Analysis on Manifolds

A Solution Manual of Selected Exercise Problems

Analysis on Manifolds
A Solution Manual of Selected Exercise Problems

Yan Zeng

Version 1.0, last revised on 2014-06-25.

This book is for sale at https://leanpub.com/munkres_solman

This version was published on 2022-08-20

This is a Leanpub book. Leanpub empowers authors and publishers with the Lean Publishing process. Lean Publishing is the act of publishing an in-progress ebook using lightweight tools and many iterations to get reader feedback, pivot until you have the right book and build traction once you do.

Abstract

This is a solution manual of selected exercise problems from *Analysis on Manifolds*, by James R. Munkres [**?**]. If you find any typos/errors, please email me at quantsummaries@gmail.com.

Contents

Chapter 1

The Algebra and Topology of $\mathbb{R}^n$

1 Review of Linear Algebra

A good textbook on linear algebra from the viewpoint of finite-dimensional spaces is Lax [?]. In the below, we make connections between the results presented in this textbook and that reference.

Theorem 1.1 (page 2) corresponds to Lax [?, page 5], Chapter 1, Lemma 1.

Theorem 1.2 (page 3) corresponds to Lax [?, page 6], Chapter 1, Theorem 4.

Theorem 1.5 (page 7) corresponds to Lax [?, page 37], Chapter 4, Theorem 2 and the paragraph below Theorem 2.

▶ 2. (Theorem 1.3, page 5)

Proof. Recall the matrix norm $|\cdot|$ is defined as the maximum of the absolute values of matrix entries (page 5).

For any $i = 1, \cdots n, j = 1, \cdots, p$, we have

$$\left|\sum_{k=1}^{m} a_{ik}b_{kj}\right| \le \sum_{k=1}^{m} |a_{ik}b_{kj}| \le |A| \sum_{k=1}^{m} |b_{kj}| \le m|A||B|.$$

Therefore,

$$|A \cdot B| = \max\left\{\left|\sum_{k=1}^{m} a_{ik}b_{kj}\right|; i = 1, \cdots n, j = 1, \cdots, p\right\} \le m|A||B|.$$

□

▶ 3.

Proof. Suppose $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^2$ having the property that $|x| = \langle x, x\rangle^{\frac{1}{2}}$, where $|x|$ is the sup norm. By the equality $\langle x, y\rangle = \frac{1}{4}(|x+y|^2 - |x-y|^2)$, we have

$$\begin{aligned}
\langle e_1, e_1 + e_2\rangle &= \frac{1}{4}(|2e_1 + e_2|^2 - |e_2|^2) = \frac{1}{4}(4-1) = \frac{3}{4},\\
\langle e_1, e_2\rangle &= \frac{1}{4}(|e_1 + e_2|^2 - |e_1 - e_2|^2) = \frac{1}{4}(1-1) = 0,\\
\langle e_1, e_1\rangle &= |e_1|^2 = 1.
\end{aligned}$$

So $\langle e_1, e_1 + e_2\rangle \ne \langle e_1, e_2\rangle + \langle e_1, e_1\rangle$, which implies $\langle \cdot, \cdot \rangle$ cannot be an inner product. Therefore, our assumption is not true and the sup norm on $\mathbb{R}^2$ is not derived from an inner product on $\mathbb{R}^2$. □

2 Matrix Inversion and Determinants

▶ 1.

(a)

Proof. Suppose $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ is a left inverse for A. Then $BA = \begin{bmatrix} b_{11}+b_{12} & 2b_{11}-b_{12}+b_{13} \\ b_{21}+b_{22} & 2b_{21}-b_{12}+b_{23} \end{bmatrix}$. So $BA = I_2$ if and only if

$$\begin{cases} b_{11}+b_{12}=1 \\ b_{21}+b_{22}=0 \\ 2b_{11}-b_{12}+b_{13}=0 \\ 2b_{21}-b_{22}+b_{23}=1. \end{cases}$$

Plug $-b_{12}=b_{11}-1$ and $-b_{22}=b_{21}$ into the las two equations, we have

$$\begin{cases} 3b_{11}+b_{13}=1 \\ 3b_{21}+b_{23}=1. \end{cases}$$

So we can have the following two different left inverses for A: $B_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & -2 \end{bmatrix}$. □

(b)

Proof. By Theorem 2.2, A has no right inverse. □

▶ 2.

(a)

Proof. By Theorem 1.5, $n \geq m$ and among the n row vectors of A, there are exactly m of them are linearly independent. By applying elementary row operations to A, we can reduce A to the echelon form $\begin{bmatrix} I_m \\ 0 \end{bmatrix}$. So we can find a matrix D that is a product of elementary matrices such that $D \cdot A = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$. □

(b)

Proof. If $\text{rank}A = m$, by part (a) there exists a matrix D that is a product of elementary matrices such that

$$DA = \begin{bmatrix} I_m \\ 0 \end{bmatrix}.$$

Let $B = [I_m, 0]D$, then $BA = I_m$, i.e. B is a left inverse of A. Conversely, if B is a left inverse of A, it is easy to see that A as a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ is injective. This implies the column vectors of A are linearly independent, i.e. $\text{rank}A = m$. □

(c)

Proof. A has a right inverse if and only if A^{tr} has a left inverse. By part (b), this implies $\text{rank}A = \text{rank}A^{tr} = n$. □

▶ 4. (a)

Proof. Suppose $(D_k)_{k=1}^K$ is a sequence of elementary matrices such that $D_K \cdots D_2 D_1 A = I_n$. Note $D_K \cdots D_2 D_1 A = D_K \cdots D_2 D_1 I_n A$, we can conclude $A^{-1} = D_K \cdots D_2 D_1 I_n$. □

▶ 5.

Proof. $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{d-bc}$ by Theorem 2.14. □

3 Review of Topology in $\mathbb{R}^n$

▶ 2.

Proof. $X = \mathbb{R}$, $Y = (0, 1]$, and $A = Y$. □

▶ 6.

Proof. For any closed subset C of Y, $f^{-1}(C) = [f^{-1}(C) \cap A] \cup [f^{-1}(C) \cap B]$. Since $f^{-1}(C) \cap A$ is a closed subset of A, there must be a closed subset D_1 of X such that $f^{-1}(C) \cap A = D_1 \cap A$. Similarly, there is a closed subset D_2 of X such that $f^{-1}(C) \cap B = D_2 \cap B$. So $f^{-1}(C) = [D_1 \cap A] \cup [D_2 \cap B]$. A and B are closed in X, so $D_1 \cap A$, $D_2 \cap B$ and $[D_1 \cap A] \cup [D_2 \cap B]$ are all closed in X. This shows f is continuous. □

▶ 7.
(a)

Proof. Take $f(x) \equiv y_0$ and let g be such that $g(y_0) \neq z_0$ but $g(y) \to z_0$ as $y \to y_0$. □

4 Compact Subspaces and Connected Subspace of $\mathbb{R}^n$

▶ 1.
(b)

Proof. Let $x_n = \left(2n\pi + \frac{\pi}{2}\right)^{-1}$ and $y_n = \left(2n\pi - \frac{\pi}{2}\right)^{-1}$. Then as $n \to \infty$, $|x_n - y_n| \to 0$ but $\left|\sin \frac{1}{x_n} - \sin \frac{1}{y_n}\right| = 2$. □

▶ 3.

Proof. The boundedness of X is clear. Since for any $i \neq j$, $\|e_i - e_j\| = 1$, the sequence $(e_i)_{i=1}^\infty$ has no accumulation point. So X cannot be compact. Also, the fact $\|e_i - e_j\| = 1$ for $i \neq j$ shows each e_i is an isolated point of X. Therefore X is closed. Combined, we conclude X is closed, bounded, and non-compact. □

Chapter 2

Differentiation

5 The Derivative

▶ 1.

Proof. By definition, $\lim_{t\to 0}\frac{f(\mathbf{a}+t\mathbf{u})-f(\mathbf{a})}{t}$ exists. Consequently,

$$\lim_{t\to 0}\frac{f(\mathbf{a}+tc\mathbf{u})-f(\mathbf{a})}{t} = c\cdot\lim_{t\to 0}\frac{f(\mathbf{a}+tc\mathbf{u})-f(\mathbf{a})}{ct} = c\lim_{t\to 0}\frac{f(\mathbf{a}+t\mathbf{u})-f(\mathbf{a})}{t}$$

exists and is equal to $cf'(\mathbf{a};\mathbf{u})$. □

▶ 2.

(a)

Solution. $f(\mathbf{u}) = f(u_1,u_2) = \frac{u_1u_2}{u_1^2+u_2^2}$. So

$$\frac{f(t\mathbf{u})-f(0)}{t} = \frac{1}{t}\frac{t^2u_1u_2}{t^2(u_1^2+u_2^2)} = \frac{1}{t}\frac{u_1u_2}{u_1^2+u_2^2}.$$

In order for $\lim_{t\to 0}\frac{f(t\mathbf{u})-f(0)}{t}$ to exist, it is necessary and sufficient that $u_1u_2=0$ and $u_1^2+u_2^2\neq 0$. So for vectors $(1,0)$ and $(0,1)$, $f'(\mathbf{0};\mathbf{u})$ exists, and we have $f'(\mathbf{0};(1,0)) = f'(\mathbf{0};(0,1)) = 0$. □

(b)

Solution. Yes, $D_1f(\mathbf{0}) = D_2f(\mathbf{0}) = 0$. □

(c)

Solution. No, because f is not continuous at $\mathbf{0}$: $\lim_{(x,y)\to 0, y=kx} f(x,y) = \frac{kx^2}{x^2+k^2x^2} = \frac{k}{1+k^2}$. For $k\neq 0$, the limit is not equal to $f(\mathbf{0})$. □

(d)

Solution. See (c). □

6 Continuously Differentiable Functions

► 1.

Proof. We note

$$\frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{1}{2}\frac{x^2+y^2}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2}.$$

So $\lim_{(x,y)\to 0} \frac{|xy|}{\sqrt{x^2+y^2}} = 0$. This shows $f(x,y) = |xy|$ is differentiable at $\mathbf{0}$ and the derivative is 0. However, for any fixed $y \neq 0$, $f(x,y)$ is not a differentiable function of x at 0. So its partial derivative w.r.t. x does not exist in a neighborhood of 0, which implies f is not of class C^1 in a neighborhood of $\mathbf{0}$. □

7 The Chain Rule

8 The Inverse Function Theorem

9 The Implicit Function Theorem

Chapter 3

Integration

10 The Integral over a Rectangle

▶ 6.

(a)

Proof. Straightforward from the Riemann condition (Theorem 10.3). □

(b)

Proof. Suppose $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $P = (P_1, \cdots, P_n)$ such that P_j is a partition of $[a_j, b_j]$ for each j. Without loss of generality, assume $P^{''}$ is obtained by adjoining t^* to $P_1 = \{t_0^1, t_1^1, \cdots, t_k^1\}$ with $a_1 = t_0^1 < \cdots < t_{l_1-1}^1 < t^* < t_{l_1}^1 < \cdots < t_k^1 = b_1$. Let

$$\begin{cases} R_1(l_2, \cdots, l_n) = [t_{l_1-1}^1, t^*] \times [t_{l_2-1}^2, t_{l_2}^2] \times \cdots \times [t_{l_n-1}^n, t_{l_n}^n] \\ R_2(l_2, \cdots, l_n) = [t^*, t_{l_1}^1] \times [t_{l_2-1}^2, t_{l_2}^2] \times \cdots \times [t_{l_n-1}^n, t_{l_n}^n] \\ R(l_2, \cdots, l_n) = [t_{l_1-1}^1, t_{l_1}^1] \times [t_{l_2-1}^2, t_{l_2}^2] \times \cdots \times [t_{l_n-1}^n, t_{l_n}^n] \end{cases}$$

Then (we omit the $l_2, \cdots, l_n$ indexes for simplicity of notation)

$$\begin{aligned} & L(f, P^{''}) - L(f, P) \\ = & \sum_{l_2, \cdots, l_n} [m_{R_1}(f) \cdot v(R_1) + m_{R_2}(f) \cdot v(R_2) - m_R(f) \cdot v(R)] \\ = & \sum_{l_2, \cdots, l_n} \{[m_{R_1}(f) - m_R(f)] \cdot v(R_1) + [m_{R_2}(f) - m_R(f)] \cdot v(R_2)\} \end{aligned}$$

We note

$$0 \le m_{R_1}(f) - m_R(f) \le 2M, \ 0 \le m_{R_2}(f) - m_R(f) \le 2M$$

and

$$v(R_1) + v(R_2) = (t_{l_1}^1 - t_{l_1-1}^1)(t_{l_2}^2 - t_{l_2-1}^2) \cdots (t_{l_n}^n - t_{l_n-1}^n)$$

Therefore

$$\begin{aligned} 0 &\leq L(f,P'') - L(f,P) \\ &\leq \sum_{l_2,\cdots,l_n} [2M \cdot v(R_1) + 2M \cdot v(R_2)] \\ &\leq 2M \cdot (t^1_{l_1} - t^1_{l_1-1}) \cdot \sum_{l_2,\cdots,l_n} (t^2_{l_2} - t^2_{l_2-1}) \cdots (t^n_{l_n} - t^n_{l_n-1}) \\ &\leq 2M \cdot \text{mesh}(P) \cdot (\text{width}(Q))^{n-1} \end{aligned}$$

where the last inequality comes from $(t^1_{l_1} - t^1_{l_1-1}) \leq \text{mesh}(P)$ and

$$\begin{aligned} & \sum_{l_2,\cdots,l_n} (t^2_{l_2} - t^2_{l_2-1}) \cdots (t^n_{l_n} - t^n_{l_n-1}) \\ = & \sum_{l_3,\cdots,l_n} \left(\sum_{l_2} (t^2_{l_2} - t^2_{l_2-1}) \right) \cdot (t^3_{l_3} - t^3_{l_3-1}) \cdots (t^n_{l_n} - t^n_{l_n-1}) \\ \leq & \sum_{l_3,\cdots,l_n} \text{width}(Q) \cdot (t^3_{l_3} - t^3_{l_3-1}) \cdots (t^n_{l_n} - t^n_{l_n-1}) \\ \vdots & \\ \leq & (\text{width}(Q))^{n-1} \end{aligned}$$

The result for upper sums can be derived similarly. □

(c)

Proof. Given $\varepsilon > 0$, choose a partition P' such that $U(f,P') - L(f,P') < \frac{\varepsilon}{2}$. Let N be the number of partition points in P' and let

$$\delta = \frac{\varepsilon}{8MN(\text{width}Q)^{n-1}}.$$

Suppose P has mesh less than δ, the common refinement P'' of P and P' is obtained by adjoining at most N points to P. So by part (b)

$$\begin{aligned} 0 &\leq L(f,P'') - L(f,P) \leq N \cdot 2M(\text{mesh}P)(\text{width}Q)^{n-1} \\ &\leq 2MN \cdot \frac{\varepsilon}{8MN(\text{width}Q)^{n-1}} \cdot (\text{width}Q)^{n-1} \\ &= \frac{\varepsilon}{4}. \end{aligned}$$

Similarly, we can show $0 \leq U(f,P) - U(f,P'') \leq \frac{\varepsilon}{4}$. So

$$\begin{aligned} U(f,P) - L(f,P) &= [U(f,P) - U(f,P'')] + [L(f,P'') - L(f,P)] + [U(f,P'') - L(f,P'')] \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + [U(f,P') - L(f,P')] \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

This shows for any given $\varepsilon > 0$, there is a $\delta > 0$ such that $U(f,P) - L(f,P) < \varepsilon$ for every partition P of mesh less than δ. □

► 7.

Proof. (Sufficiency) Note $|\sum_R f(x_R)v(R) - A| < \varepsilon$ can be written as

$$A - \varepsilon < \sum_R f(x_R)v(R) < A + \varepsilon.$$

This shows $U(f,P) \le A + \varepsilon$ and $L(f,P) \ge A - \varepsilon$. So $U(f,P) - L(f,P) \le 2\varepsilon$. By Problem 6, we conclude f is integrable over Q, with $\int_Q f \in [A - \varepsilon, A + \varepsilon]$. Since ε is arbitrary, we conclude $\int_Q f = A$.

(Necessity) By Problem 6, for any given $\varepsilon > 0$, there is a $\delta > 0$ such that $U(f,P) - L(f,P) < \varepsilon$ for every partition P of mesh less than δ. For any such partition P, if for each sub-rectangle R determined by P, x_R is a point of R, we must have

$$L(f,P) - A \le \sum_R f(x_R)v(R) - A \le U(f,P) - A.$$

Since $L(f,P) \le A \le U(f,P)$, we conclude

$$\left| \sum_R f(x_R)v(R) - A \right| \le U(f,P) - L(f,P) < \varepsilon.$$

□

11 Existence of the Integral

12 Evaluation of the Integral

13 The Integral over a Bounded Set

14 Rectifiable Sets

15 Improper Integrals