A Clear Path to General Relativity

An ebook for those who are committed to understanding General Relativity

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Chapter 1

Vector Spaces, Tangent Spaces, and General Relativity

The aim of this chapter is to bridge the gap between the abstract mathematics of vector spaces and tangent spaces and their physical meaning in general relativity. In developing general relativity, Albert Einstein was guided by deep physical insights, particularly the equivalence principle and the idea that gravity can be locally transformed away in a freely falling frame. These insights were later expressed precisely using the language of differential geometry. In this chapter we explain how the concepts of vector spaces and tangent spaces (T_pM) provide the mathematical framework that captures the physical ideas underlying general relativity.
We assume that the reader has already encountered abstract vector spaces, either in linear algebra or in an introduction to general relativity[1]. This chapter assumes that the reader is familiar with the abstract concept of vector spaces, the tangent space at a point on a manifold (T_pM), and the notion of a manifold. We use the Einstein summation convention where applicable. Throughout this chapter, the terms flat spacetime and spacetime of special relativity will be used interchangeably.

1.1 Vector spaces, tangent spaces and the mathematical model of General Relativity

Einstein Equivalence Principle (EEP): In a box that is freely falling in an idealized homogeneous gravitational field [2], all experiments are indistinguishable from experiments conducted in an inertial system at rest.

[1]We will not cover the mathematical formulation of abstract vector spaces and tangent spaces here, assuming the reader is already familiar with these concepts. For a refresher on vector spaces, refer to [**schutz1985first**] A First Course in General Relativity by Schutz (Appendix A: Summary of Linear Algebra) or [**Schuller3**] Lecture 3: Multilinear Algebra by Prof. Dr. Frederic P. Schuller, available on YouTube (International Winter School on Gravity and Light, 2015). For an introduction to tangent spaces, see [**Schuller5**] Lecture 5: Tangent Spaces by Prof. Dr. Schuller. These topics are also covered in many standard textbooks on linear algebra and general relativity.

[2]Pure homogeneous gravitational field doesn't exist in nature due to the presence of tidal forces. A tidal force is the difference in strength of gravity between two points.

The EEP states that the laws of physics are the same in a freely falling box in homogeneous "gravitational" field[3] as they are in an inertial frame. An inertial frame can be imagined as a frame located in deep space far from any gravitational field or more simply, in the absence of gravity. According to EEP, inside an elevator falling freely towards the ground, you will feel weightless, just as you would in space. As a consequence of the equivalence principle, it is impossible for any physical measurement to tell uniform "gravitational" field from a reference frame undergoing constant linear acceleration. In other words, when "gravity" acts with a uniform strength throughout a region , one can always choose a an accelerating coordinate system in which the effects of both gravity and acceleration cancel out completely at every point, making the region appear entirely free of gravity. Put more simply: in a perfectly uniform "gravitational" field, physics behaves as if gravity were absent.

In the real world, the gravitational field is not homogeneous due to tidal forces, which arise from variations in the field across a finite region of space. However, within a sufficiently small region of spacetime, the gravitational field can be approximated as uniform. In such regions (for example, inside a small freely falling box), the effects of gravity can be effectively eliminated by adopting a free-falling reference frame.

The essential point is not the motion of the observer or the use of a freely falling frame, but the uniformity of the “gravitational” field within a sufficiently small region. When the field is uniform, one can always choose an accelerating coordinate system in which the effects of the “gravitational” field disappear locally. A freely falling frame provides a natural physical realization of such a coordinate system.
It is important to note that this is only true within a sufficiently small region of spacetime. The non-uniformity of real gravitational fields cannot be removed by any choice of frame. These tidal forces are the true signature of gravity and form the physical foundation of General Relativity.

> In sufficiently small regions of spacetime, the effects of gravity can be approximated as negligible, allowing the spacetime to be described by the flat Minkowski spacetime of special relativity.

This physical insight, that spacetime is locally flat, is precisely captured by the mathematical concept of a tangent space. **The tangent space T_pM on a smooth curved manifold translates this idea into a precise mathematical model**. At each point, P on a smooth manifold a four dimensions tangent space is defined. Locally, at any given point on the manifold, the tangent space is a flat approximation of the manifold at that point.
For an ant living on the surface of a sphere, any tiny patch it stands on will look flat. That tiny flat patch is the ant’s ’tangent space.’ The ant can use simple, flat geometry (like a plane) to describe its immediate surroundings, even though the whole world (the manifold) is curved. Just as the ant uses a flat plane locally, observers in

[3]The word “gravitational” appears in quotation marks because, in general relativity, true gravitation is always accompanied by tidal forces. A perfectly uniform “gravitational” field is an idealized theoretical construct used purely for pedagogical purposes. It should be understood as such.

spacetime use tangent space to describe local physics.

There is an analogy between gravity in spacetime and curved geometry. Imagine a 2D surface, a plane (flat) versus a sphere (curved). On a plane, parallel lines never meet, and the geometry is Euclidean. On a sphere, “parallel” lines converge at the poles, indicating curvature. Similarly, in flat Minkowski spacetime, when adding Tidal forces, they cause relative accelerations between nearby particles in a gravitational field, corresponding to the curvature of spacetime. In a spacetime with no gravity, "freely falling" particles are inertial frames that move along straight lines and maintain constant relative distances. In nature, where the gravitational field is not uniform, test particles released along parallel lines will approach each other due to tidal force.
In General Relativity, spacetime curvature is intrinsic, meaning it can be measured by observers within spacetime without reference to a higher dimension. Similarly on a sphere, you can detect curvature by measuring angles of triangles with no need to move to a higher dimension (3D).

This analogy led Einstein to propose the geometric model of curvature described by differential geometry as the mathematical basis of General Relativity. Of course, the geometric analogy is not a 2-dimensional manifold (surface) but it must be to a 4-dimentions ($4D$) manifold. General Relativity modeled spacetime as $4D$ curved and smooth manifold. To each point on the manifold a $4D$ vector space is attached that forms a $4D$ tangent space at this point.

The vectors, covectors and tensors in this tangent space can be interpreted in different ways depending on the context. We will now refer to the interpretations in physics of vectors in this tangent space[4].

1.2 Tangent Vectors and Their Physical Interpretations

On a smooth manifold, each point is equipped with a vector space called the tangent space, which contains all possible tangent vectors at that point. These vectors can be interpreted in various ways, particularly in physics, depending on the context of the manifold and the physical system under study. This section explores these interpretations, focusing on their roles in physics.

1.2.1 Tangent Vectors: Displacements and Derivatives

The tangent space at a point on a manifold provides a local, linear approximation of the manifold’s geometry. Tangent vectors within this space can be understood in two complementary ways: as infinitesimal displacements or as operators that compute directional derivatives.

[4] The interpretations of covectors in the cotangent space (T_p^*M) is provided in chapter 2 section 2.2.1

1.2.1.1 Tangent Vectors as Displacements vectors

Tangent vectors can be visualized as small shifts in position along the manifold's surface.

- **Conceptual Insight**: A manifold generalizes the notion of a curved space, such as a sphere or spacetime. At any point, the tangent space acts like a flat plane touching the manifold, capturing possible directions of motion. A tangent vector represents a tiny displacement from that point in a specific direction.

- **Coordinate Form**: In local coordinates $(x^1, \ldots, x^n)$, a tangent vector at a point p is written as:

$$v = v^i \frac{\partial}{\partial x^i}\Big|_p,$$

  where v^i are the vector's components, and $\frac{\partial}{\partial x^i}\big|_p$ form the basis of the tangent space, reflecting changes along coordinate directions.

1.2.1.2 Tangent Vectors as Directional Derivatives

Alternatively, tangent vectors function as operators that measure how scalar functions change along specific directions.

- **Operational Role**: For a smooth function f defined on the manifold, a tangent vector $v = v^i \frac{\partial}{\partial x^i}\big|_p$ computes the directional derivative of f at p:

$$v(f) = v^i \frac{\partial f}{\partial x^i}\Big|_p.$$

  This perspective views tangent vectors as tools for probing the rate of change of functions.

- **Connection to Displacements**: The derivative interpretation stems from the displacement view, as the vector's direction dictates the path along which the function's change is measured.

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