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Fundamental Theorems of Asset Pricing in a Nutshell

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Fundamental Theorems of Asset Pricing in a Nutshell: With a View toward Numéraire Change

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Abstract

A survey of the Fundamental Theorems of Asset Pricing in mathematical finance.

Contents

Chapter 1

Introduction

This note serves as a summary of the numéraire change techniques as presented in Brigo and Mercurio [3] §2.2 and §2.3. The idea is to present the main results in a logically "natural" order so that we can easily remember them.

Roughly speaking, we are concerned with the following question: *Is the Fundamental Theorem of Asset Pricing (FTAP) invariant under numéraire change?*

The answer is *negative*. The key idea of this presentation is therefore the following: the version of FTAP as formulated in Delbaen and Schachermayer [8] starts from relaxing the no-arbitrage condition, so that it states its most general results in terms of local martingales or even σ-martingales. This is insufficient for the practical usage of risk-neutral pricing, as we really need martingale property, not *local* martingale property. We should instead start from the other way around: insist on martingale property and derive the right formulation of no-arbitrage.

In Section 2 and 4, we review the classical formulation of the first and second Fundamental Theorem of Asset Pricing (FTAP). Such a formulation is not invariant under numéraire change. In Section 3 and 5, we review the "right" formulation of Fundamental Theorem of Asset Pricing, insisting on martingale measures instead of local martingales measures. Such a formulation turns out to be invariant under numéraire change. In Section 6, we give concrete formulas for the case of Itô processes.

This note is based on a series of papers by Delbaen and Schachermayer ([8], [9], [10], [11], [12]), Schachermayer [24], Geman et al. [16], Yan et al. ([29], [22], [30], [28]), and Shiryaev [25], as well as the references therein.

Chapter 2

Fundamental Theorem of Asset Pricing: classical formulation

We first summarize the state of the art before Delbaen and Schachermayer [8]. The case when the time set is finite is completely settled in Dalang et al. [5] and the use of simple or even elementary integrands as trading strategies is no restrction at all. For the case of discrete but infinite time sets, the problem is solved in Schachermayer [24]; the case of continuous and bounded processes in continuous time is solved in Delbaen [6]. In these two cases the theorems are stated in terms of simple integrands and limits of sequences and by using the concept of *no free lunch with bounded risk*.

To state the results of Delbaen and Schachermayer [8], we consider a probability space $(\Omega, \mathcal{F}, P)$ and a right-continuous filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ $(T \leq \infty)$. In the given economy, $(K+1)$ non-dividend paying securities are traded continuously from time 0 until time T. Their prices are modeled by a $(K+1)$-dimensional adapted, positive semimartingale $S = \{(S_t^0, S_t^1, \cdots, S_t^K) : 0 \leq t \leq T\}$. We assume $S^0 \equiv 1$.[1]

Definition 2.1. *(Yan [29] p.661, p.662) A* **trading strategy** *is an $\mathbb{R}^{K+1}$-valued predictable process $\phi = \{\phi_t : 0 \leq t \leq T\}$ which is integrable w.r.t semimartingale S.*[2] *The* **value process** *associated with a strategy ϕ is defined by*

$$V_t(\phi) = \phi_t S_t = \sum_{k=0}^{K} \phi_t^k S_t^k, 0 \leq t \leq T,$$

and the **gains process** *associated with a strategy ϕ is defined by*

$$G_t(\phi) = (\phi \cdot S)_t := \int_0^t \phi_u dS_u = \sum_{k=0}^{K} \int_0^t \phi_u^k dS_u^k, 0 \leq t \leq T.$$

A trading strategy ϕ is $\mathbf{self-financing}$ *if*

$$V_t(\phi) = V_0(\phi) + G_t(\phi), 0 \leq t < T.$$

Definition 2.2. *(Delbaen and Schachermayer [8] Definition 2.7) A trading strategy ϕ is* **admissible** *if $G_t(\phi)$ is bounded from below, i.e. there is a constant M such that $G_t(\phi) \geq -M$ a.s. for all $t \geq 0$.*

Definition 2.3. *(Delbaen and Schachermayer [10] or Shiryaev [25] p.650, VII §2a.2 Definition 1) We say that the vector of price processes S satisfies the condition of* **no arbitrage** *(NA) at time T if for each self-financing strategy ϕ, we have*

$$P(G_T(\phi) \geq 0) = 1 \Rightarrow P(G_T(\phi) = 0) = 1.$$

[1] This assumption hides the fact that we need a numéraire (i.e. a positive discounter), since 1 is used as the numéraire.

[2] We need here the notion of integration w.r.t. a vector-valued semimartingale, which is is defined globally, not componentwisely (see Jacod [19]). This is because the notion of componentwise stochastic integral is insufficient for stating FTAP in the most general setting (see Shiryaeve [25] p.635, Shiryaev and Cherny [26]). However, when ϕ is locally bounded, componentwise integration is sufficient for stating FTAP. See Cherny [4] for more details.

The above concept of no arbitrage is already sufficient for stating FTAP in the discrete time case, but for continuous time case, we need the following concept:

Definition 2.4. *(Delbaen and Schachermayer [8] Definition 2.8 or Shiryaev [25] p.650, VII §2a.2 Definition 3) Let*

$$K = \{G_T(\phi) | \phi \text{ admissible and } G_\infty(\phi) = \lim_{t\to\infty} G_t(\phi) \text{ exists a.s. if } T = \infty\}$$

and

$$C = \{g \in L^\infty(\Omega, \mathcal{F}_T, P) |\ g \leq f \text{ for some } f \in K\}.$$

We say that S satisfies the condition of **no free lunch with vanishing risk** *(NFLVR) for admissible strategies, if*

$$\bar{C} \cap L^\infty_+(\Omega, \mathcal{F}_T, P) = \{0\},$$

where $\bar{C}$ denotes the closure of C with respect to the norm topology of $L^\infty(\Omega, \mathcal{F}_T, P)$.

To understand intuitively the NFLVR condition, we note S allows for a free lunch with vanishing risk, if there is $f \in L^\infty_+(\Omega, \mathcal{F}_T, P) \setminus \{0\}$, a sequence $(G_T(\phi_n))_{n=0}^\infty \subset K$, where $(\phi^n)_{n=0}^\infty$ is a sequence of admissible integrands, and $(g_n)_{n=0}^\infty \subset L^\infty(\Omega, \mathcal{F}_T, P)$ satisfying $g_n \leq G_T(\phi_n)$, such that

$$\lim_{n\to\infty} \|f - g_n\|_{L^\infty(\Omega, \mathcal{F}_T, P)} = 0.$$

In particular the negative parts $(G_T^-(\phi_n))_{n=0}^\infty$ and $(g_n^-)_{n=0}^\infty$ tend to zero uniformly, which explains the term "vanishing risk".

The last piece of our vocabulary for stating FTAP is the following one.

Definition 2.5. *(Shiryaev [25] p.652, VII §2b.1 and p.656, VII §2c.2) An* **equivalent martingale measure** *(EMM) is a probability measure equivalent to P and under which S is a martingale. An* **equivalent local martingale measure** *(ELMM) is a probability measure equivalent to P and under which S is a local martingale. An* **equivalent σ-martingale measure** *(EσMM) is a probability measure equivalent to P and under which S is a σ-martingale, i.e. $S = S_0 + H \cdot M$ with M a martingale and H a positive predictable process integrable w.r.t M.*

Now we are ready to state a list of results on the classical formulation of Fundamental Theorem of Asset Pricing:

Theorem 2.1. *(Shiryaev [25] p.655, VII §2c, Theorem 1, 2, and Corollary) Let S be defined as above.*

a) If S is bounded, then

$$NFLVR \Leftrightarrow EMM.$$

b) If S is locally bounded, then

$$NFLVR \Leftrightarrow ELMM.$$

c) If S is a general semimartingale, then

$$NFLVR \Leftrightarrow E\sigma MM.$$

For a clearer insight into the connection between the above results and the corresponding results in the discrete-time case (see Theorem 2.2), we reformulate the theorem as follows.

In general semimartingale models $S_t = (1, S_t^1, \cdots, S_t^K)_{0\leq t\leq T}$, $T < \infty$, we have

$$EMM \Rightarrow ELMM \Rightarrow E\sigma MM \Leftrightarrow NFLVR.$$

When S is moreover locally bounded, we have

$$EMM \Rightarrow ELMM \Leftrightarrow E\sigma MM \Leftrightarrow NFLVR.$$

When S is further assumed to be bounded, we have

$$EMM \Leftrightarrow ELMM \Leftrightarrow E\sigma MM \Leftrightarrow NFLVR.$$

As a comparison, we recall FTAP in the discrete-time case.

Theorem 2.2. *(Dalang et al. [5]. Also see Delbaen and Schachermayer [10] Theorem 15) In the discrete- and finite-time case (i.e. $t = 0, 1, \cdots, T < \infty$) , we have*

$$EMM \Leftrightarrow ELMM \Leftrightarrow E\sigma MM \Leftrightarrow NA.$$

Remark 1. *For Theorem 2.2 to hold, we need no additional assumptions on trading strategies beside predictability. We also comment that $T < \infty$ is essential for Theorem 2.2; otherwise a counter example exists (see Shiryaev [25] p.415, V §2b.3). For the case of $T = \infty$, the NA condition needs to be modified to "no free lunch with bounded risk" (see Schachermayer [24]).*

In the above statement of FTAP, we have set $S^0 \equiv 1$. In practice, we usually do not have an asset whose price is identically 1. So FTAP as stated in Theorem 2.1 and 2.2 is really a mathematical simplification: instead of a general semimartingale $S = (S^0, S^1, \cdots, S^K)$, we considered the discounted process:

$$\frac{1}{S_0}\left(S^0, S^1, \cdots, S^K\right) = \left(1, \frac{S^1}{S^0}, \cdots, \frac{S^N}{S^0}\right).$$

Therefore we implicitly used S^0 as a numéraire, and EMM or ELMM should be understood as w.r.t a numéraire: discounted by this numéraire, S is a martingale or local martingale. So measure and numéraire appear in a dual pair. Similarly, the notion of *admissibility* should be understood as "in given numéraire". That is, we require $G_t(\phi)$ denominated in the numéraire is bounded from below. Formally, we have

Definition 2.6. *A* **numéraire** *is any strictly positive semimartingale.*

Definition 2.7. *An* **equivalent martingale measure** Q^N **associated with the numéraire** N *is a probability measure equivalent to P such that S/N is a martingale under Q^N.*

Definition 2.8. *A trading strategy ϕ is* **admissible under the numéraire** N *if there is a constant M such that $G_t(\phi)/N_t \geq -M$ a.s. for all $t \geq 0$.*

To reconcile any potential conceptual conflicts, we need the following

Lemma 2.1. *ϕ is a self-financing strategy if and only if for any numéraire N, we have*

$$d\left(\frac{V_t(\phi)}{N_t}\right) = \sum_{k=0}^{K} \phi_t^k d\left(\frac{S_t^k}{N_t}\right).$$

Proof. Sufficiency is obvious as we can take $N_t \equiv 1$. For necessity, we note by integration-by-part formula

$$\begin{aligned}
& d\left(\frac{V_t(\phi)}{N_t}\right) \\
= \quad & \frac{dV_t(\phi)}{N_{t-}} + V_{t-}(\phi) d\left(\frac{1}{N_t}\right) + d[V(\phi), 1/N]_t = \sum_{k=0}^{K} \frac{\phi_t^k dS_t^k}{N_{t-}} + \sum_{k=0}^{K} \phi_t^k S_{t-}^k d\left(\frac{1}{N_t}\right) + \sum_{k=0}^{K} \phi_t^k d[S^k, 1/N]_t \\
= \quad & \sum_{k=0}^{K} \phi_t^k \left\{\frac{dS_t^k}{N_{t-}} + S_{t-}^k d\left(\frac{1}{N_t}\right) + d[S^k, 1/N]_t\right\} = \sum_{k=0}^{K} \phi_t^k d\left(\frac{S_t^k}{N_t}\right),
\end{aligned}$$

where the second "=" has used the observation that

$$V_{t-}(\phi) = V_t(\phi) - \Delta V_t(\phi) = \sum_{k=0}^{K} \phi_t^k S_t^k - \sum_{k=0}^{K} \phi_t^k \Delta S_t^k = \sum_{k=0}^{K} \phi_t^k S_{t-}^k.$$

□

As we change numéraire, a question naturally arises: *Is Fundamental Theorem of Asset Pricing invariant under a numéraire change?* This question can be more precisely stated as follows:[3]

1) Under numéraire change, is NFLVR preserved?
2) Under numéraire change, is the existence of EMM or ELMM preserved?
3) Under numéraire change, if the existence of EMM or ELMM is preserved, is the uniqueness of such a measure also preserved?
4) How are the equivalent martingale measures related to each other? For example, can we represent the Radon-Nikodym derivatives in terms of the numéraires?
5) If we change the numéraire N to another numéraire U, does the risk-neutral pricing formula still holds? That is, for an $\mathcal{F}_T$-measurable random variable ξ satisfying suitable integrability conditions, do we have $(t < T)$

$$N_t E^N \left[\frac{\xi}{N_T} \middle| \mathcal{F}_t \right] = U_t E^U \left[\frac{\xi}{U_T} \middle| \mathcal{F}_t \right]?$$

For the sake of risk-neutral pricing, we shall focus on EMM. The following example justifies our choice: *NFLVR and existence of ELMM are not preserved under numéraire change.*

Example 1. *(Delbaen [7] or Delbaen and Schachermayer [9], Corollary 5) Let R be the Bessel(3) process starting from* 1*, i.e. $R_t = \|B_t\|$ where B is a 3-dimensional Brownian motion starting at some point $x_0 \in \mathbb{R}^3 \setminus \{0\}$ with $\|x\| = 1$ and $\|\cdot\|$ is the Euclidean norm. Then R hits origin with probability* 0 *and there exists a 1-dimensional Brownian motion W such that R satisfies the SDE*

$$dR_t = dW_t + \frac{dt}{R_t},$$

where $dW_t = \sum_{i=1}^3 \frac{B_i(t)}{R_t} dB_i(t)$.

For the asset pair $(\frac{1}{R_t}, 1)$ over a time horizon $[0, T]$ $(T < \infty)$, $(\frac{1}{R_t}, 1)$ is a pair of local martingales under the original probability P, as it's easy to verify $d\left(\frac{1}{R_t}\right) = -\frac{dW_t}{R_t^2}$. Since $\frac{1}{R_t}$ is locally bounded, by Theorem 2.1 b), the system $(\frac{1}{R_t}, 1)$ satisfies NFLVR.

Suppose we now take $\frac{1}{R}$ as the numéraire. Discounted by this numéraire, the asset system becomes $(1, R_t)$. We show R_t cannot be a local martingale under a probability measure P' equivalent to P. Indeed, assume such a probability measure P' exists. Define $M_t = E^P\left[\frac{dP'}{dP}|\mathcal{F}_t\right]$. By Lemma 3.1 and localization, we can conclude MR is a local martingale under P. But $\frac{1}{R}$ is the only local martingale X such that $X_0 = 1$ and such that XR is a local martingale. So $M_t = \frac{1}{R_t}$ is a strict P-local martingale, not a martingale. This contradiction shows ELMM does not exist, and hence NFLVR property is not preserved under numéraire change.

The above example gives negative answer to Question 1) and 2) for ELMM. However, for EMM, things are much better, as we shall see in Section 3.

[3] Here, EMM and ELMM are always understood as being associated with a given numéraire.

Bibliography

[1] J. P. Ansel and C. Stricker. Couverture des actifs contingents. *Annales de l'Institut Henri Poincaré (Probabilité et Statistique)*, 1994, **30**(2): 303-315. 10

[2] A. Bensoussan. On the theory of option pricing. *Acta Appl. Math.* **2**, 139-158 (1984). 11

[3] Damiano Brigo and Fabio Mercurio. *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*, Second Edition (corrected 3rd printing), Springer, 2007. 2

[4] A. S. Cherny. Vector stochastic integrals in the Fundamental Theorem of Asset Pricing. Preprint. 3

[5] R. C. Dalang, A. Morton, and W. Willinger. Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics and Stochastics Reports* **29**, pp. 185-202, (1989). 3, 5

[6] F. Delbaen. Representing martingale measures when asset prices are continuous and bounded. *Math. Finance* **2**, 107-130. 3

[7] F. Delbaen. Arbitrage possibilities in Bessel processes and their relations to local martingales. *Probab. Theor. Relat. Fields* **102** (1995), 357-366. 6

[8] F. Delbaen and W. Schachermayer. A general version of the Fundamental Theorem of Asset Pricing. *Math. Ann.* **300** (1994), 463-520. 2, 3, 4

[9] F. Delbaen and W. Schachermayer. The no-arbitrage property under a change of numéraire. *Stochatics and Stochastics Reports*, **53**, pp. 213-226, (1995). 2, 6, 10

[10] F. Delbaen and W. Schachermayer. Non-arbitrage and the Fundamental Thoerem of Asset Pricing: Sumamry of main results. *Proceedings of Symposia in Applied Mathematics*, **00**, 1997. 2, 3, 5

[11] F. Delbaen and W. Schachermayer. The Fundamental Theorem of Asset Pricing for unbounded stochastic processes. *Math. Ann.* **321**, pp. 215-250, (1998). 2

[12] F. Delbaen and W. Schachermayer. A simple counter-example to several problems in the theory of asset pricing. *Math. Finance*, **8**, pp. 1-12, (1998). 2

[13] N. El Karoui and M. C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control and Optimization* **33**, 29-66 (1995). 10

[14] H. Föllmer and Yu. Kabanov. Optional decomposition theorem and Lagrange multipliers. *Finance and Stochastics* **2** (1), 69-81 (1998). 9

[15] M. Fujisaki, G. Kallianpur, and H. Kunita. Stochastic differential equations for the nonlinear filtering problem, *Osaka J. Math* **9** (1972), 19-40. 15

[16] Hélyette Geman, Nicole El Karoui, and Jean-Charles Rochet. Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability*, Vol. 32, No. 2 (June 1995), pp. 443-458. 2

[17] Sheng-Wu He, Jia-Gang Wang and Jia-An Yan. *Semimartingale theory and stochastic calculus.* CRC, 1992.

[18] S. D. Jacka. A martingale representation result and an application to incomplete financial markets. *Math. Finance* **2**(4), 239-250 (1992). 10

[19] J. Jacod. Integrales stochastiques par rarport a une semimartingale vectorielle et changements de filtrations, Sém. Probab. XIV, LN in Math. Vol. 784, Springer, pp. 161-172, 1980. 3

[20] I. Karatzas. On the pricing of American options. *Appl. Math. Optim.* **17**, 37-60 (1988). 11

[21] D. Kramkov. Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Prob. Theory Relat. Fields* **105**, 459-479 (1996). 9, 10, 11

[22] Shun-Long Luo and Jia-An Yan. Numeraires and martingale measures in the Black-Scholes models. Preprint. 2

[23] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, 3rd Edition. Springer, 2004. 15, 16

[24] W. Schachermayer. Martingale measures for discrete time processes with infinite horizon. *Math. Finance*, **4**, pp. 25-56, (1994). 2, 3, 5

[25] A. N. Shiryaev. *Essentials of stochastic finance: facts, models, theory.* World Scientific, Singapore, 1999. 2, 3, 4, 5, 9

[26] A. N. Shiryaev and A. S. Cherny. Vector stochastic integrals and the Fundamental Theorem of Asset Pricing. *Tr. Mat. Inst. Steklova*, 2002, Volume 237, pp. 12-56. 3

[27] S. Shreve. *Stochastic calculus for finance I. The binomial asset pricing model.* Springer-Verlag, New York, 2004. 19

[28] Jian-Ming Xia and Jia-An Yan. A new look at some basic concepts in arbitrage pricing theory. *Science in China Series A: Mathematics*, Volume 46, No. 6, 2003. 2, 12, 13

[29] Jia-An Yan. A new look at the Fundamental Theorem of Asset Pricing. *J. Korean Mat. Soc.* **35** (1998), No. 3, pp. 659-673. 2, 3, 7, 8, 14

[30] Jian-An Yan. A numéraire-free and original probability based framework for financial markets. ICM 2002, Vol. Ⅲ 1-3. arxiv.org/pdf/math/0305017. 2