

MULTIVARIABLE CALCULUS AND OPTIMIZATION TECHNIQUES

Partial and Total Differentiation, Jacobians, Extrema and Taylor Series, Classical Optimization, Vector Calculus, and Multiple and Vector Integrals — with Engineering Applications Throughout

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Sample Preview

This preview contains 5 worked-example pages from each of the book's five chapters, offering a glimpse of the range of topics and problem types covered across the full text.

Multivariable Calculus and Optimization Techniques

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Chapter 1: Multivariable Calculus and its Applications

Sample Worked Examples

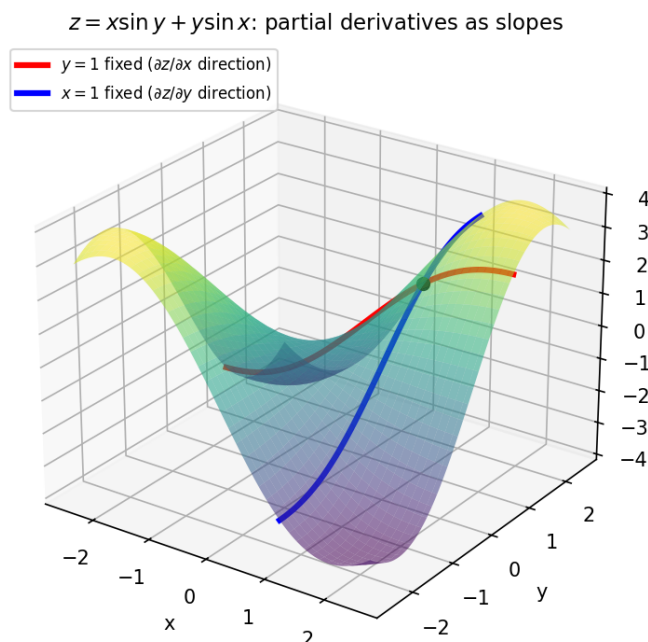


Figure 1.1: The partial derivative $\partial z / \partial x$ is the slope of the red curve (slicing the surface $z = f(x, y)$ with the plane $y = 1$); $\partial z / \partial y$ is the slope of the blue curve (slicing with $x = 1$).

1.2 Worked Examples

Example 1.1: If $z = x \sin y + y \sin x$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

Solution:

Let $z = x \sin y + y \sin x \dots (1)$. Differentiating (1) partially w.r.t. x , then w.r.t. y ,

$$\frac{\partial z}{\partial x} = \sin y + y \cos x \dots (2), \quad \frac{\partial^2 z}{\partial y \partial x} = \cos y + \cos x \dots (3)$$

Differentiating (1) partially w.r.t. y , then w.r.t. x ,

$$\frac{\partial z}{\partial y} = x \cos y + \sin x \dots (4), \quad \frac{\partial^2 z}{\partial x \partial y} = \cos y + \cos x \dots (5)$$

From (3) and (5),

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \cos x + \cos y}$$

Example 1.2: If $u = f(x + ct) + g(x - ct)$, prove that $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.

Surface $z = x^2 + y^2$ (blue-ish) and tangent plane (red) at $(1, 1)$
 $dz \approx \partial z / \partial x dx + \partial z / \partial y dy$

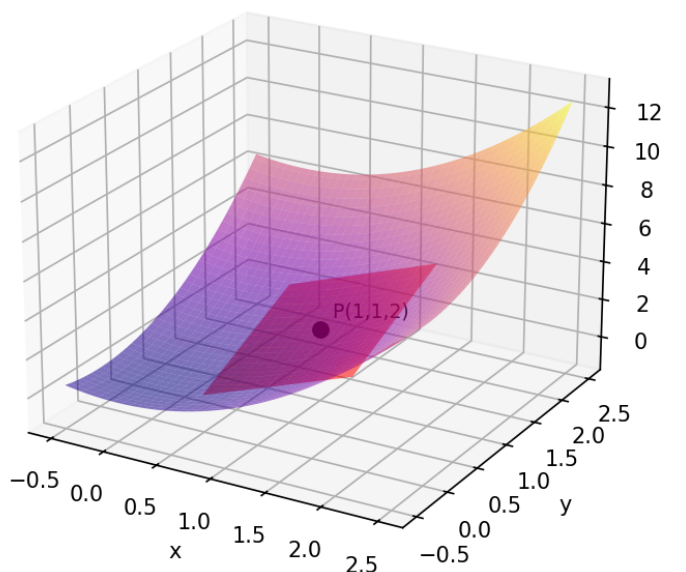
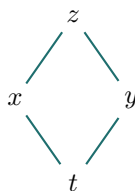


Figure 2.1: The tangent plane to $z = x^2 + y^2$ at $(1, 1, 2)$. Moving a small distance (dx, dy) from $(1, 1)$, the true surface height changes by approximately $dz = 2 dx + 2 dy$, the change predicted by the tangent plane.

2.2 Worked Examples

Example 2.1: If $z = \tan^{-1}(x/y)$, $x = 2t$, $y = 1 - t^2$, show that $\frac{dz}{dt} = \frac{2}{1 + t^2}$.

Solution:



Let $z = \tan^{-1}(x/y)$. By the total derivative formula,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \quad \dots (1)$$

Differentiating z partially,

$$\frac{\partial z}{\partial x} = \frac{y}{x^2 + y^2}, \quad \frac{\partial z}{\partial y} = \frac{-x}{x^2 + y^2} \quad \dots (2)$$

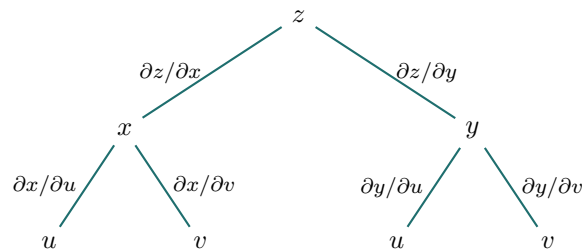
3. Composite Functions and the Chain Rule

3.1 Introduction

A “function of a function” is called a composite function. If $z = f(x, y)$ where $x = \phi(u, v)$ and $y = \psi(u, v)$, then z becomes a composite function of (u, v) , and the chain rule gives

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}.$$

The dependency between the variables is easiest to keep track of with a small *tree diagram*: z branches down to its immediate arguments x, y , and each of those branches down again to the ultimate independent variables u, v . Every path from z down to a given bottom variable contributes one product term to the derivative with respect to that variable.



Reading the diagram: to reach u from z , there are two routes ($z \rightarrow x \rightarrow u$ and $z \rightarrow y \rightarrow u$); multiplying the labels along each route and adding the routes together reproduces exactly the formula for $\partial z/\partial u$ above. The same tree, with p, q, r or r, s, t in place of x, y , governs every worked example in this section.

More generally, if $u = f(r, s, t)$ where r, s, t are each functions of x, y, z , then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x}$$

and similarly for $\partial u/\partial y$ and $\partial u/\partial z$.

Where the multivariable chain rule is used in engineering

- **Coordinate transformations:** converting a governing equation from Cartesian to polar, cylindrical, or spherical coordinates (heat conduction in a pipe, stress around a circular hole, potential flow around a cylinder) all use exactly this chain rule.
- **Fluid mechanics:** resolving a velocity or potential field into radial and tangential components uses the chain rule for the transformation $x = r \cos \theta$, $y = r \sin \theta$.
- **Thermodynamics:** changing the independent variables of a state function (e.g. expressing entropy in terms of (T, P) instead of (T, V)) is a direct application of the chain rule, used in deriving relations between C_P and C_V .

Figure 3.1 shows the geometric picture behind Example 3.3 below: the rectangular grid of constant r and constant θ lines is mapped by $x = r \cos \theta$, $y = r \sin \theta$ onto a grid of concentric circles and radial lines in the xy -plane.

4. Jacobians and their Properties

4.1 Introduction

If u, v are functions of x, y , the Jacobian of (u, v) with respect to (x, y) is the determinant

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix},$$

and similarly for three variables u, v, w of x, y, z ,

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

The Jacobian is a single number (or function) that measures how a small area (or volume) element is scaled and distorted when we change from the variables (x, y) to the variables (u, v) . Two key properties are used repeatedly:

Property 1 (Chain rule for Jacobians): $\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)}$

Property 2: If J is the Jacobian of (u, v) w.r.t. (x, y) , and J' is the Jacobian of (x, y) w.r.t. (u, v) , then $J \cdot J' = 1$.

Where Jacobians are used in engineering

- **Change of variables in multiple integrals:** converting an area or volume integral to polar, cylindrical, or spherical coordinates always introduces the Jacobian as an extra factor (e.g. $dA = r \, dr \, d\theta$) – essential for computing moments of inertia, centroids, and fields with circular or spherical symmetry.
- **Verifying invertibility:** a coordinate transformation is locally invertible exactly where its Jacobian is non-zero – used when checking that a proposed change of variables is valid.
- **Robotics and mechanism design:** the Jacobian matrix relating joint velocities to end-effector velocity is the higher-dimensional analogue of the Jacobians studied here.

Figure 4.1 shows the Jacobian geometrically for the polar transformation $x = r \cos \theta$, $y = r \sin \theta$: a small rectangle of area $dr \, d\theta$ in (r, θ) -space maps to a small curved region of area $r \, dr \, d\theta$ in the xy -plane – the extra factor r is the Jacobian.

4.2 Worked Examples

Example 4.1: If $u = x + y + z$, $v = y + z$, $w = z$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

Solution:

5.3 Worked Examples

Example 5.1: Find the maximum and minimum values of $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

Solution:

$$\frac{\partial f}{\partial x} = 3x^2 - 3, \quad \frac{\partial f}{\partial y} = 3y^2 - 12$$

Setting both to zero: $x = \pm 1$, $y = \pm 2$, giving four critical points $(1, 2), (1, -2), (-1, 2), (-1, -2)$. Here $r = 6x$, $s = 0$, $t = 6y$.

	$(1, 2)$	$(1, -2)$	$(-1, 2)$	$(-1, -2)$
$r = 6x$	6	6	-6	-6
$t = 6y$	12	-12	12	-12
$rt - s^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Result	minimum	saddle	saddle	maximum

$$f(-1, -2) = -1 - 8 + 3 + 24 + 20 = 38, \quad f(1, 2) = 1 + 8 - 3 - 24 + 20 = 2$$

Maximum = 38 at $(-1, -2)$; Minimum = 2 at $(1, 2)$

Example 5.2: Find the extreme values of $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Solution:

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y, \quad \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

Setting both to zero and adding gives $x^3 + y^3 = 0 \Rightarrow (x+y)(x^2 - xy + y^2) = 0$. Since $x^2 - xy + y^2 \geq 0$ with equality only at the origin, we need $y = -x$; substituting back gives $x^3 - 2x = 0$, so $x = 0, \pm\sqrt{2}$. The critical points are $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, with $r = 12x^2 - 4$, $s = 4$, $t = 12y^2 - 4$.

At $(0, 0)$: $r = -4$, $t = -4$, $rt - s^2 = 0$ (test fails). Checking f along $y = x$ and $y = -x$ through the origin shows f takes both signs near $(0, 0)$, so $(0, 0)$ is a **saddle point**. At $(\pm\sqrt{2}, \mp\sqrt{2})$: $r = 20$, $t = 20$, $rt - s^2 = 384 > 0$ with $r > 0$, so both are **minima**, with

$$f(\sqrt{2}, -\sqrt{2}) = 4 + 4 - 4 - 8 - 4 = -8$$

Minimum = -8 at $(\pm\sqrt{2}, \mp\sqrt{2})$; $(0, 0)$ is a saddle point

Example 5.3: Find the extreme values of $f(x, y) = x^3y^2(1 - x - y)$.

Solution:

$$\frac{\partial f}{\partial x} = x^2y^2(3 - 4x - 3y), \quad \frac{\partial f}{\partial y} = x^3y(2 - 2x - 3y)$$

The non-trivial stationary point (excluding $x = 0$ or $y = 0$, where $f \equiv 0$ trivially) solves $4x + 3y = 3$ and $2x + 3y = 2$, giving $x = \frac{1}{2}$, $y = \frac{1}{3}$. Computing

$$r = -\frac{1}{9}, \quad s = -\frac{1}{12}, \quad t = -\frac{1}{8} \quad \text{at } \left(\frac{1}{2}, \frac{1}{3}\right)$$

Chapter 2: Optimization Techniques

Sample Worked Examples

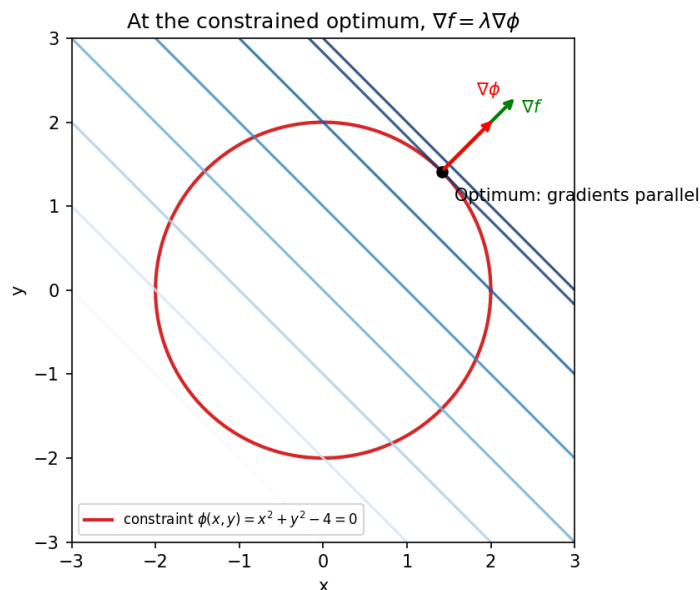


Figure 1: At the constrained optimum, the objective's level curve is tangent to the constraint curve, so ∇f and $\nabla \phi$ are parallel.

2.1.2 Worked Examples

Example 2.1: A rectangular box, open at the top, is to have volume 32 cubic ft. Find the dimensions of the box requiring the least material for its construction, using Lagrange's method.

Solution. Let x, y, z ft be the edges (base $x \times y$, height z). Since the box is open at the top (no lid), the four sides and the base contribute to the surface, giving $S = xy + 2yz + 2zx$, subject to the volume constraint $\phi \equiv xyz - 32 = 0$.

Form the Lagrangian:

$$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

Differentiate F partially with respect to each variable and set each result to zero:

$$F_x = \frac{\partial}{\partial x}(xy + 2yz + 2zx + \lambda xyz) = y + 2z + \lambda yz = 0 \quad (1)$$

$$F_y = \frac{\partial}{\partial y}(xy + 2yz + 2zx + \lambda xyz) = x + 2z + \lambda xz = 0 \quad (2)$$

$$F_z = \frac{\partial}{\partial z}(xy + 2yz + 2zx + \lambda xyz) = 2y + 2x + \lambda xy = 0 \quad (3)$$

To eliminate λ , multiply equation (1) by x , equation (2) by y , and equation (3) by z (multiplying each equation by the variable that is missing from its own λ -term makes every λ -term become λxyz):

$$xy + 2xz + \lambda xyz = 0 \quad (1')$$

$$xy + 2yz + \lambda xyz = 0 \quad (2')$$

$$2xy + 2xz + \lambda xyz = 0 \quad (3')$$

feasible region is convex (an intersection of half-planes), the KKT point found above is the global maximum. Substituting $x_1 = 4.25$, $x_2 = 5.75$ into Z :

$$\begin{aligned} Z^* &= 12(4.25) + 21(5.75) + 2(4.25)(5.75) - 2(4.25)^2 - 2(5.75)^2 \\ &= 51 + 120.75 + 48.875 - 36.125 - 66.125 = 118.375 \\ x_1^* &= 4.25, \quad x_2^* = 5.75, \quad \lambda_1 = 6.5, \quad \lambda_2 = 0; \quad Z^* = 118.375 \end{aligned}$$

Example 2.17: Minimize $f(x_1, x_2) = x_1^2 + x_2^2$ subject to $x_1 + x_2 \geq 4$.

Solution. Here $g(x) = x_1 + x_2 \geq 4$. First check whether the constraint is even needed: the unconstrained minimum of $f = x_1^2 + x_2^2$ is at $x_1 = x_2 = 0$ (where $\nabla f = \vec{0}$), but this point gives $x_1 + x_2 = 0 < 4$, which violates the constraint. So the constraint must be active at the true constrained minimum, i.e. $x_1 + x_2 = 4$.

For a minimisation problem with a $\geq$ constraint, the KKT stationarity condition is $\nabla f = \lambda \nabla g$ with $\lambda \geq 0$. Differentiate f and g :

$$\frac{\partial f}{\partial x_1} = 2x_1, \quad \frac{\partial f}{\partial x_2} = 2x_2, \quad \frac{\partial g}{\partial x_1} = 1, \quad \frac{\partial g}{\partial x_2} = 1$$

So stationarity reads

$$2x_1 = \lambda(1) = \lambda \quad (1)$$

$$2x_2 = \lambda(1) = \lambda \quad (2)$$

From (1), $x_1 = \frac{\lambda}{2}$; from (2), $x_2 = \frac{\lambda}{2}$; so $x_1 = x_2$. Substitute both into the active constraint $x_1 + x_2 = 4$:

$$\frac{\lambda}{2} + \frac{\lambda}{2} = 4 \Rightarrow \lambda = 4$$

Since $\lambda = 4 \geq 0$, dual feasibility is satisfied. Substituting back:

$$x_1^* = \frac{\lambda}{2} = \frac{4}{2} = 2, \quad x_2^* = \frac{4}{2} = 2$$

Since f is convex (its Hessian is $2I$, positive definite, as both eigenvalues equal $2 > 0$) and the constraint $g(x) \geq 4$ defines a convex feasible region (a half-plane), this KKT point is the global minimum:

$$\begin{aligned} f^* &= x_1^{*2} + x_2^{*2} = 2^2 + 2^2 = 4 + 4 = 8 \\ x_1^* &= x_2^* = 2; \quad f^* = 8 \end{aligned}$$

Example 2.18: Maximize $f(x_1, x_2) = 2x_1 + 3x_2 - x_1^2 - x_2^2$ subject to $x_1 + x_2 \leq 4$, $x_1, x_2 \geq 0$.

Solution. Differentiate $f = 2x_1 + 3x_2 - x_1^2 - x_2^2$:

$$\frac{\partial f}{\partial x_1} = 2 - 2x_1, \quad \frac{\partial f}{\partial x_2} = 3 - 2x_2, \quad g_1 = x_1 + x_2 \leq 4$$

The stationarity condition $\frac{\partial f}{\partial x_j} - \lambda_1 \frac{\partial g_1}{\partial x_j} = 0$ gives

$$2 - 2x_1 - \lambda_1 = 0 \quad (1)$$

$$3 - 2x_2 - \lambda_1 = 0 \quad (2)$$

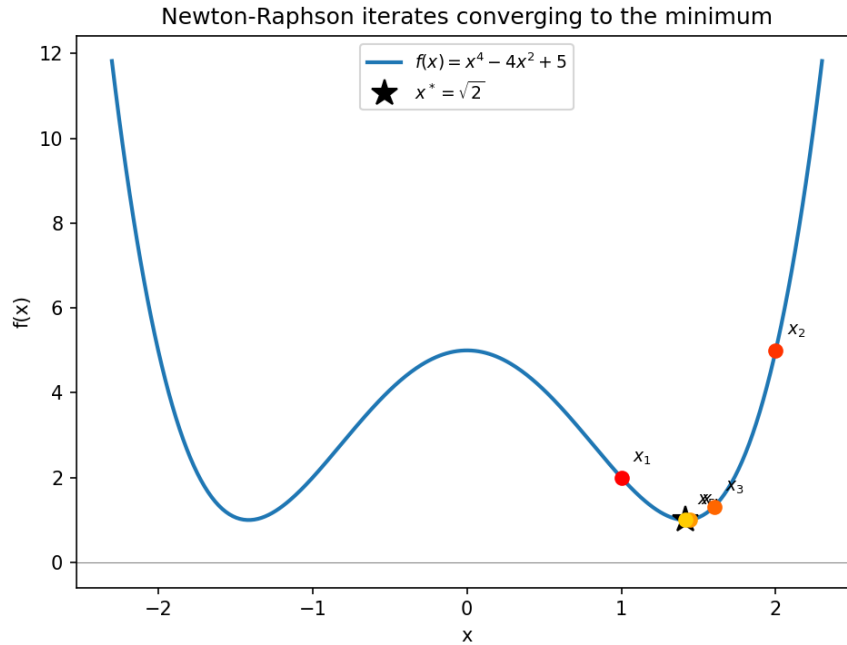


Figure 2: Newton–Raphson iterates for $f(x) = x^4 - 4x^2 + 5$ converge rapidly (quadratically) to the minimum at $x^* = \sqrt{2}$.

2.3.2 Worked Examples

Example 2.20: Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ starting from $\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Solution:

First write the gradient vector and Hessian matrix in general form, for $f(x_1, x_2)$:

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{pmatrix}, \quad H(\mathbf{x}) = \begin{pmatrix} \partial^2 f / \partial x_1^2 & \partial^2 f / \partial x_1 \partial x_2 \\ \partial^2 f / \partial x_2 \partial x_1 & \partial^2 f / \partial x_2^2 \end{pmatrix}, \quad \mathbf{x}_{k+1} = \mathbf{x}_k - H^{-1}(\mathbf{x}_k) \nabla f(\mathbf{x}_k). \quad \dots (1)$$

Differentiate $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ to get the gradient (treating the cross term $2x_1x_2$ by the product rule: it contributes $2x_2$ to $\partial f / \partial x_1$ and $2x_1$ to $\partial f / \partial x_2$):

$$\nabla f(\mathbf{x}) = \begin{pmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{pmatrix} \quad \dots (2)$$

Differentiate again to get the Hessian (each entry $H_{ij} = \partial^2 f / \partial x_i \partial x_j$):

$$H(\mathbf{x}) = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \quad (\text{constant, since } f \text{ is quadratic}) \quad \dots (3)$$

To invert this 2×2 matrix, first compute its determinant: $|H| = (4)(2) - (2)(2) = 8 - 4 = 4$. For

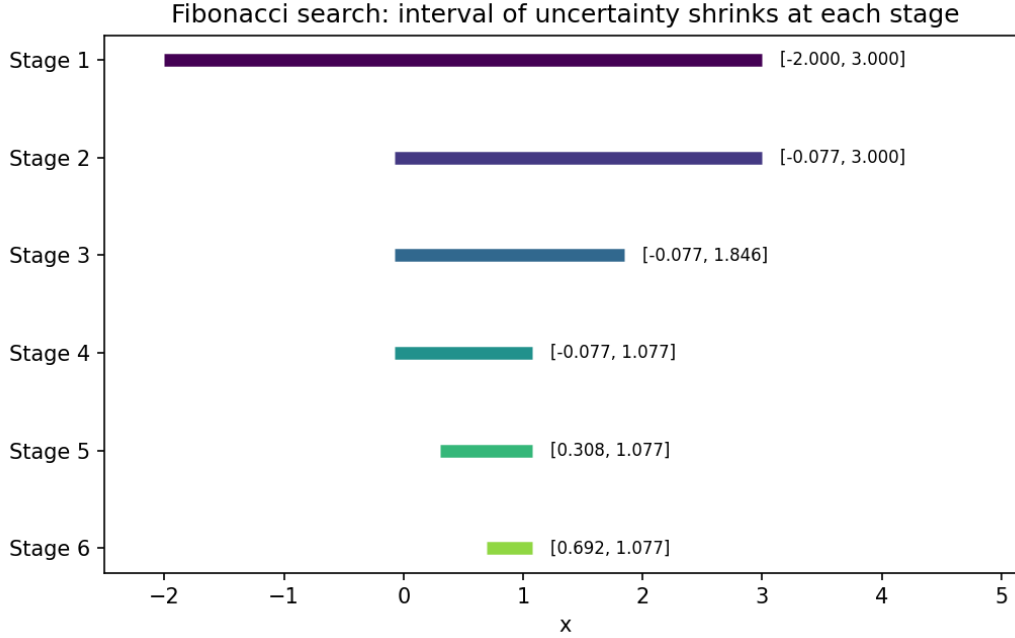


Figure 3: The Fibonacci search interval of uncertainty shrinks at every stage, bracketing the minimum ever more tightly.

2.4.2 Worked Examples

Example 2.24: Minimize $f(x) = x^2 - 1.6x + 2$, $x \in [-2, 3]$ using Fibonacci search. Perform 6 iterations ($n = 6$).

Solution. $L_0 = b - a = 3 - (-2) = 5$. Fibonacci numbers used: $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $F_6 = 13$.

Stage 1 ($m = 6$, interval $[-2, 3]$, $L = 5$): compute the two interior points using $x_{\text{lower}} = a + \frac{F_{m-2}}{F_m}L$ and $x_{\text{upper}} = a + \frac{F_{m-1}}{F_m}L$:

$$x_{\text{lower}} = -2 + \frac{F_4}{F_6}(5) = -2 + \frac{5}{13}(5) = -2 + 1.9231 = -0.0769$$

$$x_{\text{upper}} = -2 + \frac{F_5}{F_6}(5) = -2 + \frac{8}{13}(5) = -2 + 3.0769 = 1.0769$$

Evaluate f at both points:

$$f(-0.0769) = (-0.0769)^2 - 1.6(-0.0769) + 2 = 0.0059 + 0.1230 + 2 = 2.1290$$

$$f(1.0769) = (1.0769)^2 - 1.6(1.0769) + 2 = 1.1597 - 1.7230 + 2 = 1.4367$$

Since $f(x_{\text{lower}}) = 2.1290 > f(x_{\text{upper}}) = 1.4367$, the minimum lies in $[x_{\text{lower}}, b]$; discard $[a, x_{\text{lower}})$, retaining $[-0.0769, 3.0000]$.

Stage 2 ($m = 5$, interval $[-0.0769, 3.0000]$, $L = 3.0769$): the point $x = 1.0769$ was already evaluated ($f = 1.4367$) and is retained as one of the two new points; compute the other:

$$x_{\text{lower}} = 1.0769 \text{ (retained)}$$

minimizing a two- or higher-dimensional cost function.

- **Process engineering:** tuning two operating parameters (e.g. temperature and pressure set-points) to minimize an operating-cost function evaluated by simulation, where derivatives of the simulation are available but second derivatives are not.

Figure 4 shows the characteristic zig-zag path of steepest descent as it converges toward the minimum of a quadratic function.

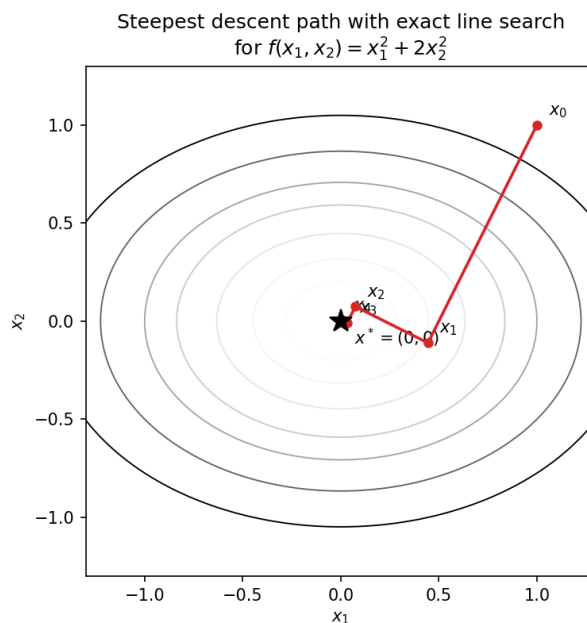


Figure 4: Steepest descent with exact line search on $f(x_1, x_2) = x_1^2 + 2x_2^2$: successive gradient directions are always perpendicular to each other (since each step exactly minimizes f along the current search direction), producing a zig-zag path toward the minimum.

2.5.2 Worked Examples

Example 2.28: Minimize $f(x_1, x_2) = x_1^2 + 2x_2^2$ by the steepest descent method with the optimal (exact line-search) step length, starting from $\mathbf{x}_0 = (1, 1)$. Perform 4 iterations.

Solution. Write $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T H \mathbf{x}$ with $H = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ (so that $\nabla f(\mathbf{x}) = H\mathbf{x}$, matching $\nabla f = (2x_1, 4x_2)^T$). For a quadratic with constant Hessian H , the exact-line-search step length is $\alpha_k = \frac{g_k^T g_k}{g_k^T H g_k}$, where $g_k = \nabla f(\mathbf{x}_k) = H\mathbf{x}_k$.

Iteration 1: at $\mathbf{x}_0 = (1, 1)$: $g_0 = H\mathbf{x}_0 = (2(1), 4(1)) = (2, 4)$.

$$g_0^T g_0 = 2^2 + 4^2 = 4 + 16 = 20$$

$$H g_0 = (2(2), 4(4)) = (4, 16), \quad g_0^T H g_0 = (2)(4) + (4)(16) = 8 + 64 = 72$$

Chapter 3: Vector Calculus and its Applications

Sample Worked Examples

1 Gradient, Directional Derivative, and Normal Derivative

3.1.1 Introduction

If $\phi(x, y, z)$ is a scalar point function, its gradient is the vector field

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}.$$

At any point, $\nabla\phi$ points in the direction in which ϕ increases most rapidly, and $|\nabla\phi|$ is the magnitude of that maximum rate of increase. The unit vector normal to the level surface $\phi(x, y, z) = c$ at a point is $\hat{n} = \nabla\phi/|\nabla\phi|$, and the directional derivative of ϕ in the direction of a unit vector $\hat{n}$ is $\nabla\phi \cdot \hat{n}$ — the rate of change of ϕ as we move in that particular direction. Setting $\hat{n} = \nabla\phi/|\nabla\phi|$ gives the normal derivative, the maximum possible directional derivative at that point.

Where this is used in engineering

- **Steepest-ascent problems:** on a terrain, temperature field, or cost surface, $\nabla\phi$ gives the direction and rate of fastest increase — used in path planning, hill-climbing optimisation, and contour-based navigation.
- **Electrostatics:** the electric field is the negative gradient of the potential, $\vec{E} = -\nabla V$ — used throughout circuit, antenna, and capacitor design.
- **Heat flow:** heat flows in the direction of $-\nabla T$, from hot to cold — the basis of Fourier's law of conduction, $\vec{q} = -k\nabla T$.
- **Surface normals:** $\nabla\phi/|\nabla\phi|$ gives the outward unit normal to a surface $\phi = c$ — needed in computer graphics (shading), CFD (boundary conditions), and CAD/CAM (surface machining).

Figure 1 shows the geometric picture: the gradient at a point is always perpendicular to the level curve (or surface) through that point, and points toward higher values of ϕ .

3.1.2 Worked Examples

Example 3.1: Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.

Solution.

Write the surface as a level set of a scalar function: let $\phi = xy^3z^2 - 4$, so the surface is $\phi = 0$, and its normal direction at any point is $\nabla\phi$.

Differentiate ϕ with respect to each variable in turn, holding the others fixed:

$$\frac{\partial\phi}{\partial x} = y^3z^2, \quad \frac{\partial\phi}{\partial y} = 3xy^2z^2, \quad \frac{\partial\phi}{\partial z} = 2xy^3z.$$

Substitute the point $(-1, -1, 2)$, where $y^3 = (-1)^3 = -1$, $y^2 = 1$, $z^2 = 4$:

$$\phi_x = (-1)(4) = -4, \quad \phi_y = 3(-1)(1)(4) = -12, \quad \phi_z = 2(-1)(-1)(2) = 4.$$

Assemble the gradient vector at the point:

$$\nabla\phi = -4\hat{i} - 12\hat{j} + 4\hat{k}.$$

3.2.2 Worked Examples

Example 3.15: Find the divergence of $\vec{F} = (2x - y^2)\hat{i} + (3z + x^2)\hat{j} + (4y - z^2)\hat{k}$ at $(1, 2, 3)$.

Solution.

Identify the components: $f_1 = 2x - y^2$, $f_2 = 3z + x^2$, $f_3 = 4y - z^2$.

Differentiate each component with respect to its own coordinate: $\frac{\partial f_1}{\partial x} = 2$, $\frac{\partial f_2}{\partial y} = 0$, $\frac{\partial f_3}{\partial z} = -2z$.

Add them: $\nabla \cdot \vec{F} = 2 + 0 - 2z = 2 - 2z$.

Substitute $z = 3$: $\nabla \cdot \vec{F} = 2 - 6 = -4$.

$$\nabla \cdot \vec{F} = -4$$

Example 3.16: Find the curl of $\vec{F} = (2x - y^2)\hat{i} + (3z + x^2)\hat{j} + (4y - z^2)\hat{k}$ at $(1, 2, 3)$.

Solution.

Identify $f_1 = 2x - y^2$, $f_2 = 3z + x^2$, $f_3 = 4y - z^2$, and set up the curl as the determinant

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y^2 & 3z + x^2 & 4y - z^2 \end{vmatrix}.$$

Expand this determinant along the first row: for each cofactor, strike out the row and column of that entry and take the 2×2 determinant of what remains.

$\hat{i}$ -cofactor (strike row 1, column 1):

$$\hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z + x^2 & 4y - z^2 \end{vmatrix} = \hat{i} \left(\frac{\partial}{\partial y}(4y - z^2) - \frac{\partial}{\partial z}(3z + x^2) \right) = \hat{i}(4 - 3) = \hat{i}.$$

$\hat{j}$ -cofactor (strike row 1, column 2; this cofactor carries a minus sign):

$$-\hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2x - y^2 & 4y - z^2 \end{vmatrix} = -\hat{j} \left(\frac{\partial}{\partial x}(4y - z^2) - \frac{\partial}{\partial z}(2x - y^2) \right) = -\hat{j}(0 - 0) = \vec{0}.$$

$\hat{k}$ -cofactor (strike row 1, column 3):

$$\hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2x - y^2 & 3z + x^2 \end{vmatrix} = \hat{k} \left(\frac{\partial}{\partial x}(3z + x^2) - \frac{\partial}{\partial y}(2x - y^2) \right) = \hat{k}(2x - (-2y)) = (2x + 2y)\hat{k}.$$

Adding the three cofactors: $\nabla \times \vec{F} = \hat{i} + (2x + 2y)\hat{k}$, valid at every point.

Substitute $(1, 2, 3)$: $2x + 2y = 2(1) + 2(2) = 2 + 4 = 6$.

$$\nabla \times \vec{F} = \hat{i} + 6\hat{k}$$

Example 3.17: If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that (i) $\nabla \cdot \vec{r} = 3$ (ii) $\nabla \times \vec{r} = \vec{0}$.

Solution.

3 Solenoidal and Irrotational Vector Fields

3.3.1 Introduction

A vector field $\vec{F}$ is **solenoidal** (or incompressible) if $\nabla \cdot \vec{F} = 0$ everywhere, and **irrotational** (or conservative) if $\nabla \times \vec{F} = \vec{0}$ everywhere. These two properties are independent of one another — a field may be one, both, or neither.

Where this is used in engineering

- **Incompressible flow (solenoidal):** liquids, and low-speed gas flows, are modelled as solenoidal velocity fields; $\nabla \cdot \vec{V} = 0$ is the mass-conservation constraint used in pump, turbine, and pipeline design.
- **Conservative force fields (irrotational):** gravity, electrostatic force, and ideal spring forces are irrotational; this guarantees the work done is path-independent, simplifying analysis in structural and mechanical design.
- **Magnetic vector potential:** because $\nabla \cdot \vec{B} = 0$ always, $\vec{B}$ can be written as $\vec{B} = \nabla \times \vec{A}$ for some solenoidal vector potential $\vec{A}$ — the starting point of magnetostatic field computation.
- **Groundwater and seepage flow:** Darcy flow through a saturated, incompressible porous medium is modelled as solenoidal, underlying dam-seepage and aquifer analysis.

3.3.2 Worked Examples

Example 3.29: If the velocity vector $\vec{V} = \vec{w} \times \vec{r}$, prove that $\vec{w} = \frac{1}{2} \text{curl } \vec{V}$, where $\vec{w}$ is the (constant) angular velocity vector.

Solution.

Write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$ with w_1, w_2, w_3 constant.

Expand the cross product using the determinant form:

$$\vec{V} = \vec{w} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} = (w_2z - w_3y)\hat{i} + (w_3x - w_1z)\hat{j} + (w_1y - w_2x)\hat{k}.$$

Identify components $f_1 = w_2z - w_3y$, $f_2 = w_3x - w_1z$, $f_3 = w_1y - w_2x$, and set up the curl as a determinant:

$$\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2z - w_3y & w_3x - w_1z & w_1y - w_2x \end{vmatrix}.$$

$$\hat{i}\text{-cofactor: } \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ w_3x - w_1z & w_1y - w_2x \end{vmatrix} = \frac{\partial}{\partial y}(w_1y - w_2x) - \frac{\partial}{\partial z}(w_3x - w_1z) = w_1 - (-w_1) = 2w_1.$$

$$\hat{j}\text{-cofactor (with its minus sign): } - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ w_2z - w_3y & w_1y - w_2x \end{vmatrix} = - \left(\frac{\partial}{\partial x}(w_1y - w_2x) - \frac{\partial}{\partial z}(w_2z - w_3y) \right) = -(-w_2 - w_2) = 2w_2.$$

$$\hat{k}\text{-cofactor: } \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ w_2z - w_3y & w_3x - w_1z \end{vmatrix} = \frac{\partial}{\partial x}(w_3x - w_1z) - \frac{\partial}{\partial y}(w_2z - w_3y) = w_3 - (-w_3) = 2w_3.$$

$$\nabla^2(f(r)) = f''(r) + \frac{2}{r}f'(r)$$

3.4.3 Engineering Applications

Application 3.7 (Gravitation — Inverse-Square Field): The gravitational field of a point mass M at the origin is $\vec{g} = -\frac{GM}{r^2}\hat{r} = -GM\frac{\vec{r}}{r^3}$. Using the standard results of this section, show that $\vec{g}$ is irrotational everywhere (except the origin), and that it is solenoidal away from the origin.

Solution.

Write $\vec{g} = -GM r^{-3}\vec{r}$, which is an instance of $r^n\vec{r}$ with $n = -3$.

By Example 3.44, $\text{curl}(r^n\vec{r}) = \vec{0}$ for any n , so

$$\text{curl } \vec{g} = -GM \text{curl}(r^{-3}\vec{r}) = \vec{0} \quad (r \neq 0).$$

By Example 3.43, $\text{div}(r^n\vec{r}) = (n+3)r^n$; with $n = -3$ this becomes $(-3+3)r^{-3} = 0$.

So $\text{div } \vec{g} = -GM \cdot 0 = 0$ for $r \neq 0$.

Both properties confirm that gravitational (and, identically, electrostatic Coulomb) fields are conservative and source-free everywhere except at the point mass itself, where the field is singular.

$$\text{curl } \vec{g} = \vec{0} \text{ and } \text{div } \vec{g} = 0 \text{ for } r \neq 0$$

Application 3.8 (Steady Radial Heat Conduction): Steady-state temperature around a small spherical heat source, far from any boundary, is modelled by $T(r) = \frac{Q}{4\pi k r}$ for constants Q, k . Using the identity of Example 3.48, confirm that T satisfies Laplace's equation $\nabla^2 T = 0$ away from the source, as required for a steady-state (source-free) temperature field.

Solution.

Write $f(r) = \frac{Q}{4\pi k}r^{-1}$, so that $T = f(r)$.

Differentiate: $f'(r) = -\frac{Q}{4\pi k}r^{-2}$, and differentiating again, $f''(r) = \frac{2Q}{4\pi k}r^{-3}$.

Substitute into the identity from Example 3.48, $\nabla^2 T = f''(r) + \frac{2}{r}f'(r)$:

$$\nabla^2 T = \frac{2Q}{4\pi k}r^{-3} + \frac{2}{r}\left(-\frac{Q}{4\pi k}r^{-2}\right) = \frac{2Q}{4\pi k}r^{-3} - \frac{2Q}{4\pi k}r^{-3}.$$

The two terms are identical in magnitude and opposite in sign, so they cancel exactly, leaving $\nabla^2 T = 0$.

This confirms that, away from the source at $r = 0$, the temperature field is harmonic — exactly the physical requirement for steady-state conduction with a single point source and no heat generated or absorbed elsewhere.

$$\nabla^2 T = 0 \text{ for } r \neq 0, \text{ confirming Laplace's equation is satisfied away from the source}$$

7 Line Integrals and Work Done

3.7.1 Introduction

For a vector field $\vec{F}$ representing a force, the work done in moving a particle along a curve C from A to B is the line integral $\int_C \vec{F} \cdot d\vec{r}$. When $\vec{F}$ is conservative, with scalar potential ϕ (so $\vec{F} = \nabla\phi$), this integral collapses to a simple difference of potential values:

$$\int_A^B \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A),$$

independent of the path taken from A to B — a direct consequence of the fundamental theorem for line integrals.

Where this is used in engineering

- **Mechanical work and energy methods:** the work-energy theorem in structural and machine design relies on exactly this shortcut to avoid evaluating a line integral explicitly.
- **Electrical engineering:** the voltage (potential difference) between two points in a conservative electric field is computed this way, without needing to know the exact wire path.
- **Robotics and trajectory planning:** because the work done by a conservative end-effector force is path-independent, energy budgets for a motion can be computed from endpoints alone, simplifying planning.

3.7.2 Worked Examples

Example 3.56: Show that $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ is irrotational; find its scalar potential, hence evaluate $\int_{(1,2)}^{(2,1)} \vec{F} \cdot d\vec{r}$.

Solution.

As in Example 3.54, with $f_1 = x^2 - y^2 + x$, $f_2 = -2xy - y$: $\frac{\partial f_2}{\partial x} = -2y$ and $\frac{\partial f_1}{\partial y} = -2y$, so $\nabla \times \vec{F} = \vec{0}$ and $\vec{F}$ is irrotational.

Integrate $\phi_x = x^2 - y^2 + x$ with respect to x : $\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} + f(y)$.

Differentiate with respect to y : $\phi_y = -2xy + f'(y)$, which must equal $-(2xy + y) = -2xy - y$, so $f'(y) = -y$, giving $f(y) = -\frac{y^2}{2} + C$.

Collect: $\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2} + C$.

Evaluate at $(2, 1)$: $\frac{8}{3} - 2(1) + \frac{4}{2} - \frac{1}{2} = \frac{8}{3} - 2 + 2 - \frac{1}{2} = \frac{8}{3} - \frac{1}{2} = \frac{16}{6} - \frac{3}{6} = \frac{13}{6}$.

Evaluate at $(1, 2)$: $\frac{1}{3} - 1(4) + \frac{1}{2} - \frac{4}{2} = \frac{1}{3} - 4 + \frac{1}{2} - 2 = \frac{1}{3} - \frac{11}{2} = \frac{2}{6} - \frac{33}{6} = -\frac{31}{6}$.

Apply the fundamental theorem for line integrals:

$$\int_{(1,2)}^{(2,1)} \vec{F} \cdot d\vec{r} = \phi(2, 1) - \phi(1, 2) = \frac{13}{6} - \left(-\frac{31}{6}\right) = \frac{44}{6} = \frac{22}{3}.$$

Chapter 4: Multiple Integrals and its Applications

Sample Worked Examples

4.2 Evaluation of Double Integrals

Consider the double integral of the form

$$\int_{x=x_1}^{x_2} \int_{y=y_1}^{y_2} f(x, y) dx dy.$$

Case I. When y_1, y_2 are functions of x and x_1, x_2 are constants, integrate first w.r.t. y (holding x fixed) between y_1, y_2 , then w.r.t. x between x_1, x_2 :

$$I_1 = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dy dx.$$

Here the integration is carried out from the inner rectangle to the outer.

Case II. When x_1, x_2 are functions of y and y_1, y_2 are constants, integrate first w.r.t. x , then w.r.t. y :

$$I_2 = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) dx dy.$$

Case III. When all four limits are constants ($x_1 = a, x_2 = b, y_1 = c, y_2 = d$), the order of integration is immaterial:

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

4.3 Worked Examples — Double Integrals

Example 4.1: Evaluate $\int_0^1 \int_0^6 xy dx dy$.

Solution:

Integrating first w.r.t. x (keeping y constant),

$$I = \int_{y=0}^1 \int_{x=0}^6 xy dx dy = \int_0^1 y \left[\frac{x^2}{2} \right]_0^6 dy \quad \dots(1)$$

$$= \int_0^1 y \cdot \frac{36}{2} dy = \int_0^1 18y dy \quad \dots(2)$$

$$= 18 \left[\frac{y^2}{2} \right]_0^1 = 18 \cdot \frac{1}{2}. \quad \dots(3)$$

$$\boxed{I = 9}$$

Example 4.2: Evaluate $\int_0^3 \int_1^2 xy(1+x+y) dx dy$.

Now integrating this w.r.t. x from 0 to 2:

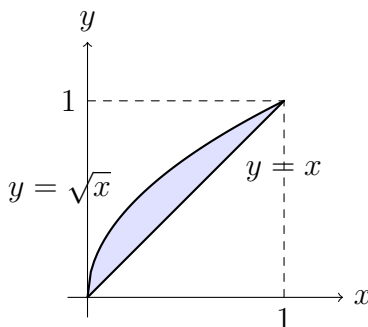
$$I = \int_0^2 (3x^3 + 9x) dx = 3 \left[\frac{x^4}{4} \right]_0^2 + 9 \left[\frac{x^2}{2} \right]_0^2 = 3(4) + 9(2) = 12 + 18. \quad \dots(3)$$

$$\boxed{I = 30}$$

Example 4.4: Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$.

Solution:

For $0 \leq x \leq 1$ we have $x \leq \sqrt{x}$, so the region R lies between the line $y = x$ and the curve $y = \sqrt{x}$.



Integrating first w.r.t. y (keeping x constant):

$$\int_x^{\sqrt{x}} xy \, dy = x \left[\frac{y^2}{2} \right]_x^{\sqrt{x}} = x \left(\frac{x}{2} - \frac{x^2}{2} \right) = \frac{x^2}{2} - \frac{x^3}{2}. \quad \dots(1)$$

Now integrate (1) w.r.t. x from 0 to 1:

$$I = \int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{2} \right) dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_0^1 - \frac{1}{2} \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{2} \cdot \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{6} - \frac{1}{8}. \quad \dots(2)$$

$$\boxed{I = \frac{1}{24}}$$

Example 4.5: Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$.

Solution:

The region of integration is the quarter of the unit disc $x^2 + y^2 \leq 1$ lying in the first quadrant.

Integrating (1) w.r.t. x from 0 to $2a$:

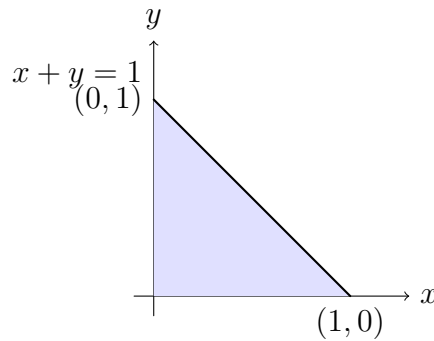
$$I = \int_0^{2a} \frac{x^5}{32a^2} dx = \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a} = \frac{(2a)^6}{192a^2} = \frac{64a^6}{192a^2}. \quad \dots(2)$$

$$\boxed{I = \frac{a^4}{3}}$$

Example 4.7: Evaluate $\iint_R xy \, dx \, dy$ over the region R in the first quadrant with $x + y \leq 1$.

Solution:

R is the triangle with vertices $(0, 0), (1, 0), (0, 1)$.



For fixed $x \in [0, 1]$, y runs from 0 to $1 - x$. Integrating first w.r.t. y :

$$\int_0^{1-x} xy \, dy = x \left[\frac{y^2}{2} \right]_0^{1-x} = \frac{x(1-x)^2}{2}. \quad \dots(1)$$

Expanding $(1 - x)^2 = 1 - 2x + x^2$, (1) becomes $\frac{x - 2x^2 + x^3}{2}$. Integrating w.r.t. x from 0 to 1:

$$I = \frac{1}{2} \int_0^1 (x - 2x^2 + x^3) \, dx = \frac{1}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{2} \cdot \frac{6 - 8 + 3}{12} = \frac{1}{2} \cdot \frac{1}{12}. \quad \dots(2)$$

$$\boxed{I = \frac{1}{24}}$$

Example 4.8: Evaluate $\iint xy \, dx \, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

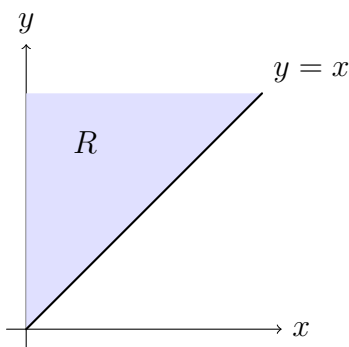
Solution:

4.4 Worked Problems — Change of Order of Integration

Example 4.11: By changing the order of integration, evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$.

Solution:

In the given integral, for fixed x , y ranges from x to ∞ , while x ranges from 0 to ∞ . Hence the region is $\{(x, y) : 0 \leq x < \infty, x \leq y < \infty\}$, the region above the line $y = x$ in the first quadrant.



Changing the order of integration (y from 0 to ∞ ; for fixed y , x ranges from 0 to y):

$$I = \int_{y=0}^{\infty} \int_{x=0}^y \frac{e^{-y}}{y} dx dy = \int_0^{\infty} \frac{e^{-y}}{y} [x]_0^y dy = \int_0^{\infty} \frac{e^{-y}}{y} \cdot y dy. \quad \dots(1)$$

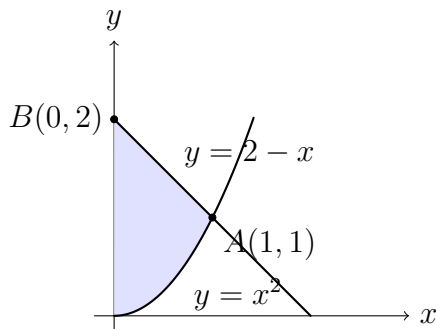
$$I = \int_0^{\infty} e^{-y} dy = \left[-e^{-y} \right]_0^{\infty} = 0 - (-1). \quad \dots(2)$$

$$\boxed{I = 1}$$

Example 4.12: Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy dy dx$ and hence evaluate it.

Solution:

The curves $y = x^2$ and $y = 2 - x$ meet where $x^2 + x - 2 = 0$, i.e. at $x = 1, y = 1$ (and $x = -2$, outside $[0, 1]$). At $x = 0$: $y = x^2 = 0$ and $y = 2 - x = 2$.



Integrating first w.r.t. y (which simply contributes a factor b , since x^2 does not depend on y):

$$\int_0^b x^2 dy = x^2 [y]_0^b = bx^2. \quad \dots(1)$$

Integrating (1) w.r.t. x from 0 to a :

$$I_y = b \int_0^a x^2 dx = b \left[\frac{x^3}{3} \right]_0^a. \quad \dots(2)$$

$$\boxed{I_y = \frac{a^3 b}{3}}$$

4.6 Evaluation of Triple Integrals

Consider a triple integral of the form

$$\iiint_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z) dx dy dz.$$

If x_1, x_2 are constants, y_1, y_2 are constants or functions of x , and z_1, z_2 are constants or functions of x, y , the integral is evaluated from the innermost bracket outward: first integrate w.r.t. z (holding x, y fixed), then w.r.t. y (holding x fixed), and finally w.r.t. x :

$$I = \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) dz \right] dy dx.$$

4.7 Worked Examples — Triple Integrals

Example 4.15: Evaluate $\int_0^1 \int_0^x \int_{x-z}^{x+z} (x + y + z) dy dz dx.$

Solution:

Integrating first w.r.t. y , holding x, z fixed:

$$\int_{x-z}^{x+z} (x + y + z) dy = (x + z) [y]_{x-z}^{x+z} + \left[\frac{y^2}{2} \right]_{x-z}^{x+z} = (x + z)(2z) + \frac{(x + z)^2 - (x - z)^2}{2}. \quad \dots(1)$$

Since $(x + z)^2 - (x - z)^2 = 4xz$, (1) simplifies to

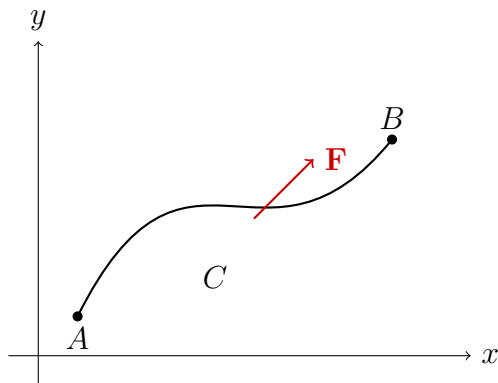
$$2xz + 2z^2 + 2xz = 4xz + 2z^2. \quad \dots(2)$$

Now integrate (2) w.r.t. z from 0 to x :

$$\int_0^x (4xz + 2z^2) dz = 4x \left[\frac{z^2}{2} \right]_0^x + 2 \left[\frac{z^3}{3} \right]_0^x = 2x \cdot x^2 + \frac{2}{3} x^3 = \frac{8}{3} x^3. \quad \dots(3)$$

Chapter 5: Vector Integral Calculus

Sample Worked Examples



If $\mathbf{F}$ represents a force, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the work done moving a particle along C ; if $\mathbf{F} = \nabla\phi$ is conservative, this work depends only on the endpoints, and $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C . If $\mathbf{F}$ is a fluid velocity field, $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is the *circulation* of $\mathbf{F}$ around C .

5.3 Problems on Line Integrals

Example 5.1: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$, where C is the straight segment from $(0,0)$ to $(1,2)$.

Solution:

Parametrise C by $\mathbf{r}(t) = (t, 2t)$, $0 \leq t \leq 1$, so

$$d\mathbf{r} = (1, 2) dt, \quad \mathbf{F} = (y, x) = (2t, t). \quad \dots(1)$$

Then

$$\mathbf{F} \cdot d\mathbf{r} = (2t)(1) dt + (t)(2) dt = 4t dt. \quad \dots(2)$$

Integrating (2) from $t = 0$ to $t = 1$:

$$I = \int_0^1 4t dt = 4 \left[\frac{t^2}{2} \right]_0^1 = 2. \quad \dots(3)$$

(Check: $\mathbf{F} = \nabla(xy)$ is conservative, so $I = [xy]_{(0,0)}^{(1,2)} = 1 \cdot 2 - 0 = 2$, confirming the result.)

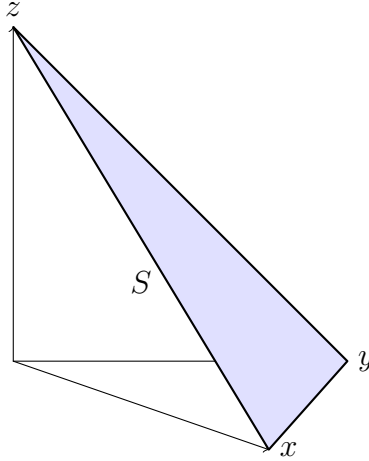
$$\boxed{I = 2}$$

Example 5.2: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$, where C is the arc $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution:

Parametrise by $x = t$, $y = t^2$, $t : 0 \rightarrow 1$, so

$$d\mathbf{r} = (1, 2t) dt, \quad \mathbf{F} = (x^2, xy) = (t^2, t \cdot t^2) = (t^2, t^3). \quad \dots(1)$$



Solution:

Here $z = 1 - x - y$, so $\partial z / \partial x = -1$, $\partial z / \partial y = -1$, giving

$$dS = \sqrt{1 + (-1)^2 + (-1)^2} dA = \sqrt{3} dA, \quad \dots(1)$$

with R the triangle $x + y \leq 1$, $x, y \geq 0$ in the xy -plane. Using (1) and substituting $z = 1 - x - y$,

$$\iint_S z dS = \sqrt{3} \iint_R (1 - x - y) dA = \sqrt{3} \int_{x=0}^1 \int_{y=0}^{1-x} (1 - x - y) dy dx. \quad \dots(2)$$

Integrating w.r.t. y first, with $u = 1 - x$:

$$\int_0^u (u - y) dy = \left[uy - \frac{y^2}{2} \right]_0^u = u^2 - \frac{u^2}{2} = \frac{u^2}{2}. \quad \dots(3)$$

Integrating (3) w.r.t. x from 0 to 1 (so $u = 1 - x$ runs from 1 to 0):

$$\int_0^1 \frac{(1-x)^2}{2} dx = \frac{1}{2} \left[-\frac{(1-x)^3}{3} \right]_0^1 = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}. \quad \dots(4)$$

Combining (2) and (4),

$$\boxed{\iint_S z dS = \sqrt{3} \cdot \frac{1}{6} = \frac{\sqrt{3}}{6}}$$

Example 5.4: Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the part of the plane $x + y + z = 1$ in the first octant.

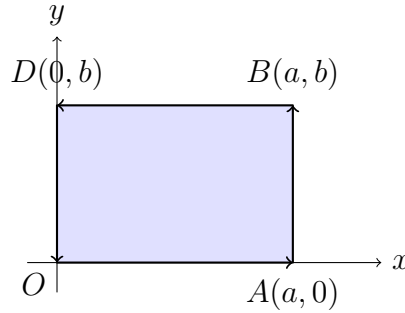
Solution:

The plane $g \equiv x + y + z = 1$ has gradient $\nabla g = (1, 1, 1)$, so the unit normal is

$$\hat{\mathbf{n}} = \frac{(1, 1, 1)}{\sqrt{3}}. \quad \dots(1)$$

5.11 Problems on Green's Theorem

Example 5.5: Verify Green's theorem for $\mathbf{F} = (x^2 + y^2)\mathbf{i} - 2xy\mathbf{j}$ over the rectangle $0 \leq x \leq a, 0 \leq y \leq b$.



Solution:

Here $\phi = x^2 + y^2$, $\psi = -2xy$, so

$$\frac{\partial \psi}{\partial x} = -2y, \quad \frac{\partial \phi}{\partial y} = 2y, \quad \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = -4y. \quad \dots(1)$$

Double integral (RHS). Using (1),

$$\iint_R (-4y) dx dy = \int_0^a \int_0^b (-4y) dy dx = \int_0^a \left[-2y^2 \right]_0^b dx = \int_0^a (-2b^2) dx = -2ab^2. \quad \dots(2)$$

Line integral (LHS). Traverse $O(0, 0) \rightarrow A(a, 0) \rightarrow B(a, b) \rightarrow D(0, b) \rightarrow O$.

Along OA ($y = 0, dy = 0, x : 0 \rightarrow a$): $\int_{OA} (x^2 + y^2) dx = \int_0^a x^2 dx = \frac{a^3}{3}$.

Along AB ($x = a, dx = 0, y : 0 \rightarrow b$): $\int_{AB} (-2xy) dy = \int_0^b (-2ay) dy = -ab^2$.

Along BD ($y = b, dy = 0, x : a \rightarrow 0$): $\int_{BD} (x^2 + b^2) dx = \int_a^0 (x^2 + b^2) dx = -\left(\frac{a^3}{3} + ab^2\right)$.

Along DO ($x = 0, dx = 0, y : b \rightarrow 0$): $\int_{DO} (-2 \cdot 0 \cdot y) dy = 0$.

Adding all four,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0 = -2ab^2. \quad \dots(3)$$

Since (2) and (3) agree, Green's theorem is verified.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -2ab^2$$

Solution:

By Green's theorem, for ϕ, ψ continuously differentiable in R ,

$$\oint_C (\phi dx + \psi dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx dy. \quad \dots(1)$$

Choose

$$\phi = -\frac{y}{2}, \quad \psi = \frac{x}{2}. \quad \dots(2)$$

Then

$$\frac{\partial \psi}{\partial x} = \frac{1}{2}, \quad \frac{\partial \phi}{\partial y} = -\frac{1}{2}, \quad \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = \frac{1}{2} + \frac{1}{2} = 1. \quad \dots(3)$$

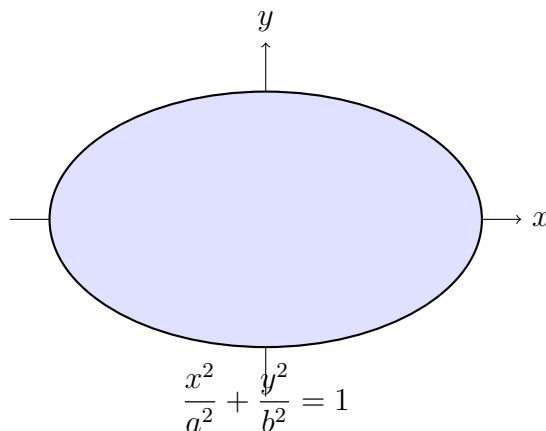
Substituting (2) and (3) into (1),

$$\oint_C \left(-\frac{y}{2} dx + \frac{x}{2} dy \right) = \iint_R 1 dx dy = \text{Area}(R). \quad \dots(4)$$

That is,

$$\text{Area} = \frac{1}{2} \oint_C (x dy - y dx)$$

Example 5.10: Find the area of an ellipse of semi-axes a, b using Green's theorem.



Solution:

Parametrise the ellipse by $x = a \cos \theta$, $y = b \sin \theta$, $0 \leq \theta \leq 2\pi$, so

$$dx = -a \sin \theta d\theta, \quad dy = b \cos \theta d\theta. \quad \dots(1)$$

Using the area formula of Example 5.9 and (1),

$$x dy - y dx = a \cos \theta \cdot b \cos \theta d\theta - b \sin \theta \cdot (-a \sin \theta) d\theta = ab (\cos^2 \theta + \sin^2 \theta) d\theta = ab d\theta. \quad \dots(2)$$

Therefore, integrating (2) around the full ellipse,

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} ab d\theta = \frac{1}{2} \cdot ab \cdot 2\pi. \quad \dots(3)$$

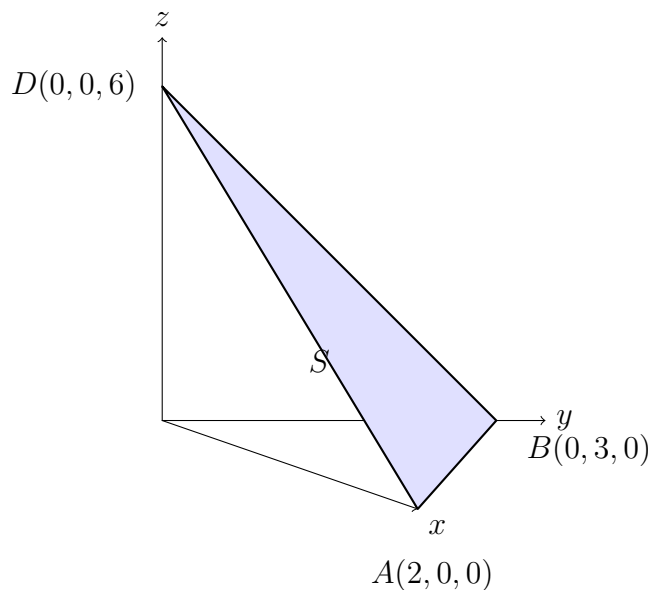
(When $a = b = r$ this reduces to the familiar area of a circle, πr^2 .)

$$\text{Area} = \pi ab$$

(Direct line integration along OA , AB , BO gives 1, 0, and $-1 + \frac{\pi}{4} + \frac{2}{\pi}$ respectively, which sum to the same total, confirming the result.)

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{2}{\pi} + \frac{\pi}{4} \approx 1.422$$

Example 5.12: Use Stokes' theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = (x + y)\mathbf{i} + (2x - z)\mathbf{j} + (y + z)\mathbf{k}$, C the triangle $A(2, 0, 0)$, $B(0, 3, 0)$, $D(0, 0, 6)$.



Solution:

First compute the curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + y & 2x - z & y + z \end{vmatrix} = \mathbf{i}(1 + 1) - \mathbf{j}(0 - 0) + \mathbf{k}(2 - 1) = 2\mathbf{i} + \mathbf{k}. \quad \dots(1)$$

The plane through A , B , D has intercept form $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$, i.e. $g \equiv 3x + 2y + z = 6$, with

$$\nabla g = (3, 2, 1), \quad |\nabla g| = \sqrt{14}, \quad \hat{\mathbf{n}} = \frac{(3, 2, 1)}{\sqrt{14}}. \quad \dots(2)$$

Using $dS = \frac{dx dy}{|\hat{\mathbf{n}} \cdot \mathbf{k}|}$, where R is the projection of the triangle on the xy -plane, and combining with (1) and (2), the surface integral becomes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dS = \iint_R (\nabla \times \mathbf{F}) \cdot \nabla g dA = \iint_R (2, 0, 1) \cdot (3, 2, 1) dA. \quad \dots(3)$$