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Convexity Adjustment in Interest Rate Derivative Pricing

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Convexity Adjustment in Interest Rate Derivative Pricing: A Practical Guide

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Abstract

Elements of convexity adjustment in interest rate derivative pricing.

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Chapter 1

Introduction

In this note, we summarize various results on convexity adjustment. The exposition is based on Boenkost and Schmidt [1], [2], Hagan [3], Hull [5], Hunt and Kennedy [4], Lesniewski [6], and Pelsser [8].

Denote by $P(t,T)$ $(0 \leq t \leq T)$ the time-t value of a zero coupon bond with maturity T. $\tau(S,T)$ is the year fraction between time S and time T $(S < T)$. The *simply-compounded forward interest rate* $F(t;S,T)$ is defined as

$$F(t;S,T) = \frac{1}{\tau(S,T)}\left(\frac{P(t,S)}{P(t,T)} - 1\right).$$

Suppose $T_\alpha < T_{\alpha+1} < \cdots < T_\beta$ is a set of future times such that the LIBOR rate is reset at $T_\alpha, \cdots, T_{\beta-1}$ and is paid at $T_{\alpha+1}, \cdots, T_\beta$ for a floating-rate note. The *forward swap rate* $S_{\alpha,\beta}(t)$ at time t for the set of times $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, \cdots, T_\beta\}$ $(t \leq T_\alpha)$ and year fractions $\tau = \{\tau_{\alpha+1}, \cdots, \tau_\beta\}$ $(\tau_i = \tau(T_{i-1}, T_i))$ is defined as

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)}.$$

These two rates often appear as the underlyings in interest rate derivatives, and will serve as the prototype for convexity adjustment.

Chapter 2

Convexity adjusted interest rates

2.1 LIBOR

The LIBOR rate $L(S,T) = F(S;S,T)$ for the interval $[S,T]$ is given by

$$L(S,T) = \frac{1}{\tau(S,T)}\left(\frac{1}{P(S,T)} - 1\right).$$

Under the forward measure Q_T for which $P(\cdot,T)$ is the numeraire, $F(t;S,T)$ is a martingale and therefore $E^{Q_T}[L(S,T)] = F(0;S,T)$. This leads to the pricing formula of a floater, which resets LIBOR at time S and makes payment at time T.

LIBOR-in-arrears

For LIBOR-in-arrears, we need to evaluate $E^{Q_S}[L(S,T)]$, where Q_S is the forward measure for which $P(\cdot,S)$ is the numeraire. The goal is to express $E^{Q_S}[L(S,T)]$ in terms of the forward rate $F(0;S,T)$ plus some "convexity" adjustment (recall $E^{Q_T}[L(S,T)] = F(0;S,T)$):

$$\begin{aligned}
E^{Q_S}[L(S,T)] &= E^{Q_T}\left[L(S,T)\frac{P(S,S)/P(0,S)}{P(S,T)/P(0,T)}\right] \\
&= E^{Q_T}\left[L(S,T)\cdot(1+\tau(S,T)L(S,T))\cdot\frac{P(0,T)}{P(0,S)}\right] \\
&= E^{Q_T}\left[L(S,T)\cdot\frac{1+\tau(S,T)L(S,T)}{1+\tau(S,T)F(0;S,T)}\right] \\
&= \frac{F(0;S,T)+\tau(S,T)E^{Q_T}[L^2(S,T)]}{1+\tau(S,T)F(0;S,T)}
\end{aligned}$$

Note $E^{Q_T}[L^2(S,T)] = \mathrm{Var}_{Q_T}(L(S,T)) + (E^{Q_T}[L(S,T)])^2$, we conclude

$$\boxed{E^{Q_S}[L(S,T)] = F(0;S,T) + \frac{\tau(S,T)\mathrm{Var}_{Q_T}(L(S,T))}{1+\tau(S,T)F(0;S,T)}} \tag{2.1}$$

Under the so-called market model which is the model underlying the market valuation for caps, the LIBOR $L(S,T)$ is lognormal under Q_T with volatility σ,

$$L(S,T) = F(S;S,T) = F(0;S,T)\exp\left\{\sigma W_S - \frac{1}{2}\sigma^2 S\right\},$$

where W is a standard Brownian motion. In this case, $\mathrm{Var}_{Q_T}(L(S,T)) = F^2(0;S,T)(e^{\sigma^2 S}-1)$ and formula (2.1) becomes

$$E^{Q_S}[L(S,T)] = F(0;S,T)\left[1+\frac{\tau(S,T)F(0;S,T)(e^{\sigma^2 S}-1)}{1+\tau(S,T)F(0;S,T)}\right]. \tag{2.2}$$

LIBOR paid at arbitrary time under the linear rate model

Suppose the payment is made at an arbitrary time $T' \in [S, T]$. This is the case of Asian floater, where S and T are the starting time and ending time of a coupon period, respectively. Then

$$E^{Q_{T'}}[L(S,T)] = E^{Q_T}\left[\frac{P(S,T')/P(0,T')}{P(S,T)/P(0,T)}L(S,T)\right]$$

The *linear rate model* assumes

$$\frac{P(S,T')}{P(S,T)} = a + b(T')L(S,T), \ \forall T' \in [S,T]$$

which requires $a = 1$ by setting $T' = T$. This is effectively equivalent to assuming

$$L(T',T) = \frac{b(T')}{\tau(T',T)}L(S,T), \ \forall T' \in [S,T].$$

Moreover, the martingale property dictates

$$\frac{P(0,T')}{P(0,T)} = E^{Q_T}\left[\frac{P(S,T')}{P(S,T)}\right] = a + b(T')F(0;S,T).$$

So we have $b(T') = \left(\frac{P(0,T')}{P(0,T)} - 1\right)/F(0;S,T) = \frac{\tau(T',T)F(0;T',T)}{F(0;S,T)}$. In summary, the **linear rate model** assumes

$$\boxed{\frac{L(T',T)}{L(S,T)} = \frac{F(0;T',T)}{F(0;S,T)}, \ \forall T' \in [S,T]} \tag{2.3}$$

which can be summarized in words as

> *The ratio of LIBOR rates over the interval $[T', T]$ and $[S, T]$ is equal to the ratio of time-zero forward rates over the same intervals.*

Note the case of LIBOR-in-arrears, where $T' = S$, satisfies the assumption of linear rate model.

Under the linear rate model assumption, we easily deduce that

$$\boxed{E^{Q_{T'}}[L(S,T)] = F(0;S,T)\left[1 + \frac{1 - P(0,T)/P(0,T')}{F^2(0;S,T)}\mathrm{Var}_{Q_T}(L(S,T))\right], \ \forall T' \in [S,T]} \tag{2.4}$$

Remark 1. *The original motivation for the linear rate model is probably the consideration that the "natural rate" under T-forward measure Q_T is $L(S,T)$. So one would like to use $L(S,T)$ to approximate $P(S,T')/P(S,T)$, and linear function is obviously the simplest. This idea can be generalized to that of making the Radon-Nikodym derivative a function of the payout rate.*

Remark 2. *For $T' = S$, formula (2.4) reduces to formula (2.1).*

Under the market model where $L(S,T)$ is lognormal under Q_T with volatility σ,

$$L(S,T) = F(0;S,T)e^{\sigma W_S - \frac{1}{2}\sigma^2 S}$$

and formula (2.4) becomes more explicit:

$$E^{Q_{T'}}[L(S,T)] = F(0;S,T)\left[1 + \left(1 - \frac{P(0,T)}{P(0,T')}\right)(e^{\sigma^2 S} - 1)\right].$$

2.2 CMS

From the definition of forward swap rate $S_{\alpha,\beta}(t)$, if we choose the annuity $N_t^{\alpha,\beta} = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$ as numeraire and denote by $Q^{\alpha,\beta}$ the associated martingale measure (the "swap measure"), we have by martingale property

$$E^{Q^{\alpha,\beta}}\left[S_{\alpha,\beta}(T_\alpha)\right] = S_{\alpha,\beta}(0).$$

If the payment is to be paid at some time $T' > T_\alpha$, we need to compute under the T'-forward measure $Q_{T'}$

$$E^{Q_{T'}}\left[S_{\alpha,\beta}(T_\alpha)\right] = E^{Q^{\alpha,\beta}}\left[\frac{P(T_\alpha,T')/P(0,T')}{N_{T_\alpha}^{\alpha,\beta}/N_0^{\alpha,\beta}} S_{\alpha,\beta}(T_\alpha)\right] = \frac{N_0^{\alpha,\beta}}{P(0,T')} E^{Q^{\alpha,\beta}}\left[\frac{P(T_\alpha,T')}{N_{T_\alpha}^{\alpha,\beta}} S_{\alpha,\beta}(T_\alpha)\right].$$

The goal is to express $E^{Q_{T'}}\left[S_{\alpha,\beta}(T_\alpha)\right]$ in terms of the time-zero swap rate $S^{\alpha,\beta}(0)$ plus some "convexity" adjustment.

CMS paid at arbitrary time under the linear swap rate model

Under the swap measure $Q^{\alpha,\beta}$ associated with the annuity numeraire $N_t^{\alpha,\beta} = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$, the entity most convenient for computation is the swap rate $S_{\alpha,\beta}(T_\alpha)$. Therefore, a natural assumption for the so-called *linear swap rate model* is

$$\frac{P(T_\alpha,T')}{N_{T_\alpha}^{\alpha,\beta}} = a + b(T')S_{\alpha,\beta}(T_\alpha),\ T' \geq T_\alpha.$$

To determine a and b, we first take expectation of both sides under the swap measure and use the martingale property to get

$$\frac{P(0,T')}{N_0^{\alpha,\beta}} = a + b(T')S_{\alpha,\beta}(0).$$

This gives $b(T') = \frac{1}{S_{\alpha,\beta}(0)}\left[\frac{P(0,T')}{N_0^{\alpha,\beta}} - a\right]$. To deduce the second equation for a and b, we note

$$1 \quad = \quad \frac{\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha,T_i)}{N_{T_\alpha}^{\alpha,\beta}} = \sum_{i=\alpha+1}^{\beta} \tau_i\left[a + b(T_i)S_{\alpha,\beta}(T_\alpha)\right] = a\left(1 - \frac{S_{\alpha,\beta}(T_\alpha)}{S_{\alpha,\beta}(0)}\right)\sum_{i=\alpha+1}^{\beta}\tau_i + \frac{S_{\alpha,\beta}(T_\alpha)}{S_{\alpha,\beta}(0)}.$$

Therefore, we can solve for a: $a = \frac{1}{\sum_{i=\alpha+1}^{\beta}\tau_i}$. In summary, the **linear swap rate model** makes the assumption

$$\begin{cases} \frac{P(T_\alpha,T')}{N_{T_\alpha}^{\alpha,\beta}} = a + b(T')S_{\alpha,\beta}(T_\alpha),\ T' \geq T_\alpha \\ a = \frac{1}{\sum_{i=\alpha+1}^{\beta}\tau_i} \\ b(T') = \frac{1}{S_{\alpha,\beta}(0)}\left[\frac{P(0,T')}{N_0^{\alpha,\beta}} - \frac{1}{\sum_{i=\alpha+1}^{\beta}\tau_i}\right],\ T' \geq T_\alpha \end{cases} \tag{2.5}$$

and consequently

$$E^{Q_{T'}}\left[S_{\alpha,\beta}(T_\alpha)\right] = S_{\alpha,\beta}(0)\left[1 + \frac{1 - \frac{P(0,T_\alpha)-P(0,T_\beta)}{S_{\alpha,\beta}(0)P(0,T')\sum_{i=\alpha+1}^{\beta}\tau_i}}{S_{\alpha,\beta}^2(0)}\mathrm{Var}_N(S_{\alpha,\beta}(T_\alpha))\right] \tag{2.6}$$

where $\mathrm{Var}_N(S_{\alpha,\beta}(T_\alpha))$ is the variance of the swap rate $S_{\alpha,\beta}(T_\alpha)$ under the swap measure $Q^{\alpha,\beta}$.

Under the so-called market model for swpation, it's assumed the swap rate $S_{\alpha,\beta}(t)$ satisfies

$$dS_{\alpha,\beta}(t) = \sigma_{\alpha,\beta}S_{\alpha,\beta}(t)dW_t^{\alpha,\beta},\ t \leq T_\alpha$$

where $W^{\alpha,\beta}$ is a standard Brownian motion under the swap measure $Q^{\alpha,\beta}$. The variance of the swap rate $S_{\alpha,\beta}(T_\alpha)$ under the swap measure is therefore $S^2_{\alpha,\beta}(0)\left(e^{\sigma^2_{\alpha,\beta}T_\alpha}-1\right)$. Then

$$E^{Q_{T'}}\left[S_{\alpha,\beta}(T_\alpha)\right] = S_{\alpha,\beta}(0)\left[1+\left(1-\frac{P(0,T_\alpha)-P(0,T_\beta)}{S_{\alpha,\beta}(0)P(0,T')\sum_{i=\alpha+1}^{\beta}\tau_i}\right)\left(e^{\sigma^2_{\alpha,\beta}T_\alpha}-1\right)\right].$$

Remark 3. *The linear rate model for Libor and CMS can be generalized as follows. Write Y_S for a floating rate which is set at time S. Let N, Q_N denote the natural ("market") numeraire pair associated with Y_S and all we need is*

$$E^{Q_N}[Y_S] = Y_0,$$

where Y_0 is known and a function of the yield curve $P(0,\cdot)$ today.

We are interested in today's price of the rate Y_S to be paid at some time $T' \geq S$,

$$P(0,T')E^{Q_{T'}}[Y_S] = N_0 E^{Q_N}\left[\frac{P(S,T')}{N_S}Y_S\right].$$

Assume a linear rate model of the form

$$\boxed{\frac{P(S,T')}{N_S} = a + b(T')Y_S} \tag{2.7}$$

with some deterministic a, $b(T')$ which have to be determined accordingly to make the model consistent. We then have

$$\boxed{E^{Q_{T'}}[Y_S] = Y_0\left[1+\frac{b(T')}{Y_0(a+b(T')Y_0)}Var_{Q_N}(Y_S)\right]} \tag{2.8}$$

If in addition, the distribution of Y_S under Q_N is lognormal with volatility σ_Y: $Y_S = Y_0 e^{\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S}$, then

$$E^{Q_{T'}}[Y_S] = Y_0\left[1+\frac{b(T')Y_0}{a+b(T')Y_0}(e^{\sigma_Y^2 S}-1)\right].$$

Under the linear rate model for Libor, $Y_S = L(S,T)$ and $N_S = P(S,T)$; under the linear rate mode for CMS, $Y_S = S_{\alpha,\beta}(T_\alpha)$ and $N_S = N^{\alpha,\beta}_{T_\alpha}$.

As a last comment, the linear approximation of linear rate model does seem very crude at first, but can be justified by the following argument. Convexity corrections only become sizeable for large maturities. However, for large maturities the term structure almost moves in parallel. Hence, a change in the level of the long end of the curve is well described by the rate Y. Furthermore, for parallel moves in the curve, the ratio $\frac{P(S,T')}{N_S}$ is closely approximated by a linear function of Y, which is exactly what the linear rate model does. Hence, exactly for long maturities the assumptions of the linear rate model become quite accurate. This leads to a good approximation of the convexity correction for long maturities.

CMS paid at arbitrary time under Hagan's model

As seen in the previous section, the key to the convexity adjustment involving CMS rate, is to express $\frac{P(T_\alpha,T')}{N^{\alpha,\beta}_{T_\alpha}}$ as a function of swap rate $S_{\alpha,\beta}(T_\alpha)$:

$$\frac{P(T_\alpha,T')}{N^{\alpha,\beta}_{T_\alpha}} = G(S_{\alpha,\beta}(T_\alpha)).$$

In this section, we explain several such models proposed by Hagan [3].

Model 1: Standard model. The standard method for computing convexity corrections uses bond math approximations: payments are discounted at a flat rate, and the coverage (day count fraction) for each period is assumed to be $1/q$, where q is the number of periods per year. At any date $t \le T_\alpha$, the annuity is approximated by

$$N_t^{\alpha,\beta} = P(t,T_\alpha) \sum_{i=\alpha+1}^{\beta} \tau_i \frac{P(t,T_i)}{P(t,T_\alpha)} \approx P(t,T_\alpha) \sum_{j=1}^{\beta-\alpha} \frac{1/q}{[1+S_{\alpha,\beta}(t)/q]^j} = \frac{P(t,T_\alpha)}{S_{\alpha,\beta}(t)} \left[1 - \frac{1}{(1+S_{\alpha,\beta}(t)/q)^n}\right]$$

Here the forward swap rate $S_{\alpha,\beta}(t)$ is used as the discount rate, since it represents the average rate over the life of the reference swap. In a similar spirit, the zero coupon bond for the pay date T' is approximated as

$$P(t,T') \approx \frac{P(t,T_\alpha)}{(1+S_{\alpha,\beta}(t)/q)^\Delta}$$

where $\Delta = \frac{T'-T_\alpha}{T_{\alpha+1}-T_\alpha}$. Combined, the standard "bond math model" leads to the approximation

$$\boxed{G(S_{\alpha,\beta}(t)) = \frac{P(t,T')}{N_t^{\alpha,\beta}} \approx \frac{S_{\alpha,\beta}(t)}{(1+S_{\alpha,\beta}(t)/q)^\Delta} \cdot \frac{1}{1-\frac{1}{(1+S_{\alpha,\beta}(t)/q)^n}}}$$

Model 2: "Exact yield" model. We can account for the reference swap's schedule and day count exactly by approximating

$$\frac{P(t,T_i)}{P(t,T_\alpha)} \approx \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j S_{\alpha,\beta}(t)}$$

and

$$P(t,T') \approx \frac{P(t,T_\alpha)}{(1+\tau_{\alpha+1} S_{\alpha,\beta}(t))^\Delta}$$

where $\Delta = \frac{T'-T_\alpha}{T_{\alpha+1}-T_\alpha}$. Therefore

$$N_t^{\alpha,\beta} \approx P(t,T_\alpha) \sum_{i=\alpha+1}^{\beta} \tau_i \left(\prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j S_{\alpha,\beta}(t)} \right) = \frac{P(t,T_\alpha)}{S_{\alpha,\beta}(t)} \left(1 - \prod_{i=\alpha+1}^{\beta} \frac{1}{1+\tau_i S_{\alpha,\beta}(t)} \right)$$

and

$$\boxed{G(S_{\alpha,\beta}(t)) = \frac{P(t,T')}{N_t^{\alpha,\beta}} \approx \frac{S_{\alpha,\beta}(t)}{(1+\tau_{\alpha+1} S_{\alpha,\beta}(t))^\Delta} \frac{1}{1-\prod_{i=\alpha+1}^{\beta} \frac{1}{1+\tau_i S_{\alpha,\beta}(t)}}}$$

This approximates the yield curve as flat and only allows parallel shifts, but has the schedule right.

Model 3: Parallel shifts. This model takes into account the initial yield curve shape, which can be significant in steep yield curve environments. We still only allow parallel yield curve shifts, so we approximate

$$\frac{P(t,T_i)}{P(t,T_\alpha)} = \frac{P(0,T_i)}{P(0,T_\alpha)} e^{-(T_i-T_\alpha)s}$$

where s is the amount of the parallel shift to be determined. To determine s, note

$$N_t^{\alpha,\beta} = P(t,T_\alpha) \sum_{i=\alpha+1}^{\beta} \tau_i \frac{P(t,T_i)}{P(t,T_\alpha)} = P(t,T_\alpha) \sum_{i=\alpha+1}^{\beta} \tau_i \frac{P(0,T_i)}{P(0,T_\alpha)} e^{-(T_i-T_\alpha)s}$$

and hence

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha)-P(t,T_\beta)}{N_t^{\alpha,\beta}} = \frac{P(0,T_\alpha)-P(0,T_\beta)e^{-(T_\beta-T_\alpha)s}}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i) e^{-(T_i-T_\alpha)s}}.$$

This equation implicitly determines s as a function of $S_{\alpha,\beta}(t)$. Therefore

$$\frac{N_t^{\alpha,\beta}}{P(t,T_\alpha)} = \sum_{i=\alpha+1}^{\beta} \tau_i \frac{P(0,T_i)}{P(0,T_\alpha)} e^{-(T_i-T_\alpha)s} = \frac{1-\frac{P(0,T_\beta)}{P(0,T_\alpha)}e^{-(T_\beta-T_\alpha)s}}{S_{\alpha,\beta}(t)}$$

and

$$\boxed{G(S_{\alpha,\beta}(t)) = \frac{P(t,T')/P(t,T_\alpha)}{N_t^{\alpha,\beta}/P(t,T_\alpha)} = \frac{e^{-(T'-T_\alpha)s}}{\left[1-\frac{P(0,T_\beta)}{P(0,T_\alpha)}e^{-(T_\beta-T_\alpha)s}\right]/S_{\alpha,\beta}(t)} = \frac{S_{\alpha,\beta}(t)e^{-(T'-T_\alpha)s}}{1-\frac{P(0,T_\beta)}{P(0,T_\alpha)}e^{-(T_\beta-T_\alpha)s}}}$$

where s is determined implicity in terms of $S_{\alpha,\beta}(t)$, by

$$S_{\alpha,\beta}(t) \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i) e^{-(T_i-T_\alpha)s} = P(0,T_\alpha) - P(0,T_\beta)e^{-(T_\beta-T_\alpha)s}.$$

This model's limitations are that it allows only parallel shifts of the yield curve and it presumes perfect correlation between long and short term rates.

Model 4: Non-parallel shifts. We can allow non-parallel shifts by approximating

$$\frac{P(t,T_i)}{P(t,T_\alpha)} = \frac{P(0,T_i)}{P(0,T_\alpha)} e^{-[h(T_i)-h(T_\alpha)]s}$$

Then similar to Model 3, we have

$$\boxed{G(S_{\alpha,\beta}(t)) = \frac{S_{\alpha,\beta}(t)e^{-[h(T')-h(T_\alpha)]s}}{1-\frac{P(0,T_\beta)}{P(0,T_\alpha)}e^{-[h(T_\beta)-h(T_\alpha)]s}}}$$

where s is determined implicity in terms of $S_{\alpha,\beta}(t)$, by

$$S_{\alpha,\beta}(t) \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i) e^{-[h(T_i)-h(T_\alpha)]s} = P(0,T_\alpha) - P(0,T_\beta)e^{-[h(T_\beta)-h(T_\alpha)]s}.$$

To complete the model, we need to select the function $h(\cdot)$ which determines the shape of the non-parallel shift. This is often done by postulating a constant mean reversion

$$h((T) - h(T_\alpha) = \frac{1}{\kappa}\left[1-e^{-\kappa(T-T_\alpha)}\right].$$

Alternatively, one can choose $h(\cdot)$ by calibrating the vanilla swaptions which have the same start date T_α and varying end dates to their market prices.

In either case, under the assumption $\frac{P(T_\alpha,T')}{N_{T_\alpha}^{\alpha,\beta}} = G(S_{\alpha,\beta}(T_\alpha))$, we have

$$\boxed{E^{Q_{T'}}[S_{\alpha,\beta}(T_\alpha)] = S_{\alpha,\beta}(0) + E^{Q_{\alpha,\beta}}\left\{\left[\frac{G(S_{\alpha,\beta}(T_\alpha))}{G(S_{\alpha,\beta}(0))}-1\right]S_{\alpha,\beta}(T_\alpha)\right\}} \tag{2.9}$$

2.3 Hull's approach to convexity adjustment (LIBOR-in-arrears)

This section is based on Hull [5], Chapter 20. We recall the relation between forward LIBOR rate $F(t;S,T)$ and zero coupon bond price $P(t,\cdot)$ is given by

$$\frac{P(t,T)}{P(t,S)} = \frac{1}{1+\tau(S,T)F(t;S,T)}.$$

Write y_t for $F(t;S,T)$ and define $G(y) = \frac{1}{1+\tau(S,T)y}$. Then Taylor expansion gives

$$\frac{P(t,T)}{P(t,S)} = G(y_t) \approx G(y_0) + G'(y_0)(y_t - y_0) + \frac{1}{2}G''(y_0)(y_t - y_0)^2$$

Under the S-forward measure Q_S, $E^{Q_S}\left[\frac{P(t,T)}{P(t,S)}\right] = \frac{P(0,T)}{P(0,S)} = G(y_0)$. So taking expectation of both sides of the Taylor expansion, we have $G(y_0) \approx G(y_0) + G'(y_0)\left(E^{Q_S}[y_t] - y_0\right) + \frac{1}{2}G''(y_0)E^{Q_S}[(y_t - y_0)^2]$. This gives

$$E^{Q_S}[y_t] \approx y_0 - \frac{1}{2}\frac{G''(y_0)}{G'(y_0)}E^{Q_S}[(y_t - y_0)^2].$$

Let $t = S$ and approximate $E^{Q_S}[(y_S - y_0)^2]$ by $\sigma^2 y_0^2 S$ with σ the volatility of y. We then have

$$E^{Q_S}[L(S,T)] \approx F(0;S,T)\left[1 + \frac{\tau(S,T)F(0;S,T)\sigma^2 S}{1+\tau(S,T)F(0;S,T)}\right]$$

This is the first order approximation of convexity adjustment formula (2.2). Note the approximation of $E^{Q_S}[(y_S - y_0)^2]$ by $\sigma^2 y_0^2 S$ is more or less equivalent to assuming y_S is lognormally distributed.

2.4 Option on interest rates paid at arbitrary time under linear rate model

In this section, we investigate European options on interest rates like LIBOR $L(S,T)$ for period $[S,T]$ or CMS rates $S_{\alpha,\beta}(T_\alpha)$ for tenure structure $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, \cdots, T_\beta\}$. The payment date of the option is an arbitrary time point T' with $T' \geq S$ or $T' \geq T_\alpha$, respectively.

Of particular interest are caps and floors or binaries. For standard caps and floors on LIBOR, we have $T' = T$ and the standard market model postulates a lognormal distribution of $L(S,T)$ under the forward measure Q_T. For standard option on a swap rate $S_{\alpha,\beta}(T_\alpha)$, i.e. swaptions, the market uses a lognormal distribution for $S_{\alpha,\beta}(T_\alpha)$ under the swap measure $Q^{\alpha,\beta}$. However in the general case, i.e. for options on LIBOR or CMS with arbitrary payment date T', a lognormal model would be inconsistent with the market model for standard options. For example, if $L(S,T)$ is lognormal under Q_T, it cannot be lognormal under Q_S in general:

$$E^{Q_T}[f(L(S,T))] = \frac{P(0,S)}{P(0,T)}E^{Q_S}\left[\frac{f(L(S,T))}{1+\tau(S,T)L(S,T)}\right].$$

We follow the general setup of linear rate model. Y_S is a floating interest rate which is set at time S and (N, Q_N) denotes the "market" numeraire pair associated with Y_S. We assume that the distribution of Y_S under Q_N is lognormal with volatility σ_Y,

$$Y_S = Y_0 \exp\left\{\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S\right\},$$

where W is a standard Brownian motion. For a payment date $T' \geq S$, we further assume a linear rate model of the form (2.7)

$$\frac{P(S,T')}{N_S} = a + b(T')Y_S.$$

Recall that for the case of $Y_S = L(S,T)$, $N_S = P(S,T)$, and $T' = S$, i.e. LIBOR-in-arrears, the assumption of a linear rate model is trivially satisfied and there is no restriction.

We first consider the valuation of standard options on the rate Y_S but the option payout is at some arbitrary time $T' \geq S$. The value of a call option with strike K is then

$$\begin{aligned} P(0,T')E^{Q_{T'}}\left[(Y_S - K)^+\right] &= N_0 E^{Q_N}\left[(Y_S - K)^+\frac{P(S,T')}{N_S}\right] = N_0 E^{Q_N}\left[(Y_S - K)^+(a + b(T')Y_S)\right] \\ &= P(0,T')\frac{Y_0\Phi(d_1)(a - b(T')K) - aK\Phi(d_2) + b(T')Y_0^2 e^{\sigma_Y^2 S}\Phi(d_1 + \sigma_Y\sqrt{S})}{a + b(T')Y_0} \end{aligned}$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution,

$$d_1 = \frac{\ln\left(\frac{Y_0}{K}\right) + \frac{1}{2}\sigma_Y^2 S}{\sigma_Y \sqrt{S}}, \; d_2 = \frac{\ln\left(\frac{Y_0}{K}\right) - \frac{1}{2}\sigma_Y^2 S}{\sigma_Y \sqrt{S}}.$$

It is desirable to be able to use standard valuation formula (i.e. Black's formula) also for options on interest rates which are irregularly paid. For this purpose, we assume Y_S is lognormally distributed under $Q_{T'}$, with adjusted volatility.

Using formula (2.8), i.e. $E^{Q_{T'}}[Y_S] = Y_0 \left[1 + \frac{b(T')Y_0}{a+b(T')Y_0}(e^{\sigma_Y^2 S} - 1)\right]$, and moment matching, we can derive

$$Y_S \approx E^{Q_{T'}}[Y_S] \exp\left\{\sigma_Y^* \widehat{W}_S - \frac{1}{2}(\sigma_Y^*)^2 S\right\},$$

where $\widehat{W}$ is a standard Brownian motion under $Q_{T'}$ and

$$(\sigma_Y^*)^2 = \sigma_Y^2 + \ln\left[\frac{(a + b(T')Y_0)(a + b(T')Y_0 e^{2\sigma_Y^2 S})}{(a + b(T')Y_0 e^{\sigma_Y^2 S})^2}\right] / S. \tag{2.10}$$

This will give option price via Black's formula.

With the above lognormal approximation, we can consider an exchange option involving two interest rates, Y_1 and Y_2. We assume Y_1 and Y_2 are set (fixed) at times S_1 and S_2, respectively, with $S_1 \leq S_2$. For example, Y_1 and Y_2 could be LIBOR rates $L(S_1, T_1)$ and $L(S_2, T_2)$ referring to different fixing dates S_1, S_2 (e.g. LIBOR and LIBOR-in-arrears). One could also think of two CMS rates to be set at the same date but with different tenors.

Suppose the option's payoff is $(Y_2 - Y_1)^+$, paid at time $T' \geq \max\{S_1, S_2\}$. In view of the above lognormal approximation, we can assume that both interest rates are lognormal under $Q_{T'}$ $(i = 1, 2)$

$$Y_i = Y_i^0 e^{\sigma_i W_{S_i}^i - \frac{1}{2}\sigma_i^2 S_i}, \; Y_i^0 = E^{Q_{T'}}[Y_i],$$

with $E^{Q_{T'}}[Y_i]$ given by formula (2.4), (2.6), or (2.8) and W^i standard Brownian motion under $Q_{T'}$. Suppose the instantaneous correlation between W^1 and W^2 is ρ, the fair price of the exchange option is then given by

$$P(0, T')[Y_2^0 N(b_1) - Y_1^0 N(b_2)],$$

where

$$b_1 = \frac{\ln\left(\frac{Y_2^0}{Y_1^0}\right) + \frac{1}{2}(\sigma_1^2 S_1 + \sigma_2^2 S_2 - 2\sigma_1\sigma_2\rho S_1)}{\sqrt{\sigma_1^2 S_1 + \sigma_2^2 S_2 - 2\sigma_1\sigma_2\rho S_1}}, \; b_2 = \frac{\ln\left(\frac{Y_2^0}{Y_1^0}\right) - \frac{1}{2}(\sigma_1^2 S_1 + \sigma_2^2 S_2 - 2\sigma_1\sigma_2\rho S_1)}{\sqrt{\sigma_1^2 S_1 + \sigma_2^2 S_2 - 2\sigma_1\sigma_2\rho S_1}}$$

For details of the computation, see Boenkost and Schmidt [1], Section 4.4, Proposition 7.

Remark 4. *The market standard method of valuing options on convexity adjusted rates is to apply the Black formula using the convexity adjusted rate as the forward rate. But to be conceptually correct, we should also convexity adjust volatility by formula (2.10). These two adjustments combined is equivalent to assuming the rate is lognormal under the T'-forward measure $Q_{T'}$.*

Bibliography

[1] W. Boenkost and W. Schmidt. Notes on convexity and quanto adjustments for interest rates and related options. Working paper. October 2003. 2, 10

[2] W. Boenkost and W. Schmidt. Interest rate convexity and the volatility smile. Working paper. May 2006. 2

[3] P. Hagan. Convexity conundrums: Pricing CMS swaps, cpas, and floors. *Wilmott Magazine*, March, p.38-44. 2, 6

[4] P. J. Hunt and J. E. Kennedy. *Financial derivatives in theory and practice*. Revised Edition, Wiley, 2004. 2

[5] J. Hull. *Options, futures, and other derivatives*. Fourth Edition. Prentice-Hall, 2000. 2, 8

[6] A. Lesniewski. Convexity. 2

[7] Constantin P. Niculescu and Lars-Erik Persson. *Convex functions and their applications: A contemporary approach*. Springer. 32

[8] A. Pelsser. Mathematical foundation of convexity correction. Wroking paper. April 8, 2001. 2