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The SABR Model: An Implementation Guide

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Abstract

This is an implementation guide to various approximation formulas for the SABR model. Some general facts on implied volatility and an estimation of convergence rate of one of the approximation formulas are also provided.

Contents

Chapter 1

SABR model

The Stochastic Alpha Beta Rho (SABR) model is a stochastic volatility model that attempts to model the volatility smile in the context of derivative pricing. In this model, the forward price F and the volatility α are given by

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} dW_1(t),\ F_0 = f \\ d\alpha_t = \nu \alpha_t dW_2(t),\ \alpha_0 = \alpha \end{cases}$$

under the forward measure $\mathbb{P}$, where the two processes are correlated by

$$dW_1(t)dW_2(t) = \rho dt.$$

The price of a European call option on F with exercise date t_{ex}, settlement date t_{set}, and strike K is given by

$$V_{call} = D(t_{set})\mathbb{E}[(F_{t_{ex}} - K)^+]$$

where $\mathbb{E}[\cdot]$ is under the forward measure. Market convention often quotes the price through Black's formula

$$V_{call} = D(t_{set})[f\Phi(d_+) - K\Phi(d_-)],\ d_\pm = \frac{\log(f/K) \pm \frac{1}{2}\sigma_B^2 t_{ex}}{\sigma_B \sqrt{t_{ex}}}.$$

The remaining problem is how to obtain $\sigma_B = \sigma_B(K, f)$.

Chapter 2

Hagan's formula

2.1 Formula

Hagan [2] gives the following formula

$$\sigma_B(K,f) \approx \frac{\alpha}{(fK)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\log^2(f/K)+\frac{(1-\beta)^4}{1920}\log^4(f/K)+\cdots\right]}\cdot\frac{z}{x(z)}$$
$$\cdot\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}+\cdots\right\}$$

where

$$z=\frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log(f/K),\ x(z)=\log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}$$

For the special case of at-the-money options, options struck at $K=f$, this formula reduces to

$$\sigma_{ATM}=\sigma_B(f,f)\approx\frac{\alpha}{f^{(1-\beta)}}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{(1-\beta)}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}+\cdots\right\}$$

An alternative representation is through normal volatility:

$$\begin{aligned}&\sigma_N(K,f)\\ \approx\ &\frac{1}{\alpha}\cdot\frac{f^{1-\beta}-K^{1-\beta}}{(1-\beta)(f-K)}\cdot\frac{x(z)}{z}\\ &\cdot\left\{\frac{1+\frac{\beta(2-\beta)}{24}\frac{1-\frac{2-2\beta+\beta^2}{120}\log^2(f/K)}{1+\frac{(1-\beta)^2}{12}\log^2(f/K)}\frac{\alpha^2 t_{ex}}{(fK)^{(1-\beta)}}+\frac{\beta(2-\beta)}{80}[(1-\beta)^2+\frac{1}{72}\beta(2-\beta)]\frac{\alpha^4t_{ex}^2}{(fK)^{2-2\beta}}}{1+\frac{\beta\rho}{4}\frac{\alpha\nu t_{ex}}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2t_{ex}}\right\}\end{aligned}$$

For the special case of at-the-money options, this formula reduces to

$$\sigma_N(f,f)\approx\frac{1}{\alpha f^{\beta}}\left\{\frac{1+\frac{\beta(2-\beta)}{24}\frac{\alpha^2t_{ex}}{f^{2-2\beta}}+\frac{\beta(2-\beta)}{80}\left[(1-\beta)^2+\frac{1}{72}\beta(2-\beta)\right]\frac{\alpha^4t_{ex}^2}{f^{4-4\beta}}}{1+\frac{\beta\rho}{4}\cdot\frac{\alpha\nu t_{ex}}{f^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2t_{ex}}\right\}.$$

In actual implementation, the ATM case is unified under the case of "near ATM": $x(z)/z$ is approximated by $1+\frac{1}{2}\rho z-\frac{1-3\rho^2}{6}z^2$ and

$$\frac{f^{1-\beta}-K^{1-\beta}}{(1-\beta)(f-K)}\approx\frac{1}{(fK)^{(1-\beta)/2}}\cdot\frac{1+\frac{(1-\beta)^2}{24}\log^2(f/K)+\frac{(1-\beta)^2}{1920}\log^4(f/K)}{1+\frac{1}{24}\log^2(f/K)+\frac{1}{1920}\log^4(f/K)}$$

2.2 Asymptotics

Assuming $\beta < 1$, then $\lim_{K\to\infty} z = -\infty$ and $x(z) \sim -\log(-z)$. So as $K \to \infty$,

$$\sigma_B(K,f) \sim \frac{\nu \log(f/K)}{1 + \frac{(1-\beta)^2}{24}\log^2(f/K) + \frac{(1-\beta)^4}{1920}\log^4(f/K)} \cdot \frac{1}{-\log(-z)} \cdot \left(1 + \frac{2-3\rho^2}{24}\nu^2 t_{ex}\right).$$

Since $\log(-z) \sim \frac{1-\beta}{2}\log K$,

$$\lim_{K\to\infty} \frac{\nu \log(f/K)}{-\log(-z)} = \frac{2\nu}{1-\beta},$$

and we conclude

$$\lim_{K\to\infty} \sigma(K,f) = 0.$$

Assuming $\beta = 1$, Hagan's formula becomes

$$\sigma_B(K,f) \approx \frac{\nu \log(f/K)}{x(z)} \left[1 + \left(\frac{\rho\nu\alpha}{4} + \frac{2-3\rho^2}{24}\nu^2\right) t_{ex}\right],$$

where

$$z = \frac{\nu}{\alpha}\log(f/K),\ x(z) = \log\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}.$$

We still have $\lim_{K\to\infty} z = -\infty$ and $x(z) \sim -\log(-z)$. Hence $\lim_{K\to\infty} \sigma_B(K,f) = \infty$.

In summary,

$$\lim_{K\to\infty} \sigma_B(K,f) = \begin{cases} 0 & \beta < 1 \\ \infty & \beta = 1 \end{cases}$$

We further note when $\beta = 1$, Hagan's formula satisfies

$$\sigma_B(K,f) \sim \frac{\nu \log K}{\log(\log K)} \left[1 + \left(\frac{\rho\nu\alpha}{4} + \frac{2-3\rho^2}{24}\nu^2\right) t_{ex}\right],\ \text{as } K \to \infty,$$

This implies

$$\frac{\sigma_B^2(K,f)t_{ex}}{\log(K/f)} \sim \frac{(\nu \log K)^2 t_{ex}}{\log^2(\log K)\cdot \log K} \left[1 + \left(\frac{\rho\nu\alpha}{4} + \frac{2-3\rho^2}{24}\nu^2\right) t_{ex}\right]^2 \overset{K\to\infty}{\longrightarrow} \infty,$$

which contradicts with the moment formula (see Section 5). So Hagan's formula when $\beta = 1$ cannot be an accurate approximation for large strikes.

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