac{d}{dt}E\left[Z_t^2\right] = 2E[Z_tX_t] = 2c_{XZ}(t) = 2\nu^h(t)$, we have $v_Z^2(t) = E[Z_t^2] = 2\int_0^t \nu^h(s)ds = 2\nu^H(t)$. In summary, we have

$$\begin{cases} v_X^2(t) = E[X_t^2] = \nu(t) \\ v_Z^2(t) = E[Z_t^2] = 2\nu^H(t) \\ c_{XZ}(t) = E[X_tZ_t] = \nu^h(t) \\ \rho_{XZ}^2(t) = \frac{(E[X_tZ_t])^2}{v_X^2(t)v_Z^2(t)} = \frac{(\nu^h(t))^2}{2\nu(t)\nu^H(t)} \end{cases}$$

Therefore, the covariance matrix Σ_t of the pair (Z_t, X_t) is

$$\Sigma_t = \begin{pmatrix} E[Z_t^2] & E[X_tZ_t] \\ E[X_tZ_t] & E[X_t^2] \end{pmatrix} = \begin{pmatrix} 2\nu^H(t) & \nu^h(t) \\ \nu^h(t) & \nu(t) \end{pmatrix}$$

and its inverse is

$$\Sigma_t^{-1} = \frac{1}{1-\rho_{XZ}^2(t)} \begin{pmatrix} \frac{1}{v_Z^2(t)} & -\frac{\rho_{XZ}(t)}{v_X(t)v_Z(t)} \\ -\frac{\rho_{XZ}(t)}{v_X(t)v_Z(t)} & \frac{1}{v_X^2(t)} \end{pmatrix}$$

The joint density function of the pair (X_t, Z_t) is therefore

$$\begin{aligned} g(x,z;t) &= \frac{|\Sigma_t|^{-\frac{1}{2}}}{2\pi} e^{-\frac{1}{2}(x,z)\Sigma_t^{-1}(x,z)'} \\ &= \boxed{\frac{1}{2\pi v_X(t)v_Z(t)\sqrt{1-\rho_{XZ}^2(t)}} \exp\left\{-\frac{1}{2\left(1-\rho_{XZ}^2(t)\right)}\left[\frac{x^2}{v_X^2(t)} - 2\rho_{XZ}(t)\frac{xz}{v_X(t)v_Z(t)} + \frac{z^2}{v_Z^2(t)}\right]\right\}} \end{aligned} \quad (1.3)$$

To compute the other quantity, note when conditioning on $X_t = x$,

$$Z_t \sim N\left(\frac{c_{XZ}(t)}{v_X^2(t)}x, v_Z^2(t) - \frac{c_{XZ}^2(t)}{v_X^2(t)}\right) = N\left(x\frac{\nu^h(t)}{\nu(t)}, 2\nu^H(t) - \frac{(\nu^h(t))^2}{\nu(t)}\right).$$

So

$$\boxed{E^Q\left[e^{-Z_t}\middle| X_t\right] = \exp\left\{-X_t\frac{\nu^h(t)}{\nu(t)} + \nu^H(t) - \frac{(\nu^h(t))^2}{2\nu(t)}\right\}} \quad (1.4)$$

1.4 Pricing formula of European contingent claim

To price a European contingent claim with payoff $f(X_T)$ at terminal time T, we note by formula (1.4)

$$\begin{aligned} V_0 &= E^Q\left[e^{-\int_0^T r_s ds}f(X_T)\right] = e^{-\int_0^T \theta_s ds}E^Q\left[e^{-\int_0^T X_s ds}f(X_T)\right] \\ &= e^{-\int_0^T \theta_s ds}E^Q\left[E^Q\left[e^{-\int_0^T X_s ds}\middle| X_T\right]f(X_T)\right] \\ &= \exp\left\{-\int_0^T \theta_s ds + \nu^H(T) - \frac{(\nu^h(T))^2}{2\nu(T)}\right\}E^Q\left[f(X_T)\exp\left\{-X_T\frac{\nu^h(T)}{\nu(T)}\right\}\right] \\ &= \boxed{P(0,T)\exp\left\{-\frac{(\nu^h(T))^2}{2\nu(T)}\right\}E^Q\left[f(X_T)\exp\left\{-X_T\frac{\nu^h(T)}{\nu(T)}\right\}\right]} \end{aligned} \quad (1.5)$$

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