Relativity & Black Holes

Yuval Rosen

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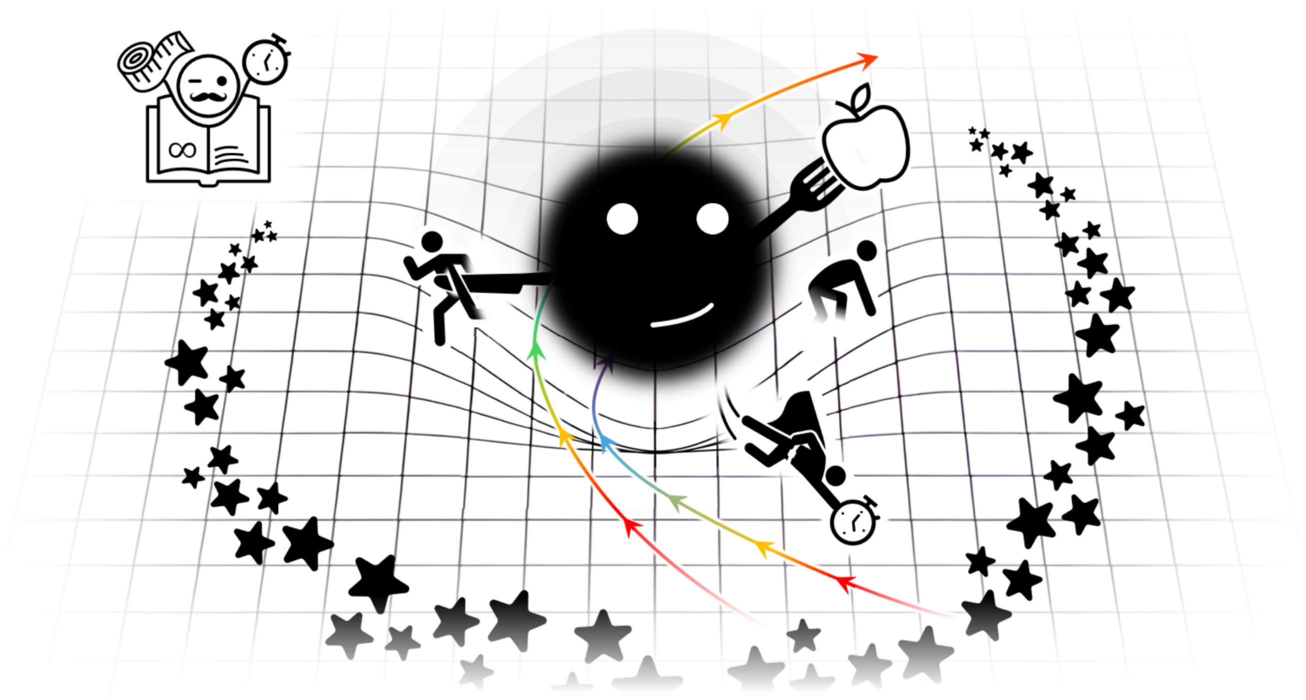
Preface

In this document, I discuss a portion of my studies on special relativity and gravity, focusing on its greatest natural phenomenon - black holes' effect on time and space. The goal, evidently, is not to equip the reader with a proficient understanding of general relativity, but to expose them to the counter-intuitive thinking process uniquely attributed to physics. To establish a general comprehension of the subject, mathematical tools are also included, made as accessible as possible. Again, all discussions are highly simplified, though they introduce the right amount of intuition.

Much of the material presented in this work is based on the concepts and explanations found in the following excellent books:

- *Classical Electrodynamics* by John David Jackson [1] (special relativity)
- *Exploring Black Holes: Introduction to General Relativity* by Edwin F. Taylor and John Archibald Wheeler [2] (general relativity of black holes)



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1 Special Relativity

1.1 Basic Observables

From special relativity (SR), we now understand that time and space are interconnected and influence each other when objects move at high speeds. How high? When close enough to the speed of light, which is postulated to be constant in every frame of reference (every observer measures the speed of light to be the same, regardless of the motion of the source or the observer). Under this assumption, it is clear that classical physics requires adjustments to accommodate this principle, as space and time must adapt their definitions to maintain a constant speed across various circumstances.

So, beginning the discussion on space-time curvature and its influence on physical observables, one must first ask: what do we mean by *length*? This question may seem a little odd at first, but already here, we take the opportunity to build a stronger (or more physically grounded) intuition about something that has always felt universally agreed upon. In fact, we will see that length is not always a universal measure - it can depend on the observer's position, and in SR, it depends on the observer's velocity.

Note that we distinguish between "speed" and "velocity" when defining a rate of motion. These are not the same: Speed provides no information about the direction of motion, as shown on a speedometer, while velocity does! This is because velocity is a *vector*, meaning it has both magnitude and direction (like position, momentum, etc.), while speed is a *scalar*, meaning it has only magnitude (like time, mass, etc.). These are purely geometrical definitions, while advanced topics in physics extend these concepts into a much broader framework (which we do not explore here).

We define length as follows:

"Length is a quantity with a dimension of distance."

This definition avoids the conflict we associated with the global nature of length, as it is conveniently general as needed. We also provide the common definition of time for completeness:

"Time is the continuous progression of events."

These definitions are intentionally flavored to prepare for the incoming conclusions.

1.2 Minkowski Metric

Now, diving into what space-time curvature means, we introduce the Minkowski Metric:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{1.1}$$

Before simplifying the terror that was written above, take a look at 3.1 for a (really) short reminder on basic math notation, if needed. The variable s stands for the length (not quite the way we know it - as it is now time-considered) that is measured by the *observer*, and x, y, z are the coordinates in each axis that we live in (we see the world in 3D, of course), t stands for the time. Both the spatial x, y, z and temporal t are measured in an inertial frame of reference. What does it mean? Inertial frames of reference remains at rest or in constant velocity relative to another frame unless acted upon by external forces. Physically, we may define an inertial frame of reference as such that no corrections (due to acceleration) are needed when transforming to it.

By frame of reference we equivalently argue, who measures? (from which position and what velocity). Now, we explain the structure of that Metric - some may identify the similarity to Pythagorean theorem (defined in 3.2 as a reminder). In fact, it is based on the same principles - Measurements are made by constructing right triangles in space and calculating the hypotenuse. However, here we apply this method on *position-time* space, or more common, *spacetime*. This means that now, time constitutes an essential role in measuring distances! Here, ds is the hypotenuse, which stands for a length unit in spacetime. The unexpected minus sign on all spatial coordinates will become clear later on[1]. c is the speed of light.

The only symbol left to describe is the prefix $'d'$ in front of each term. It is there to notify that the equation holds for *infinitesimal translations* of every variable - only for small translations of each coordinate. The term "small" is a bit unclear, but it must stay so for now. This in fact is the differential operator, not a multiplied variable! (Some might have seen it in derivatives or integrals. But if not, that's okay!). To consider a finite path, integrals come into play. Note that the square operand acts after 'd', namely $dx^2 = (dx)^2$ and not $d(x)^2$.

1.3 The Power of Symmetry - Lorentz Transformation

Firstly, we turn to provide some key insights on some of the processes developing new physics, as popular today. Modern research in theoretical physics engages in investigating *symmetries*. These are transformations that, when applied, do not change the physical properties of the system. The following example may clarify this proposition:

> Imagine standing in the middle of a vast, barren desert under a star-studded night sky. The sandy surface stretches endlessly, identical in every direction, offering no landmarks to guide your way. As you walk in a straight line, nothing seems to change - the scenery remains unwavering. The desert, or more formally its 3D subspace, could be described as *invariant under translations*. But now, picture turning your gaze upward and rotating your perspective. The stars above shift their positions, painting a different view with every turn. In this way, the desert's subspace reveals itself to be *variant under rotations*.

There are many applications for utilizing symmetries, like simplifying calculations, but the most profound use is preserved for finding new physics. In fact, Noether's theorem states that every continuous symmetry of the *action* of a physical system with conservative forces has a corresponding conservation law. Simplified, an action is a defined functional containing information on the dynamics of the system (what forces act on it). For instance, one famous law that can be derived from this theorem is *conservation of energy*. But what physical observable yields a symmetry of the action (namely, the system for simplicity) to output this law? The answer to that question is formally accompanied with intricate calculations, but they won't be given here. This part is where one may ponder a little before exposed to the solution (avoiding math does not supply an excuse to suppress creative thinking!).

[1] It distinguishes spacetime intervals from Euclidean distances and ensures that time and space behave differently in relativity. We could also put a " $-$ " on the time term and " $+$ " on all spatial coordinates instead, but this is only a matter of convention. Here, we use what is called the *mostly minus* convention.