An Introduction to Second-Order Partial Differential Equations and the Method of Characteristics

A Foundation Course in Classical PDE Theory

Contents of the Course

Preface

A partial differential equation is an equation that an unknown function must satisfy, involving its rates of change in several independent directions simultaneously, and the challenge it poses is to find the function—or, given additional conditions, the unique function—satisfying the relation at every point of a region. Second-order partial differential equations, in which the highest derivatives appearing are of order two, are the equations from which the mathematical description of physical reality is most extensively built: the vibration of a string, the diffusion of heat, the distribution of gravitational potential—all governed by second-order equations, each behaving in a manner entirely unlike the others.

The central question this course sets out to answer is what structural feature of a second-order equation determines the behaviour of its solutions and the correct way to find them. The answer emerges from the method of characteristics. In the first-order setting, the characteristics are curves along which the equation reduces to an ordinary differential equation, and finding the solution reduces to following those curves from where the data is known. For a second-order equation, the same question—along which curves does prescribed data fail to propagate into the domain?—leads to a condition that is quadratic rather than linear in the slope of the candidate curve. A quadratic has two real solutions, one, or none, and this trichotomy is the classification of second-order linear equations into three fundamental types, arising directly and inevitably from the attempt to apply the characteristic method at second order.

The course traces that development from beginning to end, introducing every term it uses, deriving every condition it applies, and working every example to an explicit solution. The reader it is written for has met ordinary differential equations and the calculus of several variables; no prior knowledge of partial differential equations is assumed, and everything else is developed here in a careful and unhurried manner, with the aim of equipping the reader to take a second-order linear PDE, determine its type, find its characteristics, reduce it to canonical form, and construct its solution.

1. Origins and History of Partial Differential Equations

1.1. Functions of Several Variables and the Notion of a PDE

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1.2. Three Equations, Three Behaviours

By the early nineteenth century, three equations had come to dominate the mathematical description of physical reality, each arising from a different branch of physics and each demanding a different mathematical treatment.

The first was the wave equation, derived by Jean le Rond d'Alembert in 1746 to describe the transverse motion of a vibrating string. If $u(x,t)$ denotes the displacement of the string at position x and time t, d'Alembert showed that u satisfies

$$u_{tt} = c^2 u_{xx}, \tag{1}$$

where c is a constant determined by the tension and density of the string. The general solution takes the form $u(x,t) = f(x+ct) + g(x-ct)$ for two arbitrary functions f and g, expressing the motion as the superposition of two waveforms travelling in opposite directions at speed c, each maintaining its shape exactly. The solution is determined by prescribing both the initial displacement $u(x,0)$ and the initial velocity $u_t(x,0)$; two pieces of data are required, one for each arbitrary function.

The second was the heat equation, introduced by Joseph Fourier in his 1822 treatise on the conduction of heat. If $u(x,t)$ denotes the temperature at position x and time t in a conducting medium, Fourier showed that u satisfies

$$u_t = \kappa\, u_{xx}, \tag{2}$$

where $\kappa > 0$ is the thermal diffusivity of the medium. The behaviour of solutions is entirely unlike that of the wave equation. Rather than propagating waveforms, the heat equation smooths them: an initial temperature distribution, however rough or irregular, becomes infinitely smooth for all positive time, and the smoothing is irreversible. The solution is determined by prescribing only the initial temperature profile $u(x,0)$; a single piece of data suffices, and there is no analogue of the initial velocity. The direction of time is distinguished: the equation can be solved forward but not backward.

The third was the Laplace equation, arising in the work of Pierre-Simon Laplace on gravitational and electrostatic potential. If $u(x,y)$ denotes the potential at a point

(x, y) in a region free of sources, Laplace showed that u satisfies

$$u_{xx} + u_{yy} = 0. \tag{3}$$

The solutions of the Laplace equation, called harmonic functions, have no time variable at all. They are determined not by initial data in the interior of the domain but by their values on the boundary, and the solution at any interior point is the average of its values on any circle centred there. Harmonic functions satisfy a maximum principle: no interior point can be a strict maximum or minimum of the solution. The equation treats all spatial directions symmetrically, with no preferred direction of propagation.

Three equations, all second-order and linear, yet with solution theories so different that they seemed to belong to entirely separate subjects. The wave equation propagated data along two families of curves; the heat equation evolved an initial profile forward in time with irreversible smoothing; the Laplace equation was determined by boundary data and admitted no preferred directions at all. What structural feature of each equation was responsible for its characteristic behaviour? That question is the one this course sets out to answer.

1.3. Cauchy, Riemann, and the Classification

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2. General Second-Order Equations and Their Classification

2.1. The General Second-Order Linear Equation

The historical equations of Section 1 are all instances of a single general concept, and both the classification and the characteristic condition rest on having that concept stated precisely.

Definition 2.1. A *partial differential equation* (PDE) of order k in n independent variables $x = (x_1, \ldots, x_n)$ is an equation of the form

$$F\big(x,\, u(x),\, Du(x),\, D^2u(x),\, \ldots,\, D^k u(x)\big) = 0,$$

where $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is the unknown function, $Du = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ is its gradient, and $D^j u$ denotes the collection of all partial derivatives of u of order exactly j. The *order* of the PDE is the order of the highest derivative actually appearing in F.

The definition is deliberately general, encompassing equations of any order, in any number of variables, linear or nonlinear. What singles out the setting of this course is the restriction to *second-order* equations in *two* independent variables, where the unknown function u depends on two variables and the equation involves u, its first partial derivatives u_x and u_y, and its three second partial derivatives u_{xx}, u_{xy}, and u_{yy}. A *classical solution* of such an equation on an open set $\Omega \subset \mathbb{R}^2$ is a function $u \in C^2(\Omega)$, twice continuously differentiable, that satisfies the equation at every point of Ω. Throughout this course, all solutions are classical.

The most general second-order linear PDE in two independent variables x and y is

$$A(x,y)\,u_{xx}+2B(x,y)\,u_{xy}+C(x,y)\,u_{yy}+D(x,y)\,u_x+E(x,y)\,u_y+F(x,y)\,u=G(x,y), \tag{4}$$

where A, B, C, D, E, F, and G are given smooth functions on an open domain $\Omega \subset \mathbb{R}^2$, and $u : \Omega \to \mathbb{R}$ is the unknown. The factor of 2 in front of the mixed-derivative term is a convention ensuring the matrix of second-order coefficients is symmetric, placing the classification within the natural framework of real symmetric matrices.

Definition 2.2. The *principal part* of equation (4) is its collection of second-order terms,

$$A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy}.$$

The *coefficient matrix* of the principal part is

$$\mathcal{A}(x,y) = \begin{pmatrix} A(x,y) & B(x,y) \\ B(x,y) & C(x,y) \end{pmatrix}.$$

The lower-order terms in (4)—those involving u_x, u_y, u, and G—affect the quantitative form of solutions but not the characteristic geometry of the equation. The characteristics, and therefore the classification, are determined entirely by the principal-part coefficients A, B, C, as will become clear in the derivation of the characteristic condition that follows.

2.2. Hypersurfaces, Hyperplanes, and the Cauchy Problem

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2.3. The Characteristic Condition and Its Geometric Meaning

The characteristic question asks along which curves in Ω the Cauchy data fails to determine the second derivatives of u. A curve along which this failure occurs is a *characteristic curve* of the equation, and finding these curves requires analysing what the Cauchy problem demands of the equation.

Let $\Gamma = \{\phi(x, y) = 0\}$ be a smooth curve, and suppose u and $\partial_\nu u$ are known on Γ. Knowing u on Γ means knowing all tangential derivatives of u along Γ. Both u_x and u_y are therefore known on Γ: their tangential components come from differentiating u_0 along the curve, and their normal component comes from $u_1 = \partial_\nu u$. What cannot be recovered from the Cauchy data alone is the second normal derivative u_{nn}.

Equation (4), evaluated on Γ, together with conditions obtained from the known first-derivative data, produces a system of linear relations for u_{xx}, u_{xy}, u_{yy}. The Cauchy data fails to determine the second derivatives precisely when this system fails to have a unique solution—that is, when its determinant vanishes. Computing this determinant and simplifying, the condition reduces to

$$A\,\phi_x^2 + 2B\,\phi_x\,\phi_y + C\,\phi_y^2 = 0. \tag{5}$$

This is the *characteristic condition*. A curve $\phi(x, y) = 0$ is a characteristic curve of (4) if and only if ϕ satisfies (5) at every point of the curve. Only the principal-part coefficients A, B, C enter the condition, because the lower-order terms contribute only known quantities to the linear system for the second derivatives and do not alter its coefficient matrix.

The complementary condition—that the initial curve is not characteristic—is what makes the Cauchy problem solvable.

Definition 2.3. Let $\Gamma = \{\phi(x, y) = 0\}$ be a smooth curve. The curve Γ is *noncharacteristic* for equation (4) at a point $(x_0, y_0) \in \Gamma$ if

$$A\phi_x^2 + 2B\phi_x\phi_y + C\phi_y^2 \neq 0 \quad \text{at } (x_0, y_0).$$

This condition is equivalently written as $\mathcal{A}(x_0, y_0)\,\nabla\phi(x_0, y_0) \cdot \nabla\phi(x_0, y_0) \neq 0$. If Γ is noncharacteristic at every one of its points, it is called a *noncharacteristic initial curve*.

When the initial curve is a coordinate hyperplane such as $\{x = 0\}$, $\{y = 0\}$, or $\{t = 0\}$, the noncharacteristic condition reduces to the corresponding coefficient among u_{xx}, u_{yy}, and u_{tt} in the principal part being nonzero on that curve.

The geometric content of this condition is that the characteristics are transverse to Γ, crossing the initial curve at a finite angle rather than running along it. When the noncharacteristic condition holds, the characteristics pass through Γ and carry information from the initial data into the domain. When it fails—when Γ is itself a characteristic—the characteristics run along the initial curve, the data prescribed on Γ

is redundant in one direction and insufficient in the other, and the Cauchy problem is either inconsistent or underdetermined. The noncharacteristic condition is the single geometric requirement on which the solvability of the Cauchy problem rests.

2.4. The Discriminant and the Classification

The characteristic condition (5) can be rewritten as a quadratic in the slope $\lambda = dy/dx$ of the characteristic curve. Assuming $\phi_y \neq 0$ and setting $\lambda = -\phi_x/\phi_y$ (the slope of the level curve $\phi = 0$), condition (5) becomes

$$A\,\lambda^2 - 2B\,\lambda + C = 0. \tag{6}$$

This is a quadratic in λ whose discriminant is

$$\Delta(x, y) = B(x, y)^2 - A(x, y)\,C(x, y). \tag{7}$$

The number of real solutions of (6) is determined entirely by the sign of Δ: two distinct real characteristic slopes when $\Delta > 0$, exactly one when $\Delta = 0$, and none when $\Delta < 0$. This trichotomy is the classification.

Definition 2.4. The equation (4) is called, at a point $(x_0, y_0) \in \Omega$: *hyperbolic* if $\Delta(x_0, y_0) > 0$, *parabolic* if $\Delta(x_0, y_0) = 0$, *elliptic* if $\Delta(x_0, y_0) < 0$. If the type is the same at every point of Ω, the equation is *uniformly* of that type on Ω.

The names are borrowed from the classification of conic sections, which share the same discriminant. The connection to the coefficient matrix $\mathcal{A}$ is direct: $\det \mathcal{A} = AC - B^2 = -\Delta$. The equation is hyperbolic when $\mathcal{A}$ has eigenvalues of opposite sign (indefinite), parabolic when $\mathcal{A}$ is singular, and elliptic when $\mathcal{A}$ is definite.

The wave equation (1) has $A = 1$, $B = 0$, $C = -c^2$, giving $\Delta = c^2 > 0$: hyperbolic. The heat equation (2), viewed in the (x, t)-plane with $A = \kappa$, $B = 0$, $C = 0$ (no u_{tt} term), gives $\Delta = 0$: parabolic. The Laplace equation (3) has $A = 1$, $B = 0$, $C = 1$, giving $\Delta = -1 < 0$: elliptic. The three historical equations are the canonical representatives of the three types.

When $\Delta > 0$, the two real roots of (6),

$$\lambda_{\pm} = \frac{B \pm \sqrt{\Delta}}{A} \qquad (A \neq 0), \tag{8}$$

give two families of ODEs $dy/dx = \lambda_{\pm}$, each defining a family of characteristic curves filling the domain. When $\Delta = 0$, the two roots coincide and there is a single family. When $\Delta < 0$, there are no real roots and no real characteristic curves, so the equation has no preferred directions of propagation in the plane. These three cases are what the method of characteristics produces when applied to the general second-order equation, and they govern everything that follows.

3. Canonical Forms and How to Reach Them

3.1. Why Canonical Forms Are the Goal

The classification identifies the type, but what the type implies for finding solutions is the deeper question. The bridge between the algebraic fact of the discriminant's sign and the practical business of solving is the *canonical form*, a standard form that every equation of a given type can be transformed into by a change of independent variables built from the characteristic curves, making the equation's structure as simple as possible.

The canonical form is what the method of characteristics produces when applied to a second-order equation: in the hyperbolic case, the mixed-derivative form $w_{\xi\eta} = \Phi$; in the parabolic case, the single second-derivative form $w_{\eta\eta} = \Phi$; in the elliptic case, the Laplacian form $w_{\xi\xi} + w_{\eta\eta} = \Phi$. Once an equation is in canonical form, its solution theory is transparent. The procedure for reaching canonical form is the same in all three cases: find the characteristic curves, use them (or their complex analogues in the elliptic case) to define new coordinates, introduce a new unknown w in those coordinates, and compute the transformed equation satisfied by w.

The legitimacy of this procedure rests on one geometric condition. Given a smooth change of variables $(x, y) \mapsto (\xi(x, y), \eta(x, y))$, define $w(\xi, \eta)$ by the relation

$$w\big(\xi(x, y),\, \eta(x, y)\big) = u(x, y).$$

This definition makes w and u carry identical information, provided the map $(x, y) \mapsto (\xi, \eta)$ is locally invertible: given w, the function u is recovered by composing with the inverse map. The condition for local invertibility is that the *Jacobian*

$$J(x, y) = \xi_x(x, y)\, \eta_y(x, y) - \xi_y(x, y)\, \eta_x(x, y)$$

is nonzero throughout the domain. This is the *transversality condition* on the coordinate change: the two families of level curves $\xi = \text{const}$ and $\eta = \text{const}$ must cross one another transversally, never running parallel. When the transversality condition holds, u and w are in one-to-one correspondence, and solving the canonical equation for w is exactly the same as solving the original equation for u.

Under such a change of variables, the principal part $Au_{xx} + 2Bu_{xy} + Cu_{yy}$ transforms into $\tilde{A}w_{\xi\xi} + 2\tilde{B}w_{\xi\eta} + \tilde{C}w_{\eta\eta}$, where the new coefficients are given by

$$\begin{aligned}
\tilde{A} &= A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2, \\
\tilde{B} &= A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y, \\
\tilde{C} &= A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2.
\end{aligned}$$

Two facts about these formulas are central to everything that follows, both being consequences of the discriminant identity $\tilde{B}^2 - \tilde{A}\tilde{C} = J^2(B^2 - AC)$, which can be verified directly from the expressions above. First, the sign of the discriminant is preserved: $\tilde{\Delta} = J^2\Delta$, so the type of the equation is an invariant unchanged by any smooth invertible change of variables. Second, the expressions for $\tilde{A}$ and $\tilde{C}$ are exactly the characteristic condition (5) evaluated at ξ and η respectively, so if ξ satisfies the characteristic condition then $\tilde{A} = 0$, and if η satisfies it then $\tilde{C} = 0$. These two facts are what make the reductions below work, their proofs being direct computations from the formulas above, and the worked examples that follow make the mechanism visible in each case.

3.2. Reduction in the Hyperbolic, Parabolic, and Elliptic Cases

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3.3. Worked Examples of Reduction to Canonical Form

Example 3.1. Consider $u_{xx} - 4u_{yy} = 0$.

Here $A = 1$, $B = 0$, $C = -4$, and $\Delta = 0 - (1)(-4) = 4 > 0$: hyperbolic. The characteristic slopes satisfy $\lambda^2 - 4 = 0$, giving $\lambda = \pm 2$, so the two characteristic ODEs are $dy/dx = 2$ and $dy/dx = -2$. Integrating, the two characteristic families are $y - 2x = c_1$ and $y + 2x = c_2$.

Set $\xi = y - 2x$ and $\eta = y + 2x$, and define $w(\xi, \eta)$ by $w(\xi(x, y), \eta(x, y)) = u(x, y)$. The Jacobian is $J = \xi_x\eta_y - \xi_y\eta_x = (-2)(1) - (1)(2) = -4 \neq 0$, so the transversality condition holds and the change of variables is legitimate. We have $\xi_x = -2$, $\xi_y = 1$, $\eta_x = 2$, $\eta_y = 1$. By the chain rule:

$$u_x = w_\xi\,\xi_x + w_\eta\,\eta_x = -2w_\xi + 2w_\eta, \qquad u_y = w_\xi\,\xi_y + w_\eta\,\eta_y = w_\xi + w_\eta,$$

$$u_{xx} = \frac{\partial}{\partial x}(-2w_\xi + 2w_\eta) = -2(w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x) + 2(w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x) = 4w_{\xi\xi} - 8w_{\xi\eta} + 4w_{\eta\eta},$$

$$u_{yy} = \frac{\partial}{\partial y}(w_\xi + w_\eta) = (w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y) + (w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y) = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}.$$

Substituting into the equation:

$$u_{xx} - 4u_{yy} = (4w_{\xi\xi} - 8w_{\xi\eta} + 4w_{\eta\eta}) - 4(w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}) = -16\,w_{\xi\eta}.$$

Setting $-16w_{\xi\eta} = 0$ gives $w_{\xi\eta} = 0$, the canonical form (**??**) with $\Phi = 0$. The general solution for w is $w(\xi, \eta) = f(\xi) + g(\eta)$ for arbitrary smooth f and g, and returning to the original variables:

$$u(x, y) = f(y - 2x) + g(y + 2x).$$

Example 3.2. Consider $u_{xx} - 2u_{xy} + u_{yy} + u_x = 0$. Here $A = 1$, $B = -1$, $C = 1$, and $\Delta = (-1)^2 - (1)(1) = 0$: parabolic. The characteristic slope satisfies $\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$, giving the single characteristic ODE $dy/dx = -1$. Integrating, the characteristic family is $y + x = c_1$.

Set $\xi = y + x$ and choose $\eta = x$ as the transverse coordinate. Define $w(\xi, \eta)$ by $w(\xi(x, y), \eta(x, y)) = u(x, y)$. Then $\xi_x = 1$, $\xi_y = 1$, $\eta_x = 1$, $\eta_y = 0$, and the Jacobian is $J = \xi_x\eta_y - \xi_y\eta_x = (1)(0) - (1)(1) = -1 \neq 0$: the transversality condition holds. By the chain rule:

$$u_x = w_\xi\, \xi_x + w_\eta\, \eta_x = w_\xi + w_\eta, \qquad u_y = w_\xi\, \xi_y + w_\eta\, \eta_y = w_\xi,$$

$$u_{xx} = \frac{\partial}{\partial x}(w_\xi + w_\eta) = (w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x) + (w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x) = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta},$$

$$u_{yy} = \frac{\partial}{\partial y}(w_\xi) = w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y = w_{\xi\xi},$$

$$u_{xy} = \frac{\partial}{\partial x}(w_\xi) = w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x = w_{\xi\xi} + w_{\xi\eta}.$$

Substituting into the equation and collecting terms:

$$\begin{aligned} u_{xx} - 2u_{xy} + u_{yy} + u_x &= (w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}) - 2(w_{\xi\xi} + w_{\xi\eta}) + w_{\xi\xi} + (w_\xi + w_\eta) \\ &= w_{\eta\eta} + w_\xi + w_\eta. \end{aligned}$$

Setting this to zero gives the canonical form (**??**):

$$w_{\eta\eta} + w_\xi + w_\eta = 0,$$

second-order in η and first-order in ξ, as the parabolic type requires.

Example 3.3. Consider $u_{xx} + 2u_{xy} + 5u_{yy} = 0$. Here $A = 1$, $B = 1$, $C = 5$, and $\Delta = (1)^2 - (1)(5) = -4 < 0$: elliptic. The complex characteristic slopes satisfy $\lambda^2 - 2\lambda + 5 = 0$, giving $\lambda = 1 \pm 2i$: complex roots. The complex characteristic functions are obtained by integrating $dy/dx = 1 \pm 2i$, giving $\zeta = y - (1 + 2i)x$. Taking the real and imaginary parts: $\xi = y - x$ and $\eta = -2x$.

Define $w(\xi, \eta)$ by $w(\xi(x, y), \eta(x, y)) = u(x, y)$. Then $\xi_x = -1$, $\xi_y = 1$, $\eta_x = -2$, $\eta_y = 0$, and the Jacobian is $J = \xi_x\eta_y - \xi_y\eta_x = (-1)(0) - (1)(-2) = 2 \neq 0$: the transversality condition holds. By the chain rule:

$$u_x = w_\xi\, \xi_x + w_\eta\, \eta_x = -w_\xi - 2w_\eta, \qquad u_y = w_\xi\, \xi_y + w_\eta\, \eta_y = w_\xi,$$

$$u_{xx} = \frac{\partial}{\partial x}(-w_\xi - 2w_\eta) = -(w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x) - 2(w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x) = w_{\xi\xi} + 4w_{\xi\eta} + 4w_{\eta\eta},$$

$$u_{yy} = \frac{\partial}{\partial y}(w_\xi) = w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y = w_{\xi\xi},$$

$$u_{xy} = \frac{\partial}{\partial x}(w_\xi) = w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x = -w_{\xi\xi} - 2w_{\xi\eta}.$$

Substituting into the equation:

$$\begin{aligned} u_{xx} + 2u_{xy} + 5u_{yy} &= (w_{\xi\xi} + 4w_{\xi\eta} + 4w_{\eta\eta}) + 2(-w_{\xi\xi} - 2w_{\xi\eta}) + 5w_{\xi\xi} \\ &= (1 - 2 + 5)w_{\xi\xi} + (4 - 4)w_{\xi\eta} + 4w_{\eta\eta} = 4w_{\xi\xi} + 4w_{\eta\eta}. \end{aligned}$$

Dividing by 4 gives the canonical form (**??**):

$$w_{\xi\xi} + w_{\eta\eta} = 0,$$

the Laplace equation in the new coordinates.

Exercise 3.1. Consider $4u_{xx} + 12u_{xy} + 9u_{yy} - u_y = 0$. Compute Δ, determine the type, find the characteristic family, and choose a transverse coordinate η. Then define $w(\xi, \eta)$ by $w(\xi(x, y), \eta(x, y)) = u(x, y)$, verify that the transversality condition holds, and reduce the equation to canonical form by direct substitution using the chain rule.

In each case, the canonical form produced reveals the equation's structure in the simplest possible terms. In the hyperbolic case, $w_{\xi\eta} = 0$ has an immediately visible general solution; in the parabolic case, the equation is second-order in one direction and first-order in the other; in the elliptic case, the two directions appear symmetrically. These structural features are what the classification predicts, and they govern how the equation is solved.

4. Hyperbolic Equations and the Method of Characteristics

4.1. Two Characteristic Families and the Cauchy Problem

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4.2. Worked Examples of Solving Hyperbolic Equations

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5. Parabolic Equations and the Structure of Their Solutions

5.1. One Characteristic Family, One Free Coordinate

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5.2. Worked Examples of Solving Parabolic Equations

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6. Elliptic Equations and the Structure of Their Solutions

6.1. Complex Characteristics and the Laplacian Canonical Form

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6.2. Worked Examples of Solving Elliptic Equations

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7. Closing Remarks on the Method of Characteristics

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