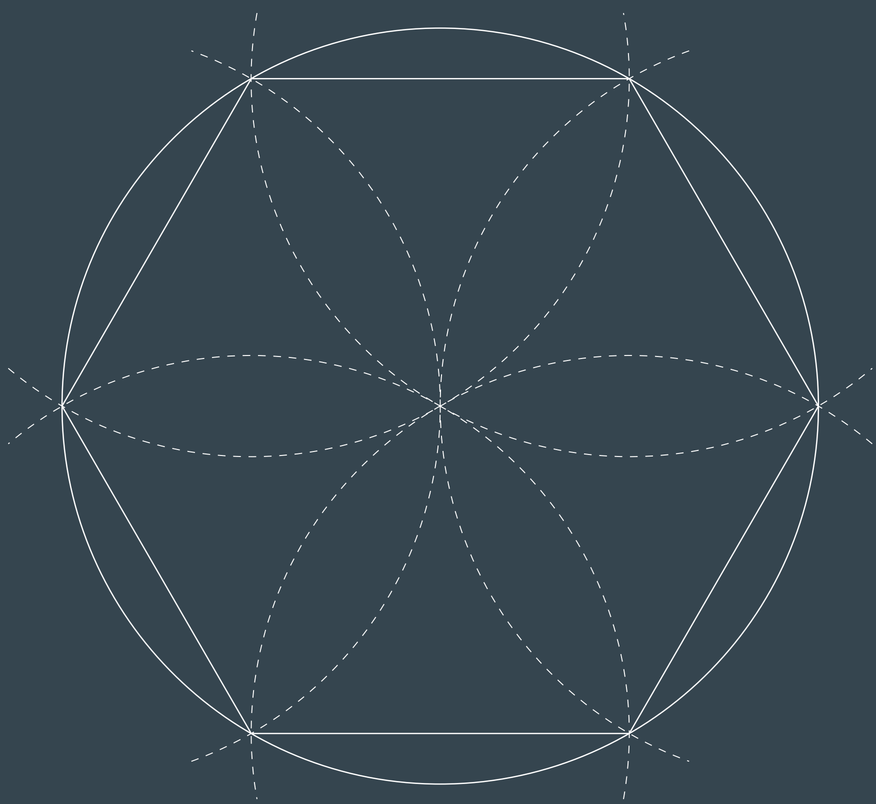
Mathematics for the IGCSE

And a bit more



Lucas Virgili

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Updated 08/04/2021

Introduction

This book may be different from what you are used to. It does not have exercises! What is its goal, you should ask?

As a teacher, I found that many textbooks with a good selection of exercises are available already. However, I believe they miss two things: a more structured explanation of the topics and some extension for those that are interested in understanding more about *why* things work.

Thus, I started writing this book, which tries to emulate how I approach the topics while teaching them to my students. Not only that, I have added, whenever possible, justification and more mathematical theory for the facts used. This extension is usually at the end of the chapter in a section called "Formality after taste", which you can skip if you are not interested, as it will not be accessed.

I hope this book can help you in achieving your goals.

Acknowledgments

I would like to thank all the students that proof read and gave feedback on initial versions of this project. In particular to Laura, Sofia, Victoria, Gabriela and Luiza. I also thank all the motivation my students have given me to continue writing!

More importantly, I thank my wife for, well, everything. Not only the book would not be here if not for you.

Updates on this version

- Added an example of quadratic sequences.

Contents

[1] I highly recommend you to ignore this. It's always better to think than to memorize.

41. Circle theorems

41.1. Why learn circle theorems

I honestly don't know how to justify learning this in a 'practical' sense. It is fun to solve these kind of problems, and is definitely useful to prove the theorems, but I have no justification here. Sorry.

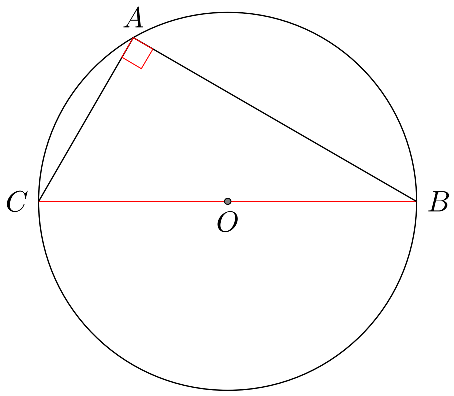
Given that proving the theorems is the only thing outside memorising them (you *do have* to memorise them!), I highly recommend reading the 'Formality after taste' session to see how each theorem is proved.

41.2. The theorems

41.2.1. Inscribed triangles with a side on a diameter

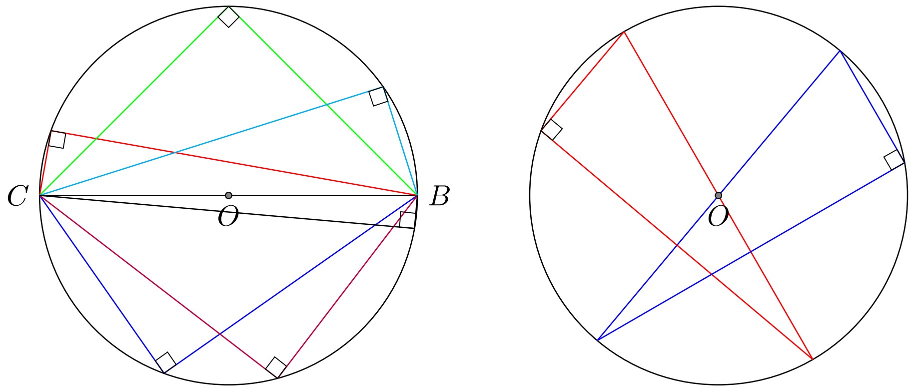
Theorem. Every inscribed triangle in which the largest side is a diameter of a circle is a right triangle.

For instance, in the triangle ABC below, where BC is a diameter of the circle with centre O, the angle $BAC = 90°$:



I have colored BC red to show you the important part: the largest side *has to be* a diameter of the circle.

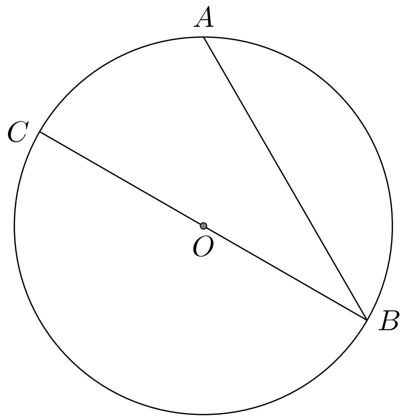
Notice that it does not matter *where* the vertex nor in the diameter is:



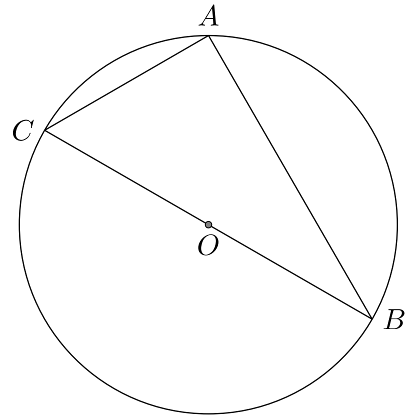
If you see a triangle inscribed in a circle and one of its sides is on a diameter, use this very useful result: it is a right triangle.

Inscribed triangle with side on a diameter example

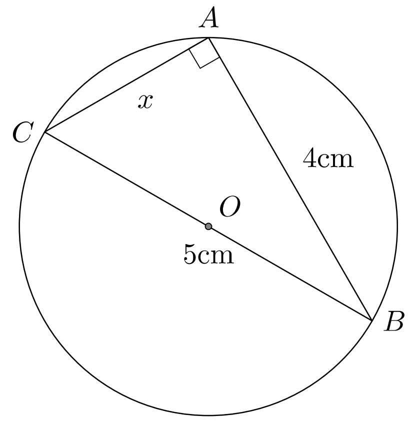
On the image below, the radius of the circle is equal to 2.5cm. The length of the segment AB is 4cm and BC is a diameter of the circle. Find the length of the segment AC.



Solution: Let's draw the segment AC:



We have that the triangle ABC has one side on the diameter BC, therefore it is a right triangle and $\angle BAC = 90°$. We know that the segment $AB = 4$cm and, given that the radius of the circle is 2.5cm, the diameter is equal to $2 \times 2.5 = 5$cm. Adding this information to the drawing and calling the length of the segment AC x:



Being a right triangle we can use our beloved Pythagoras's theorem:

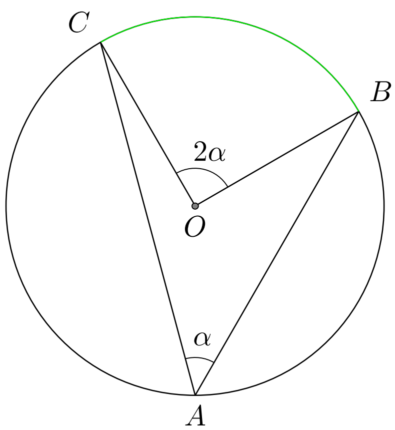
$$
\begin{aligned}
AB^2 + AC^2 &= BC^2 && \text{Pythagoras's} \\
4^2 + x^2 &= 5^2 \\
16 + x^2 &= 25 \\
16 - 16 + x^2 &= 25 - 16 && \text{Subtracting 16 on both sides} \\
x^2 &= 9 \\
\sqrt{x^2} &= \sqrt{9} && \text{Square rooting both sides} \\
x &= 3
\end{aligned}
$$

Thus, $x = AC = 3$cm.

41.2.2. Inscribed angles

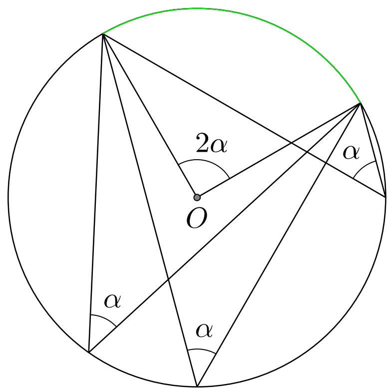
Theorem. Every inscribed angle is equal to half the size of the central angle that determines the same arc.

A picture is worth 18 words:



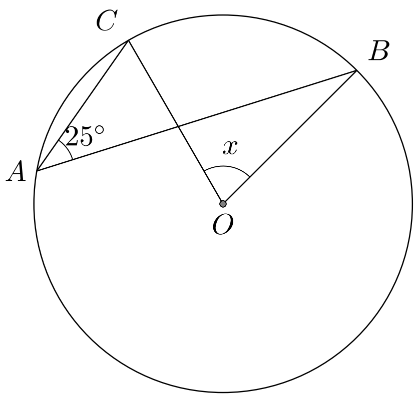
I like identifying these angles by finding the arcs on the circle each of them 'look': I have painted the arc in green in this case. Notice that both the central angle BOC and the inscribed angle BAC 'look at' the same arc.

Again, it does not matter *where* the inscribed angle is: if it is 'looking' at the same arc as a central angle, it measures half the central:

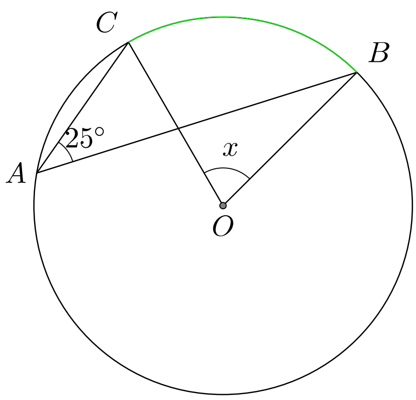


Inscribed angle with central angle example I

In the following circle, find the value of x.



Solution: Notice that both angles BAC and BOC 'look' at the same arc:



Therefore, we can use the inscribed angle theorem: we know that the central angles is twice the size of the inscribed angle:

$$\text{central} = 2 \times \text{inscribed} \qquad \text{The theorem}$$

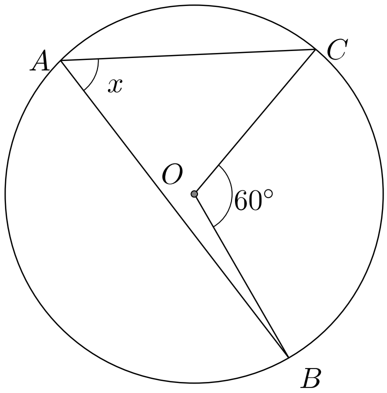
$$x = 2 \times 25$$

$$x = 50$$

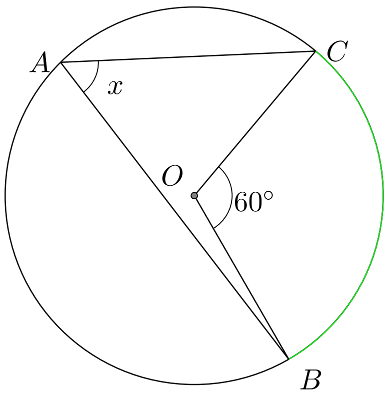
So we have that $x = 50°$.

Inscribed angle with central angle example II

In the following circle, find the value of x.



Solution: Again, notice that both angle BAC and angle OBC 'look' at the same arc:



We can, then, use the inscribed angle theorem:

$$\begin{aligned} \text{central} &= 2 \times \text{inscribed} && \text{The theorem} \\ 60 &= 2 \times x \\ \frac{60}{2} &= \frac{2x}{2} && \text{Dividing both sides by 2} \\ 30 &= x \end{aligned}$$

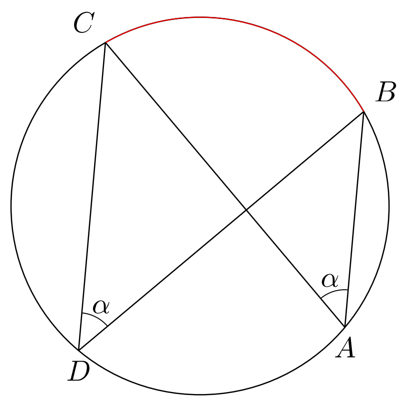
Thus, $x = 30°$.

41.2.2.1. Corollary: angles in the same arc, a.k.a. the 'bow-tie property'

Remember that a corollary is a result that follows easily from another. Here, it follows easily from the inscribed angle theorem the 'bow-tie property':

Corollary. Inscribed angles that are on the same arc are equal.

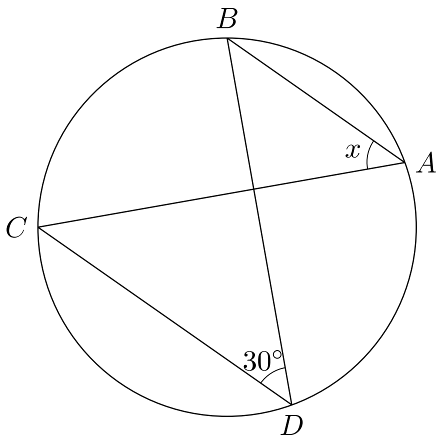
Let's see why it is called the 'bow-tie property':



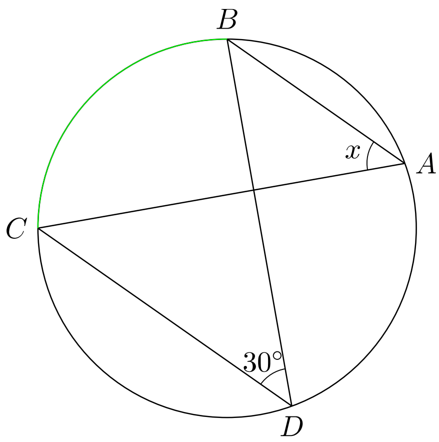
Notice that both angle BAC and angle BDC 'look' at the arc BC and are equal.

'Bow-tie property' example

Find the value of x.



Solution: I like marking the arc both angles are looking to be sure I can use the ‘bow-tie’:

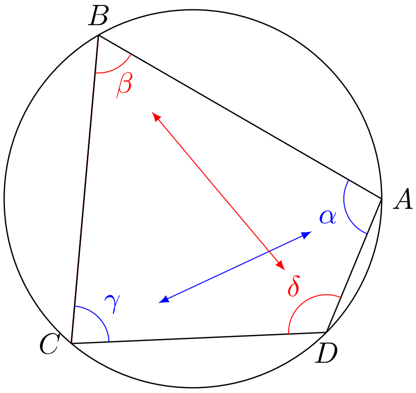


Thus, we know that $x = \angle BAC = \angle CDB$. Given that $\angle CDB = 30°$, we have that $x = 30°$.

41.2.3. Inscribed quadrilaterals (cyclic quadrilaterals)

Theorem. Opposite angles in inscribed quadrilaterals add up to $180°$.

As usual, our drawing:



In this the quadrilateral $ABCD$ is inscribed (or cyclical[1]), and the angles ABC and ADC are opposite, so we have:

$$\angle ABC + \angle ADC = 180^\circ \qquad \text{Theorem: inscribed quadrilaterals}$$

$$\beta + \delta = 180^\circ$$

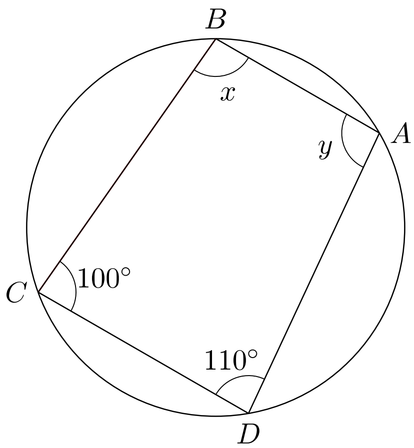
Not only, the angles BAD and BCD are opposite, thus:

$$\angle BAD + \angle BCD = 180^\circ \qquad \text{Theorem: inscribed quadrilaterals}$$

$$\alpha + \gamma = 180^\circ$$

Inscribed quadrilateral example

Find the values of x and y.



[1] I have to say I *hate* the 'cyclic quadrilateral' name. Every quadrilateral is 'cyclic', in the sense you can start walking in a vertex and reach it back.

Solution: We have an inscribed quadrilateral, so we know that opposite angles add up to 180°. Angles BAD and BCD are opposite:

$$\begin{aligned} \angle BAD + \angle BCD &= 180^\circ && \text{Inscribed quadrilateral} \\ y + 100 &= 180 \\ y + 100 - 100 &= 180 - 100 && \text{Subtrating 100 on both sides} \\ y &= 80 \end{aligned}$$

The same thing for angles ABD and ADC:

$$\begin{aligned} \angle ABD + \angle ADC &= 180^\circ && \text{Inscribed quadrilateral} \\ x + 110 &= 180 \\ x + 110 - 110 &= 180 - 110 && \text{Subtracting 110 on both sides} \\ x &= 70 \end{aligned}$$

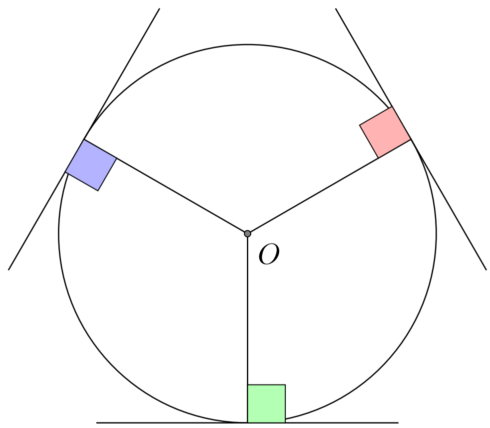
So we have that $x = 70^\circ$ and $y = 80^\circ$.

41.3. Tangent properties

These are useful to know[2].

41.3.1. The angle between a tangent to a circle and a radius is always 90°

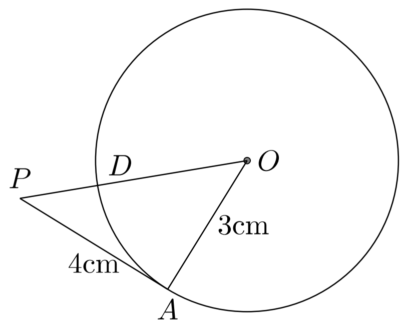
Remember that a tangent is a straight line that only touches the circle **in one point**. Drawing version:



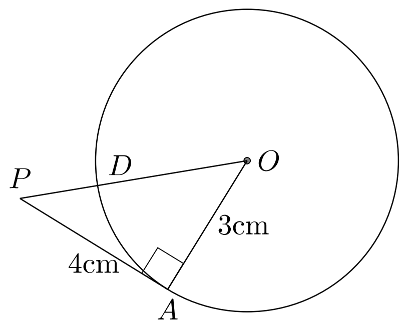
[2]I'll leave the proofs for these properties to you as exercises.

Tangents are perpendicular to radius example

Find the length of the segment PD, given that OA is a radius and has length 3cm.



Solution: We know that tangents are perpendicular to radiuses, thus we know that angle OAP is $90°$:



We can now apply Pythagoras's in triangle OAP:

$$
\begin{aligned}
OP^2 &= OA^2 + AP^2 && \text{Pythagoras's} \\
OP^2 &= 3^2 + 4^2 \\
OP^2 &= 9 + 16 \\
OP^2 &= 25 \\
\sqrt{OP^2} &= \sqrt{25} && \text{Squaring both sides} \\
OP &= 5
\end{aligned}
$$

Now, to find PB, notice that the segment OD is also a radius and measures 3. We have, then:

$$PD = OP - OD$$

$$PD = 5 - 3$$

$$PD = 2$$

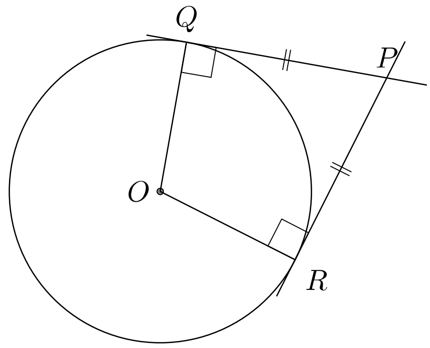
Thus, $PD = 2\text{cm}$.

41.3.2. Tangents from the same point have equal length

Given a point P outside the circle, and the two (why two?) points Q and R defined by the intersection of the tangents and the circle, we have that

$$PQ = PR$$

Our drawing:



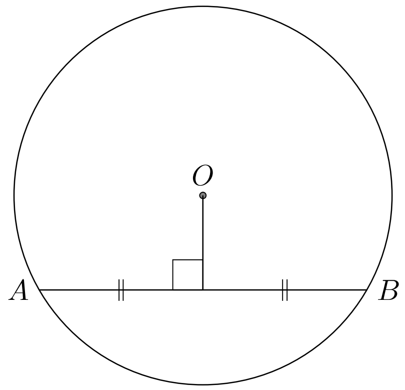
41.4. Chords properties

Also useful to know[3].

41.4.1. Chords are bisected by the perpendicular line from the centre

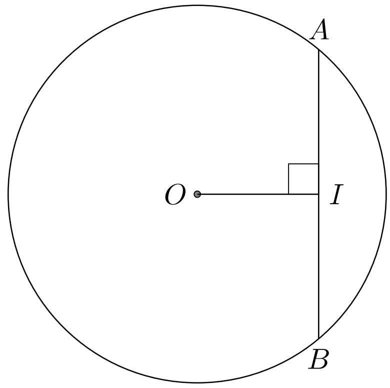
This one is very useful. It says that if you have a chord in a circle, and a radius crosses this chord in a right angle, then the radius bisects the chord. Drawing version:

[3] Also leaving the proofs to you.

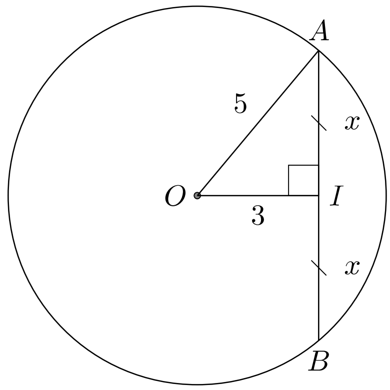


Perpendicular radius bisects chords example

Find the length of AB, given that the radius of the circle is 5cm and the segment OI measures 3cm.



Solution: This is a classic. Let's draw the radius OA and make $AI = x$ and remember that if the radius meets the chord at a right angle, it bisects it:



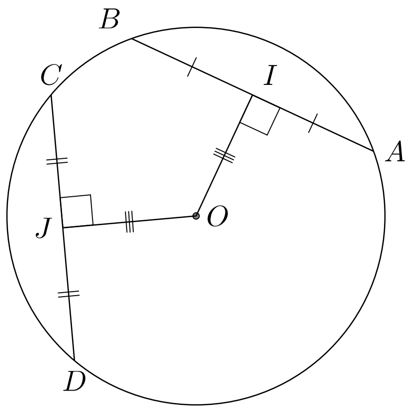
Let's use our favourite theorem, Pythagoras's, in triangle AOI:

$$\begin{aligned}
OA^2 &= AI^2 + OI^2 && \text{Pythagoras's} \\
5^2 &= x^2 + 3^2 \\
25 &= x^2 + 9 \\
25 - 9 &= x^2 + 9 - 9 && \text{Subtracting 9 on both sides} \\
16 &= x^2 \\
\sqrt{16} &= \sqrt{x^2} && \text{Square rooting both sides} \\
4 &= x
\end{aligned}$$

We know that $x = AI = 4$, but we want the length of the full chord AB. Now, given that AI is half the full chord, we have that $AB = 2x = 8$cm.

41.4.2. Chords of equal length are at the same distance from the centre

This one may sound complicated, but it is very simple. I'll give the drawing first this time.



Here, we have that if $AB = CD$, then $OI = OJ$. That's it.

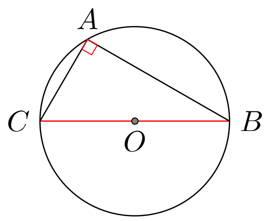
41.5. Exam hints

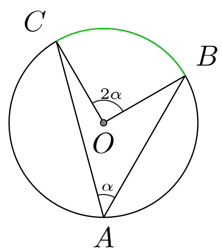
When you are presented with a circles theorem question, you have to be selectively ignorant: you almost always will be using many of the theorems and properties, and it is important to practice a lot in order to visualise which parts to focus and which parts to ignore.

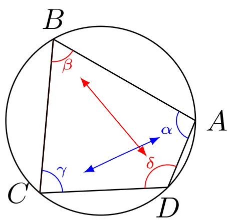
Be aware of the classic mistakes:

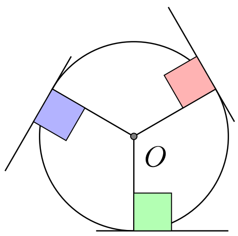
- A quadrilateral has to have **all 4** vertices on the circle to be an inscribed (cyclic) quadrilateral. If any is not on the circle, you cannot use the theorem that opposite vertices add up to 180°;
- The inscribed angle theorem only applies to **central angles**, if you have an angle which is not central it does not follow that it is twice the size of the inscribed angle;
- To be a right triangle, the largest side **has to be a diameter**.

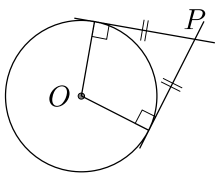
Summary

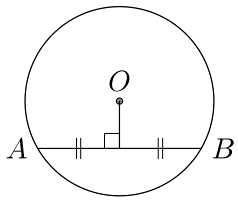


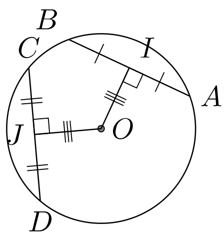










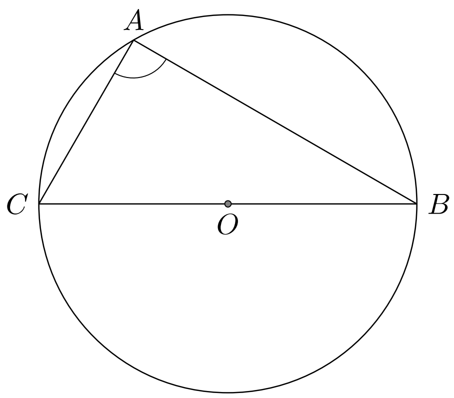


Formality after taste

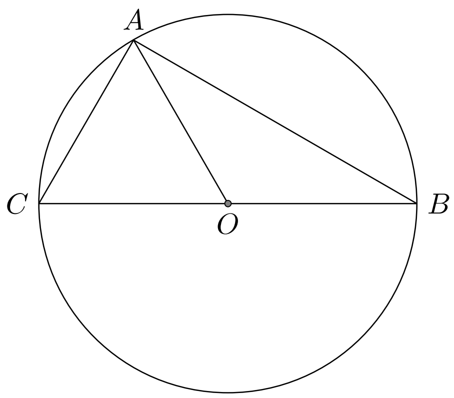
Proof of the right inscribed triangles

Theorem. Every inscribed triangle in which the largest side is a diameter of a circle is a right triangle.

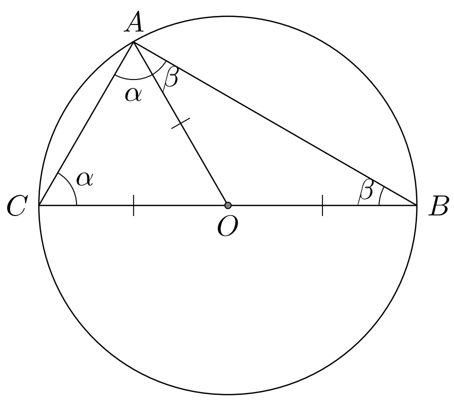
Proof. Let's take our triangle ABC with side BC on a diameter:



We want to prove that the angle CAB is a right angle. Let's start by dividing ABC into two triangles using the radius OA:



Notice that $OC = OA = OB$ as they all radiuses of the circle. Now, because of this, the triangles OAC and OAB are isosceles and, therefore, we know that $\angle OCA = \angle OAC$ and that $\angle OAB = \angle OBA$. Let's say that $\angle OCA = \angle OAC = \alpha$ and $\angle OAB = \angle OBA = \beta$:



Notice that the angle CAB is equal to $\alpha+\beta$. Now, adding all the angles in the triangle ABC we should get $180°$:

$$\begin{aligned}
\angle OAC + \angle OAC + \angle OAB + \angle OBA &= 180° && \text{Sum of angles in } \triangle ABC \\
\alpha + \alpha + \beta + \beta &= 180 \\
2\alpha + 2\beta &= 180 \\
2(\alpha + \beta) &= 180 && \text{Factorising} \\
\frac{2(\alpha + \beta)}{2} &= \frac{180}{2} \\
\alpha + \beta &= 90
\end{aligned}$$

Thus, we have shown that $\alpha + \beta = 90°$, and given that angle $CAB = \alpha + \beta$, it follows that the angle CAB is $90°$ and that the triangle ABC is a right triangle.

□

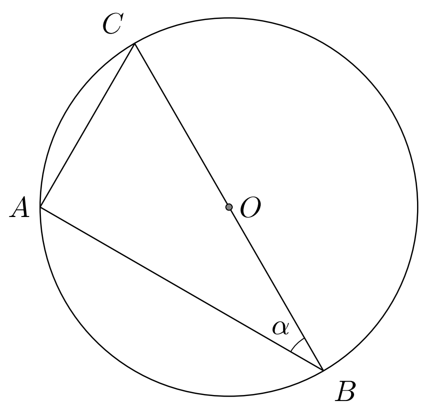
There is a way to prove this theorem using the inscribed angle theorem which is very nice. Give it a try!

Proof of the inscribed angle theorem

Theorem. Every inscribed angle is equal to half the size of the central angle that determines the same arc.[4]

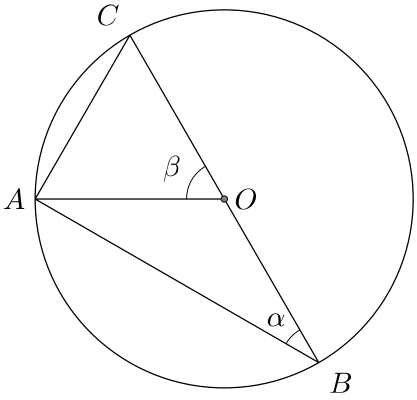
Proof. Let's first use a special case of the theorem. This means we'll start with a simpler version of what we want to prove, and later use this 'weaker' result to help us prove the 'full' one we are actually interested in.

The special case is when one of the chords that determine the arc is a diameter of the circle:

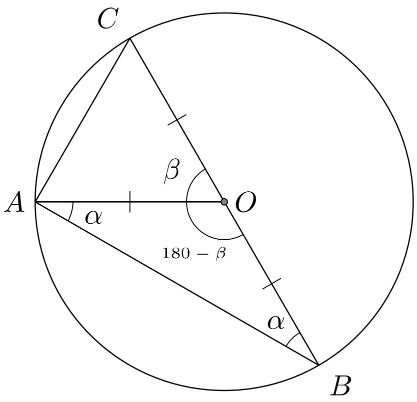


[4] Khan Academy has a great video on this proof: `https://www.khanacademy.org/math/geometry/hs-geo-circles/modal/v/inscribed-and-central-angles`

Our aim here is to prove that the angle ABC is equal to half the size of the angle AOC. Let's call $\angle AOC = \beta$:



Here $OA = OB = OC$ as they are all radiuses of the circle. Thus, we have that $\angle OAB = \angle ABO = \alpha$ and that $\angle AOB = 180 - \beta$, as the angles AOC and AOB are angles on a line:



Now, if we add all the angles in triangle OAB we obtain 180°:

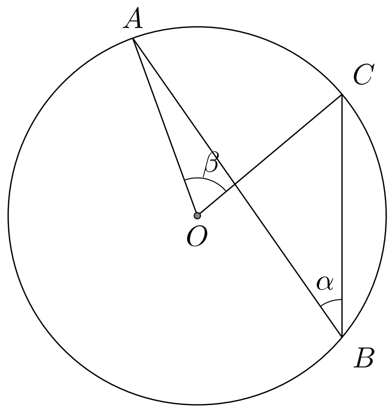
$$\angle OAB + \angle AOB + \angle OBA^\circ = 180 \qquad \text{Sum of angles in a triangle}$$

$$\alpha + 180 - \beta + \alpha = 180$$

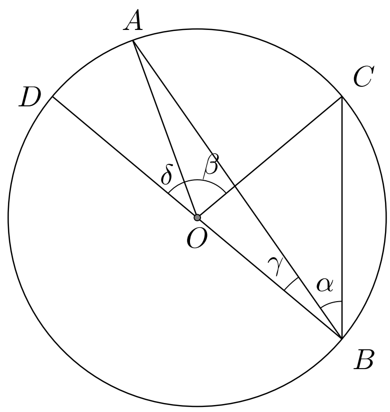
$$2\alpha - \beta + \cancel{180} = \cancel{180}$$

$$2\alpha = \beta$$

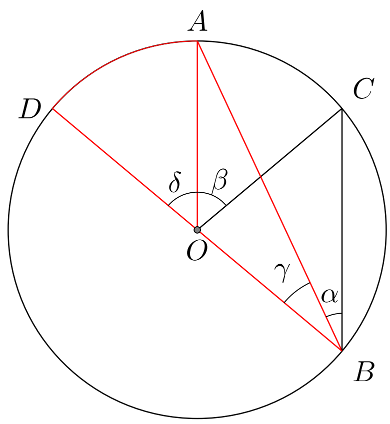
There we go: we have proved that the central angle, β, is twice as big as the inscribed angle α, in the particular case one of the legs of the angle is a diameter. Let's try a general case now, such as:



We know how to solve this problem when we have a diameter with one ends on the vertex of our inscribed angle, as we have just proved it. So let's draw a diameter and label some angles:



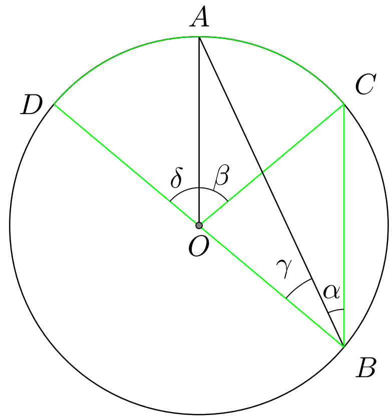
Now let's use abstraction in the sense of 'ignoring' and focus on the coloured parts:



Notice that we have both angle ABD and angle AOD are 'looking' at the same red arc, and that one of the legs of the angle ABD is the diameter BD. Therefore, we can use the result we just proved about inscribed angles with one leg being a diameter:

$$\begin{aligned} \text{central} &= 2 \times \text{inscribed} && \text{In the special case we just proved} \\ \delta &= 2\gamma \end{aligned}$$

Now focus on the green parts:



Here we have that both angle CBO and angle COD are 'looking' at the same green arc. Not only that, one of the legs of angle CBD is the diameter BD, so we can use the result we just proved again:

$$\begin{aligned} \text{central} &= 2 \times \text{inscribed} && \text{Still the special case} \\ \delta + \beta &= 2 \times (\alpha{+}\gamma) \\ \delta + \beta &= 2\alpha + 2\gamma \\ 2\gamma + \beta &= 2\alpha + 2\gamma && \delta = 2\gamma \\ \cancel{2\gamma} + \beta &= 2\alpha + \cancel{2\gamma} \\ \beta &= 2\alpha \end{aligned}$$

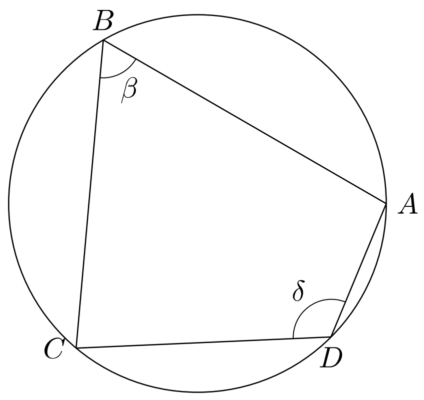
And there we go, we have proved that $\beta = 2\alpha$.

□

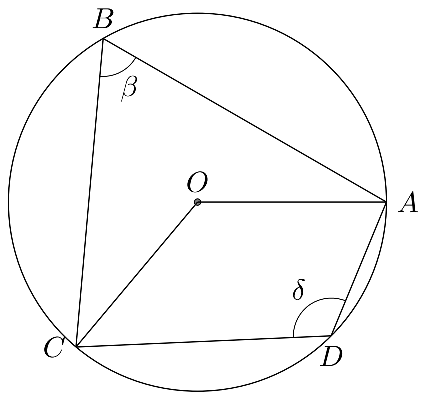
Proof of the inscribed quadrilateral theorem

Theorem. Opposite angles in inscribed quadrilaterals add up to 180°.

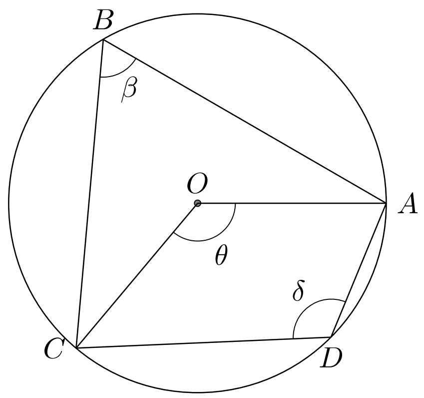
Proof. We only need to prove this for one pair of opposite angles, without loss of generality. Thus, we want to prove that $\beta + \delta = 180°$:



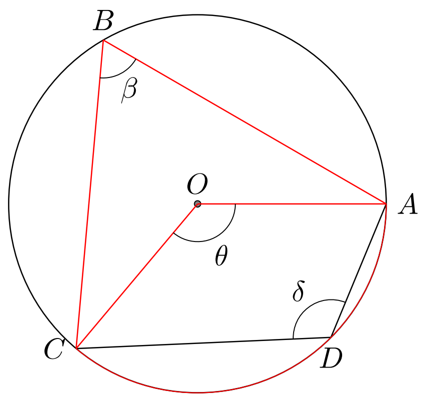
Let's draw the centre of the circle O and draw two radiuses, OA and OB:



Now, notice that we have this new central angle called θ:



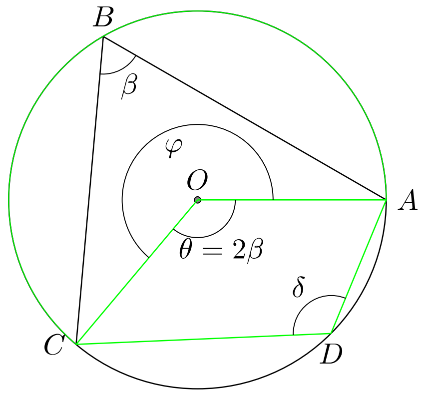
Now, let's focus on the red parts:



Notice that both angles ABC and AOC are both 'looking' at the arc ADC! Therefore, we can use the inscribed angle theorem:

$$\text{central} = 2 \times \text{inscribed} \qquad \text{Inscribed angle theorem}$$
$$\theta = 2\beta$$

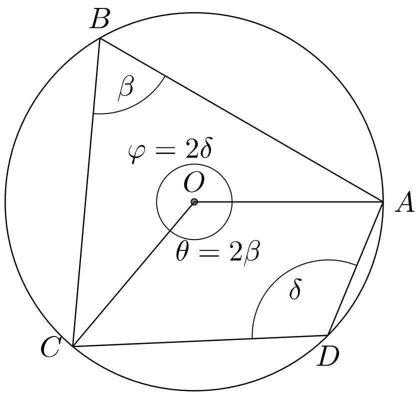
Look now at the reflex angle AOC, called φ and at the green parts:



Notice that both angles ADC and AOC are 'looking' at the same arc ABC. Again, we can use the inscribed angle theorem:

$$\text{central} = 2 \times \text{inscribed} \qquad \text{Inscribed angle theorem}$$
$$\varphi = 2\delta$$

Thus, we have



Around the centre O we need to add to 360°, so we have

$$\varphi + \theta = 360^\circ \qquad \text{Angles around a point}$$

$$2\delta + 2\beta = 360$$

$$2(\delta + \beta) = 360 \qquad \text{Factorising}$$

$$\frac{2(\delta + \beta)}{2} = \frac{360}{2}$$

$$\delta + \beta = 180$$

There we have it, $\beta + \delta = 180^\circ$.

□