Chapter 1

The mysterious movement of Mr. Brown

> I was afraid that the world around me could at any moment begin to move, to deform, first slowly and then abruptly, to disintegrate, to transform, to lose all meaning.
>
> *ERNESTO SABATO, On heroes and tombs.*

1.1 A bit of history

This story begins in the laboratory of a Scottish botanist named Robert Brown. Born in 1773 in Montrose, Scotland, Brown was a meticulous botanist known for his work documenting the Australian flora, as well as for making the first reference to the cell nucleus in studies of the microscopic structure of orchids.[1]

[1] Brown (1814, 1866).

In one of his studies of 1827, with the concise title *A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies*, Brown observed pollen grains submerged in water under the microscope, finding that they were not at rest but rather exhibited a strange zigzag movement.[2] In the words of the physicist-chemist and Nobel Prize-winning, Jean Perrin, each particle "[...] instead of sinking regularly, accelerates with an extremely agitated and totally random motion [...]. Each particle spins to and fro, rises, sinks and rises again, without ever tending to rest".[3] In figure 1.1 Perrin's graphical representation of the movement of colloidal particles of 0.52 micrometer radius can be seen. Each point represents the position of a particle every 30 seconds.

At first not much importance was given to what from that moment began to be called Brownian motion. In fact, Brown himself seemed to attach more importance to the change in shape that pollen particles underwent along their trajectories.

Brownian motion began to be an object of interest when it passed from the field of botany to that of physics. At the end of the 18th century and the beginning of the 19th century, an arduous dispute took place about the atomic-molecular nature of matter. The "atomists", led by James Clerk Maxwell and Ludwig Boltzmann, had succeeded in explaining the thermodynamic properties of gases in terms of their molecular structure, but the "energicists", among whom we can cite Ernst Mach and Pierre Duhem, rejected the atomic-molecular hypothesis

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[2] This study also appears in Brown (1866).

[3] Perrin (1909).

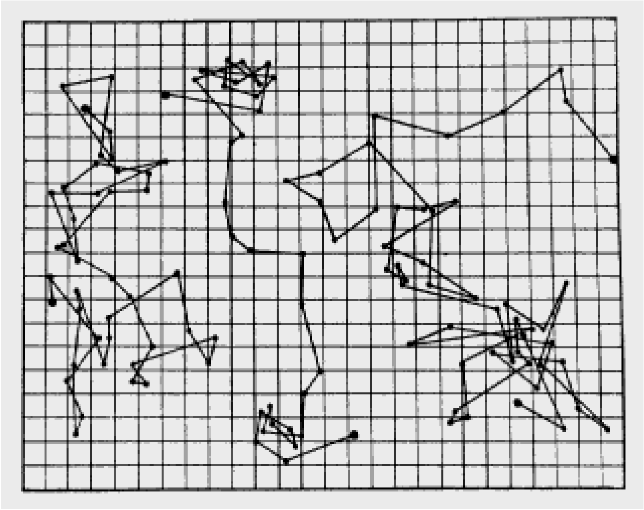

Figure 1.1: Brownian motion of particles in a suspension

arguing that, a hypothesis such as that of invisible microscopic constituents with an incessant movement, also invisible, lacked scientific guarantees.[4] The controversy ended (in favor of the atomists) with the intervention of a young employee of the Patent Office in Bern, whose name was Albert and whose last name was Einstein.

In 1905, his *annus mirabilis*, Albert Einstein published three articles that changed the development of all Physics, giving rise to the Special Theory of Relativity, Quantum Theory and, without knowing it, modern Mathematical Finance.[5] Some of the features associated with the current concept of Brownian motion

[4]Something similar is happening today with superstring theory. The description of elementary particles as vibrations of strings of incredibly small size and practically impossible to be detected experimentally, has led many physicists to believe that it exceeds the limits of scientific knowledge.

[5]The three articles can be found in Stachel (2005).

already appeared in Einstein's 1905 article. Specifically, he showed that the number of particles in suspension per unit volume was a solution to the so-called *heat equation*.[6] This equation describes the evolution of the temperature at each point of a body over time. The same equation is satisfied by the probability density associated with some cases of mathematical processes known as diffusions, which we will talk about later, and of which Brownian motion is a particular case.

Our next stage on this path to the concept of Brownian motion that is used today, goes through another genius, in this case from mathematics, named Norbert Wiener. Born in Missouri in 1894, Wiener was the son of Leo Wiener, a Russian immigrant of Jewish origin. Self-made, Leo went from making a living as a traveling salesman to being a professor of Slavic language and literature at Harvard. His confidence in the ability of human beings to progress was reflected in his concept of education, expressed in an article in American Magazine, from July 1911, in which the following could be read:

"Professor Leo Wiener of Harvard University [...] believes that the secret of early mental development lies in early training [...]. He is the father of four children ranging in age from four to sixteen, and has had the courage to express his convictions by making them the object of an educational experiment. The results have been amazing, especially in the case of his eldest son, Norbert". Whether as a result of such an "experiment" or as a result of innate talent in his son, Norbert Wiener

[6] In the one-dimensional case, which is the one originally studied by Einstein, if the number of particles per unit length is $\nu = f(x,t)$, we have that $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$, where the diffusion coefficient D is given by $D = \frac{RT}{N}\frac{1}{6\pi k P}$, and where R is the perfect gas constant, T the absolute temperature, N Avogadro's number, k the coefficient of viscosity and P the radius of the particles considered spherical.

became a child prodigy. At eleven years old he entered Tufts College where he graduated in mathematics three years later. After various comings and goings, Wiener ended up getting a PhD at Harvard with a thesis on mathematical logic. His postdoctoral studies took him to Europe, where he studied under some of the greatest minds of the time, such as Bertrand Russell, G. H. Hardy and David Hilbert. The impression made by the *wunderkinder* Wiener on Russell can be clearly seen in a letter he sent to a friend: "The youth has been flattered, and thinks himself God Almighty. There is a perpetual contest between him and me as to which is to do the teaching".[7]

Already as a professor at MIT and within his study of Measure Theory, Wiener devoted his efforts to expand the concept of measure, moving from the measure of a set of points to that of a set of trajectories. As a mathematician who liked to relate mathematics to physical phenomena, Wiener was attracted to Brownian motion and decided to apply his new research to obtaining an appropriate measure for Brownian trajectories, later known as the *Wiener measure*. In one of his works related to this topic, Wiener addresses the study of Brownian motion by directly studying the trajectories, instead of the number of particles per unit volume as Einstein did.[8] In that 1923 article, Wiener presents the most important mathematical features of Brownian motion: particle displacements are independent of their previous history and their probability distribution is normal. We will expand on the explanation of these concepts later, when we deal with the mathematical definition of Brownian motion, also called

[7] Bertrand Russell to Lucy Donnelly, 19 de octubre de 1913, cited in Grattan-Guiness (1974).

[8] Wiener (1923).

Wiener process after Norbert Wiener. This last name of the process is more used in articles on pure mathematics, and not so much in works related to finance, where the term Brownian motion is still used. I will follow the latter terminology.

Before we get to the modern definition of Brownian motion, we need some mathematical background.

1.2 Stochastic processes

Suppose you invest in shares listed on the Madrid Stock Exchange and you have the habit of checking their prices at 12:00 AM each day. You will agree with me that, before making this query, it is impossible to determine with complete certainty the price that will appear on your monitor. These types of experiments are known as *random experiments* because their result cannot be known in advance with complete certainty. This result will depend on the *state of nature* given, that is, on the number of shares supplied and demanded in the instants prior to 12:00 AM. The numerical magnitude that we measure in a random experiment (in our example the price) is known as a *random variable*. Examples of random variables are the result of a dice roll or the temperature it will be tomorrow at a certain time of day.

Sometimes we have information about the probability that the random variable is within a given range through what is known as *probability density*. For example, consider the figure 1.2, which represents the percentage of people in a given country whose height is in a given range.

If we randomly pick a citizen of that country, their height is most likely to be around 175 cm and it would be quite unlikely to find someone whose height is around 150 cm.

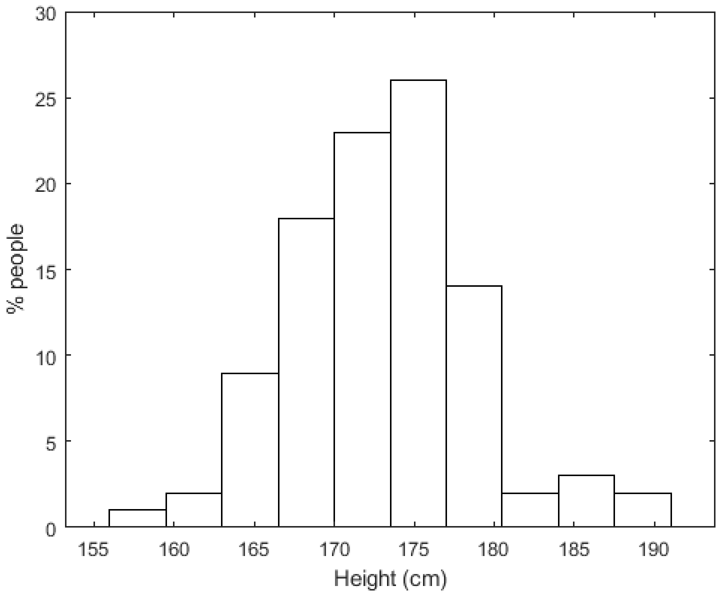

Figure 1.2: Histogram

Phenomena like this of height and many others can be approximated quite well by the *normal* or *Gaussian* density function, named after Carl Friedrich Gauss (1777-1855), the so-called Prince of Mathematics. For the previous example of the heights we would have the Gaussian density function of the figure 1.3 also known by its shape as *Gaussian bell*.

Gaussian distributions are determined by two parameters, which fully identify them and allows us to differentiate one from the other. These are its *mean* and its *variance* (or its square root called *standard deviation*). The mean refers to the value of the variable that corresponds to a greater height in the Gauss bell and that divides it into two equal halves. In the example above

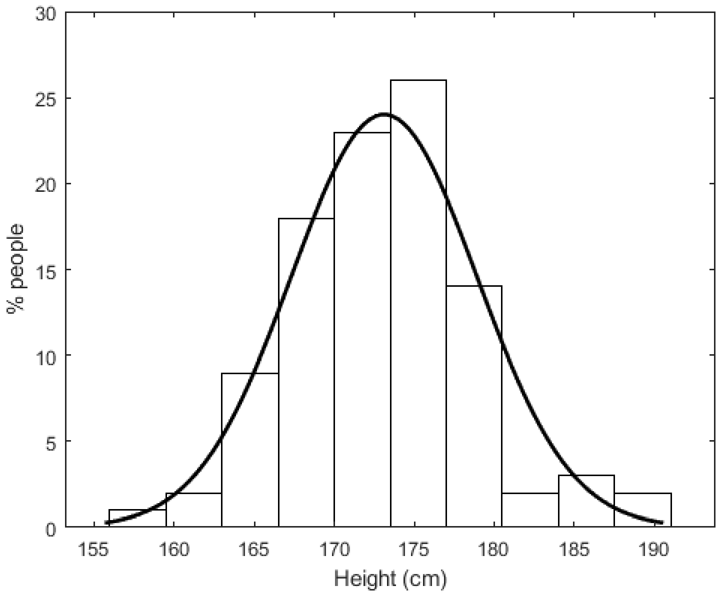

Figure 1.3: Fit to Gaussian density

the mean is between 170 and 175 cm. The variance cannot be determined by the naked eye as the mean, but it tells us how spread out the graph is. Large variances correspond to very wide graphs while small variances refer to very steep curves concentrated around the mean. In other words, if the variance is small, the probability of finding values of the variable very far from the mean (in the tails) is very low, being higher if the variance is large. Another important property of any probability density, not exclusive to normal ones, is that the probability that the variable takes a value less than a given value coincides with the area between the horizontal axis and the curve to the left of said value. Since the probability that the variable takes a value less than infinity is one, the total area between the curve and the horizontal axis is equal to one. The Gaussian distribution with mean zero and variance one is called *standard normal distribution*. To perform probability calculations with the normal distribution, the standard normal distribution is usually taken as a reference, since its values are tabulated.

Returning to our example of stock prices, as is well known, the interest of an investor is not focused exclusively on knowing the price of the shares in his portfolio at a given moment. He or she is mainly interested in knowing its evolution over time. If we give all the possible values that a random variable can take over time for all possible states of nature, we are giving what is known as a *random process* or more commonly *stochastic process*. For each fixed value of the state of nature we will have a possible path of the random process. Thus, the graph of the IBEX 35 index shown in figure 1.4, is an example of a path of a stochastic process.

Figure 1.4: IBEX 35 historical data

What is represented in figure 1.4 is one of the possible trajectories of said process, the one corresponding to the state of nature that occurred in reality. If the state of nature had been different (for example, if the evolution of the Spanish economy during that period of time had been different), the effective path followed by the stochastic process would have been different.

If you have not already tossed the book aside, you are ready to learn about the modern mathematical concept of Brownian motion.

1.3 Brownian motion

A one-dimensional *Brownian motion* or *Wiener process*, W, is a stochastic process that satisfies:

1. It is continuous.

2. Its increase from one instant to another is independent of the history of the process.

3. Said increment is a Gaussian random variable with zero mean and variance equal to the size of the time interval considered.

Before proceeding further, it is worth delving into the definition. The term one-dimensional refers to the fact that we are considering that the movement is going to be carried out in only one direction, but in both senses, for example, up and down. A graphical way to consider the time evolution of a one-dimensional Brownian motion is to imagine that it can be represented by a seismogram (figure 1.5).

Thus, the value at each instant of the Brownian motion will be given by the height of the seismogram line. Increments of W will correspond to upward movements of the pencil, while decreases would correspond to downward movements. The temporal evolution of W will be

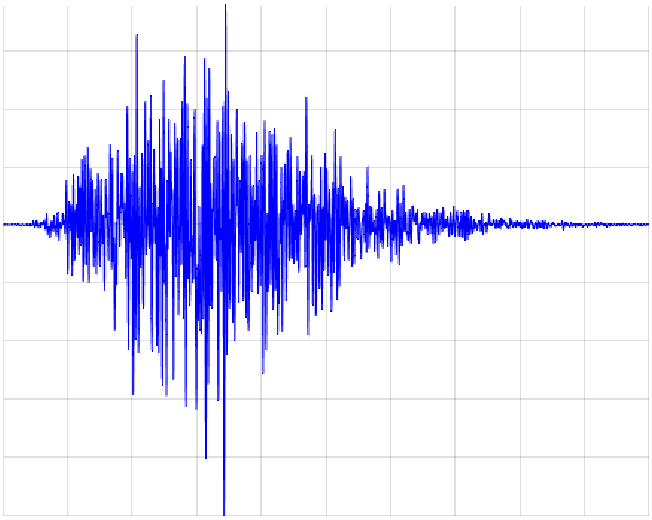

Figure 1.5: Seismogram

given by the displacement of the paper of the seismograph (figure 1.6).

The continuity of the process in the first point of the definition refers to the fact that the seismogram does not have "jumps" and is carried out without lifting the pencil from the paper. The property of the second point is that the Brownian motion is "refreshed" when it reaches each point of its trajectory, and its future displacements have nothing to do with the path already traveled. We will insist somewhat later on this characteristic known as *Markov property.*

The third point tells us that the increments of W from any moment are unpredictable (they constitute a random variable), but that all the increments "up" compensate

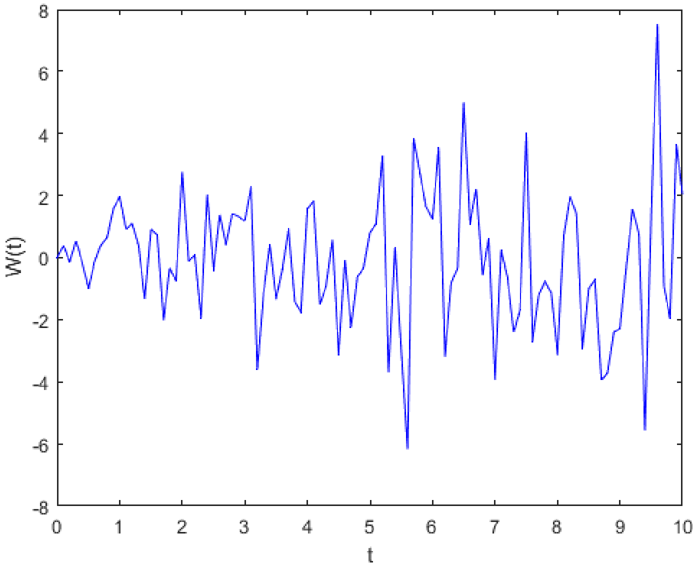

Figure 1.6: Brownian motion trajectory

in some way all those that go "down", giving zero mean. That the variance is equal to the time interval means that the more we let time pass, the greater the probability of finding larger increments (the Gauss bell widens). This property leads to a "spreading" of the paths as time goes by, as can be seen in the figure 1.7, in which different trajectories of a Brownian motion are shown.

1.4 Properties of Brownian motion

Next, we state and explore in some detail some properties of Brownian motion. The first two correspond to mathematical characteristics of the stochastic process itself.

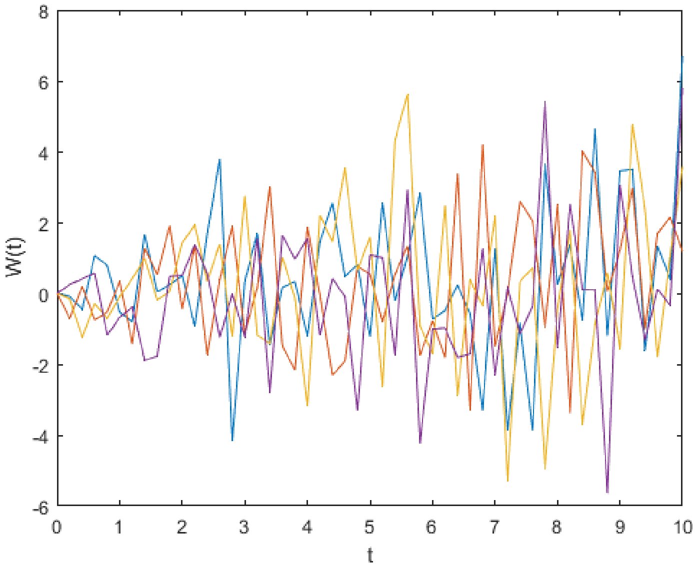

Figure 1.7: Different trajectories of a Brownian motion

The rest refer to the paths of the process.

1. **Brownian motion is a Markov process.** A stochastic process is a *Markov process* if the value of the process at a given instant does not depend on its previous history. Like Dori in *Finding Nemo*, the process only "knows" its current value at each instant, but does not "remember" how it got there.

   As an example of a Markov process, consider a dice game in which the score in each game is obtained as the maximum of the results obtained by throwing a dice a number of times equal to the score in the previous game. Thus, for example, you would start by rolling the dice only once. If the result is 2, that will be the score of the first game. In the next game the player would roll the dice twice. If the results are 3 and 5, then his score in the second game would be 5, and in the next game he would roll 5 times. The stochastic process defined by the score in each game is Markov because the probability of each result in a game can be obtained exclusively from the score of the immediately previous game. If this probability depended on past games, the process would not be Markov.

2. **Brownian motion is a martingale.** Suppose we know the entire history of a process up to the current moment. The average of the possible values of the process at a future instant, estimated with said information, is known as the *conditional expectation* of the process. If said conditional expectation coincides with the current value of the process, then the process is a *martingale*. In other words, a martingale is a stochastic process for which the best estimate we can make of its future value, with the information we have up to a certain point in time, is that it stays as it is. Knowing the history of the process, all possible future paths "up" count the same as all possible paths "down". In the realm of gambling, a martingale is what we know as "fair game".

   To illustrate the concept of martingale, let us consider a game that consists of tossing a coin with its sides labeled +1 and -1. The player starts with an initial score of 1 and the result of the game after each toss is to multiply what comes out on the coin by the sum of all the previous scores and add it to the last score. For example, if the player gets -1, +1 and +1, respectively, on the first three rolls, the scores will be:

(a) Initial: 1

(b) After the first roll: $1 - 1 \times 1 = 0$

(c) After the second roll: $0 + 1 \times (1 + 0) = 1$

(d) After the third roll: $1 + 1 \times (1 + 0 + 1) = 3$

The stochastic process determined by the scores of the game after each roll is a martingale, since once the result is known until a specific roll, for example, that of the third which is 3, there is the same probability that it will increase by five units and that decrease by that same amount, with the conditional expectation equal to the present value of 3.

The Markov process example in point 1 is not a martingale, since if the result of one game is 1, in the next game it would roll the die only once and the conditional expectation of the next result would be greater than 1. On the other hand, the martingale example that we have just seen is not Markov, since the result of each roll depends on all previous results.
As we have just seen, there are Markov processes that are not martingales, in the same way that there are martingales that are not Markov, the two concepts are independent.

3. **The paths of Brownian motion are not differentiable at any point.** From my point of view this is the most amazing property of Brownian motion and it's worth stopping slowly to enjoy it.
As the reader knows from his studies of Elementary Geometry, a *tangent to a curve* is the line that touches the curve at a single point and has the same "inclination" as the curve at that point (figure 1.8).

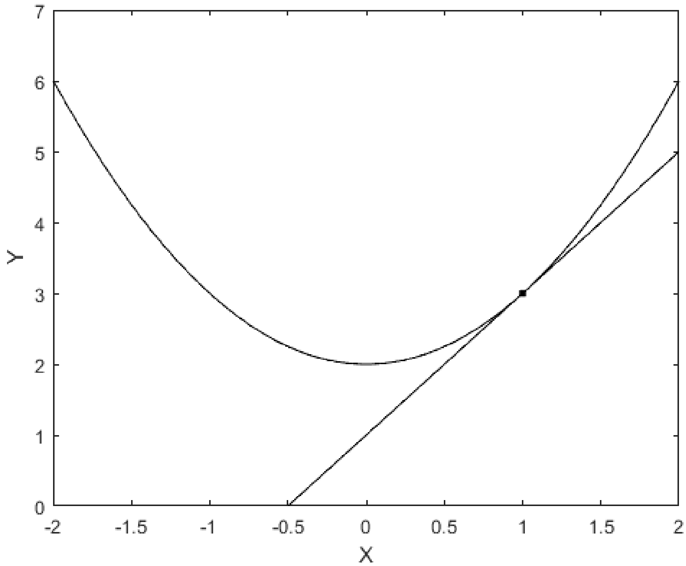

Figure 1.8: Tangent to a curve

"Smooth" curves have tangents at all points and are said to be *differentiable*. However, we can draw curves in the plane that have "peaks" where it is not possible to draw a tangent, since in them the concept of inclination is not well defined. Looking at the above graphs of Brownian motion it is evident that they are "full" of such peaks. In fact, Jean Baptiste Perrin already referred to the trajectories of Brownian motion in these terms: "The trajectories are confused and complicated, they change direction so frequently and rapidly that it is impossible to follow them [...]. It is impossible to fix a tangent, even approximately, and reminds us of the nondifferentiable continuous functions of

mathematicians".[9] As it seems evident, the paths of Brownian motion are very irregular and have "peaks" at many points. But at how many? The answer is at every point. All points on a Brownian motion path are points of abrupt change in direction. It seems incredible to think of a graph with this property, but such is mathematics. In fact, this property is the content of a theorem due to Zigmund, Wiener and Paley that appeared in 1933 in an article with the brief title: "Notes on random functions".[10]
This spectacular property is not unique to Brownian motion trajectories. Let us look at two more examples.

In the 19th century, the conjecture that every continuous graph was differentiable except at isolated points began to spread among mathematicians. Although it seems that Bolzano in 1834 presented a continuous function not differentiable at any point, the best known example that has survived to this day of a function with these properties is the so-called *Weierstrass function*, whose graph we have represented in figure 1.9.[11] Another curious example of a continuous graph with no tangent at any point is the so-called *Koch curve*. This curve was proposed by Helge von Koch in 1904 and its purpose is perfectly reflected in the title of his article: "On a continuous curve that does not have tangents and is obtained by the methods of elementary geometry". The Koch curve construction process

[9] Perrin quote in Paley y Wiener (1934).
[10] Paley, Wiener y Zygmund (1933).
[11] The Weierstrass function is given by $f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$, where a is an odd positive integer, $0 < b < 1$ and $ab > 1 + \frac{3\pi}{2}$.

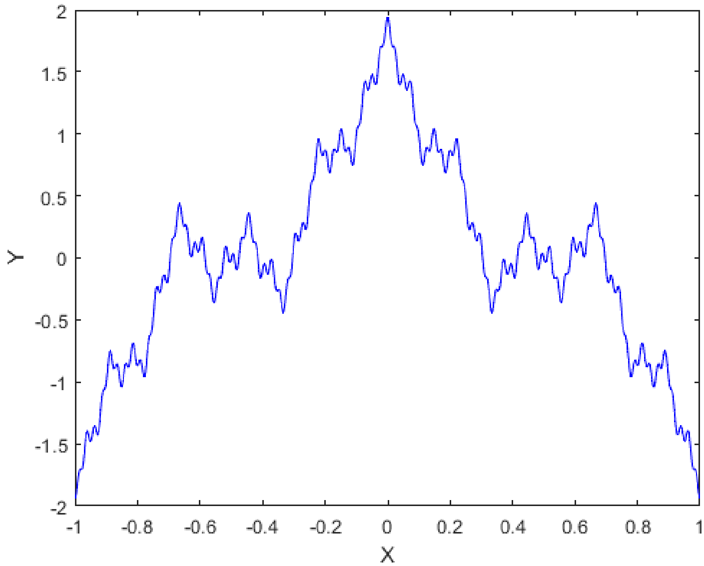

Figure 1.9: Weierstrass function

can be given in the form of a simple algorithm:

(a) It starts from a horizontal segment of a certain length. It is divided into three equal parts and the central part is removed. On it, an angle formed by two segments of the same length as the one we have removed is built as a “tent”.

(b) The same procedure of the previous section is repeated for each segment that has been formed.

(c) Repeat section b).

The first 5 steps of the construction of the Koch curve can be seen, from top to bottom, in figure 1.10. How many steps are required to complete the Koch curve?

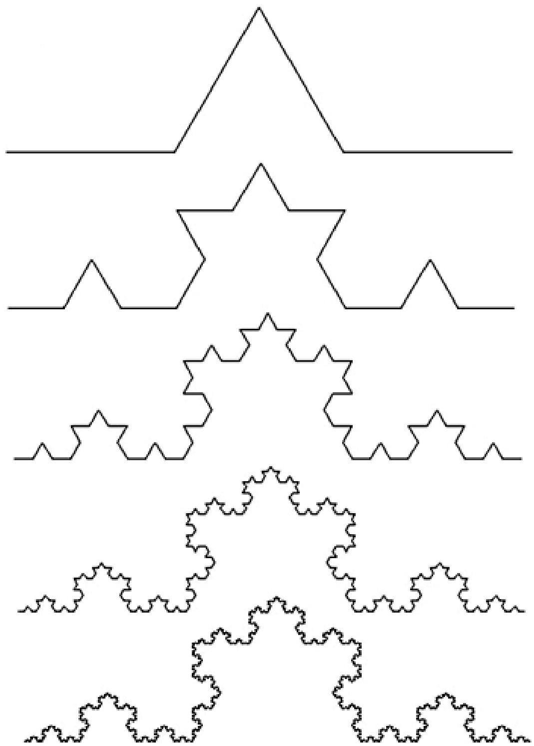

Figure 1.10: Koch curve

Well, just infinite. The Koch curve is the result of repeating this process endlessly. If instead of starting with a segment you start with an equilateral triangle, the result is the (prettier) *Koch snowflake* (figure 1.11).
Apart from its beauty and its mathematical importance, one of the interesting features of the Koch curve is that it can be studied with elementary methods. For example, it is not difficult to show that the total length of the

curve is infinite.[12] Furthermore, if we consider the Koch snowflake, we have a closed curve of infinite length within a finite area.

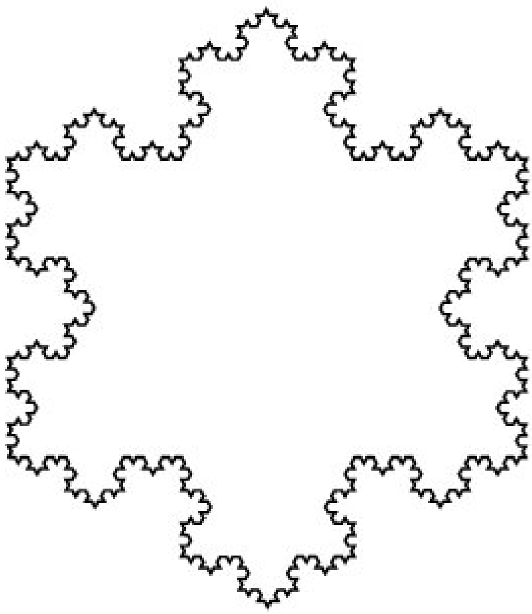

Figure 1.11: Koch snowflake

4. **Scaling property.** Consider a Brownian motion that takes the value zero at time zero, that is, a *standard Brownian motion* (see figure 1.6). Suppose we are looking at the Brownian motion path from the start to 9 seconds. Will it look the same as the path from the start to a second? The answer is

[12] If we take the length of the initial segment equal to 1, the sequence that gives the length of the curve after each step is 1, 4/3, 16/9,..., therefore, the length of the Koch curve is given by $\lim_{n\to\infty}\left(\frac{4}{3}\right)^n = \infty$.

no, because we know that the variance of the increments is proportional to time (the spreading effect we talked about before). To make it look the same, i.e., for it to remain a Brownian motion, we have to multiply its value by $3 = \sqrt{9}$. This is what the scaling property consists of: if we divide the time of a Brownian motion by a positive number, the resulting process is a Brownian motion if we multiply it by the square root of that same number.

The Brownian motion properties allow us to qualify it as a *fractal* object, in the sense of being an irregular and self-similar geometric figure, qualities that it shares with the Weierstrass function and the Koch curve.[13] The touchstone to determine if a geometric figure can be considered "officially" as a fractal is to verify that its *Haussdorf dimension* or *fractal dimension* is greater than its topological dimension. Do not be scared by the terminology, we will explain it in layman's terms.
Suppose we can (physically) take a Brownian path and stretch it as much as we want. At some point it will end up becoming a straight line, which, as is well known has only one dimension. Thus, we will say that the topological dimension of a Brownian trajectory is 1. However, if we return it to its initial state, we see that in some way, due to its irregularity, it "occupies" or "fills" more of the plane than a straight line. This is what the fractal dimension measures: curves with a fractal dimension somewhat greater than 1 will occupy little more

[13]The term fractal was coined by Benoit Mandelbrot, giving it the meaning of "rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole" (Mandelbrot, 1997).

than a straight line and curves with a fractal dimension close to 2 will occupy almost all the points of the plane. The fractal dimension of any Brownian path is exactly 1.5, so we can accept Brownian motion paths as true fractals. Also, the Brownian motion outperforms the Koch curve, which has fractal dimension 1.2618 . . ..[14] Completing the podium of this competition between fractals, and with the gold medal, appears the *Hilbert curve.* The first steps of its construction are shown in figure 1.12.

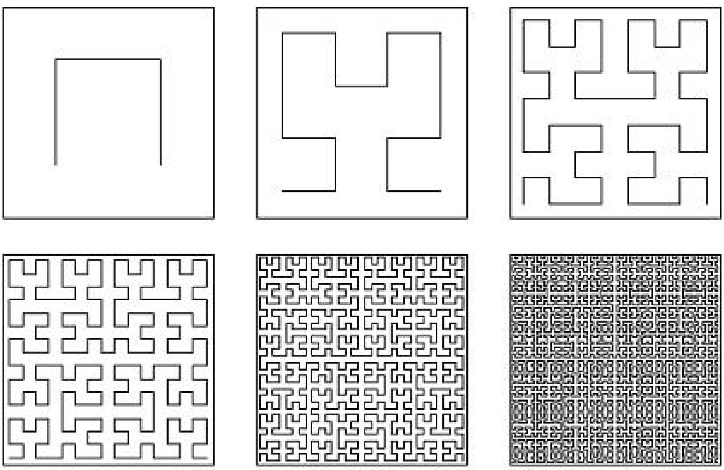

Figure 1.12: Construction of the Hilbert curve

The Hilbert curve has topological dimension 1 and fractal dimension 2, that is, it passes through all the points of the square.[15]

[14]The fractal dimension of the Koch curve (and its snowflake) is exactly $\frac{\ln 4}{\ln 3}$.

[15]I have excluded the Weierstrass function from this competition because, as far as I know, there is no exact value for its fractal

To end this section dedicated to the surprising properties of Brownian motion, we will present another property that, although less mathematical, is no less amazing.

5. **A standard Brownian motion changes sign infinitely many times between instant zero and any subsequent instant.** If you start a standard Brownian motion and stop it at any later time, looking at your graph, you will have already traversed the horizontal axis corresponding to the value zero an infinity number of times. Also, no matter how fast the clock stops, the number of sign changes will never be finite.

1.5 Bibliography of the chapter

About Brown's scientific works, see Brown (1866). A classic biography of Einstein that also delves deeply into his scientific production is Pais (2005). Another biography, of Wiener and Von Neumann, is Heims (1980), from which I have extracted most of the biographical material on Norbert Wiener. It is difficult to find literature at an informative level related to stochastic processes and Brownian motion (that is one of the objectives of this book!). For technical approaches see, for example, Mörters and Peres (2010) or Karatzas and Shreve (1991). A somewhat more affordable level book that can be used as an introduction to these concepts is Klebaner (2005). Another good reference is Martínez and Villalón (2003). On fractals, the classic reference is Mandelbrot (1982).

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dimension, there is only a lower bound value $\frac{\ln a}{2+\ln b}$ (Falconer, 2003). We have also stopped mentioning other continuous curves that fill the plane such as the Peano curve (earlier than Hilbert's) or the Gosper curve.