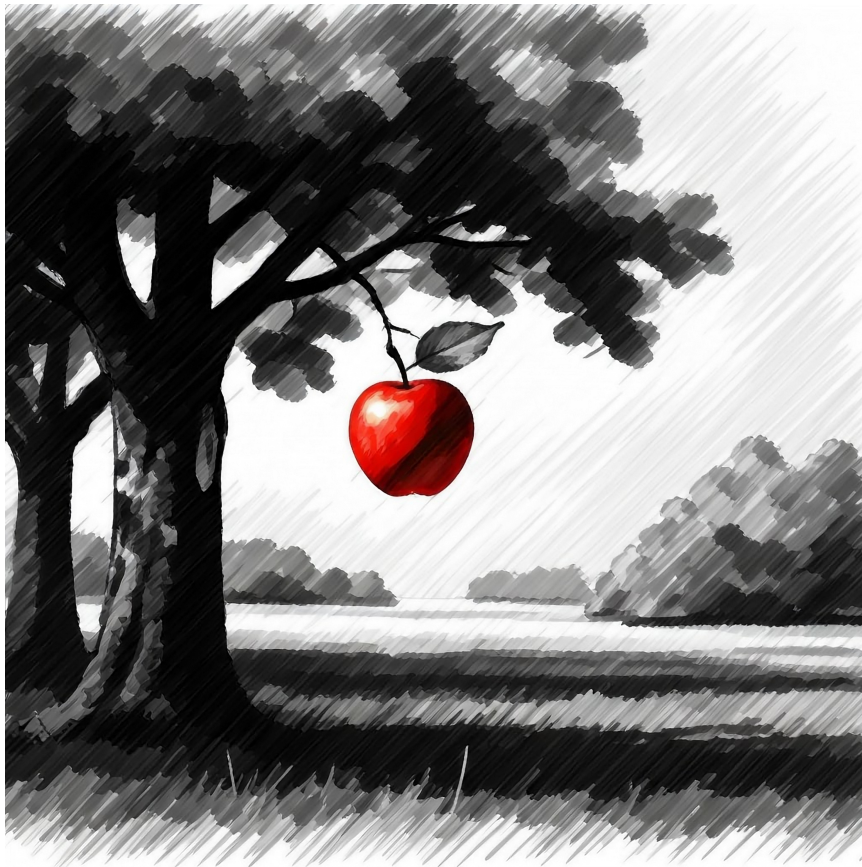


Introduction to Differential Equations

Los Medanos College Edition



by Scott Hubbard

March 25, 2025
(version 1.0)

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During the writing of this book, the author was supported by a sabbatical leave from Los Medanos College in the Contra Costa Community College District.

Edition number started at 1.0, when it was finished for the author’s sabbatical project in spring 2025.

The cover image is an AI-generated depiction of a red apple ready to fall from a tree. It’s hard to verify, but Isaac Newton himself would often tell the story that seeing an [apple fall from a tree](#) while thinking about gravity helped lead him to his formulation of the universal law of gravity. This law is written using differential equations in his masterwork *Philosophiæ Naturalis Principia Mathematica*.

Typeset in L^AT_EX. Body text is set in [New PX font](#), heading sans serif text is in [Source Sans Pro font](#), and math is set in [New PX Math font](#).

Dedication

To my wife, Nathalie. Min elskede: Du er mit livs kærlighed, min bedste ven og den største gave fra Gud!

Preface

This is a textbook designed for use in an introductory differential equations course, which is typically taught after the second semester of calculus. Specifically, I developed this book to be used for a course we teach at Los Medanos College called Math 240 — Introduction to Differential Equations, which matches the name-sake model transfer curriculum (MATH 240) in California under the Course Identification Numbering System (C-ID). This book is a combination of my own writing and ideas from teaching the course nearly every semester since 2012, as well as content from two openly licensed books that are used in many similar courses.

The material in this book completely covers the curriculum required for the C-ID transfer model course, as well as all topics in our local LMC course outline of record. There are many extra topics beyond the C-ID curriculum in this book that can be picked and chosen by an instructor when they you the course. I've included these in case you want to supplement your course with interesting topics, or have your students do a project on their own.

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Preliminaries

0.1 Overview of Differential Equations

0.1.1 Differential Equations

The laws of physics are generally written down as differential equations. Therefore, all of science and engineering use differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science and engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned. As an analogy, suppose all your classes from now on were given in Swahili. It would be important to first learn Swahili, or you would have a very tough time getting a good grade in your classes.

You saw many differential equations already without perhaps knowing about it. And you even solved simple differential equations when you took calculus. Let us see an example you may not have seen:

$$\frac{dx}{dt} + x = 2 \cos t. \quad (1)$$

Here x is the *dependent variable* and t is the *independent variable*. Equation (1) is a basic example of a *differential equation*. It is an example of a *first order differential equation*, since it involves only the first derivative of the dependent variable. This equation arises from Newton's law of cooling where the ambient temperature oscillates with time.

0.1.2 Solutions of Differential Equations

Solving the differential equation (1) means finding x in terms of t . That is, we want to find a function of t , which we call x , such that when we plug x , t , and $\frac{dx}{dt}$ into (1), the equation holds; that is, the left hand side equals the right hand side. It is the same idea as it would be for a normal (algebraic) equation of just x and t . We claim that

$$x = x(t) = \cos t + \sin t$$

is a *solution*. How do we check? We simply plug x into equation (1)! First we need to compute $\frac{dx}{dt}$. We find that $\frac{dx}{dt} = -\sin t + \cos t$. Now let us compute the left-hand side of (1).

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t)}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t)}_x = 2 \cos t.$$

Yay! We got precisely the right-hand side. But there is more! We claim $x = \cos t + \sin t + e^{-t}$ is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}.$$

We plug into the left-hand side of (1)

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t - e^{-t})}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t + e^{-t})}_x = 2 \cos t.$$

And it works yet again!

So there can be many different solutions. For this equation all solutions can be written in the form

$$x = \cos t + \sin t + Ce^{-t},$$

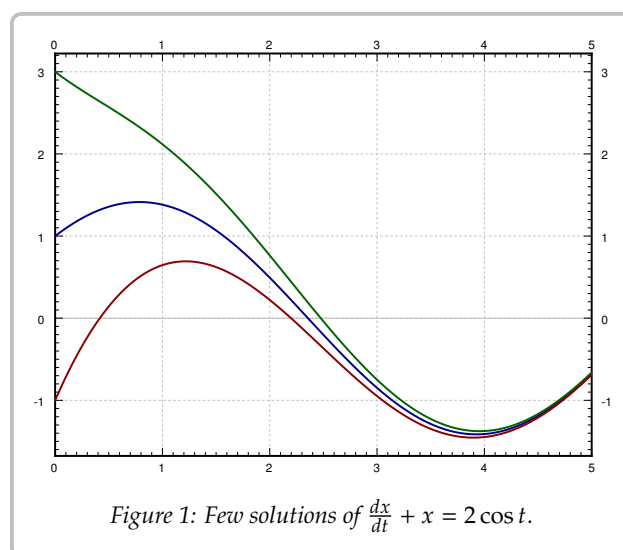
for some constant C . Different constants C will give different solutions, so there are really infinitely many possible solutions. See Figure 1 for the graph of a few of these solutions. We will see how we find these solutions a few lectures from now.

Solving differential equations can be quite hard. There is no general method that solves every differential equation. We will generally focus on how to get exact formulas for solutions of certain differential equations, but we will also spend a little bit of time on getting approximate solutions. And we will spend some time on understanding the equations without solving them.

Most of this book is dedicated to *ordinary differential equations* or ODEs, that is, equations with only one independent variable, where derivatives are only with respect to this one variable. If there are several independent variables, we get *partial differential equations* or PDEs.

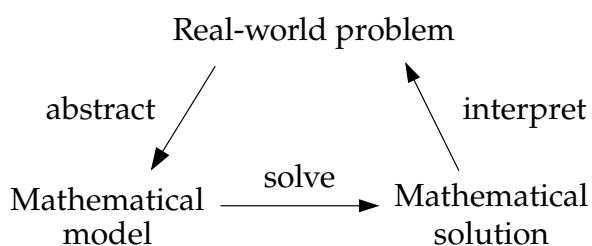
Even for ODEs, which are very well understood, it is not a simple question of turning a crank to get answers. When you can find exact solutions, they are usually preferable to approximate solutions. It is important to understand how such solutions are found. Although in real applications you will leave much of the actual calculations to computers, you need to understand what they are doing. It is often necessary to simplify or transform your equations into something that a computer can understand and solve. You may even need to make certain assumptions and changes in your model to achieve this.

To be a successful engineer or scientist, you will be required to solve problems in your job that you never saw before. It is important to learn problem solving techniques, so that you may apply those techniques to new problems. A common mistake is to expect to learn some prescription for solving all the problems you will encounter in your later career. This course is no exception.



0.1.3 Differential Equations in Practice

So how do we use differential equations in science and engineering? First, we have some *real-world problem* we wish to understand. We make some simplifying assumptions and create a *mathematical model*. That is, we translate the real-world situation into a set of differential equations. Then we apply mathematics to get some sort of a *mathematical solution*. There is still something left to do. We have to interpret the results. We have to figure out what the mathematical solution says about the real-world problem we started with.



Learning how to formulate the mathematical model and how to interpret the results is what your physics and engineering classes do. In this course, we will focus mostly on the mathematical analysis. Sometimes we will work with simple real-world examples so that we have some intuition and motivation about what we are doing.

Let us look at an example of this process. One of the most basic differential equations is the standard *exponential growth model*. Let P denote the population of some bacteria on a Petri dish. We assume that there is enough food and enough space. Then the rate of growth of bacteria is proportional to the population—a large population grows quicker. Let t denote time (say in seconds) and P the population. Our model is

$$\frac{dP}{dt} = kP,$$

for some positive constant $k > 0$.

Example 0.1.1: Suppose there are 100 bacteria at time 0 and 200 bacteria 10 seconds later. How many bacteria will there be 1 minute from time 0 (in 60 seconds)?

First we need to solve the equation. We claim that a solution is given by

$$P(t) = Ce^{kt},$$

where C is a constant. Let us try:

$$\frac{dP}{dt} = Cke^{kt} = kP.$$

And it really is a solution.

OK, now what? We do not know C , and we do not know k . But we know something. We know $P(0) = 100$, and we know $P(10) = 200$. Let us plug these conditions in and see what happens.

$$100 = P(0) = Ce^{k0} = C,$$

$$200 = P(10) = 100e^{k10}.$$

Therefore, $2 = e^{10k}$ or $k = \frac{\ln 2}{10} \approx 0.069$. So

$$P(t) = 100e^{(\ln 2)t/10} \approx 100e^{0.069t}.$$

At one minute, $t = 60$, the population is $P(60) = 6400$. See [Figure 2](#).

Let us interpret the results. Does our solution mean that there must be exactly 6400 bacteria on the plate at 60s? No! We made assumptions that might not be true exactly, just approximately. If our assumptions are reasonable, then there will be approximately 6400 bacteria. Also, in real life P is a discrete quantity, not a real number. However, our model has no problem saying that for example at 61 seconds, $P(61) \approx 6859.35$.

Normally, the k in $P' = kP$ is given, and we want to solve the equation for different *initial conditions*. What does that mean? Take $k = 1$ for simplicity: We want to solve the equation $\frac{dP}{dt} = P$ subject to $P(0) = 1000$ (the initial condition). Then the solution is (exercise)

$$P(t) = 1000e^t.$$

We call $P(t) = Ce^t$ the *general solution*, as every solution of the equation can be written in this form for some constant C . We need an initial condition to find out what C is, in order to find the *particular solution* we are looking for. Generally, when we say “particular solution,” we just mean some solution.

In real life, parameters such as k must first often be somehow computed or estimated. The example above shows how finding an analytic solution to the differential equation is useful in finding these parameters.

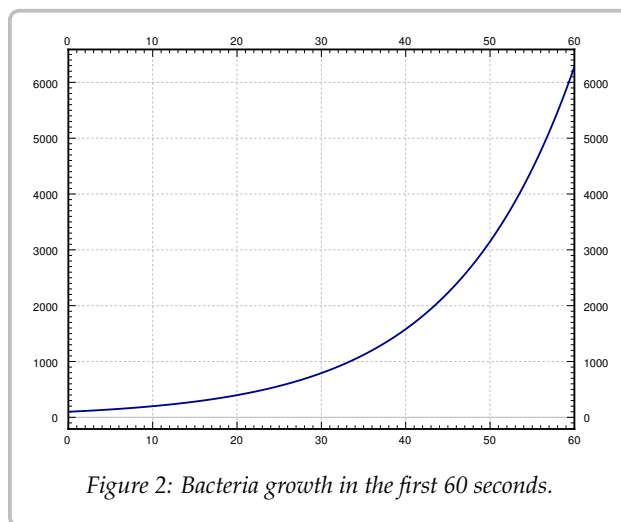


Figure 2: Bacteria growth in the first 60 seconds.

0.1.4 Four Fundamental Equations

A few equations appear often and it is useful to just memorize what their solutions are. Let us call them the four fundamental equations. Their solutions are reasonably easy to guess by recalling properties of exponentials, sines, and cosines. They are also simple to check, which is something that you should always do. No need to wonder if you remembered the solution correctly.

First such equation is

$$\frac{dy}{dx} = ky,$$

for some constant $k > 0$. Here y is the dependent and x the independent variable. The general solution for this equation is

$$y(x) = Ce^{kx}.$$

We saw above that this function is a solution, although we used different variable names.

Next,

$$\frac{dy}{dx} = -ky,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = Ce^{-kx}.$$

Exercise 0.1.1: Check that the y given is really a solution to the equation.

Next, take the second order differential equation

$$\frac{d^2y}{dx^2} = -k^2y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx).$$

Since the equation is a second order differential equation, we have two constants in our general solution.

Exercise 0.1.2: Check that the y given is really a solution to the equation.

Finally, consider the second order differential equation

$$\frac{d^2y}{dx^2} = k^2y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 e^{kx} + C_2 e^{-kx},$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(kx).$$

For those that do not know, cosh and sinh are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

They are called the *hyperbolic cosine* and *hyperbolic sine*. These functions are sometimes easier to work with than exponentials. They have some nice familiar properties such as $\cosh 0 = 1$, $\sinh 0 = 0$, and $\frac{d}{dx} \cosh x = \sinh x$ (no that is not a typo) and $\frac{d}{dx} \sinh x = \cosh x$.

Exercise 0.1.3: Check that both forms of the y given are really solutions to the equation.

Example 0.1.2: In equations of higher order, you get more constants you must solve for to get a particular solution and hence you need more initial conditions. The equation $\frac{d^2y}{dx^2} = 0$ has the general solution $y = C_1x + C_2$; simply integrate twice and don't forget about the constant of integration. Consider the initial conditions $y(0) = 2$ and $y'(0) = 3$. We plug in our general solution and solve for the constants:

$$2 = y(0) = C_1 \cdot 0 + C_2 = C_2, \quad 3 = y'(0) = C_1.$$

In other words, $y = 3x + 2$ is the particular solution we seek.

An interesting note about cosh: The graph of cosh is the exact shape of a hanging chain. This shape is called a *catenary*. Contrary to popular belief this is not a parabola. If you invert the graph of cosh, it is also the ideal arch for supporting its weight. For example, the Gateway Arch in Saint Louis is an inverted graph of cosh—if it were just a parabola it might fall. The formula used in the design is inscribed inside the arch:

$$y = -127.7 \text{ ft} \cdot \cosh(x/127.7 \text{ ft}) + 757.7 \text{ ft}.$$

0.1.5 Exercises

Exercise 0.1.4: Show that $x = e^{4t}$ is a solution to $x''' - 12x'' + 48x' - 64x = 0$.

Exercise 0.1.5: Show that $x = e^t$ is not a solution to $x''' - 12x'' + 48x' - 64x = 0$.

Exercise 0.1.6: Is $y = \sin t$ a solution to $\left(\frac{dy}{dt}\right)^2 = 1 - y^2$? Justify.

Exercise 0.1.7: Let $y'' + 2y' - 8y = 0$. Now try a solution of the form $y = e^{rx}$ for some (unknown) constant r . Is this a solution for some r ? If so, find all such r .

Exercise 0.1.8: Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.

Exercise 0.1.9: Verify that $x = C_1e^{-t} + C_2e^{2t}$ is a solution to $x'' - x' - 2x = 0$. Find C_1 and C_2 to solve for the initial conditions $x(0) = 10$ and $x'(0) = 0$.

Exercise 0.1.10: Find a solution to $(x')^2 + x^2 = 4$ using your knowledge of derivatives of functions that you know from basic calculus.

Exercise 0.1.11: Solve:

a) $\frac{dA}{dt} = -10A, \quad A(0) = 5$

b) $\frac{dH}{dx} = 3H, \quad H(0) = 1$

c) $\frac{d^2y}{dx^2} = 4y, \quad y(0) = 0, \quad y'(0) = 1$

d) $\frac{d^2x}{dy^2} = -9x, \quad x(0) = 1, \quad x'(0) = 0$

Exercise 0.1.12: Is there a solution to $y' = y$, such that $y(0) = y(1)$?

Exercise 0.1.13: The population of city X was 100 thousand 20 years ago, and the population of city X was 120 thousand 10 years ago. Assuming constant growth, you can use the exponential population model (like for the bacteria). What do you estimate the population is now?

Exercise 0.1.14: Suppose that a football coach gets a salary of one million dollars now, and a raise of 10% every year (so exponential model, like population of bacteria). Let s be the salary in millions of dollars, and t is time in years.

a) What is $s(0)$ and $s(1)$.

b) Approximately how many years will it take for the salary to be 10 million.

c) Approximately how many years will it take for the salary to be 20 million.

d) Approximately how many years will it take for the salary to be 30 million.

Note: Exercises with numbers 101 and higher have solutions in the back of the book.

Exercise 0.1.101: Show that $x = e^{-2t}$ is a solution to $x'' + 4x' + 4x = 0$.

Exercise 0.1.102: Is $y = x^2$ a solution to $x^2 y'' - 2y = 0$? Justify.

Exercise 0.1.103: Let $xy'' - y' = 0$. Try a solution of the form $y = x^r$. Is this a solution for some r ? If so, find all such r .

Exercise 0.1.104: Verify that $x = C_1 e^t + C_2$ is a solution to $x'' - x' = 0$. Find C_1 and C_2 so that x satisfies $x(0) = 10$ and $x'(0) = 100$.

Exercise 0.1.105: Solve $\frac{d\varphi}{ds} = 8\varphi$ and $\varphi(0) = -9$.

Exercise 0.1.106: Solve:

a) $\frac{dx}{dt} = -4x, \quad x(0) = 9$

b) $\frac{d^2x}{dt^2} = -4x, \quad x(0) = 1, \quad x'(0) = 2$

c) $\frac{dp}{dq} = 3p, \quad p(0) = 4$

d) $\frac{d^2T}{dx^2} = 4T, \quad T(0) = 0, \quad T'(0) = 6$

0.2 Classification of Differential Equations

There are many types of differential equations, and we classify them into different categories based on their properties. Let us quickly go over the most basic classification. We already saw the distinction between ordinary and partial differential equations:

- *Ordinary differential equations* or (ODE) are equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.
- *Partial differential equations* or (PDE) are equations that depend on partial derivatives of several variables. That is, there are several independent variables.

Let us see some examples of ordinary differential equations:

$$\begin{aligned}\frac{dy}{dt} &= ky, & (\text{Exponential growth}) \\ \frac{dy}{dt} &= k(A - y), & (\text{Newton's law of cooling}) \\ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx &= f(t). & (\text{Mechanical vibrations})\end{aligned}$$

And of partial differential equations:

$$\begin{aligned}\frac{\partial y}{\partial t} + c \frac{\partial y}{\partial x} &= 0, & (\text{Transport equation}) \\ \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, & (\text{Heat equation}) \\ \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. & (\text{Wave equation in 2 dimensions})\end{aligned}$$

If there are several equations working together, we have a so-called *system of differential equations*. For example,

$$y' = x, \quad x' = y$$

is a system of ordinary differential equations. Maxwell's equations for electromagnetics,

$$\begin{aligned}\nabla \cdot \vec{D} &= \rho, & \nabla \cdot \vec{B} &= 0, \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, & \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t},\end{aligned}$$

are a system of partial differential equations. The divergence operator $\nabla \cdot$ and the curl operator $\nabla \times$ can be written out in partial derivatives of the functions involved in the x , y , and z variables.

The next bit of information is the *order* of the equation (or system). The order is the order of the largest derivative that appears. If the highest derivative that appears is the first derivative, the equation is of first order. If the highest derivative that appears is the second derivative, then the equation is of second order. For example, Newton's law of cooling above is a first order equation, while the mechanical vibrations equation is a second order equation. The equation governing transversal vibrations in a beam,

$$a^4 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0,$$

is a fourth order partial differential equation. It is fourth order as at least one derivative is the fourth derivative. It does not matter that the derivative in t is only of second order.

In the first chapter, we will start attacking first order ordinary differential equations, that is, equations of the form $\frac{dy}{dx} = f(x, y)$. In general, lower order equations are easier to work with and have simpler behavior, which is why we start with them.

We also distinguish how the dependent variables appear in the equation (or system). In particular, an equation is *linear* if the dependent variable (or variables) and their derivatives appear linearly, that is, only as first powers, they are not multiplied together, and no other functions of the dependent variables appear. The equation is a sum of terms, where each term is a function of the independent variables or a function of the independent variables multiplied by a dependent variable or its derivative. Otherwise, the equation is called *nonlinear*. An ordinary differential equation is linear if it can be put into the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = b(x). \quad (2)$$

The functions $a_0, a_1, \dots, a_n$ are called the *coefficients*. The equation is allowed to depend arbitrarily on the independent variable. So

$$e^x \frac{d^2 y}{dx^2} + \sin(x) \frac{dy}{dx} + x^2 y = \frac{1}{x} \quad (3)$$

is a linear equation as y and its derivatives only appear linearly.

All the equations and systems above as examples are linear. It may not be immediately obvious for Maxwell's equations unless you write out the divergence and curl in terms of partial derivatives. Let us see some nonlinear equations. For example Burger's equation,

$$\frac{\partial y}{\partial t} + y \frac{\partial y}{\partial x} = \nu \frac{\partial^2 y}{\partial x^2},$$

is a nonlinear second order partial differential equation. It is nonlinear because y and $\frac{\partial y}{\partial x}$ are multiplied together. The equation

$$\frac{dx}{dt} = x^2 \quad (4)$$

is a nonlinear first order differential equation as there is a second power of the dependent variable x . Another nonlinear ODE is the pendulum equation

$$\theta'' + \sin(\theta) = 0, \quad (5)$$

which is nonlinear as the dependent variable θ appears inside a sin function. Nonlinear equations are notoriously difficult to solve and their solutions may behave in strange and unexpected ways. Perhaps you have heard of chaos theory and the butterflies in the Amazon causing hurricanes in the Atlantic, all due to nonlinear equations. So sometimes we study related linear equations, such as $\theta'' + \theta = 0$ for the pendulum, instead.

A linear equation is further called *homogeneous* if all terms depend on the dependent variable. That is, if no term is a function of the independent variables alone. Otherwise, the equation is called *nonhomogeneous* or *inhomogeneous*. For example, the exponential growth equation, the wave equation, or the transport equation above are homogeneous. The mechanical vibrations equation above is nonhomogeneous as long as $f(t)$ is not the zero function. Similarly, if the ambient temperature A is nonzero, Newton's law of cooling is nonhomogeneous. A homogeneous linear ODE can be put into the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0.$$

Compare to (2) and notice there is no function $b(x)$.

If the coefficients of a linear equation are actually constant functions, then the equation is said to have *constant coefficients*. The coefficients are the functions multiplying the dependent variable(s) or one of its

derivatives, not the function $b(x)$ standing alone. A constant coefficient nonhomogeneous ODE is an equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = b(x),$$

where $a_0, a_1, \dots, a_n$ are all constants, but b may depend on the independent variable x . The mechanical vibrations equation above is a constant coefficient nonhomogeneous second order ODE. The same nomenclature applies to PDEs, so the transport equation, heat equation and wave equation are all examples of constant coefficient linear PDEs.

Finally, an equation (or system) is called *autonomous* if the equation does not depend on the independent variable. For autonomous ordinary differential equations, the independent variable is then thought of as time. Autonomous equation means an equation that does not change with time. For example, Newton's law of cooling is autonomous, so are the equations (4) and (5). On the other hand, mechanical vibrations (as long as $f(t)$ is nonconstant) or (3) are not autonomous. A general first order autonomous ODE would have the form

$$\frac{dx}{dt} = f(x).$$

0.2.1 Exercises

Exercise 0.2.1: Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?

a) $\sin(t) \frac{d^2 x}{dt^2} + \cos(t)x = t^2$

b) $\frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial y} = xy$

c) $y'' + 3y + 5x = 0, \quad x'' + x - y = 0$

d) $\frac{\partial^2 u}{\partial t^2} + u \frac{\partial^2 u}{\partial s^2} = 0$

e) $x'' + tx^2 = t$

f) $\frac{d^4 x}{dt^4} = 0$

Exercise 0.2.2: If $\vec{u} = (u_1, u_2, u_3)$ is a vector, we have the divergence $\nabla \cdot \vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$ and curl $\nabla \times \vec{u} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right)$. Notice that curl of a vector is still a vector. Write out Maxwell's equations in terms of partial derivatives and classify the system.

Exercise 0.2.3: Suppose F is a linear function, that is, $F(x, y) = ax + by$ for constants a and b . What is the classification of equations of the form $F(y', y) = 0$.

Exercise 0.2.4: Write down an explicit example of a third order, linear, nonconstant coefficient, nonautonomous, nonhomogeneous system of two ODE such that every derivative that could appear, does appear.

Exercise 0.2.101: Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?

a) $\frac{\partial^2 v}{\partial x^2} + 3 \frac{\partial^2 v}{\partial y^2} = \sin(x)$

b) $\frac{dx}{dt} + \cos(t)x = t^2 + t + 1$

c) $\frac{d^7 F}{dx^7} = 3F(x)$

d) $y'' + 8y' = 1$

e) $x'' + t y x' = 0, \quad y'' + t x y = 0$

f) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s^2} + u^2$

Exercise 0.2.102: Write down the general zeroth order linear ordinary differential equation. Write down the general solution.

Exercise 0.2.103: For which k is $\frac{dx}{dt} + x^k = t^{k+2}$ linear. Hint: There are two answers.

Chapter 1

Introduction

1.1 Applications Leading to Differential Equations

In order to apply mathematical methods to a physical or “real life” problem, we must formulate the problem in mathematical terms; that is, we must construct a *mathematical model* for the problem. Many physical problems concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more of its derivatives. Such equations are *differential equations*. They are the subject of this book.

Much of calculus is devoted to learning mathematical techniques that are applied in later courses in mathematics and the sciences; you wouldn’t have time to learn much calculus if you insisted on seeing a specific application of every topic covered in the course. Similarly, much of this book is devoted to methods that can be applied in later courses. Only a relatively small part of the book is devoted to the derivation of specific differential equations from mathematical models, or relating the differential equations that we study to specific applications. In this section we mention a few such applications.

The mathematical model for an applied problem is almost always simpler than the actual situation being studied, since simplifying assumptions are usually required to obtain a mathematical problem that can be solved. For example, in modeling the motion of a falling object, we might neglect air resistance and the gravitational pull of celestial bodies other than Earth, or in modeling population growth we might assume that the population grows continuously rather than in discrete steps.

A good mathematical model has two important properties:

- It’s sufficiently simple so that the mathematical problem can be solved.
- It represents the actual situation sufficiently well so that the solution to the mathematical problem predicts the outcome of the real problem to within a useful degree of accuracy. If results predicted by the model don’t agree with physical observations, the underlying assumptions of the model must be revised until satisfactory agreement is obtained.

We’ll now give examples of mathematical models involving differential equations. We’ll return to these problems at the appropriate times, as we learn how to solve the various types of differential equations that occur in the models.

All the examples in this section deal with functions of time, which we denote by t . If y is a function of t , y' denotes the derivative of y with respect to t ; thus,

$$y' = \frac{dy}{dt}.$$

1.1.1 Population Growth and Decay

Although the number of members of a population (people in a given country, bacteria in a laboratory culture, wildflowers in a forest, etc.) at any given time t is necessarily an integer, models that use differential equations to describe the growth and decay of populations usually rest on the simplifying assumption that the number of members of the population can be regarded as a differentiable function $P = P(t)$. In most models it is assumed that the differential equation takes the form

$$P' = a(P)P, \quad (1.1)$$

where a is a continuous function of P that represents the rate of change of population per unit time per individual. In the *Malthusian model*, it is assumed that $a(P)$ is a constant, so (1.1) becomes

$$P' = aP. \quad (1.2)$$

(When you see a name in blue italics, just click on it for information about the person.) This model assumes that the numbers of births and deaths per unit time are both proportional to the population. The constants of proportionality are the *birth rate* (births per unit time per individual) and the *death rate* (deaths per unit time per individual); a is the birth rate minus the death rate. You learned in calculus that if c is any constant then

$$P = ce^{at} \quad (1.3)$$

satisfies (1.2), so (1.2) has infinitely many solutions. To select the solution of the specific problem that we're considering, we must know the population P_0 at an initial time, say $t = 0$. Setting $t = 0$ in (1.3) yields $c = P(0) = P_0$, so the applicable solution is

$$P(t) = P_0 e^{at}.$$

This implies that

$$\lim_{t \rightarrow \infty} P(t) = \begin{cases} \infty & \text{if } a > 0, \\ 0 & \text{if } a < 0; \end{cases}$$

that is, the population approaches infinity if the birth rate exceeds the death rate, or zero if the death rate exceeds the birth rate.

To see the limitations of the Malthusian model, suppose we're modeling the population of a country, starting from a time $t = 0$ when the birth rate exceeds the death rate (so $a > 0$), and the country's resources in terms of space, food supply, and other necessities of life can support the existing population. Then the prediction $P = P_0 e^{at}$ may be reasonably accurate as long as it remains within limits that the country's resources can support. However, the model must inevitably lose validity when the prediction exceeds these limits. (If nothing else, eventually there won't be enough space for the predicted population!)

This flaw in the Malthusian model suggests the need for a model that accounts for limitations of space and resources that tend to oppose the rate of population growth as the population increases. Perhaps the most famous model of this kind is the *Verhulst model*, where (1.2) is replaced by

$$P' = aP(1 - \alpha P), \quad (1.4)$$

where α is a positive constant. As long as P is small compared to $1/\alpha$, the ratio P'/P is approximately equal to a . Therefore the growth is approximately exponential; however, as P increases, the ratio P'/P decreases as opposing factors become significant.

Equation (1.4) is the *logistic equation*. You will learn how to solve it in Section 1.2. (See Exercise 2.2.28.) The solution is

$$P = \frac{P_0}{\alpha P_0 + (1 - \alpha P_0)e^{-at}},$$

where $P_0 = P(0) > 0$. Therefore $\lim_{t \rightarrow \infty} P(t) = 1/\alpha$, independent of P_0 .

Figure 1.1 shows typical graphs of P versus t for various values of P_0 .

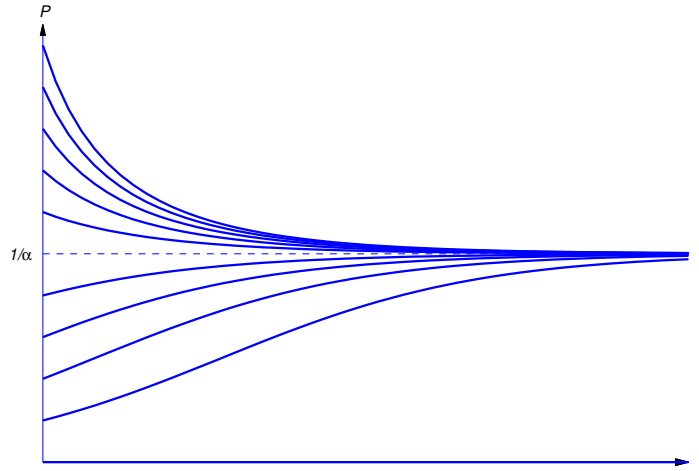


Figure 1.1: Solutions of the logistic equation

1.1.2 Newton's Law of Cooling

According to *Newton's law of cooling*, the temperature of a body changes at a rate proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Thus, if T_m is the temperature of the medium and $T = T(t)$ is the temperature of the body at time t , then

$$T' = -k(T - T_m), \quad (1.5)$$

where k is a positive constant and the minus sign indicates; that the temperature of the body increases with time if it's less than the temperature of the medium, or decreases if it's greater. We'll see in Section 4.2 that if T_m is constant then the solution of (1.5) is

$$T = T_m + (T_0 - T_m)e^{-kt}, \quad (1.6)$$

where T_0 is the temperature of the body when $t = 0$. Therefore $\lim_{t \rightarrow \infty} T(t) = T_m$, independent of T_0 . (Common sense suggests this. Why?)

Figure 1.2 shows typical graphs of T versus t for various values of T_0 .

Assuming that the medium remains at constant temperature seems reasonable if we're considering a cup of coffee cooling in a room, but not if we're cooling a huge cauldron of molten metal in the same room. The difference between the two situations is that the heat lost by the coffee isn't likely to raise the temperature of the room appreciably, but the heat lost by the cooling metal is. In this second situation we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium respectively, and let T_0 and T_{m0} be their initial values. Again, we assume that T and T_m are related by (1.5). We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T - T_0)$ and the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m - T_{m0})$, where a and a_m are positive constants depending upon the masses and thermal properties of the object and medium respectively. If we assume that the total heat of the in the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0.$$

Solving this for T_m and substituting the result into (1.6) yields the differential equation

$$T' = -k \left(1 + \frac{a}{a_m} \right) T + k \left(T_{m0} + \frac{a}{a_m} T_0 \right)$$

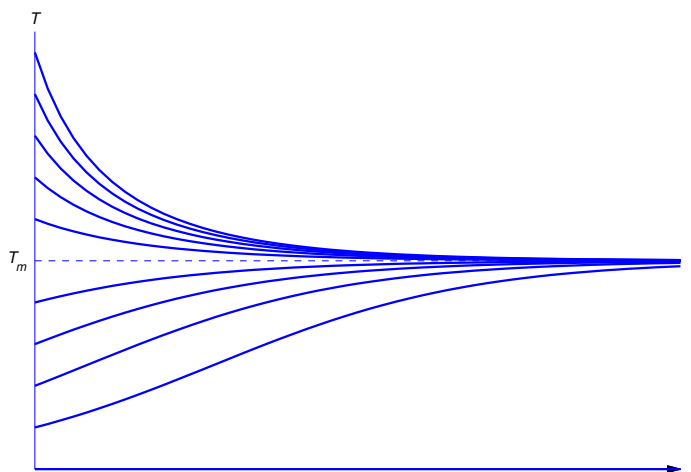


Figure 1.2: Temperature according to Newton's Law of Cooling

for the temperature of the object. After learning to solve linear first order equations, you'll be able to show (Exercise 5.2.17) that

$$T = \frac{aT_0 + a_m T_{m0}}{a + a_m} + \frac{a_m(T_0 - T_{m0})}{a + a_m} e^{-k(1+a/a_m)t}.$$

1.1.3 Glucose Absorption by the Body

Glucose is absorbed by the body at a rate proportional to the amount of glucose present in the bloodstream. Let λ denote the (positive) constant of proportionality. Suppose there are G_0 units of glucose in the bloodstream when $t = 0$, and let $G = G(t)$ be the number of units in the bloodstream at time $t > 0$. Then, since the glucose being absorbed by the body is leaving the bloodstream, G satisfies the equation

$$G' = -\lambda G. \quad (1.7)$$

From calculus you know that if c is any constant then

$$G = ce^{-\lambda t} \quad (1.8)$$

satisfies (1.7), so (1.7) has infinitely many solutions. Setting $t = 0$ in (1.8) and requiring that $G(0) = G_0$ yields $c = G_0$, so

$$G(t) = G_0 e^{-\lambda t}.$$

Now let's complicate matters by injecting glucose intravenously at a constant rate of r units of glucose per unit of time. Then the rate of change of the amount of glucose in the bloodstream per unit time is

$$G' = -\lambda G + r, \quad (1.9)$$

where the first term on the right is due to the absorption of the glucose by the body and the second term is due to the injection. After you've studied Section 2.1, you'll be able to show (Exercise 2.1.43) that the solution of (1.9) that satisfies $G(0) = G_0$ is

$$G = \frac{r}{\lambda} + \left(G_0 - \frac{r}{\lambda}\right) e^{-\lambda t}.$$

Graphs of this function are similar to those in Figure 1.2. (Why?)

1.1.4 Spread of Epidemics

One model for the spread of epidemics assumes that the number of people infected changes at a rate proportional to the product of the number of people already infected and the number of people who are susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t , then $S - I$ is the number of people who are susceptible, but not yet infected. Thus,

$$I' = rI(S - I),$$

where r is a positive constant. Assuming that $I(0) = I_0$, the solution of this equation is

$$I = \frac{SI_0}{I_0 + (S - I_0)e^{-rSt}}$$

(Exercise 2.2.29). Graphs of this function are similar to those in Figure 1.1. (Why?) Since $\lim_{t \rightarrow \infty} I(t) = S$, this model predicts that all the susceptible people eventually become infected.

1.1.5 Newton's Second Law of Motion

According to *Newton's second law of motion*, the instantaneous acceleration a of an object with constant mass m is related to the force F acting on the object by the equation $F = ma$. For simplicity, let's assume that $m = 1$ and the motion of the object is along a vertical line. Let y be the displacement of the object from some reference point on Earth's surface, measured positive upward. In many applications, there are three kinds of forces that may act on the object:

- (a) A force such as gravity that depends only on the position y , which we write as $-p(y)$, where $p(y) > 0$ if $y \geq 0$.
- (b) A force such as atmospheric resistance that depends on the position and velocity of the object, which we write as $-q(y, y')y'$, where q is a nonnegative function and we've put y' "outside" to indicate that the resistive force is always in the direction opposite to the velocity.
- (c) A force $f = f(t)$, exerted from an external source (such as a towline from a helicopter) that depends only on t .

In this case, Newton's second law implies that

$$y'' = -q(y, y')y' - p(y) + f(t),$$

which is usually rewritten as

$$y'' + q(y, y')y' + p(y) = f(t).$$

Since the second (and no higher) order derivative of y occurs in this equation, we say that it is a *second order differential equation*.

1.1.6 Interacting Species: Competition

Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time t , and assume that each population would grow exponentially if the other didn't exist; that is, in the absence of competition we would have

$$P' = aP \quad \text{and} \quad Q' = bQ, \tag{1.10}$$

where a and b are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so (1.10) is replaced by

$$\begin{aligned} P' &= aP - \alpha Q \\ Q' &= -\beta P + bQ, \end{aligned}$$

where α and β are positive constants. (Since negative population doesn't make sense, this system works only while P and Q are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$. It can be shown (Exercise 13.4.42) that there's a positive constant ρ such that if (P_0, Q_0) is above the line L through the origin with slope ρ , then the species with population P becomes extinct in finite time, but if (P_0, Q_0) is below L , the species with population Q becomes extinct in finite time. Figure 1.3 illustrates this. The curves shown there are given parametrically by $P = P(t), Q = Q(t), t > 0$. The arrows indicate direction along the curves with increasing t .

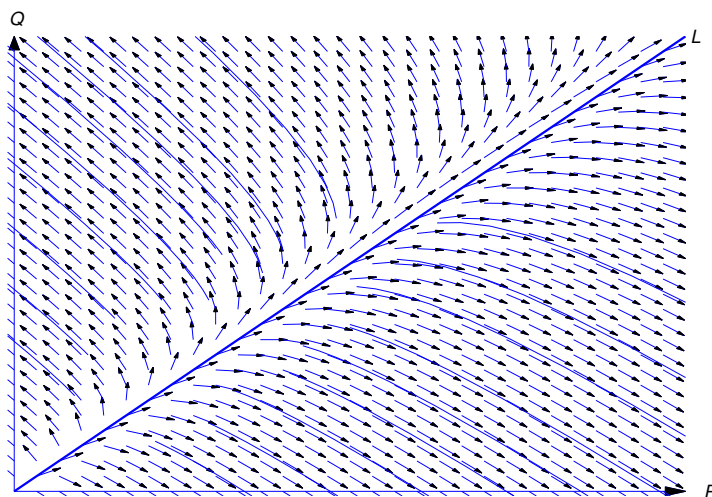


Figure 1.3: Populations of competing species

1.2 Basic Concepts

A *differential equation* is an equation that contains one or more derivatives of an unknown function. The *order* of a differential equation is the order of the highest derivative that it contains. A differential equation is an *ordinary differential equation* if it involves an unknown function of only one variable, or a *partial differential equation* if it involves partial derivatives of a function of more than one variable. For now we'll consider only ordinary differential equations, and we'll just call them *differential equations*.

Throughout this text, all variables and constants are real unless it's stated otherwise. We'll usually use x for the independent variable unless the independent variable is time; then we'll use t .

The simplest differential equations are first order equations of the form

$$\frac{dy}{dx} = f(x) \quad \text{or, equivalently,} \quad y' = f(x),$$

where f is a known function of x . We already know from calculus how to find functions that satisfy this kind of equation. For example, if

$$y' = x^3,$$

then

$$y = \int x^3 dx = \frac{x^4}{4} + c,$$

where c is an arbitrary constant. If $n > 1$ we can find functions y that satisfy equations of the form

$$y^{(n)} = f(x) \tag{1.11}$$

by repeated integration. Again, this is a calculus problem.

Except for illustrative purposes in this section, there's no need to consider differential equations like (1.11). We'll usually consider differential equations that can be written as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \tag{1.12}$$

where at least one of the functions $y, y', \dots, y^{(n-1)}$ actually appears on the right. Here are some examples:

$$\begin{aligned} \frac{dy}{dx} - x^2 &= 0 && \text{(first order),} \\ \frac{dy}{dx} + 2xy^2 &= -2 && \text{(first order),} \\ \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y &= 2x && \text{(second order),} \\ xy''' + y^2 &= \sin x && \text{(third order),} \\ y^{(n)} + xy' + 3y &= x && \text{(n-th order).} \end{aligned}$$

Although none of these equations is written as in (1.12), all of them *can* be written in this form:

$$\begin{aligned} y' &= x^2, \\ y' &= -2 - 2xy^2, \\ y'' &= 2x - 2y' - y, \\ y''' &= \frac{\sin x - y^2}{x}, \\ y^{(n)} &= x - xy' - 3y. \end{aligned}$$

1.2.1 Solutions of Differential Equations

A **solution** of a differential equation is a function that satisfies the differential equation on some open interval; thus, y is a solution of (1.12) if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all x in some open interval (a, b) . In this case, we also say that y is a **solution of (1.12) on** (a, b) . Functions that satisfy a differential equation at isolated points are not interesting. For example, $y = x^2$ satisfies

$$xy' + x^2 = 3x$$

if and only if $x = 0$ or $x = 1$, but it's not a solution of this differential equation because it does not satisfy the equation on an open interval.

The graph of a solution of a differential equation is a **solution curve**. More generally, a curve C is said to be an **integral curve** of a differential equation if every function $y = y(x)$ whose graph is a segment of C is a solution of the differential equation. Thus, any solution curve of a differential equation is an integral curve, but an integral curve need not be a solution curve.

Example 1.2.1: If a is any positive constant, the circle

$$x^2 + y^2 = a^2 \tag{1.13}$$

is an integral curve of

$$y' = -\frac{x}{y}. \tag{1.14}$$

To see this, note that the only functions whose graphs are segments of (1.13) are

$$y_1 = \sqrt{a^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{a^2 - x^2}.$$

We leave it to you to verify that these functions both satisfy (1.14) on the open interval $(-a, a)$. However, (1.13) is not a solution curve of (1.14), since it's not the graph of a function.

Example 1.2.2: Verify that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.15}$$

is a solution of

$$xy' + y = x^2 \tag{1.16}$$

on $(0, \infty)$ and on $(-\infty, 0)$.

Solution: Substituting (1.15) and

$$y' = \frac{2x}{3} - \frac{1}{x^2}$$

into (1.16) yields

$$xy'(x) + y(x) = x \left(\frac{2x}{3} - \frac{1}{x^2} \right) + \left(\frac{x^2}{3} + \frac{1}{x} \right) = x^2$$

for all $x \neq 0$. Therefore y is a solution of (1.16) on $(-\infty, 0)$ and $(0, \infty)$. However, y isn't a solution of the differential equation on any open interval that contains $x = 0$, since y is not defined at $x = 0$.

Figure 1.4 shows the graph of (1.15). The part of the graph of (1.15) on $(0, \infty)$ is a solution curve of (1.16), as is the part of the graph on $(-\infty, 0)$.

Example 1.2.3: Show that if c_1 and c_2 are constants then

$$y = (c_1 + c_2 x)e^{-x} + 2x - 4 \tag{1.17}$$

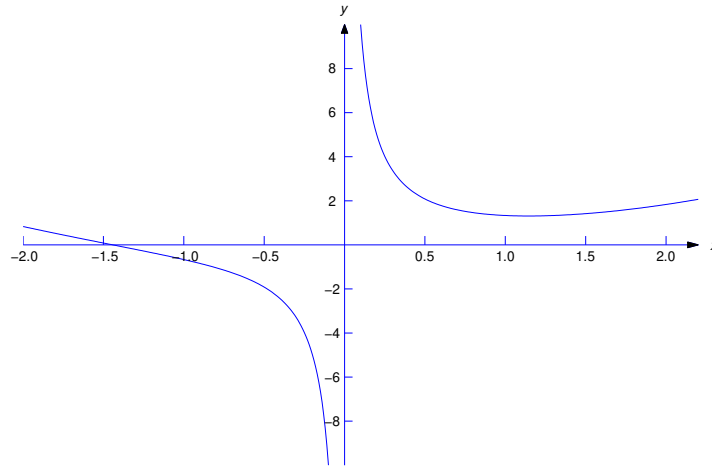


Figure 1.4: $y = \frac{x^2}{3} + \frac{1}{x}$

is a solution of

$$y'' + 2y' + y = 2x \quad (1.18)$$

on $(-\infty, \infty)$.

Solution: Differentiating (1.17) twice yields

$$y' = -(c_1 + c_2x)e^{-x} + c_2e^{-x} + 2$$

and

$$y'' = (c_1 + c_2x)e^{-x} - 2c_2e^{-x},$$

so

$$\begin{aligned} y'' + 2y' + y &= (c_1 + c_2x)e^{-x} - 2c_2e^{-x} \\ &\quad + 2[-(c_1 + c_2x)e^{-x} + c_2e^{-x} + 2] \\ &\quad + (c_1 + c_2x)e^{-x} + 2x - 4 \\ &= (1 - 2 + 1)(c_1 + c_2x)e^{-x} + (-2 + 2)c_2e^{-x} \\ &\quad + 4 + 2x - 4 = 2x \end{aligned}$$

for all values of x . Therefore y is a solution of (1.18) on $(-\infty, \infty)$.

Example 1.2.4: Find all solutions of

$$y^{(n)} = e^{2x}. \quad (1.19)$$

Solution: Integrating (1.19) yields

$$y^{(n-1)} = \frac{e^{2x}}{2} + k_1,$$

where k_1 is a constant. If $n \geq 2$, integrating again yields

$$y^{(n-2)} = \frac{e^{2x}}{4} + k_1x + k_2.$$

If $n \geq 3$, repeatedly integrating yields

$$y = \frac{e^{2x}}{2^n} + k_1 \frac{x^{n-1}}{(n-1)!} + k_2 \frac{x^{n-2}}{(n-2)!} + \cdots + k_n, \quad (1.20)$$

where $k_1, k_2, \dots, k_n$ are constants. This shows that every solution of (1.19) has the form (1.20) for some choice of the constants $k_1, k_2, \dots, k_n$. On the other hand, differentiating (1.20) n times shows that if $k_1, k_2, \dots, k_n$ are arbitrary constants, then the function y in (1.20) satisfies (1.19).

Since the constants $k_1, k_2, \dots, k_n$ in (1.20) are arbitrary, so are the constants

$$\frac{k_1}{(n-1)!}, \frac{k_2}{(n-2)!}, \dots, k_n.$$

Therefore Example 1.2.4 actually shows that all solutions of (1.19) can be written as

$$y = \frac{e^{2x}}{2^n} + c_1 + c_2 x + \cdots + c_n x^{n-1},$$

where we renamed the arbitrary constants in (1.20) to obtain a simpler formula. As a general rule, arbitrary constants appearing in solutions of differential equations should be simplified if possible. You'll see examples of this throughout the text.

1.2.2 Initial Value Problems

In Example 1.2.4 we saw that the differential equation $y^{(n)} = e^{2x}$ has an infinite family of solutions that depend upon the n arbitrary constants $c_1, c_2, \dots, c_n$. In the absence of additional conditions, there's no reason to prefer one solution of a differential equation over another. However, we'll often be interested in finding a solution of a differential equation that satisfies one or more specific conditions. The next example illustrates this.

Example 1.2.5: Find a solution of

$$y' = x^3$$

such that $y(1) = 2$.

Solution: At the beginning of this section we saw that the solutions of $y' = x^3$ are

$$y = \frac{x^4}{4} + c.$$

To determine a value of c such that $y(1) = 2$, we set $x = 1$ and $y = 2$ here to obtain

$$2 = y(1) = \frac{1}{4} + c, \quad \text{so} \quad c = \frac{7}{4}.$$

Therefore the required solution is

$$y = \frac{x^4 + 7}{4}.$$

Figure 1.5 shows the graph of this solution. Note that imposing the condition $y(1) = 2$ is equivalent to requiring the graph of y to pass through the point $(1, 2)$.

We can rewrite the problem considered in Example 1.2.5 more briefly as

$$y' = x^3, \quad y(1) = 2.$$

We call this an *initial value problem*. The requirement $y(1) = 2$ is an *initial condition*. Initial value problems can also be posed for higher order differential equations. For example,

$$y'' - 2y' + 3y = e^x, \quad y(0) = 1, \quad y'(0) = 2 \quad (1.21)$$

is an initial value problem for a second order differential equation where y and y' are required to have specified values at $x = 0$. In general, an initial value problem for an n -th order differential equation requires y and its first $n - 1$ derivatives to have specified values at some point x_0 . These requirements are the *initial conditions*.

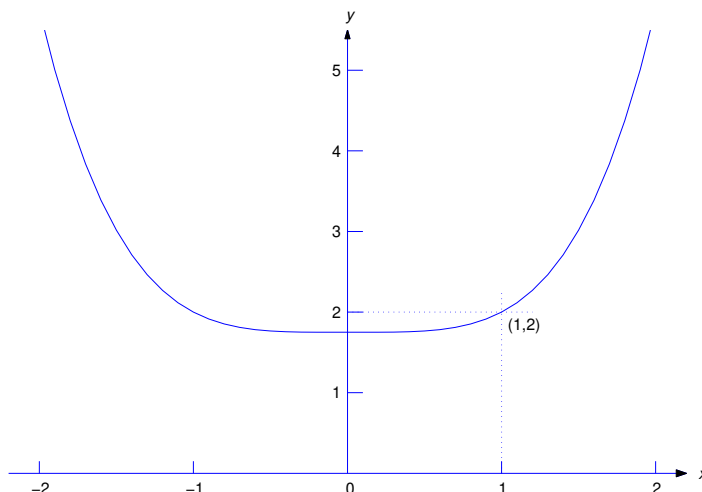


Figure 1.5: $y = \frac{x^2 + 7}{4}$

We'll denote an initial value problem for a differential equation by writing the initial conditions after the equation, as in (1.21). For example, we would write an initial value problem for (1.12) as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \quad \dots, \quad y^{(n-1)}(x_0) = k_{n-1}. \quad (1.22)$$

Consistent with our earlier definition of a solution of the differential equation in (1.22), we say that y is a solution of the initial value problem (1.22) if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all x in some open interval (a, b) that contains x_0 , and y satisfies the initial conditions in (1.22). The largest open interval that contains x_0 on which y is defined and satisfies the differential equation is the *interval of validity* of y .

Example 1.2.6: In Example 1.2.5 we saw that

$$y = \frac{x^4 + 7}{4} \quad (1.23)$$

is a solution of the initial value problem

$$y' = x^3, \quad y(1) = 2.$$

Since the function in (1.23) is defined for all x , the interval of validity of this solution is $(-\infty, \infty)$.

Example 1.2.7: In Example 1.2.2 we verified that

$$y = \frac{x^2}{3} + \frac{1}{x} \quad (1.24)$$

is a solution of

$$xy' + y = x^2$$

on $(0, \infty)$ and on $(-\infty, 0)$. By evaluating (1.24) at $x = \pm 1$, you can see that (1.24) is a solution of the initial value problems

$$xy' + y = x^2, \quad y(1) = \frac{4}{3} \quad (1.25)$$

and

$$xy' + y = x^2, \quad y(-1) = -\frac{2}{3}. \quad (1.26)$$

The interval of validity of (1.24) as a solution of (1.25) is $(0, \infty)$, since this is the largest interval that contains $x_0 = 1$ on which (1.24) is defined. Similarly, the interval of validity of (1.24) as a solution of (1.26) is $(-\infty, 0)$, since this is the largest interval that contains $x_0 = -1$ on which (1.24) is defined.

1.2.3 Free Fall Under Constant Gravity

The term *initial value problem* originated in problems of motion where the independent variable is t (representing elapsed time), and the initial conditions are the position and velocity of an object at the initial (starting) time of an experiment.

Example 1.2.8: An object falls under the influence of gravity near Earth's surface, where it can be assumed that the magnitude of the acceleration due to gravity is a constant g .

- (a) Construct a mathematical model for the motion of the object in the form of an initial value problem for a second order differential equation, assuming that the altitude and velocity of the object at time $t = 0$ are known. Assume that gravity is the only force acting on the object.
- (b) Solve the initial value problem derived in (a) to obtain the altitude as a function of time.

Solution (a): Let $y(t)$ be the altitude of the object at time t . Since the acceleration of the object has constant magnitude g and is in the downward (negative) direction, y satisfies the second order equation

$$y'' = -g,$$

where the prime now indicates differentiation with respect to t . If y_0 and v_0 denote the altitude and velocity when $t = 0$, then y is a solution of the initial value problem

$$y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0. \quad (1.27)$$

Solution (b): Integrating (1.27) twice yields

$$\begin{aligned} y' &= -gt + c_1, \\ y &= -\frac{gt^2}{2} + c_1t + c_2. \end{aligned}$$

Imposing the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ in these two equations shows that $c_1 = v_0$ and $c_2 = y_0$. Therefore the solution of the initial value problem (1.27) is

$$y = -\frac{gt^2}{2} + v_0t + y_0.$$

1.2.4 Exercises

1. Find the order of the equation.

(a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \frac{d^3y}{dx^3} + x = 0$

(b) $y'' - 3y' + 2y = x^7$

(c) $y' - y^7 = 0$

(d) $y''y - (y')^2 = 2$

2. Verify that the function is a solution of the differential equation on some interval, for any choice of the arbitrary constants appearing in the function.

(a) $y = ce^{2x}; \quad y' = 2y$

(b) $y = \frac{x^2}{3} + \frac{c}{x}; \quad xy' + y = x^2$

(c) $y = \frac{1}{2} + ce^{-x^2}; \quad y' + 2xy = x$

(d) $y = (1 + ce^{-x^2/2}); (1 - ce^{-x^2/2})^{-1} \quad 2y' + x(y^2 - 1) = 0$

(e) $y = \tan\left(\frac{x^3}{3} + c\right); \quad y' = x^2(1 + y^2)$

(f) $y = (c_1 + c_2x)e^x + \sin x + x^2; \quad y'' - 2y' + y = -2\cos x + x^2 - 4x + 2$

(g) $y = c_1e^x + c_2x + \frac{2}{x}; \quad (1 - x)y'' + xy' - y = 4(1 - x - x^2)x^{-3}$

(h) $y = x^{-1/2}(c_1 \sin x + c_2 \cos x) + 4x + 8;$
 $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 4x^3 + 8x^2 + 3x - 2$

3. Find all solutions of the equation.

(a) $y' = -x$

(b) $y' = -x \sin x$

(c) $y' = x \ln x$

(d) $y'' = x \cos x$

(e) $y'' = 2xe^x$

(f) $y'' = 2x + \sin x + e^x$

(g) $y''' = -\cos x$

(h) $y''' = -x^2 + e^x$

(i) $y''' = 7e^{4x}$

4. Solve the initial value problem.

(a) $y' = -xe^x, \quad y(0) = 1$

(b) $y' = x \sin x^2, \quad y\left(\sqrt{\frac{\pi}{2}}\right) = 1$

(c) $y' = \tan x, \quad y(\pi/4) = 3$

(d) $y'' = x^4, \quad y(2) = -1, \quad y'(2) = -1$

(e) $y'' = xe^{2x}, \quad y(0) = 7, \quad y'(0) = 1$

(f) $y'' = -x \sin x, \quad y(0) = 1, \quad y'(0) = -3$

(g) $y''' = x^2e^x, \quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 3$

(h) $y''' = 2 + \sin 2x, \quad y(0) = 1, \quad y'(0) = -6, \quad y''(0) = 3$

(i) $y''' = 2x + 1, \quad y(2) = 1, \quad y'(2) = -4, \quad y''(2) = 7$

5. Verify that the function is a solution of the initial value problem.

(a) $y = x \cos x; \quad y' = \cos x - y \tan x, \quad y(\pi/4) = \frac{\pi}{4\sqrt{2}}$

(b) $y = \frac{1 + 2 \ln x}{x^2} + \frac{1}{2}; \quad y' = \frac{x^2 - 2x^2y + 2}{x^3}, \quad y(1) = \frac{3}{2}$

(c) $y = \tan\left(\frac{x^2}{2}\right); \quad y' = x(1 + y^2), \quad y(0) = 0$

(d) $y = \frac{2}{x-2}; \quad y' = \frac{-y(y+1)}{x}, \quad y(1) = -2$

6. Verify that the function is a solution of the initial value problem.

(a) $y = x^2(1 + \ln x); \quad y'' = \frac{3xy' - 4y}{x^2}, \quad y(e) = 2e^2, \quad y'(e) = 5e$

(b) $y = \frac{x^2}{3} + x - 1; \quad y'' = \frac{x^2 - xy' + y + 1}{x^2}, \quad y(1) = \frac{1}{3}, \quad y'(1) = \frac{5}{3}$

(c) $y = (1 + x^2)^{-1/2}$; $y'' = \frac{(x^2 - 1)y - x(x^2 + 1)y'}{(x^2 + 1)^2}$, $y(0) = 1$,
 $y'(0) = 0$

(d) $y = \frac{x^2}{1 - x}$; $y'' = \frac{2(x + y)(xy' - y)}{x^3}$, $y(1/2) = 1/2$, $y'(1/2) = 3$

7. Suppose an object is launched from a point 320 feet above the earth with an initial velocity of 128 ft/sec upward, and the only force acting on it thereafter is gravity. Take $g = 32$ ft/sec².

(a) Find the highest altitude attained by the object.

(b) Determine how long it takes for the object to fall to the ground.

8. Let a be a nonzero real number.

(a) Verify that if c is an arbitrary constant then

$$y = (x - c)^a \quad (\text{A})$$

is a solution of

$$y' = ay^{(a-1)/a} \quad (\text{B})$$

on (c, ∞) .

(b) Suppose $a < 0$ or $a > 1$. Can you think of a solution of (B) that isn't of the form (A)?

9. Verify that

$$y = \begin{cases} e^x - 1, & x \geq 0, \\ 1 - e^{-x}, & x < 0, \end{cases}$$

is a solution of

$$y' = |y| + 1$$

on $(-\infty, \infty)$. *HINT: Use the definition of derivative at $x = 0$.*

10. (a) Verify that if c is any real number then

$$y = c^2 + cx + 2c + 1 \quad (\text{A})$$

satisfies

$$y' = \frac{-(x + 2) + \sqrt{x^2 + 4x + 4y}}{2} \quad (\text{B})$$

on some open interval. Identify the open interval.

(b) Verify that

$$y_1 = \frac{-x(x + 4)}{4}$$

also satisfies (B) on some open interval, and identify the open interval. (Note that y_1 can't be obtained by selecting a value of c in (A).)

1.3 Slope Fields for First Order Equations

It's impossible to find explicit formulas for solutions of some differential equations. Even if there are such formulas, they may be so complicated that they're useless. In this case we may resort to graphical or numerical methods to get some idea of how the solutions of the given equation behave.

In Section 2.3 we'll take up the question of existence of solutions of a first order equation

$$y' = f(x, y). \quad (1.1)$$

In this section we'll simply assume that (1.1) has solutions and discuss a graphical method for approximating them. In Chapter 3 we discuss numerical methods for obtaining approximate solutions of (1.1).

Recall that a solution of (1.1) is a function $y = y(x)$ such that

$$y'(x) = f(x, y(x))$$

for all values of x in some interval, and an integral curve is either the graph of a solution or is made up of segments that are graphs of solutions. Therefore, not being able to solve (1.1) is equivalent to not knowing the equations of integral curves of (1.1). However, it's easy to calculate the slopes of these curves. To be specific, the slope of an integral curve of (1.1) through a given point (x_0, y_0) is given by the number $f(x_0, y_0)$. This is the basis of *the method of slope fields*. Do note that slope fields are also called direction fields. In this book, we will strictly use the term slope field.

If f is defined on a set R , we can construct a *slope field* for (1.1) in R by drawing a short line segment through each point (x, y) in R with slope $f(x, y)$. Of course, as a practical matter, we can't actually draw line segments through *every* point in R ; rather, we must select a finite set of points in R . For example, suppose f is defined on the closed rectangular region

$$R : \{a \leq x \leq b, c \leq y \leq d\}.$$

Let

$$a = x_0 < x_1 < \cdots < x_m = b$$

be equally spaced points in $[a, b]$ and

$$c = y_0 < y_1 < \cdots < y_n = d$$

be equally spaced points in $[c, d]$. We say that the points

$$(x_i, y_j), \quad 0 \leq i \leq m, \quad 0 \leq j \leq n,$$

form a *rectangular grid* (Figure 1.6). Through each point in the grid we draw a short line segment with slope $f(x_i, y_j)$. The result is an approximation to a slope field for (1.1) in R . If the grid points are sufficiently numerous and close together, we can draw approximate integral curves of (1.1) by drawing curves through points in the grid tangent to the line segments associated with the points in the grid.

Unfortunately, approximating a slope field and graphing integral curves in this way is too tedious to be done effectively by hand. However, there is software for doing this. As you'll see, the combination of slope fields and integral curves gives useful insights into the behavior of the solutions of the differential equation even if we can't obtain exact solutions.

We'll study numerical methods for solving a single first order equation (1.1) in Chapter 3. These methods can be used to plot solution curves of (1.1) in a rectangular region R if f is continuous on R . Figures 1.7, 1.8, and 1.9 show slope fields and solution curves for the differential equations

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y' = 1 + xy^2, \quad \text{and} \quad y' = \frac{x - y}{1 + x^2},$$

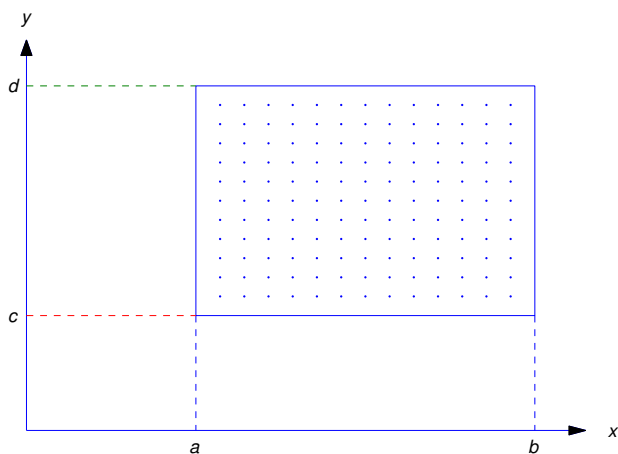
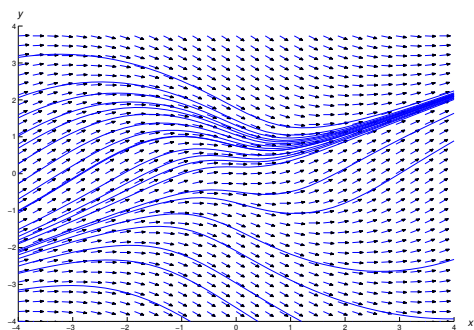
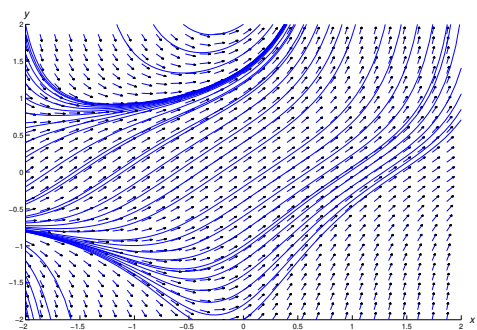


Figure 1.6: A rectangular grid

Figure 1.7: A slope field and integral curves for $y' = \frac{x^2 - y^2}{1 + x^2 + y^2}$ Figure 1.8: A slope field and integral curves for $y' = 1 + xy^2$

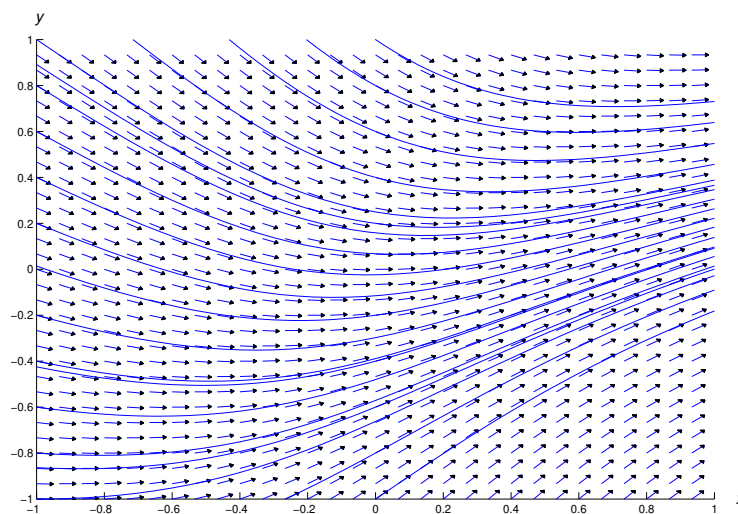


Figure 1.9: A slope field and integral curves for $y' = \frac{x - y}{1 + x^2}$

which are all of the form (1.1) with f continuous for all (x, y) .

The methods of Chapter 3 won't work for the equation

$$y' = -x/y \quad (1.2)$$

if R contains part of the x -axis, since $f(x, y) = -x/y$ is undefined when $y = 0$. Similarly, they won't work for the equation

$$y' = \frac{x^2}{1 - x^2 - y^2} \quad (1.3)$$

if R contains any part of the unit circle $x^2 + y^2 = 1$, because the right side of (1.3) is undefined if $x^2 + y^2 = 1$. However, (1.2) and (1.3) can be written as

$$y' = \frac{A(x, y)}{B(x, y)} \quad (1.4)$$

where A and B are continuous on any rectangle R . Because of this, some differential equation software is based on numerically solving pairs of equations of the form

$$\frac{dx}{dt} = B(x, y), \quad \frac{dy}{dt} = A(x, y) \quad (1.5)$$

where x and y are regarded as functions of a parameter t . If $x = x(t)$ and $y = y(t)$ satisfy these equations, then

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{A(x, y)}{B(x, y)},$$

so $y = y(x)$ satisfies (1.4).

Eqns. (1.2) and (1.3) can be reformulated as in (1.4) with

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x$$

and

$$\frac{dx}{dt} = 1 - x^2 - y^2, \quad \frac{dy}{dt} = x^2,$$

respectively. Even if f is continuous and otherwise “nice” throughout R , your software may require you to reformulate the equation $y' = f(x, y)$ as

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = f(x, y),$$

which is of the form (1.5) with $A(x, y) = f(x, y)$ and $B(x, y) = 1$.

Figure 1.10 shows a slope field and some integral curves for (1.2). As we saw in Example 1.2.1 and will verify again in Section 2.2, the integral curves of (1.2) are circles centered at the origin.

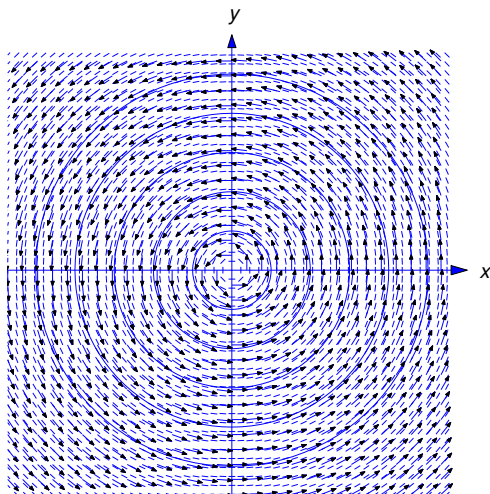


Figure 1.10: A slope field and integral curves for $y' = -\frac{x}{y}$

Figure 1.11 shows a slope field and some integral curves for (1.3). The integral curves near the top and bottom are solution curves. However, the integral curves near the middle are more complicated. For example, Figure 1.12 shows the integral curve through the origin. The vertices of the dashed rectangle are on the circle $x^2 + y^2 = 1$ ($a \approx .846$, $b \approx .533$), where all integral curves of (1.3) have infinite slope. There are three solution curves of (1.3) on the integral curve in the figure: the segment above the level $y = b$ is the graph of a solution on $(-\infty, a)$, the segment below the level $y = -b$ is the graph of a solution on $(-a, \infty)$, and the segment between these two levels is the graph of a solution on $(-a, a)$.

As you study from this book, you'll often be asked to use computer software and graphics. Exercises with this intent are marked as C (computer or calculator required), C/G (computer and/or graphics required), or L (laboratory work requiring software and/or graphics). Often you may not completely understand how the software does what it does. This is similar to the situation most people are in when they drive automobiles or watch television, and it doesn't decrease the value of using modern technology as an aid to learning. Just be careful that you use the technology as a supplement to thought rather than a substitute for it.

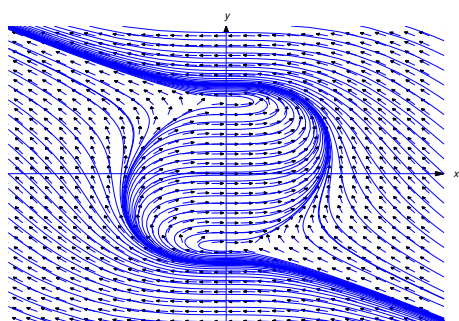


Figure 1.11: A slope field and integral curves for $y' = \frac{x^2}{1 - x^2 - y^2}$

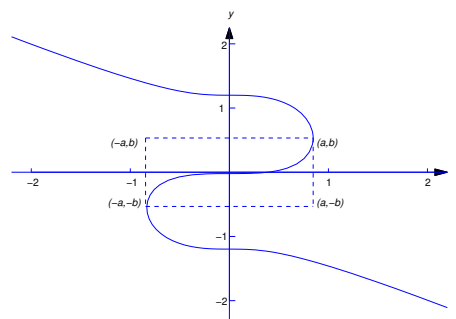
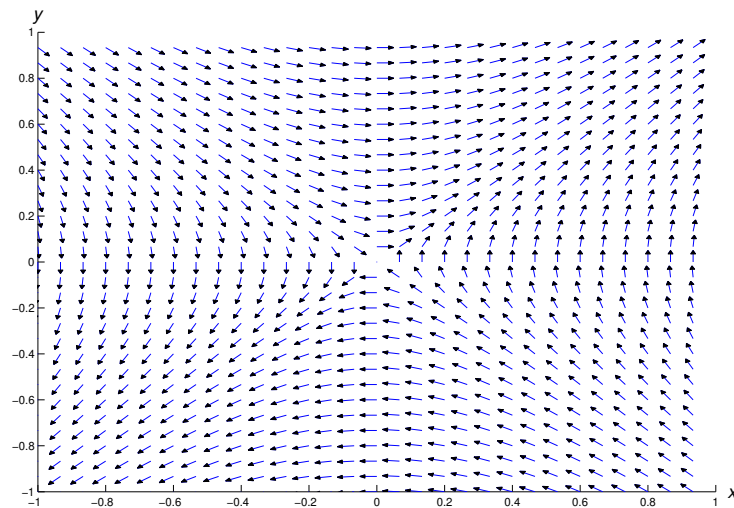


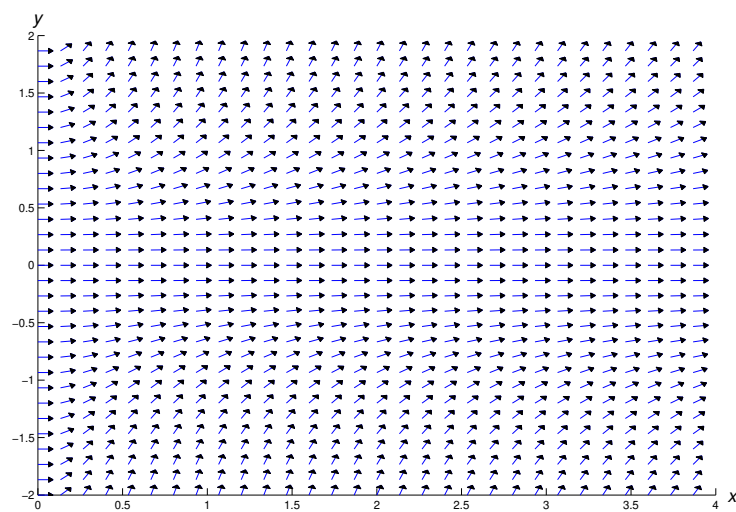
Figure 1.12

1.3.1 Exercises

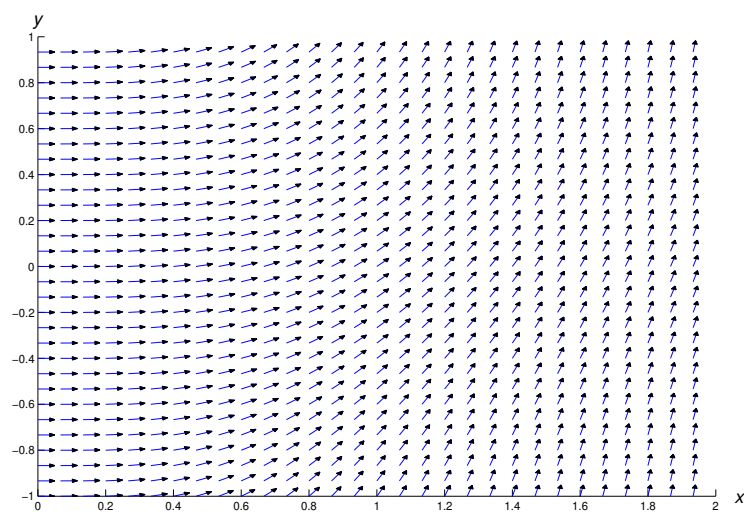
In Exercises **1–11** a slope field is drawn for the given equation. Sketch some integral curves.



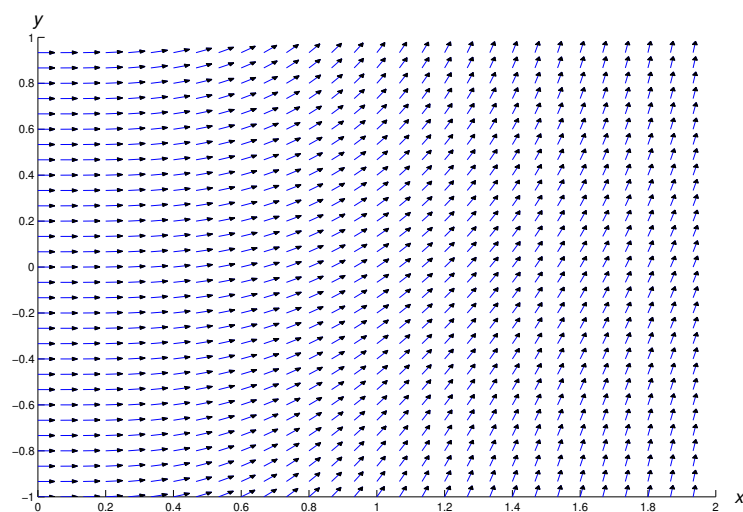
1 A slope field for $y' = \frac{x}{y}$



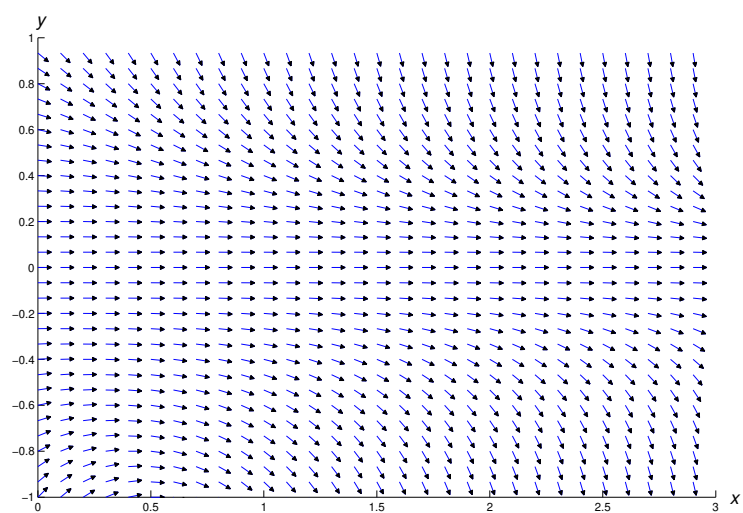
2 A slope field for $y' = \frac{2xy^2}{1+x^2}$



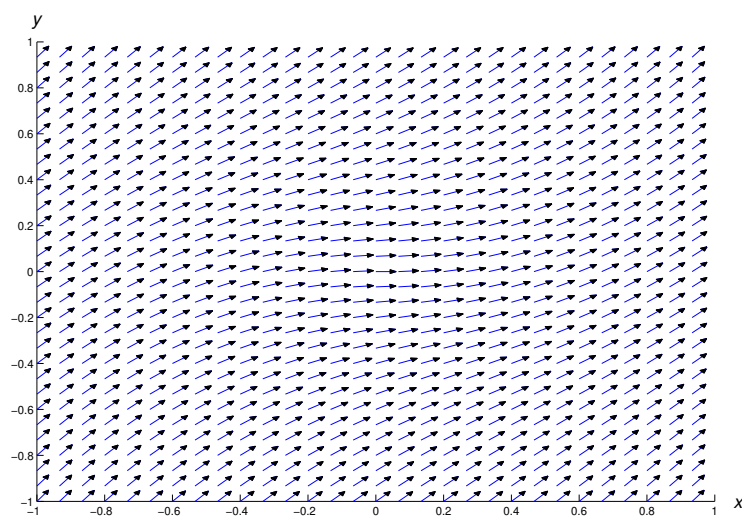
3 A slope field for $y' = x^2(1+y^2)$



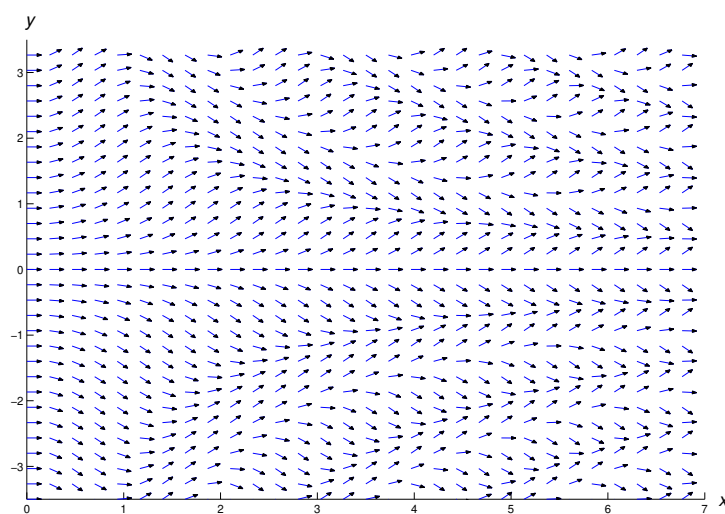
4 A slope field for $y' = \frac{1}{1+x^2+y^2}$



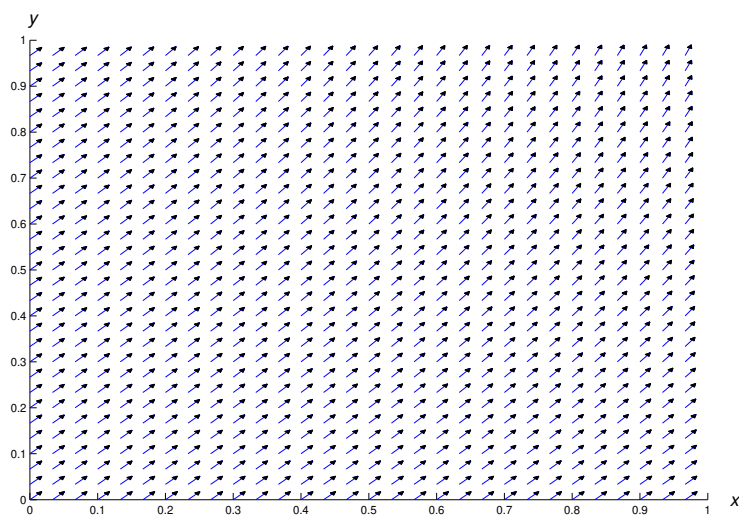
5 A slope field for $y' = -(2xy^2 + y^3)$



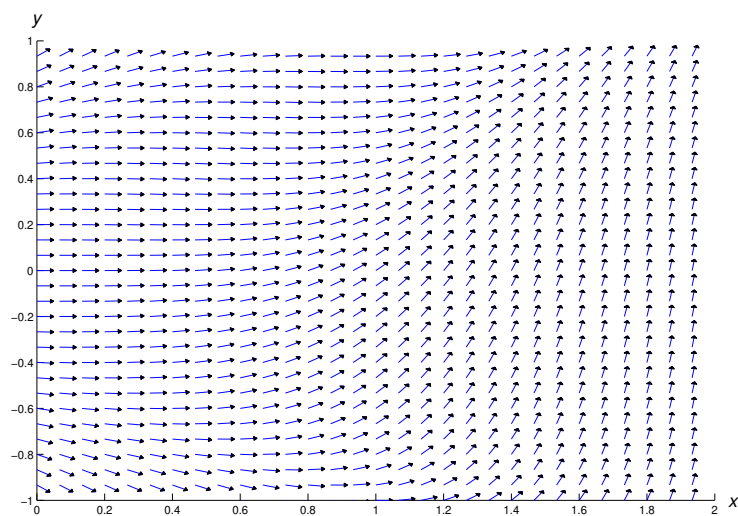
6 A slope field for $y' = (x^2 + y^2)^{1/2}$



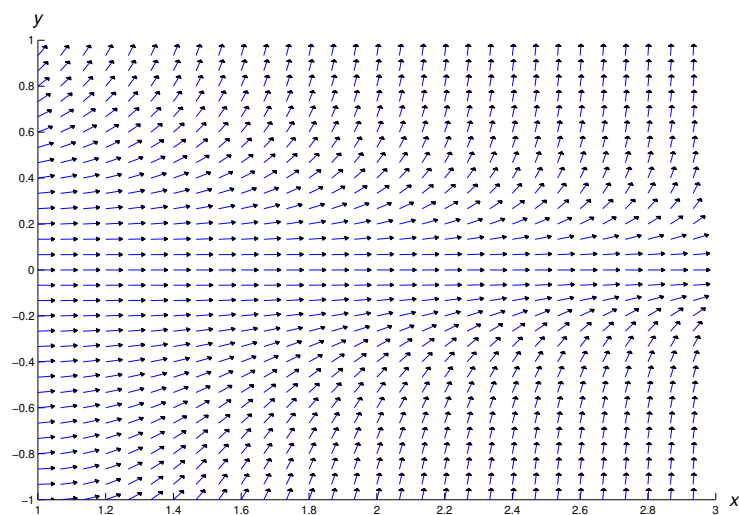
7 A slope field for $y' = \sin xy$



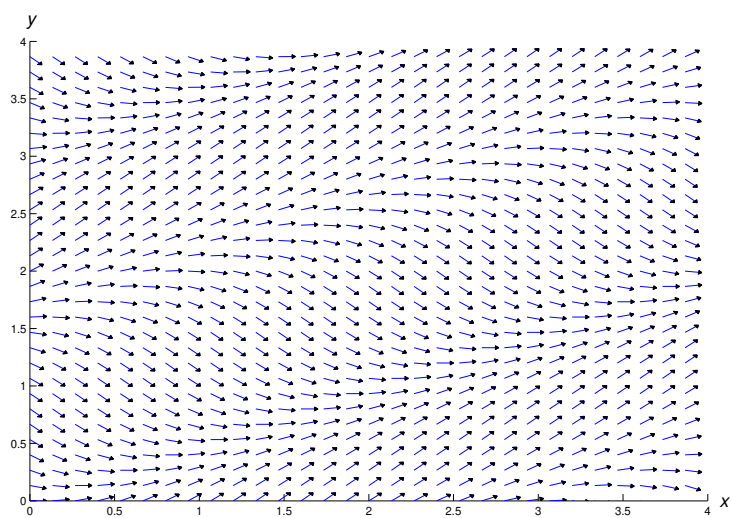
8 A slope field for $y' = e^{xy}$



9 A slope field for $y' = (x - y^2)(x^2 - y)$



10 A slope field for $y' = x^3y^2 + xy^3$



11 A slope field for $y' = \sin(x - 2y)$

In Exercises 1.3.12–1.3.22 construct a slope field and plot some integral curves in the indicated rectangular region.

12. C/G $y' = y(y - 1); \quad \{-1 \leq x \leq 2, -2 \leq y \leq 2\}$
13. C/G $y' = 2 - 3xy; \quad \{-1 \leq x \leq 4, -4 \leq y \leq 4\}$
14. C/G $y' = xy(y - 1); \quad \{-2 \leq x \leq 2, -4 \leq y \leq 4\}$
15. C/G $y' = 3x + y; \quad \{-2 \leq x \leq 2, 0 \leq y \leq 4\}$
16. C/G $y' = y - x^3; \quad \{-2 \leq x \leq 2, -2 \leq y \leq 2\}$
17. C/G $y' = 1 - x^2 - y^2; \quad \{-2 \leq x \leq 2, -2 \leq y \leq 2\}$
18. C/G $y' = x(y^2 - 1); \quad \{-3 \leq x \leq 3, -3 \leq y \leq 2\}$
19. C/G $y' = \frac{x}{y(y^2 - 1)}; \quad \{-2 \leq x \leq 2, -2 \leq y \leq 2\}$
20. C/G $y' = \frac{xy^2}{y - 1}; \quad \{-2 \leq x \leq 2, -1 \leq y \leq 4\}$
21. C/G $y' = \frac{x(y^2 - 1)}{y}; \quad \{-1 \leq x \leq 1, -2 \leq y \leq 2\}$
22. C/G $y' = -\frac{x^2 + y^2}{1 - x^2 - y^2}; \quad \{-2 \leq x \leq 2, -2 \leq y \leq 2\}$

23. L By suitably renaming the constants and dependent variables in the equations

$$T' = -k(T - T_m) \tag{A}$$

and

$$G' = -\lambda G + r \tag{B}$$

discussed in Section 1.2 in connection with Newton's law of cooling and absorption of glucose in the body, we can write both as

$$y' = -ay + b, \tag{C}$$

where a is a positive constant and b is an arbitrary constant. Thus, (A) is of the form (C) with $y = T$, $a = k$, and $b = kT_m$, and (B) is of the form (C) with $y = G$, $a = \lambda$, and $b = r$. We'll encounter equations of the form (C) in many other applications in Chapter 2.

Choose a positive a and an arbitrary b . Construct a slope field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \leq t \leq T, c \leq y \leq d\}$$

of the ty -plane. Vary T , c , and d until you discover a common property of all the solutions of (C). Repeat this experiment with various choices of a and b until you can state this property precisely in terms of a and b .

24. L By suitably renaming the constants and dependent variables in the equations

$$P' = aP(1 - \alpha P) \tag{A}$$

and

$$I' = rI(S - I) \tag{B}$$

discussed in Section 1.1 in connection with Verhulst's population model and the spread of an epidemic, we can write both in the form

$$y' = ay - by^2, \tag{C}$$

where a and b are positive constants. Thus, (A) is of the form (C) with $y = P$, $a = a$, and $b = a\alpha$, and (B) is of the form (C) with $y = I$, $a = rS$, and $b = r$. In Chapter 2 we'll encounter equations of the form (C) in other applications..

- (a) Choose positive numbers a and b . Construct a slope field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \leq t \leq T, 0 \leq y \leq d\}$$

of the ty -plane. Vary T and d until you discover a common property of all solutions of (C) with $y(0) > 0$. Repeat this experiment with various choices of a and b until you can state this property precisely in terms of a and b .

- (b) Choose positive numbers a and b . Construct a slope field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \leq t \leq T, c \leq y \leq 0\}$$

of the ty -plane. Vary a , b , T and c until you discover a common property of all solutions of (C) with $y(0) < 0$.

You can verify your results later by doing Exercise [2.2.27](#).

Chapter 2

First Order Equations

2.1 Linear First Order Equations

A first order differential equation is said to be *linear* if it can be written as

$$y' + p(x)y = f(x). \quad (2.1)$$

A first order differential equation that can't be written like this is *nonlinear*. We say that (2.1) is *homogeneous* if $f \equiv 0$; otherwise it's *nonhomogeneous*. Since $y \equiv 0$ is obviously a solution of the homogeneous equation

$$y' + p(x)y = 0,$$

we call it the *trivial solution*. Any other solution is *nontrivial*.

Example 2.1.1: The first order equations

$$\begin{aligned} x^2 y' + 3y &= x^2, \\ x y' - 8x^2 y &= \sin x, \\ x y' + (\ln x)y &= 0, \\ y' &= x^2 y - 2, \end{aligned}$$

are not in the form (2.1), but they are linear, since they can be rewritten as

$$\begin{aligned} y' + \frac{3}{x^2}y &= 1, \\ y' - 8xy &= \frac{\sin x}{x}, \\ y' + \frac{\ln x}{x}y &= 0, \\ y' - x^2 y &= -2. \end{aligned}$$

Example 2.1.2: Here are some nonlinear first order equations:

$$\begin{aligned} x y' + 3y^2 &= 2x && \text{(because } y \text{ is squared),} \\ y y' &= 3 && \text{(because of the product } y y'), \\ y' + x e^y &= 12 && \text{(because of } e^y). \end{aligned}$$

2.1.1 General Solution of a Linear First Order Equation

To motivate a definition that we'll need, consider the simple linear first order equation

$$y' = \frac{1}{x^2}. \quad (2.2)$$

From calculus we know that y satisfies this equation if and only if

$$y = -\frac{1}{x} + c, \quad (2.3)$$

where c is an arbitrary constant. We call c a *parameter* and say that (2.3) defines a *one-parameter family* of functions. For each real number c , the function defined by (2.3) is a solution of (2.2) on $(-\infty, 0)$ and $(0, \infty)$; moreover, every solution of (2.2) on either of these intervals is of the form (2.3) for some choice of c . We say that (2.3) is *the general solution* of (2.2).

We'll see that a similar situation occurs in connection with any first order linear equation

$$y' + p(x)y = f(x); \quad (2.4)$$

that is, if p and f are continuous on some open interval (a, b) then there's a unique formula $y = y(x, c)$ analogous to (2.3) that involves x and a parameter c and has the these properties:

- For each fixed value of c , the resulting function of x is a solution of (2.4) on (a, b) .
- If y is a solution of (2.4) on (a, b) , then y can be obtained from the formula by choosing c appropriately.

We'll call $y = y(x, c)$ the *general solution* of (2.4).

When this has been established, it will follow that an equation of the form

$$P_0(x)y' + P_1(x)y = F(x) \quad (2.5)$$

has a general solution on any open interval (a, b) on which P_0 , P_1 , and F are all continuous and P_0 has no zeros, since in this case we can rewrite (2.5) in the form (2.4) with $p = P_1/P_0$ and $f = F/P_0$, which are both continuous on (a, b) .

To avoid awkward wording in examples and exercises, we won't specify the interval (a, b) when we ask for the general solution of a specific linear first order equation. Let's agree that this always means that we want the general solution on every open interval on which p and f are continuous if the equation is of the form (2.4), or on which P_0 , P_1 , and F are continuous and P_0 has no zeros, if the equation is of the form (2.5). We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0 , P_1 , and F are all continuous on an open interval (a, b) , but P_0 *does* have a zero in (a, b) , then (2.5) may fail to have a general solution on (a, b) in the sense just defined. Since this isn't a major point that needs to be developed in depth, we won't discuss it further; however, see Exercise 2.1.44 for an example.

2.1.2 Homogeneous Linear First Order Equations

We begin with the problem of finding the general solution of a homogeneous linear first order equation. The next example recalls a familiar result from calculus.

Example 2.1.3: Let a be a constant.

(a) Find the general solution of

$$y' - ay = 0. \quad (2.6)$$

(b) Solve the initial value problem

$$y' - ay = 0, \quad y(x_0) = y_0.$$

Solution (a): You already know from calculus that if c is any constant, then $y = ce^{ax}$ satisfies (2.6). However, let's pretend you've forgotten this, and use this problem to illustrate a general method for solving a homogeneous linear first order equation.

We know that (2.6) has the trivial solution $y \equiv 0$. Now suppose y is a nontrivial solution of (2.6). Then, since a differentiable function must be continuous, there must be some open interval I on which y has no zeros. We rewrite (2.6) as

$$\frac{y'}{y} = a$$

for x in I . Integrating this shows that

$$\ln|y| = ax + k, \quad \text{so} \quad |y| = e^k e^{ax},$$

where k is an arbitrary constant. Since e^{ax} can never equal zero, y has no zeros, so y is either always positive or always negative. Therefore we can rewrite y as

$$y = ce^{ax} \tag{2.7}$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

This shows that every nontrivial solution of (2.6) is of the form $y = ce^{ax}$ for some nonzero constant c . Since setting $c = 0$ yields the trivial solution, *all* solutions of (2.6) are of the form (2.7). Conversely, (2.7) is a solution of (2.6) for every choice of c , since differentiating (2.7) yields $y' = ace^{ax} = ay$.

Solution (b): Imposing the initial condition $y(x_0) = y_0$ yields $y_0 = ce^{ax_0}$, so $c = y_0 e^{-ax_0}$ and

$$y = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

Figure 2.1 show the graphs of this function with $x_0 = 0$, $y_0 = 1$, and various values of a .

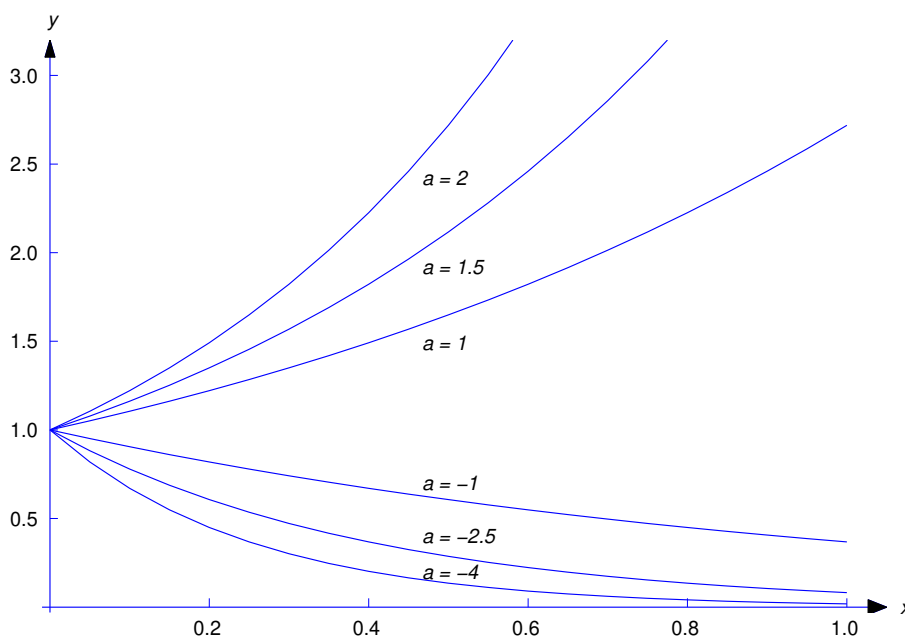


Figure 2.1: Solutions of $y' - ay = 0$, $y(0) = 1$

Example 2.1.4:

(a) Find the general solution of

$$xy' + y = 0. \quad (2.8)$$

(b) Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3. \quad (2.9)$$

Solution (a): We rewrite (2.8) as

$$y' + \frac{1}{x}y = 0, \quad (2.10)$$

where x is restricted to either $(-\infty, 0)$ or $(0, \infty)$. If y is a nontrivial solution of (2.10), there must be some open interval I on which y has no zeros. We can rewrite (2.10) as

$$\frac{y'}{y} = -\frac{1}{x}$$

for x in I . Integrating shows that

$$\ln |y| = -\ln |x| + k, \quad \text{so} \quad |y| = \frac{e^k}{|x|}.$$

Since a function that satisfies the last equation can't change sign on either $(-\infty, 0)$ or $(0, \infty)$, we can rewrite this result more simply as

$$y = \frac{c}{x} \quad (2.11)$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

We've now shown that every solution of (2.10) is given by (2.11) for some choice of c . (Even though we assumed that y was nontrivial to derive (2.11), we can get the trivial solution by setting $c = 0$ in (2.11).) Conversely, any function of the form (2.11) is a solution of (2.10), since differentiating (2.11) yields

$$y' = -\frac{c}{x^2},$$

and substituting this and (2.11) into (2.10) yields

$$\begin{aligned} y' + \frac{1}{x}y &= -\frac{c}{x^2} + \frac{1}{x} \frac{c}{x} \\ &= -\frac{c}{x^2} + \frac{c}{x^2} = 0. \end{aligned}$$

Figure 2.2 shows the graphs of some solutions corresponding to various values of c

Solution (b): Imposing the initial condition $y(1) = 3$ in (2.11) yields $c = 3$. Therefore the solution of (2.9) is

$$y = \frac{3}{x}.$$

The interval of validity of this solution is $(0, \infty)$.

The results in Examples 2.1.3(a) and 2.1.4(b) are special cases of the next theorem.

Theorem 2.1.1. *If p is continuous on (a, b) , then the general solution of the homogeneous equation*

$$y' + p(x)y = 0 \quad (2.12)$$

on (a, b) is

$$y = ce^{-P(x)},$$

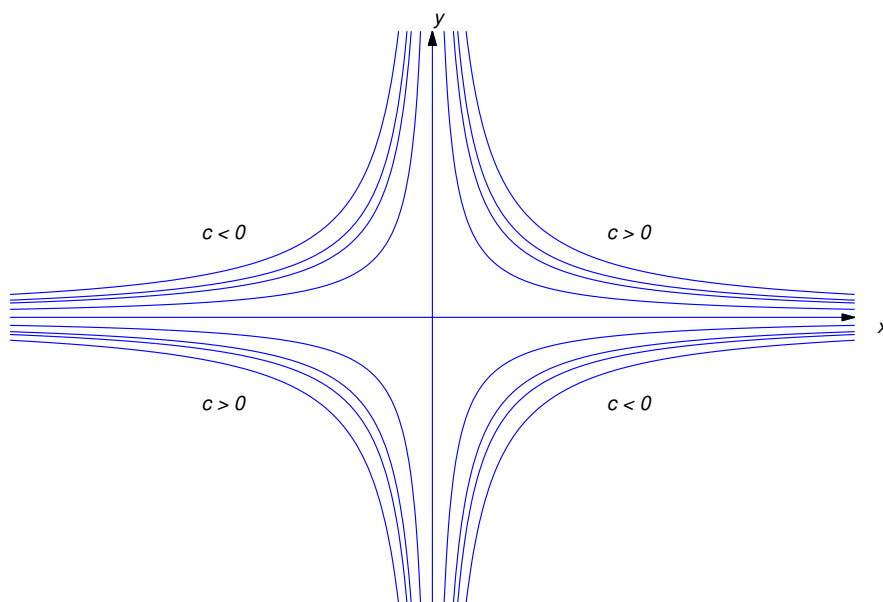


Figure 2.2: Solutions of $xy' + y = 0$ on $(0, \infty)$ and $(-\infty, 0)$

where

$$P(x) = \int p(x) dx \quad (2.13)$$

is any antiderivative of p on (a, b) ; that is,

$$P'(x) = p(x), \quad a < x < b. \quad (2.14)$$

Proof. If $y = ce^{-P(x)}$, differentiating y and using (2.14) shows that

$$y' = -P'(x)ce^{-P(x)} = -p(x)ce^{-P(x)} = -p(x)y,$$

so $y' + p(x)y = 0$; that is, y is a solution of (2.12), for any choice of c .

Now we'll show that any solution of (2.12) can be written as $y = ce^{-P(x)}$ for some constant c . The trivial solution can be written this way, with $c = 0$. Now suppose y is a nontrivial solution. Then there's an open subinterval I of (a, b) on which y has no zeros. We can rewrite (2.12) as

$$\frac{y'}{y} = -p(x) \quad (2.15)$$

for x in I . Integrating (2.15) and recalling (2.13) yields

$$\ln |y| = -P(x) + k,$$

where k is a constant. This implies that

$$|y| = e^k e^{-P(x)}.$$

Since P is defined for all x in (a, b) and an exponential can never equal zero, we can take $I = (a, b)$, so y has no zeros on (a, b) , so we can rewrite the last equation as $y = ce^{-P(x)}$, where

$$c = \begin{cases} e^k & \text{if } y > 0 \text{ on } (a, b), \\ -e^k & \text{if } y < 0 \text{ on } (a, b). \end{cases}$$

Remark 2.1.1: Rewriting a first order differential equation so that one side depends only on y and y' and the other depends only on x is called *separation of variables*. We did this in Examples 2.1.3 and 2.1.4, and in rewriting (2.12) as (2.15). We'll apply this method to nonlinear equations in Section 2.2.

2.1.3 Linear Nonhomogeneous First Order Equations

We'll now solve the nonhomogeneous equation

$$y' + p(x)y = f(x). \quad (2.16)$$

When considering this equation we call

$$y' + p(x)y = 0$$

the *complementary equation*.

We'll find solutions of (2.16) in the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation and u is to be determined. This method of using a solution of the complementary equation to obtain solutions of a nonhomogeneous equation is a special case of a method called *variation of parameters*, which you'll encounter several times in this book. (Obviously, u can't be constant, since if it were, the left side of (2.16) would be zero. Recognizing this, the early users of this method viewed u as a "parameter" that varies; hence, the name "variation of parameters.")

If

$$y = uy_1, \quad \text{then} \quad y' = u'y_1 + uy_1'.$$

Substituting these expressions for y and y' into (2.16) yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x),$$

which reduces to

$$u'y_1 = f(x), \quad (2.17)$$

since y_1 is a solution of the complementary equation; that is,

$$y_1' + p(x)y_1 = 0.$$

In the proof of Theorem 2.2.1 we saw that y_1 has no zeros on an interval where p is continuous. Therefore we can divide (2.17) through by y_1 to obtain

$$u' = f(x)/y_1(x).$$

We can integrate this (introducing a constant of integration), and multiply the result by y_1 to get the general solution of (2.16). Before turning to the formal proof of this claim, let's consider some examples.

Example 2.1.5: Find the general solution of

$$y' + 2y = x^3e^{-2x}. \quad (2.18)$$

By applying (a) of Example 2.1.3 with $a = -2$, we see that $y_1 = e^{-2x}$ is a solution of the complementary equation $y' + 2y = 0$. Therefore we seek solutions of (2.18) in the form $y = ue^{-2x}$, so that

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y' + 2y = u'e^{-2x} - 2ue^{-2x} + 2ue^{-2x} = u'e^{-2x}. \quad (2.19)$$

Therefore y is a solution of (2.18) if and only if

$$u'e^{-2x} = x^3e^{-2x} \quad \text{or, equivalently,} \quad u' = x^3.$$

Therefore

$$u = \frac{x^4}{4} + c,$$

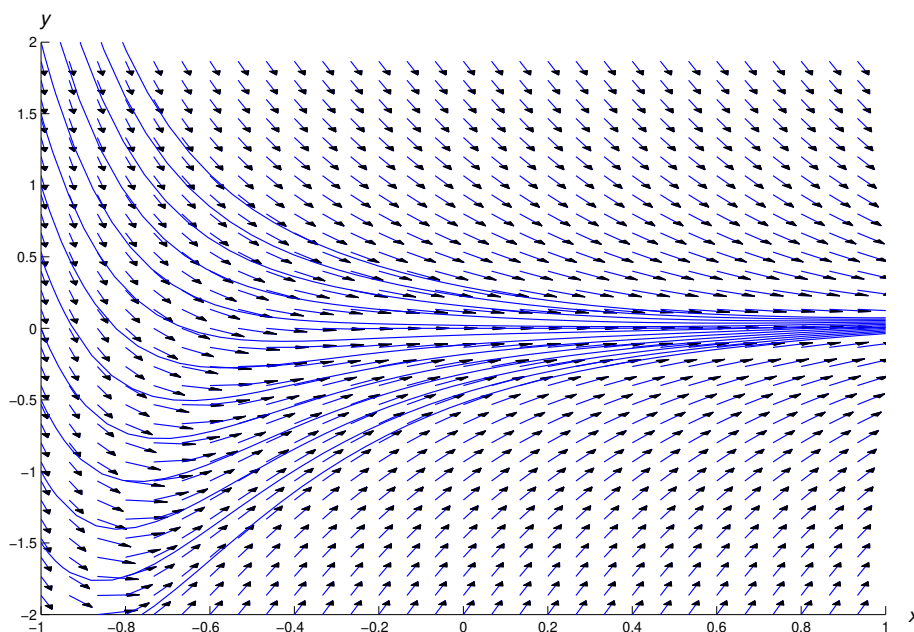


Figure 2.3: A direction field and integral curves for $y' + 2y = x^2 e^{-2x}$

and

$$y = u e^{-2x} = e^{-2x} \left(\frac{x^4}{4} + c \right)$$

is the general solution of (2.18).

Figure 2.3 shows a direction field and some integral curves for (2.18).

Example 2.1.6:

(a) Find the general solution

$$y' + (\cot x)y = x \csc x. \quad (2.20)$$

(b) Solve the initial value problem

$$y' + (\cot x)y = x \csc x, \quad y(\pi/2) = 1. \quad (2.21)$$

Solution (a): Here $p(x) = \cot x$ and $f(x) = x \csc x$ are both continuous except at the points $x = r\pi$, where r is an integer. Therefore we seek solutions of (2.20) on the intervals $(r\pi, (r+1)\pi)$. We need a nontrivial solution y_1 of the complementary equation; thus, y_1 must satisfy $y_1' + (\cot x)y_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = -\cot x = -\frac{\cos x}{\sin x}. \quad (2.22)$$

Integrating this yields

$$\ln |y_1| = -\ln |\sin x|,$$

where we take the constant of integration to be zero since we need only *one* function that satisfies (2.22). Clearly $y_1 = 1/\sin x$ is a suitable choice. Therefore we seek solutions of (2.20) in the form

$$y = \frac{u}{\sin x},$$

so that

$$y' = \frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x} \quad (2.23)$$

and

$$\begin{aligned} y' + (\cot x)y &= \frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x} + \frac{u \cot x}{\sin x} \\ &= \frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x} + \frac{u \cos x}{\sin^2 x} \\ &= \frac{u'}{\sin x}. \end{aligned} \quad (2.24)$$

Therefore y is a solution of (2.20) if and only if

$$u'/\sin x = x \csc x = x/\sin x \quad \text{or, equivalently,} \quad u' = x.$$

Integrating this yields

$$u = \frac{x^2}{2} + c, \quad \text{and} \quad y = \frac{u}{\sin x} = \frac{x^2}{2 \sin x} + \frac{c}{\sin x}. \quad (2.25)$$

is the general solution of (2.20) on every interval $(r\pi, (r+1)\pi)$ (r =integer).

Solution (b): Imposing the initial condition $y(\pi/2) = 1$ in (2.25) yields

$$1 = \frac{\pi^2}{8} + c \quad \text{or} \quad c = 1 - \frac{\pi^2}{8}.$$

Thus,

$$y = \frac{x^2}{2 \sin x} + \frac{(1 - \pi^2/8)}{\sin x}$$

is a solution of (2.21). The interval of validity of this solution is $(0, \pi)$; Figure 2.4 shows its graph.

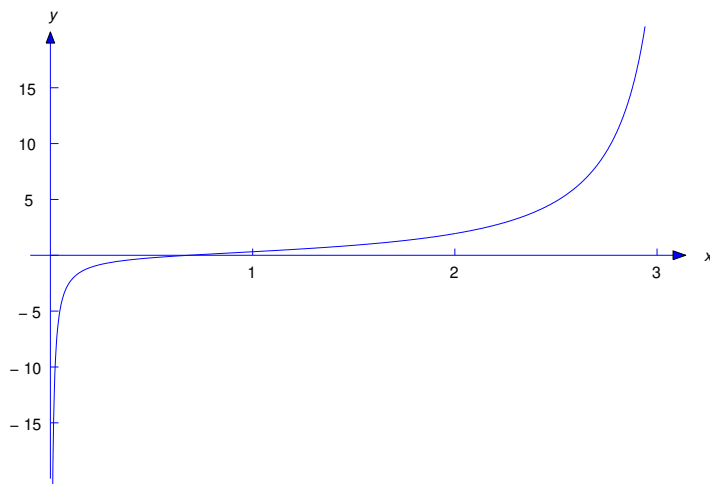


Figure 2.4: Solution of $y' + (\cot x)y = x \csc x$, $y(\pi/2) = 1$

Remark 2.1.2: It wasn't necessary to do the computations (2.23) and (2.24) in Example 2.1.6, since we showed in the discussion preceding Example 2.1.5 that if $y = u y_1$ where $y_1' + p(x)y_1 = 0$, then $y' + p(x)y = u' y_1$. We did these computations so you would see this happen in this specific example. We recommend that you include these "unnecessary" computations in doing exercises, until you're confident that you really understand the method. After that, omit them.

We summarize the method of variation of parameters for solving

$$y' + p(x)y = f(x) \quad (2.26)$$

as follows:

- (a) Find a function y_1 such that

$$\frac{y_1'}{y_1} = -p(x).$$

For convenience, take the constant of integration to be zero.

- (b) Write

$$y = uy_1 \quad (2.27)$$

to remind yourself of what you're doing.

- (c) Write $u'y_1 = f$ and solve for u' ; thus, $u' = f/y_1$.
 (d) Integrate u' to obtain u , with an arbitrary constant of integration.
 (e) Substitute u into (2.27) to obtain y .

To solve an equation written as

$$P_0(x)y' + P_1(x)y = F(x),$$

we recommend that you divide through by $P_0(x)$ to obtain an equation of the form (2.26) and then follow this procedure.

2.1.4 Solutions in Integral Form

Sometimes the integrals that arise in solving a linear first order equation can't be evaluated in terms of elementary functions. In this case the solution must be left in terms of an integral.

Example 2.1.7:

- (a) Find the general solution of

$$y' - 2xy = 1.$$

- (b) Solve the initial value problem

$$y' - 2xy = 1, \quad y(0) = y_0. \quad (2.28)$$

Solution (a): To apply variation of parameters, we need a nontrivial solution y_1 of the complementary equation; thus, $y_1' - 2xy_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = 2x.$$

Integrating this and taking the constant of integration to be zero yields

$$\ln|y_1| = x^2, \quad \text{so} \quad |y_1| = e^{x^2}.$$

We choose $y_1 = e^{x^2}$ and seek solutions of (2.28) in the form $y = ue^{x^2}$, where

$$u'e^{x^2} = 1, \quad \text{so} \quad u' = e^{-x^2}.$$

Therefore

$$u = c + \int e^{-x^2} dx,$$

but we can't simplify the integral on the right because there's no elementary function with derivative equal to e^{-x^2} . Therefore the best available form for the general solution of (2.28) is

$$y = ue^{x^2} = e^{x^2} \left(c + \int e^{-x^2} dx \right). \quad (2.29)$$

Solution (b): Since the initial condition in (2.28) is imposed at $x_0 = 0$, it is convenient to rewrite (2.29) as

$$y = e^{x^2} \left(c + \int_0^x e^{-t^2} dt \right), \quad \text{since} \quad \int_0^0 e^{-t^2} dt = 0.$$

Setting $x = 0$ and $y = y_0$ here shows that $c = y_0$. Therefore the solution of the initial value problem is

$$y = e^{x^2} \left(y_0 + \int_0^x e^{-t^2} dt \right). \quad (2.30)$$

For a given value of y_0 and each fixed x , the integral on the right can be evaluated by numerical methods. An alternate procedure is to apply the numerical integration procedures discussed in Chapter 3 directly to the initial value problem (2.28). Figure 2.5 shows graphs of (2.30) for several values of y_0 .

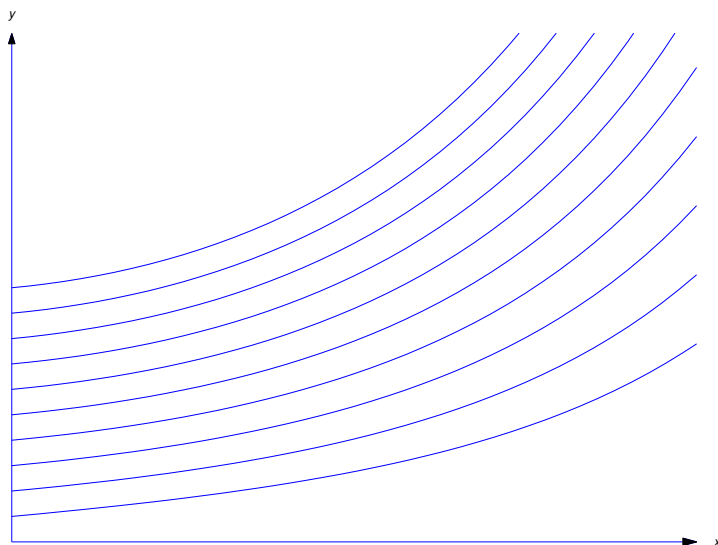


Figure 2.5: Solutions of $y' - 2xy = 1$, $y(0) = y_0$

2.1.5 An Existence and Uniqueness Theorem

The method of variation of parameters leads to this theorem.

Theorem 2.1.2. Suppose p and f are continuous on an open interval (a, b) , and let y_1 be any nontrivial solution of the complementary equation

$$y' + p(x)y = 0$$

on (a, b) . Then:

(a) *The general solution of the nonhomogeneous equation*

$$y' + p(x)y = f(x) \quad (2.31)$$

on (a, b) is

$$y = y_1(x) \left(c + \int f(x)/y_1(x) dx \right). \quad (2.32)$$

(b) *If x_0 is an arbitrary point in (a, b) and y_0 is an arbitrary real number, then the initial value problem*

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has the unique solution

$$y = y_1(x) \left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^x \frac{f(t)}{y_1(t)} dt \right)$$

on (a, b) .

Proof. (a) To show that (2.32) is the general solution of (2.31) on (a, b) , we must prove that:

(i) If c is any constant, the function y in (2.32) is a solution of (2.31) on (a, b) .

(ii) If y is a solution of (2.31) on (a, b) then y is of the form (2.32) for some constant c .

To prove (i), we first observe that any function of the form (2.32) is defined on (a, b) , since p and f are continuous on (a, b) . Differentiating (2.32) yields

$$y' = y_1'(x) \left(c + \int f(x)/y_1(x) dx \right) + f(x).$$

Since $y_1' = -p(x)y_1$, this and (2.32) imply that

$$\begin{aligned} y' &= -p(x)y_1(x) \left(c + \int f(x)/y_1(x) dx \right) + f(x) \\ &= -p(x)y(x) + f(x), \end{aligned}$$

which implies that y is a solution of (2.31).

To prove (ii), suppose y is a solution of (2.31) on (a, b) . From the proof of Theorem 2.1.1, we know that y_1 has no zeros on (a, b) , so the function $u = y/y_1$ is defined on (a, b) . Moreover, since

$$y' = -py + f \quad \text{and} \quad y_1' = -py_1,$$

$$\begin{aligned} u' &= \frac{y_1 y' - y_1' y}{y_1^2} \\ &= \frac{y_1(-py + f) - (-py_1)y}{y_1^2} = \frac{f}{y_1}. \end{aligned}$$

Integrating $u' = f/y_1$ yields

$$u = \left(c + \int f(x)/y_1(x) dx \right),$$

which implies (2.32), since $y = uy_1$.

(b) We've proved (a), where $\int f(x)/y_1(x) dx$ in (2.32) is an arbitrary antiderivative of f/y_1 . Now it's convenient to choose the antiderivative that equals zero when $x = x_0$, and write the general solution of (2.31) as

$$y = y_1(x) \left(c + \int_{x_0}^x \frac{f(t)}{y_1(t)} dt \right).$$

Since

$$y(x_0) = y_1(x_0) \left(c + \int_{x_0}^{x_0} \frac{f(t)}{y_1(t)} dt \right) = cy_1(x_0),$$

we see that $y(x_0) = y_0$ if and only if $c = y_0/y_1(x_0)$.

2.1.6 Exercises

In Exercises 2.1.1–2.1.5 find the general solution.

1. $y' + ay = 0$ ($a = \text{constant}$)
2. $y' + 3x^2y = 0$
3. $xy' + (\ln x)y = 0$
4. $xy' + 3y = 0$
5. $x^2y' + y = 0$

In Exercises 2.1.6–2.1.11 solve the initial value problem.

6. $y' + \left(\frac{1+x}{x}\right)y = 0, \quad y(1) = 1$
7. $xy' + \left(1 + \frac{1}{\ln x}\right)y = 0, \quad y(e) = 1$
8. $xy' + (1 + x \cot x)y = 0, \quad y\left(\frac{\pi}{2}\right) = 2$
9. $y' - \left(\frac{2x}{1+x^2}\right)y = 0, \quad y(0) = 2$
10. $y' + \frac{k}{x}y = 0, \quad y(1) = 3 \quad (k = \text{constant})$
11. $y' + (\tan kx)y = 0, \quad y(0) = 2 \quad (k = \text{constant})$

In Exercises 2.1.12–2.1.15 find the general solution. Also, plot a direction field and some integral curves on the rectangular region $\{-2 \leq x \leq 2, -2 \leq y \leq 2\}$.

12. C/G $y' + 3y = 1$
13. C/G $y' + \left(\frac{1}{x} - 1\right)y = -\frac{2}{x}$
14. C/G $y' + 2xy = xe^{-x^2}$
15. C/G $y' + \frac{2x}{1+x^2}y = \frac{e^{-x}}{1+x^2}$

In Exercises 2.1.16–2.1.24 find the general solution.

16. $y' + \frac{1}{x}y = \frac{7}{x^2} + 3$
17. $y' + \frac{4}{x-1}y = \frac{1}{(x-1)^5} + \frac{\sin x}{(x-1)^4}$
18. $xy' + (1 + 2x^2)y = x^3e^{-x^2}$
19. $xy' + 2y = \frac{2}{x^2} + 1$
20. $y' + (\tan x)y = \cos x$
21. $(1+x)y' + 2y = \frac{\sin x}{1+x}$
22. $(x-2)(x-1)y' - (4x-3)y = (x-2)^3$
23. $y' + (2 \sin x \cos x)y = e^{-\sin^2 x}$
24. $x^2y' + 3xy = e^x$

In Exercises 2.1.25–2.1.29 solve the initial value problem and sketch the graph of the solution.

25. C/G $y' + 7y = e^{3x}, \quad y(0) = 0$
26. C/G $(1+x^2)y' + 4xy = \frac{2}{1+x^2}, \quad y(0) = 1$

27. C/G $xy' + 3y = \frac{2}{x(1+x^2)}, \quad y(-1) = 0$

28. C/G $y' + (\cot x)y = \cos x, \quad y\left(\frac{\pi}{2}\right) = 1$

29. C/G $y' + \frac{1}{x}y = \frac{2}{x^2} + 1, \quad y(-1) = 0$

In Exercises 2.1.30–2.1.37 solve the initial value problem.

30. $(x-1)y' + 3y = \frac{1}{(x-1)^3} + \frac{\sin x}{(x-1)^2}, \quad y(0) = 1$

31. $xy' + 2y = 8x^2, \quad y(1) = 3$

32. $xy' - 2y = -x^2, \quad y(1) = 1$

33. $y' + 2xy = x, \quad y(0) = 3$

34. $(x-1)y' + 3y = \frac{1+(x-1)\sec^2 x}{(x-1)^3}, \quad y(0) = -1$

35. $(x+2)y' + 4y = \frac{1+2x^2}{x(x+2)^3}, \quad y(-1) = 2$

36. $(x^2-1)y' - 2xy = x(x^2-1), \quad y(0) = 4$

37. $(x^2-5)y' - 2xy = -2x(x^2-5), \quad y(2) = 7$

In Exercises 2.1.38–2.1.42 solve the initial value problem and leave the answer in a form involving a definite integral. (You can solve these problems numerically by methods discussed in Chapter 3.)

38. $y' + 2xy = x^2, \quad y(0) = 3$

39. $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2$

40. $y' + y = \frac{e^{-x} \tan x}{x}, \quad y(1) = 0$

41. $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1$

42. $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2$

43. Experiments indicate that glucose is absorbed by the body at a rate proportional to the amount of glucose present in the bloodstream. Let λ denote the (positive) constant of proportionality. Now suppose glucose is injected into a patient's bloodstream at a constant rate of r units per unit of time. Let $G = G(t)$ be the number of units in the patient's bloodstream at time $t > 0$. Then

$$G' = -\lambda G + r,$$

where the first term on the right is due to the absorption of the glucose by the patient's body and the second term is due to the injection. Determine G for $t > 0$, given that $G(0) = G_0$. Also, find $\lim_{t \rightarrow \infty} G(t)$.

44. (a) L Plot a direction field and some integral curves for

$$xy' - 2y = -1 \tag{A}$$

on the rectangular region $\{-1 \leq x \leq 1, -0.5 \leq y \leq 1.5\}$. What do all the integral curves have in common?

- (b) Show that the general solution of (A) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{1}{2} + cx^2.$$

- (c) Show that y is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} \frac{1}{2} + c_1 x^2, & x \geq 0, \\ \frac{1}{2} + c_2 x^2, & x < 0, \end{cases}$$

where c_1 and c_2 are arbitrary constants.

- (d) Conclude from (c) that all solutions of (A) on $(-\infty, \infty)$ are solutions of the initial value problem

$$xy' - 2y = -1, \quad y(0) = \frac{1}{2}.$$

- (e) Use (b) to show that if $x_0 \neq 0$ and y_0 is arbitrary, then the initial value problem

$$xy' - 2y = -1, \quad y(x_0) = y_0$$

has infinitely many solutions on $(-\infty, \infty)$. Explain why this doesn't contradict Theorem 2.1.1(b).

45. Suppose f is continuous on an open interval (a, b) and α is a constant.

- (a) Derive a formula for the solution of the initial value problem

$$y' + \alpha y = f(x), \quad y(x_0) = y_0, \tag{A}$$

where x_0 is in (a, b) and y_0 is an arbitrary real number.

- (b) Suppose $(a, b) = (a, \infty)$, $\alpha > 0$ and $\lim_{x \rightarrow \infty} f(x) = L$. Show that if y is the solution of (A), then $\lim_{x \rightarrow \infty} y(x) = L/\alpha$.

46. Assume that all functions in this exercise are defined on a common interval (a, b) .

- (a) Prove: If y_1 and y_2 are solutions of

$$y' + p(x)y = f_1(x)$$

and

$$y' + p(x)y = f_2(x)$$

respectively, and c_1 and c_2 are constants, then $y = c_1 y_1 + c_2 y_2$ is a solution of

$$y' + p(x)y = c_1 f_1(x) + c_2 f_2(x).$$

(This is the principle of superposition.)

- (b) Use (a) to show that if y_1 and y_2 are solutions of the nonhomogeneous equation

$$y' + p(x)y = f(x), \tag{A}$$

then $y_1 - y_2$ is a solution of the homogeneous equation

$$y' + p(x)y = 0. \tag{B}$$

- (c) Use (a) to show that if y_1 is a solution of (A) and y_2 is a solution of (B), then $y_1 + y_2$ is a solution of (A).

47. Some nonlinear equations can be transformed into linear equations by changing the dependent variable. Show that if

$$g'(y)y' + p(x)g(y) = f(x)$$

where y is a function of x and g is a function of y , then the new dependent variable $z = g(y)$ satisfies the linear equation

$$z' + p(x)z = f(x).$$

48. Solve by the method discussed in Exercise 2.1.47.

(a) $(\sec^2 y)y' - 3 \tan y = -1$

(b) $e^{y^2} \left(2yy' + \frac{2}{x} \right) = \frac{1}{x^2}$

(c) $\frac{xy'}{y} + 2 \ln y = 4x^2$

(d) $\frac{y'}{(1+y)^2} - \frac{1}{x(1+y)} = -\frac{3}{x^2}$

49. We've shown that if p and f are continuous on (a, b) then every solution of

$$y' + p(x)y = f(x) \tag{A}$$

on (a, b) can be written as $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation for (A) and $u' = f/y_1$. Now suppose $f, f', \dots, f^{(m)}$ and $p, p', \dots, p^{(m-1)}$ are continuous on (a, b) , where m is a positive integer, and define

$$\begin{aligned} f_0 &= f, \\ f_j &= f'_{j-1} + pf_{j-1}, \quad 1 \leq j \leq m. \end{aligned}$$

Show that

$$u^{(j+1)} = \frac{f_j}{y_1}, \quad 0 \leq j \leq m.$$

2.2 Separable Equations

A first order differential equation is *separable* if it can be written as

$$h(y)y' = g(x), \quad (2.33)$$

where the left side is a product of y' and a function of y and the right side is a function of x . Rewriting a separable differential equation in this form is called *separation of variables*. In Section 2.1 we used separation of variables to solve homogeneous linear equations. In this section we'll apply this method to nonlinear equations.

To see how to solve (2.33), let's first assume that y is a solution. Let $G(x)$ and $H(y)$ be antiderivatives of $g(x)$ and $h(y)$; that is,

$$H'(y) = h(y) \quad \text{and} \quad G'(x) = g(x). \quad (2.34)$$

Then, from the chain rule,

$$\frac{d}{dx}H(y(x)) = H'(y(x))y'(x) = h(y)y'(x).$$

Therefore (2.33) is equivalent to

$$\frac{d}{dx}H(y(x)) = \frac{d}{dx}G(x).$$

Integrating both sides of this equation and combining the constants of integration yields

$$H(y(x)) = G(x) + c. \quad (2.35)$$

Although we derived this equation on the assumption that y is a solution of (2.33), we can now view it differently: Any differentiable function y that satisfies (2.35) for some constant c is a solution of (2.33). To see this, we differentiate both sides of (2.35), using the chain rule on the left, to obtain

$$H'(y(x))y'(x) = G'(x),$$

which is equivalent to

$$h(y(x))y'(x) = g(x)$$

because of (2.34).

In conclusion, to solve (2.33) it suffices to find functions $G = G(x)$ and $H = H(y)$ that satisfy (2.34). Then any differentiable function $y = y(x)$ that satisfies (2.35) is a solution of (2.33).

Example 2.2.1: Solve the equation

$$y' = x(1 + y^2).$$

Solution: Separating variables yields

$$\frac{y'}{1 + y^2} = x.$$

Integrating yields

$$\tan^{-1} y = \frac{x^2}{2} + c$$

Therefore

$$y = \tan\left(\frac{x^2}{2} + c\right).$$

Example 2.2.2:

(a) Solve the equation

$$y' = -\frac{x}{y}. \quad (2.36)$$

(b) Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = 1. \quad (2.37)$$

(c) Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = -2. \quad (2.38)$$

Solution (a): Separating variables in (2.36) yields

$$yy' = -x.$$

Integrating yields

$$\frac{y^2}{2} = -\frac{x^2}{2} + c, \quad \text{or, equivalently,} \quad x^2 + y^2 = 2c.$$

The last equation shows that c must be positive if y is to be a solution of (2.36) on an open interval. Therefore we let $2c = a^2$ (with $a > 0$) and rewrite the last equation as

$$x^2 + y^2 = a^2. \quad (2.39)$$

This equation has two differentiable solutions for y in terms of x :

$$y = \sqrt{a^2 - x^2}, \quad -a < x < a, \quad (2.40)$$

and

$$y = -\sqrt{a^2 - x^2}, \quad -a < x < a. \quad (2.41)$$

The solution curves defined by (2.40) are semicircles above the x -axis and those defined by (2.41) are semicircles below the x -axis (Figure 2.6).

Solution (b): The solution of (2.37) is positive when $x = 1$; hence, it is of the form (2.40). Substituting $x = 1$ and $y = 1$ into (2.39) to satisfy the initial condition yields $a^2 = 2$; hence, the solution of (2.37) is

$$y = \sqrt{2 - x^2}, \quad -\sqrt{2} < x < \sqrt{2}.$$

Solution (c): The solution of (2.38) is negative when $x = 1$ and is therefore of the form (2.41). Substituting $x = 1$ and $y = -2$ into (2.39) to satisfy the initial condition yields $a^2 = 5$. Hence, the solution of (2.38) is

$$y = -\sqrt{5 - x^2}, \quad -\sqrt{5} < x < \sqrt{5}.$$

2.2.1 Implicit Solutions of Separable Equations

In Examples 2.2.1 and 2.2.2 we were able to solve the equation $H(y) = G(x) + c$ to obtain explicit formulas for solutions of the given separable differential equations. As we'll see in the next example, this isn't always possible. In this situation we must broaden our definition of a solution of a separable equation. The next theorem provides the basis for this modification. We omit the proof, which requires a result from advanced calculus called as the *implicit function theorem*.

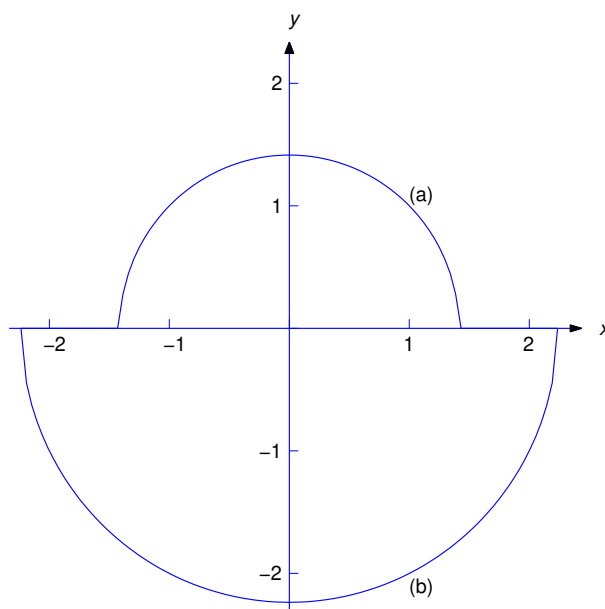


Figure 2.6: (a) $y = \sqrt{2 - x^2}$, $-\sqrt{2} < x < \sqrt{2}$; (b) $y = -\sqrt{5 - x^2}$, $-\sqrt{5} < x < \sqrt{5}$

Theorem 2.2.1. Suppose $g = g(x)$ is continuous on (a, b) and $h = h(y)$ are continuous on (c, d) . Let G be an antiderivative of g on (a, b) and let H be an antiderivative of h on (c, d) . Let x_0 be an arbitrary point in (a, b) , let y_0 be a point in (c, d) such that $h(y_0) \neq 0$, and define

$$c = H(y_0) - G(x_0). \quad (2.42)$$

Then there's a function $y = y(x)$ defined on some open interval (a_1, b_1) , where $a \leq a_1 < x_0 < b_1 \leq b$, such that $y(x_0) = y_0$ and

$$H(y) = G(x) + c \quad (2.43)$$

for $a_1 < x < b_1$. Therefore y is a solution of the initial value problem

$$h(y)y' = g(x), \quad y(x_0) = y_0. \quad (2.44)$$

It's convenient to say that (2.43) with c arbitrary is an *implicit solution* of $h(y)y' = g(x)$. Curves defined by (2.43) are integral curves of $h(y)y' = g(x)$. If c satisfies (2.42), we'll say that (2.43) is an *implicit solution of the initial value problem* (2.44). However, keep these points in mind:

- For some choices of c there may not be any differentiable functions y that satisfy (2.43).
- The function y in (2.43) (not (2.43) itself) is a solution of $h(y)y' = g(x)$.

Example 2.2.3:

(a) Find implicit solutions of

$$y' = \frac{2x + 1}{5y^4 + 1}. \quad (2.45)$$

(b) Find an implicit solution of

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \quad (2.46)$$

Solution (a): Separating variables yields

$$(5y^4 + 1)y' = 2x + 1.$$

Integrating yields the implicit solution

$$y^5 + y = x^2 + x + c. \quad (2.47)$$

of (2.45).

Solution (b): Imposing the initial condition $y(2) = 1$ in (2.47) yields $1 + 1 = 4 + 2 + c$, so $c = -4$. Therefore

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem (2.46). Although more than one differentiable function $y = y(x)$ satisfies (2.45) near $x = 1$, it can be shown that there's only one such function that satisfies the initial condition $y(1) = 2$.

Figure 2.7 shows a direction field and some integral curves for (2.45).

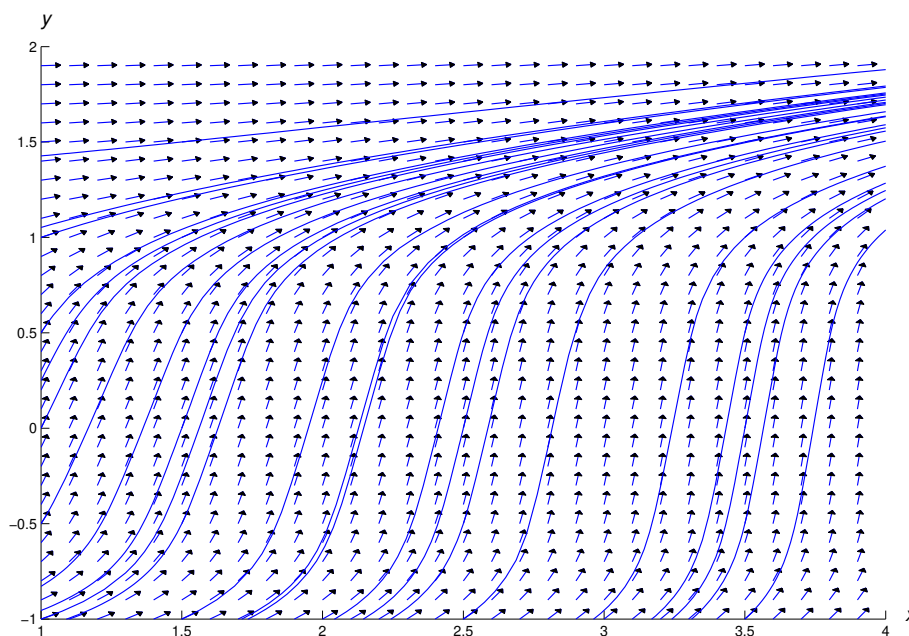


Figure 2.7: A direction field and integral curves for $y' = \frac{2x+1}{5y^4+1}$

2.2.2 Constant Solutions of Separable Equations

An equation of the form

$$y' = g(x)p(y)$$

is separable, since it can be rewritten as

$$\frac{1}{p(y)}y' = g(x).$$

However, the division by $p(y)$ is not legitimate if $p(y) = 0$ for some values of y . The next two examples show how to deal with this problem.

Example 2.2.4: Find all solutions of

$$y' = 2xy^2. \quad (2.48)$$

Solution: Here we must divide by $p(y) = y^2$ to separate variables. This isn't legitimate if y is a solution of (2.48) that equals zero for some value of x . One such solution can be found by inspection: $y \equiv 0$. Now suppose y is a solution of (2.48) that isn't identically zero. Since y is continuous there must be an interval on which y is never zero. Since division by y^2 is legitimate for x in this interval, we can separate variables in (2.48) to obtain

$$\frac{y'}{y^2} = 2x.$$

Integrating this yields

$$-\frac{1}{y} = x^2 + c,$$

which is equivalent to

$$y = -\frac{1}{x^2 + c}. \quad (2.49)$$

We've now shown that if y is a solution of (2.48) that is not identically zero, then y must be of the form (2.49). By substituting (2.49) into (2.48), you can verify that (2.49) is a solution of (2.48). Thus, solutions of (2.48) are $y \equiv 0$ and the functions of the form (2.49). Note that the solution $y \equiv 0$ isn't of the form (2.49) for any value of c .

Figure 2.8 shows a direction field and some integral curves for (2.48)

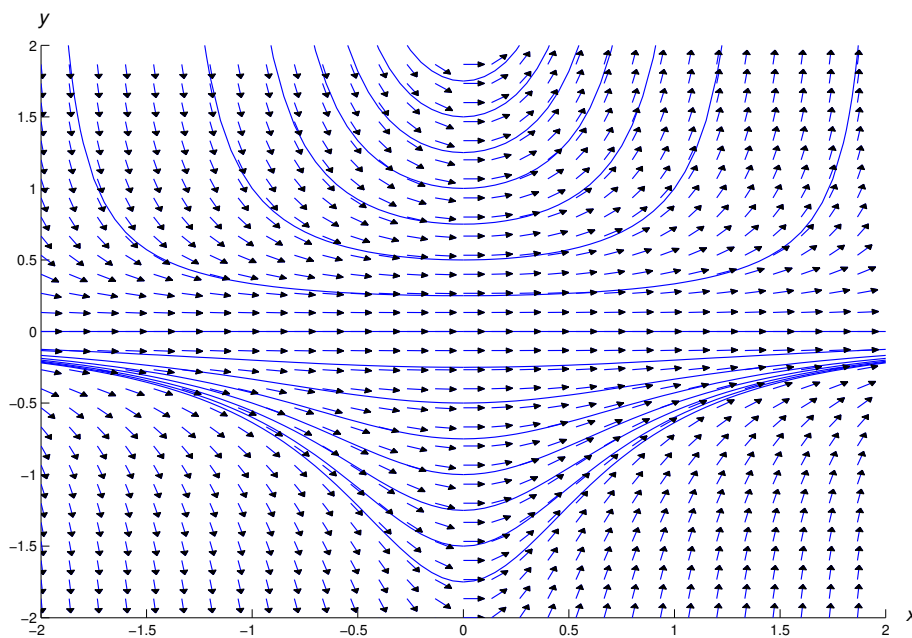


Figure 2.8: A direction field and integral curves for $y' = 2xy^2$

Example 2.2.5: Find all solutions of

$$y' = \frac{1}{2}x(1 - y^2). \quad (2.50)$$

Solution: Here we must divide by $p(y) = 1 - y^2$ to separate variables. This isn't legitimate if y is a solution of (2.50) that equals ± 1 for some value of x . Two such solutions can be found by inspection: $y \equiv 1$ and $y \equiv -1$.

Now suppose y is a solution of (2.50) such that $1 - y^2$ isn't identically zero. Since $1 - y^2$ is continuous there must be an interval on which $1 - y^2$ is never zero. Since division by $1 - y^2$ is legitimate for x in this interval, we can separate variables in (2.50) to obtain

$$\frac{2y'}{y^2 - 1} = -x.$$

A partial fraction expansion on the left yields

$$\left[\frac{1}{y-1} - \frac{1}{y+1} \right] y' = -x,$$

and integrating yields

$$\ln \left| \frac{y-1}{y+1} \right| = -\frac{x^2}{2} + k;$$

hence,

$$\left| \frac{y-1}{y+1} \right| = e^k e^{-x^2/2}.$$

Since $y(x) \neq \pm 1$ for x on the interval under discussion, the quantity $(y-1)/(y+1)$ can't change sign in this interval. Therefore we can rewrite the last equation as

$$\frac{y-1}{y+1} = c e^{-x^2/2},$$

where $c = \pm e^k$, depending upon the sign of $(y-1)/(y+1)$ on the interval. Solving for y yields

$$y = \frac{1 + c e^{-x^2/2}}{1 - c e^{-x^2/2}}. \quad (2.51)$$

We've now shown that if y is a solution of (2.50) that is not identically equal to ± 1 , then y must be as in (2.51). By substituting (2.51) into (2.50) you can verify that (2.51) is a solution of (2.50). Thus, the solutions of (2.50) are $y \equiv 1$, $y \equiv -1$ and the functions of the form (2.51). Note that the constant solution $y \equiv 1$ can be obtained from this formula by taking $c = 0$; however, the other constant solution, $y \equiv -1$, can't be obtained in this way.

Figure 2.9 shows a direction field and some integrals for (2.50).

2.2.3 Differences Between Linear and Nonlinear Equations

Theorem 2.1.2 states that if p and f are continuous on (a, b) then every solution of

$$y' + p(x)y = f(x)$$

on (a, b) can be obtained by choosing a value for the constant c in the general solution, and if x_0 is any point in (a, b) and y_0 is arbitrary, then the initial value problem

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a solution on (a, b) .

This is not true for nonlinear equations. First, we saw in Examples 2.2.4 and 2.2.5 that a nonlinear equation may have solutions that can't be obtained by choosing a specific value of a constant appearing in a one-parameter family of solutions. Second, it is in general impossible to determine the interval of validity of a solution to an initial value problem for a nonlinear equation by simply examining the equation, since the interval of validity may depend on the initial condition. For instance, in Example 2.2.2 we saw that the solution of

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(x_0) = y_0$$

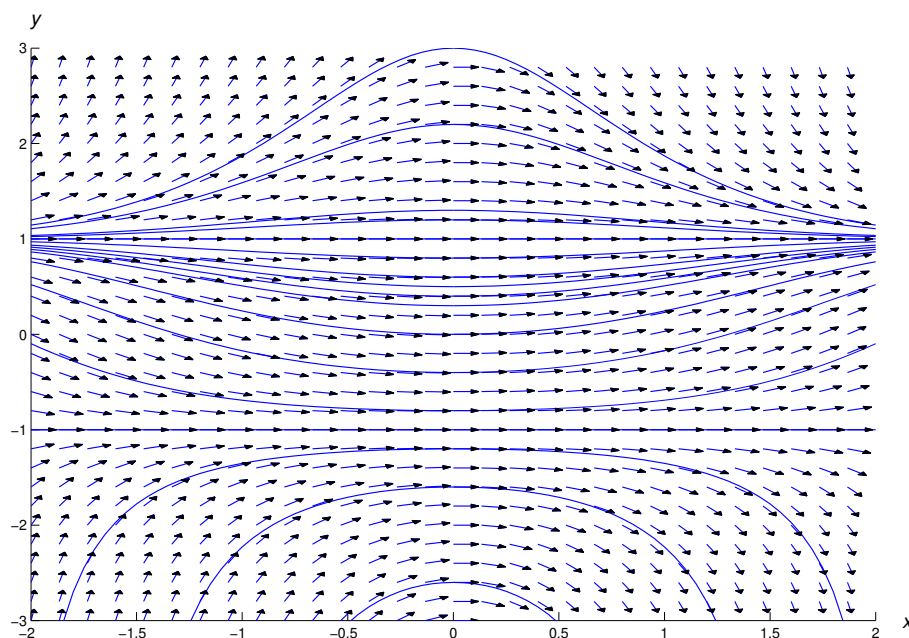


Figure 2.9: A direction field and integral curves for $y' = \frac{x(1 - y^2)}{2}$

is valid on $(-a, a)$, where $a = \sqrt{x_0^2 + y_0^2}$.

Example 2.2.6: Solve the initial value problem

$$y' = 2xy^2, \quad y(0) = y_0$$

and determine the interval of validity of the solution.

Solution: First suppose $y_0 \neq 0$. From Example 2.2.4, we know that y must be of the form

$$y = -\frac{1}{x^2 + c}. \quad (2.52)$$

Imposing the initial condition shows that $c = -1/y_0$. Substituting this into (2.52) and rearranging terms yields the solution

$$y = \frac{y_0}{1 - y_0 x^2}.$$

This is also the solution if $y_0 = 0$. If $y_0 < 0$, the denominator isn't zero for any value of x , so the the solution is valid on $(-\infty, \infty)$. If $y_0 > 0$, the solution is valid only on $(-1/\sqrt{y_0}, 1/\sqrt{y_0})$.

2.2.4 Exercises

In Exercises 2.2.1–2.2.6 find all solutions.

1. $y' = \frac{3x^2 + 2x + 1}{y - 2}$
2. $(\sin x)(\sin y) + (\cos y)y' = 0$
3. $xy' + y^2 + y = 0$
4. $y' \ln |y| + x^2 y = 0$

$$5. \quad (3y^3 + 3y \cos y + 1)y' + \frac{(2x+1)y}{1+x^2} = 0$$

$$6. \quad x^2 y y' = (y^2 - 1)^{3/2}$$

In Exercises 2.2.7–2.2.10 find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

$$7. \quad \boxed{\text{C/G}} \quad y' = x^2(1 + y^2); \quad \{-1 \leq x \leq 1, -1 \leq y \leq 1\}$$

$$8. \quad \boxed{\text{C/G}} \quad y'(1 + x^2) + xy = 0; \quad \{-2 \leq x \leq 2, -1 \leq y \leq 1\}$$

$$9. \quad \boxed{\text{C/G}} \quad y' = (x-1)(y-1)(y-2); \quad \{-2 \leq x \leq 2, -3 \leq y \leq 3\}$$

$$10. \quad \boxed{\text{C/G}} \quad (y-1)^2 y' = 2x+3; \quad \{-2 \leq x \leq 2, -2 \leq y \leq 5\}$$

In Exercises 2.2.11 and 2.2.12 solve the initial value problem.

$$11. \quad y' = \frac{x^2 + 3x + 2}{y - 2}, \quad y(1) = 4$$

$$12. \quad y' + x(y^2 + y) = 0, \quad y(2) = 1$$

In Exercises 2.2.13–2.2.16 solve the initial value problem and graph the solution.

$$13. \quad \boxed{\text{C/G}} \quad (3y^2 + 4y)y' + 2x + \cos x = 0, \quad y(0) = 1$$

$$14. \quad \boxed{\text{C/G}} \quad y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0$$

$$15. \quad \boxed{\text{C/G}} \quad y' + 2x(y+1) = 0, \quad y(0) = 2$$

$$16. \quad \boxed{\text{C/G}} \quad y' = 2xy(1 + y^2), \quad y(0) = 1$$

In Exercises 2.2.17–2.2.23 solve the initial value problem and find the interval of validity of the solution.

$$17. \quad y'(x^2 + 2) + 4x(y^2 + 2y + 1) = 0, \quad y(1) = -1$$

$$18. \quad y' = -2x(y^2 - 3y + 2), \quad y(0) = 3$$

$$19. \quad y' = \frac{2x}{1+2y}, \quad y(2) = 0$$

$$20. \quad y' = 2y - y^2, \quad y(0) = 1$$

$$21. \quad x + yy' = 0, \quad y(3) = -4$$

$$22. \quad y' + x^2(y+1)(y-2)^2 = 0, \quad y(4) = 2$$

$$23. \quad (x+1)(x-2)y' + y = 0, \quad y(1) = -3$$

$$24. \quad \text{Solve } y' = \frac{(1+y^2)}{(1+x^2)} \text{ explicitly. HINT: Use the identity } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$$25. \quad \text{Solve } y'\sqrt{1-x^2} + \sqrt{1-y^2} = 0 \text{ explicitly. HINT: Use the identity } \sin(A-B) = \sin A \cos B - \cos A \sin B.$$

$$26. \quad \text{Solve } y' = \frac{\cos x}{\sin y}, \quad y(\pi) = \frac{\pi}{2} \text{ explicitly. HINT: Use the identity } \cos(x + \pi/2) = -\sin x \text{ and the periodicity of the cosine.}$$

$$27. \quad \text{Solve the initial value problem}$$

$$y' = ay - by^2, \quad y(0) = y_0.$$

Discuss the behavior of the solution if (a) $y_0 \geq 0$; (b) $y_0 < 0$.

28. The population $P = P(t)$ of a species satisfies the logistic equation

$$P' = aP(1 - \alpha P)$$

and $P(0) = P_0 > 0$. Find P for $t > 0$, and find $\lim_{t \rightarrow \infty} P(t)$.

29. An epidemic spreads through a population at a rate proportional to the product of the number of people already infected and the number of people susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t , then

$$I' = rI(S - I),$$

where r is a positive constant. Assuming that $I(0) = I_0$, find $I(t)$ for $t > 0$, and show that $\lim_{t \rightarrow \infty} I(t) = S$.

30. L The result of Exercise 2.2.29 is discouraging: if any susceptible member of the group is initially infected, then in the long run all susceptible members are infected! On a more hopeful note, suppose the disease spreads according to the model of Exercise 2.2.29, but there's a medication that cures the infected population at a rate proportional to the number of infected individuals. Now the equation for the number of infected individuals becomes

$$I' = rI(S - I) - qI \tag{A}$$

where q is a positive constant.

- (a) Choose r and S positive. By plotting direction fields and solutions of (A) on suitable rectangular grids

$$R = \{0 \leq t \leq T, 0 \leq I \leq d\}$$

in the (t, I) -plane, verify that if I is any solution of (A) such that $I(0) > 0$, then $\lim_{t \rightarrow \infty} I(t) = S - q/r$ if $q < rS$ and $\lim_{t \rightarrow \infty} I(t) = 0$ if $q \geq rS$.

- (b) To verify the experimental results of (a), use separation of variables to solve (A) with initial condition $I(0) = I_0 > 0$, and find $\lim_{t \rightarrow \infty} I(t)$. HINT: There are three cases to consider: (i) $q < rS$; (ii) $q > rS$; (iii) $q = rS$.

31. L Consider the differential equation

$$y' = ay - by^2 - q, \tag{A}$$

where a, b are positive constants, and q is an arbitrary constant. Suppose y denotes a solution of this equation that satisfies the initial condition $y(0) = y_0$.

- (a) Choose a and b positive and $q < a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids

$$R = \{0 \leq t \leq T, c \leq y \leq d\} \tag{B}$$

in the (t, y) -plane, discover that there are numbers y_1 and y_2 with $y_1 < y_2$ such that if $y_0 > y_1$ then $\lim_{t \rightarrow \infty} y(t) = y_2$, and if $y_0 < y_1$ then $y(t) = -\infty$ for some finite value of t . (What happens if $y_0 = y_1$?)

- (b) Choose a and b positive and $q = a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids of the form (B), discover that there's a number y_1 such that if $y_0 \geq y_1$ then $\lim_{t \rightarrow \infty} y(t) = y_1$, while if $y_0 < y_1$ then $y(t) = -\infty$ for some finite value of t .

- (c) Choose positive a, b and $q > a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids of the form (B), discover that no matter what y_0 is, $y(t) = -\infty$ for some finite value of t .

- (d) Verify your results experiments analytically. Start by separating variables in (A) to obtain

$$\frac{y'}{ay - by^2 - q} = 1.$$

To decide what to do next you'll have to use the quadratic formula. This should lead you to see why there are three cases. Take it from there!

Because of its role in the transition between these three cases, $q_0 = a^2/4b$ is called a *bifurcation value* of q . In general, if q is a parameter in any differential equation, q_0 is said to be a bifurcation value of q if the nature of the solutions of the equation with $q < q_0$ is qualitatively different from the nature of the solutions with $q > q_0$.

32. L By plotting direction fields and solutions of

$$y' = qy - y^3,$$

convince yourself that $q_0 = 0$ is a bifurcation value of q for this equation. Explain what makes you draw this conclusion.

33. Suppose a disease spreads according to the model of Exercise 2.2.29, but there's a medication that cures the infected population at a constant rate of q individuals per unit time, where $q > 0$. Then the equation for the number of infected individuals becomes

$$I' = rI(S - I) - q.$$

Assuming that $I(0) = I_0 > 0$, use the results of Exercise 2.2.31 to describe what happens as $t \rightarrow \infty$.

34. Assuming that $p \not\equiv 0$, state conditions under which the linear equation

$$y' + p(x)y = f(x)$$

is separable. If the equation satisfies these conditions, solve it by separation of variables and by the method developed in Section 2.1.

Solve the equations in Exercises 2.2.35–2.2.38 using variation of parameters followed by separation of variables.

35. $y' + y = \frac{2xe^{-x}}{1 + ye^x}$

36. $xy' - 2y = \frac{x^6}{y + x^2}$

37. $y' - y = \frac{(x+1)e^{4x}}{(y+e^x)^2}$

38. $y' - 2y = \frac{xe^{2x}}{1 - ye^{-2x}}$

39. Use variation of parameters to show that the solutions of the following equations are of the form $y = uy_1$, where u satisfies a separable equation $u' = g(x)p(u)$. Find y_1 and g for each equation.

(a) $xy' + y = h(x)p(xy)$

(b) $xy' - y = h(x)p\left(\frac{y}{x}\right)$

(c) $y' + y = h(x)p(e^xy)$

(d) $xy' + ry = h(x)p(x^ry)$

(e) $y' + \frac{v'(x)}{v(x)}y = h(x)p(v(x)y)$

2.3 Existence and Uniqueness of Solutions of Nonlinear Equations

Although there are methods for solving some nonlinear equations, it's impossible to find useful formulas for the solutions of most. Whether we're looking for exact solutions or numerical approximations, it's useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations. In this section we state such a condition and illustrate it with examples.

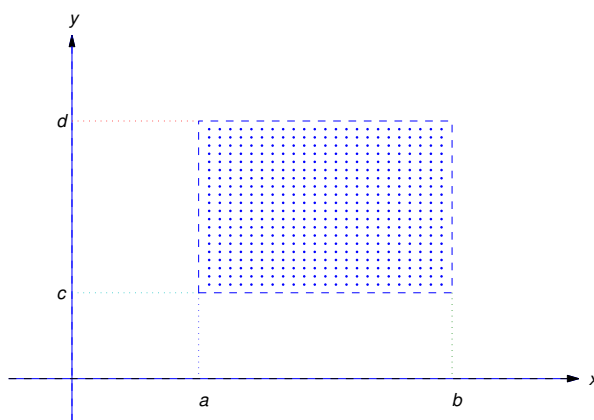


Figure 2.10: An open rectangle

Some terminology: an *open rectangle* R is a set of points (x, y) such that

$$a < x < b \quad \text{and} \quad c < y < d$$

(Figure 2.10). We'll denote this set by $R : \{a < x < b, c < y < d\}$. "Open" means that the boundary rectangle (indicated by the dashed lines in Figure 2.10) isn't included in R .

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order nonlinear differential equations. We omit the proof, which is beyond the scope of this book.

Theorem 2.3.1.

- (a) If f is continuous on an open rectangle

$$R : \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.53}$$

has at least one solution on some open subinterval of (a, b) that contains x_0 .

- (b) If both f and f_y are continuous on R then (2.53) has a unique solution on some open subinterval of (a, b) that contains x_0 .

It's important to understand exactly what Theorem 2.3.1 says.

- **(a)** is an *existence theorem*. It guarantees that a solution exists on some open interval that contains x_0 , but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, **(a)** provides no information on the number of solutions that (2.53) may have. It leaves open the possibility that (2.53) may have two or more solutions that differ for values of x arbitrarily close to x_0 . We will see in Example 2.3.6 that this can happen.
- **(b)** is a *uniqueness theorem*. It guarantees that (2.53) has a unique solution on some open interval (a, b) that contains x_0 . However, if $(a, b) \neq (-\infty, \infty)$, (2.53) may have more than one solution on a larger interval that contains (a, b) . For example, it may happen that $b < \infty$ and all solutions have the same values on (a, b) , but two solutions y_1 and y_2 are defined on some interval (a, b_1) with $b_1 > b$, and have different values for $b < x < b_1$; thus, the graphs of the y_1 and y_2 “branch off” in different directions at $x = b$. (See Example 2.3.7 and Figure 2.12). In this case, continuity implies that $y_1(b) = y_2(b)$ (call their common value $\bar{y}$), and y_1 and y_2 are both solutions of the initial value problem

$$y' = f(x, y), \quad y(b) = \bar{y} \quad (2.54)$$

that differ on every open interval that contains b . Therefore f or f_y must have a discontinuity at some point in each open rectangle that contains $(b, \bar{y})$, since if this were not so, (2.54) would have a unique solution on some open interval that contains b . We leave it to you to give a similar analysis of the case where $a > -\infty$.

Example 2.3.1: Consider the initial value problem

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0. \quad (2.55)$$

Since

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

are continuous for all (x, y) , Theorem 2.3.1 implies that if (x_0, y_0) is arbitrary, then (2.55) has a unique solution on some open interval that contains x_0 .

Example 2.3.2: Consider the initial value problem

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0. \quad (2.56)$$

Here

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{4x^2y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0, 0)$. If $(x_0, y_0) \neq (0, 0)$, there's an open rectangle R that contains (x_0, y_0) that does not contain $(0, 0)$. Since f and f_y are continuous on R , Theorem 2.3.1 implies that if $(x_0, y_0) \neq (0, 0)$ then (2.56) has a unique solution on some open interval that contains x_0 .

Example 2.3.3: Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \quad (2.57)$$

Here

$$f(x, y) = \frac{x + y}{x - y} \quad \text{and} \quad f_y(x, y) = \frac{2x}{(x - y)^2}$$

are continuous everywhere except on the line $y = x$. If $y_0 \neq x_0$, there's an open rectangle R that contains (x_0, y_0) that does not intersect the line $y = x$. Since f and f_y are continuous on R , Theorem 2.3.1 implies that if $y_0 \neq x_0$, (2.57) has a unique solution on some open interval that contains x_0 .

Example 2.3.4: In Example 2.2.4 we saw that the solutions of

$$y' = 2xy^2 \quad (2.58)$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2 + c},$$

where c is an arbitrary constant. In particular, this implies that no solution of (2.58) other than $y \equiv 0$ can equal zero for any value of x . Show that Theorem 2.3.1(b) implies this.

Solution: We'll obtain a contradiction by assuming that (2.58) has a solution y_1 that equals zero for some value of x , but isn't identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0 . This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \quad (2.59)$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of x on every open interval that contains x_0 . This contradicts Theorem 2.3.1(b), since in (2.58) the functions

$$f(x, y) = 2xy^2 \quad \text{and} \quad f_y(x, y) = 4xy.$$

are both continuous for all (x, y) , which implies that (2.59) has a unique solution on some open interval that contains x_0 .

Example 2.3.5: Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \quad (2.60)$$

- (a) For what points (x_0, y_0) does Theorem 2.3.1(a) imply that (2.60) has a solution?
- (b) For what points (x_0, y_0) does Theorem 2.3.1(b) imply that (2.60) has a unique solution on some open interval that contains x_0 ?

Solution (a): Since

$$f(x, y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x, y) , Theorem 2.3.1 implies that (2.60) has a solution for every (x_0, y_0) .

Solution (b): Here

$$f_y(x, y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x, y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 2.3.1 implies that (2.60) has a unique solution on some open interval that contains x_0 .

If $y = 0$ then $f_y(x, y)$ is undefined, and therefore discontinuous; hence, Theorem 2.3.1 does not apply to (2.60) if $y_0 = 0$.

Example 2.3.6: Example 2.3.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \quad (2.61)$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution: By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3}xy^{2/5}. \quad (2.62)$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it's a solution of (2.61).

Now suppose y is a solution of (2.62) that isn't identically zero. Separating variables in (2.62) yields

$$y^{-2/5} y' = \frac{10}{3} x$$

on any open interval where y has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3} y^{3/5} = \frac{5}{3} (x^2 + c).$$

Therefore

$$y = (x^2 + c)^{5/3}. \quad (2.63)$$

Since we divided by y to separate variables in (2.62), our derivation of (2.63) is legitimate only on open intervals where y has no zeros. However, (2.63) actually defines y for all x , and differentiating (2.63) shows that

$$y' = \frac{10}{3} x (x^2 + c)^{2/3} = \frac{10}{3} x y^{2/5}, \quad -\infty < x < \infty.$$

Therefore (2.63) satisfies (2.62) on $(-\infty, \infty)$ even if $c \leq 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in (2.63) yields

$$y = x^{10/3}$$

as a second solution of (2.61). Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (see Figure 2.11.) In fact, there are *four* distinct solutions of (2.61) defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two?

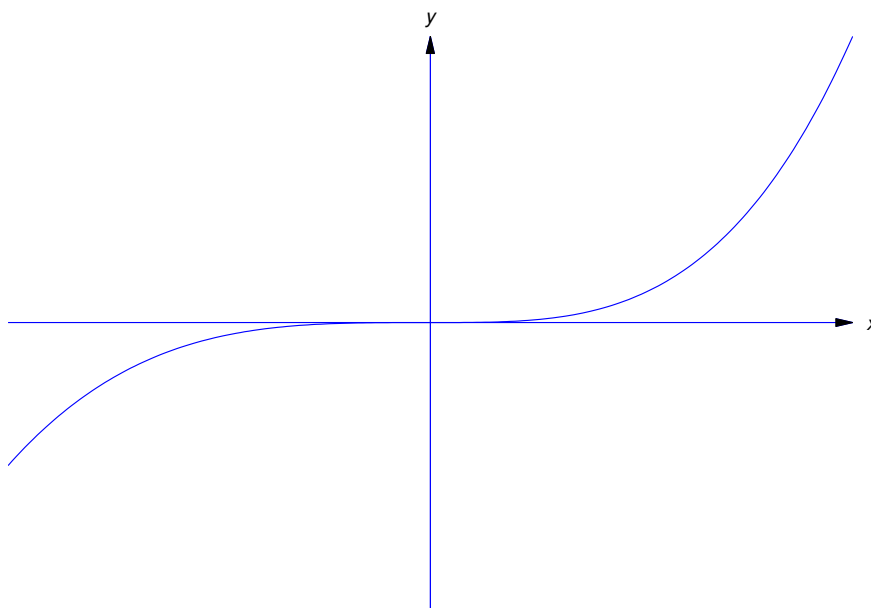


Figure 2.11: Two solutions ($y = 0$ and $y = x^{1/2}$) of (2.61) that differ on every interval containing $x_0 = 0$

Example 2.3.7: From Example 2.3.5, the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = -1 \quad (2.64)$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it's unique.

Solution: Let y be any solution of (2.64). Because of the initial condition $y(0) = -1$ and the continuity of y , there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form (2.63). Setting $x = 0$ and $y = -1$ in (2.63) yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \quad (2.65)$$

for x in I . Therefore every solution of (2.64) differs from zero and is given by (2.65) on $(-1, 1)$; that is, (2.65) is the unique solution of (2.64) on $(-1, 1)$. This is the largest open interval on which (2.64) has a unique solution. To see this, note that (2.65) is a solution of (2.64) on $(-\infty, \infty)$. From Exercise 2.2.15, there are infinitely many other solutions of (2.64) that differ from (2.65) on every open interval larger than $(-1, 1)$. One such solution is

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 \leq x \leq 1, \\ 0, & |x| > 1. \end{cases}$$

(Figure 2.12).

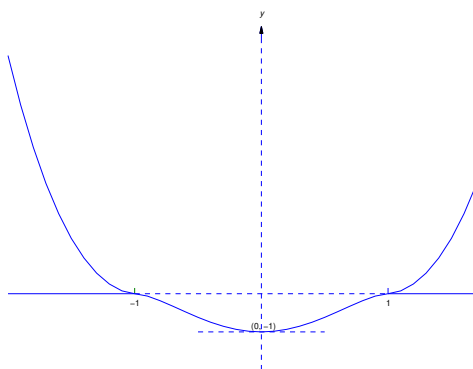


Figure 2.12: Two solutions of (2.64) on $(-\infty, \infty)$ that coincide on $(-1, 1)$, but on no larger open interval

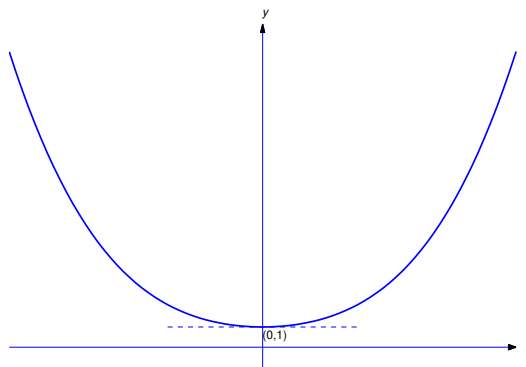


Figure 2.13: The unique solution of (2.66)

Example 2.3.8: From Example 2.3.5, the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 1 \quad (2.66)$$

has a unique solution on some open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it's unique.

Solution: Let y be any solution of (2.66). Because of the initial condition $y(0) = 1$ and the continuity of y , there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form (2.63). Setting $x = 0$ and $y = 1$ in (2.63) yields $c = 1$, so

$$y = (x^2 + 1)^{5/3} \quad (2.67)$$

for x in I . Therefore every solution of (2.66) differs from zero and is given by (2.67) on $(-\infty, \infty)$; that is, (2.67) is the unique solution of (2.66) on $(-\infty, \infty)$. Figure 2.13 shows the graph of this solution.

2.3.1 Exercises

In Exercises 2.3.1-2.3.13 find all (x_0, y_0) for which Theorem 2.3.1 implies that the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has (a) a solution (b) a unique solution on some open interval that contains x_0 .

1. $y' = \frac{x^2 + y^2}{\sin x}$

2. $y' = \frac{e^x + y}{x^2 + y^2}$

3. $y' = \tan xy$

4. $y' = \frac{x^2 + y^2}{\ln xy}$

5. $y' = (x^2 + y^2)y^{1/3}$

6. $y' = 2xy$

7. $y' = \ln(1 + x^2 + y^2)$

8. $y' = \frac{2x + 3y}{x - 4y}$

9. $y' = (x^2 + y^2)^{1/2}$

10. $y' = x(y^2 - 1)^{2/3}$

11. $y' = (x^2 + y^2)^2$

12. $y' = (x + y)^{1/2}$

13. $y' = \frac{\tan y}{x - 1}$

14. Apply Theorem 2.3.1 to the initial value problem

$$y' + p(x)y = q(x), \quad y(x_0) = y_0$$

for a linear equation, and compare the conclusions that can be drawn from it to those that follow from Theorem 2.1.2.

15. (a) Verify that the function

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 < x < 1, \\ 0, & |x| \geq 1, \end{cases}$$

is a solution of the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1$$

on $(-\infty, \infty)$. HINT: You'll need the definition

$$y'(\bar{x}) = \lim_{x \rightarrow \bar{x}} \frac{y(x) - y(\bar{x})}{x - \bar{x}}$$

to verify that y satisfies the differential equation at $\bar{x} = \pm 1$.

(b) Verify that if $\epsilon_i = 0$ or 1 for $i = 1, 2$ and $a, b > 1$, then the function

$$y = \begin{cases} \epsilon_1(x^2 - a^2)^{5/3}, & -\infty < x < -a, \\ 0, & -a \leq x \leq -1, \\ (x^2 - 1)^{5/3}, & -1 < x < 1, \\ 0, & 1 \leq x \leq b, \\ \epsilon_2(x^2 - b^2)^{5/3}, & b < x < \infty, \end{cases}$$

is a solution of the initial value problem of (a) on $(-\infty, \infty)$.

16. Use the ideas developed in Exercise 2.3.15 to find infinitely many solutions of the initial value problem

$$y' = y^{2/5}, \quad y(0) = 1$$

on $(-\infty, \infty)$.

17. Consider the initial value problem

$$y' = 3x(y - 1)^{1/3}, \quad y(x_0) = y_0. \quad (\text{A})$$

- (a) For what points (x_0, y_0) does Theorem 2.3.1 imply that (A) has a solution?
 (b) For what points (x_0, y_0) does Theorem 2.3.1 imply that (A) has a unique solution on some open interval that contains x_0 ?

18. Find nine solutions of the initial value problem

$$y' = 3x(y - 1)^{1/3}, \quad y(0) = 1$$

that are all defined on $(-\infty, \infty)$ and differ from each other for values of x in every open interval that contains $x_0 = 0$.

19. From Theorem 2.3.1, the initial value problem

$$y' = 3x(y - 1)^{1/3}, \quad y(0) = 9$$

has a unique solution on an open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it's unique.

20. (a) From Theorem 2.3.1, the initial value problem

$$y' = 3x(y - 1)^{1/3}, \quad y(3) = -7 \quad (\text{A})$$

has a unique solution on some open interval that contains $x_0 = 3$. Determine the largest such open interval, and find the solution on this interval.

- (b) Find infinitely many solutions of (A), all defined on $(-\infty, \infty)$.

21. Prove:

- (a) If

$$f(x, y_0) = 0, \quad a < x < b, \quad (\text{A})$$

and x_0 is in (a, b) , then $y \equiv y_0$ is a solution of

$$y' = f(x, y), \quad y(x_0) = y_0$$

on (a, b) .

- (b) If f and f_y are continuous on an open rectangle that contains (x_0, y_0) and (A) holds, no solution of $y' = f(x, y)$ other than $y \equiv y_0$ can equal y_0 at any point in (a, b) .

2.4 Transformation of Nonlinear Equations into Separable Equations

In Section 2.1 we found that the solutions of a linear nonhomogeneous equation

$$y' + p(x)y = f(x)$$

are of the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation

$$y' + p(x)y = 0 \quad (2.68)$$

and u is a solution of

$$u'y_1(x) = f(x).$$

Note that this last equation is separable, since it can be rewritten as

$$u' = \frac{f(x)}{y_1(x)}.$$

In this section we'll consider nonlinear differential equations that are not separable to begin with, but can be solved in a similar fashion by writing their solutions in the form $y = uy_1$, where y_1 is a suitably chosen known function and u satisfies a separable equation. We'll say in this case that we *transformed* the given equation into a separable equation.

2.4.1 Bernoulli Equations

A *Bernoulli equation* is an equation of the form

$$y' + p(x)y = f(x)y^r, \quad (2.69)$$

where r can be any real number other than 0 or 1. (Note that (2.69) is linear if and only if $r = 0$ or $r = 1$.) We can transform (2.69) into a separable equation by variation of parameters: if y_1 is a nontrivial solution of (2.68), substituting $y = uy_1$ into (2.69) yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x)(uy_1)^r,$$

which is equivalent to the separable equation

$$u'y_1(x) = f(x) (y_1(x))^r u^r \quad \text{or} \quad \frac{u'}{u^r} = f(x) (y_1(x))^{r-1},$$

since $y_1' + p(x)y_1 = 0$.

Example 2.4.1: Solve the Bernoulli equation

$$y' - y = xy^2. \quad (2.70)$$

Solution: Since $y_1 = e^x$ is a solution of $y' - y = 0$, we look for solutions of (2.70) in the form $y = ue^x$, where

$$u'e^x = xu^2e^{2x} \quad \text{or, equivalently,} \quad u' = xu^2e^x.$$

Separating variables yields

$$\frac{u'}{u^2} = xe^x,$$

and integrating yields

$$-\frac{1}{u} = (x-1)e^x + c.$$

Hence,

$$u = -\frac{1}{(x-1)e^x + c}$$

and

$$y = -\frac{1}{x-1 + ce^{-x}}.$$

Figure 2.14 shows direction field and some integral curves of (2.70).

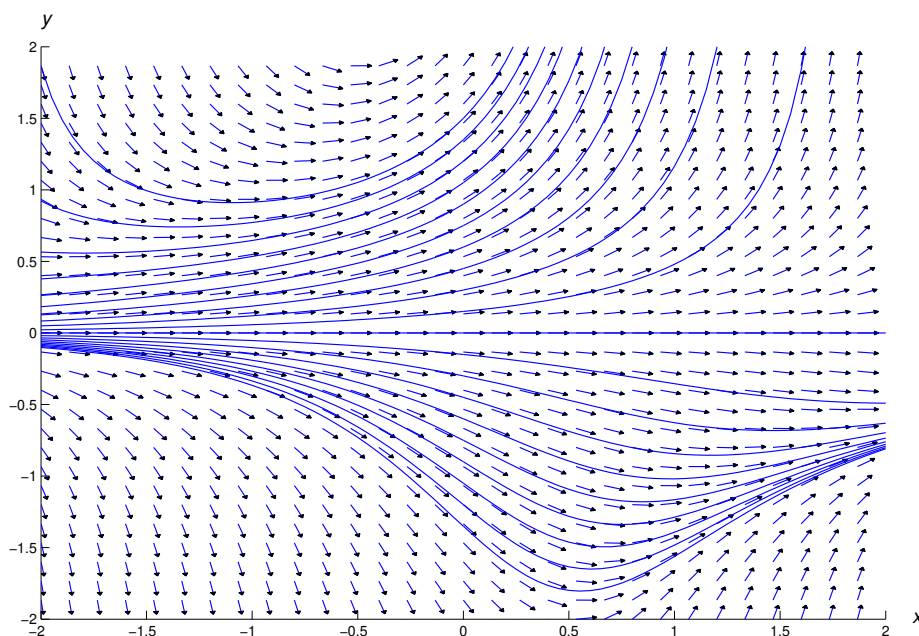


Figure 2.14: A direction field and integral curves for $y' - y = xy^2$

2.4.2 Other Nonlinear Equations That Can be Transformed Into Separable Equations

We've seen that the nonlinear Bernoulli equation can be transformed into a separable equation by the substitution $y = uy_1$ if y_1 is suitably chosen. Now let's discover a sufficient condition for a nonlinear first order differential equation

$$y' = f(x, y) \tag{2.71}$$

to be transformable into a separable equation in the same way. Substituting $y = uy_1$ into (2.71) yields

$$u'y_1(x) + uy_1'(x) = f(x, uy_1(x)),$$

which is equivalent to

$$u'y_1(x) = f(x, uy_1(x)) - uy_1'(x). \tag{2.72}$$

If

$$f(x, uy_1(x)) = q(u)y_1'(x)$$

for some function q , then (2.72) becomes

$$u'y_1(x) = (q(u) - u)y_1'(x), \tag{2.73}$$

which is separable. After checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$, we can separate variables to obtain

$$\frac{u'}{q(u) - u} = \frac{y_1'(x)}{y_1(x)}.$$

2.4.3 Homogeneous Nonlinear Equations

In the text we'll consider only the most widely studied class of equations for which the method of the preceding paragraph works. Other types of equations appear in Exercises 2.4.44–2.4.51.

The differential equation (2.71) is said to be *homogeneous* if x and y occur in f in such a way that $f(x, y)$ depends only on the ratio y/x ; that is, (2.71) can be written as

$$y' = q(y/x), \quad (2.74)$$

where $q = q(u)$ is a function of a single variable. For example,

$$y' = \frac{y + xe^{-y/x}}{x} = \frac{y}{x} + e^{-y/x}$$

and

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

are of the form (2.74), with

$$q(u) = u + e^{-u} \quad \text{and} \quad q(u) = u^2 + u - 1,$$

respectively. The general method discussed above can be applied to (2.74) with $y_1 = x$ (and therefore $y_1' = 1$). Thus, substituting $y = ux$ in (2.74) yields

$$u'x + u = q(u),$$

and separation of variables (after checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$) yields

$$\frac{u'}{q(u) - u} = \frac{1}{x}.$$

Before turning to examples, we point out something that you may've have already noticed: the definition of *homogeneous equation* given here isn't the same as the definition given in Section 2.1, where we said that a linear equation of the form

$$y' + p(x)y = 0$$

is homogeneous. We make no apology for this inconsistency, since we didn't create it historically, *homogeneous* has been used in these two inconsistent ways. The one having to do with linear equations is the most important. This is the only section of the book where the meaning defined here will apply.

Since y/x is in general undefined if $x = 0$, we'll consider solutions of nonhomogeneous equations only on open intervals that do not contain the point $x = 0$.

Example 2.4.2: Solve

$$y' = \frac{y + xe^{-y/x}}{x}. \quad (2.75)$$

Solution: Substituting $y = ux$ into (2.75) yields

$$u'x + u = \frac{ux + xe^{-ux/x}}{x} = u + e^{-u}.$$

Simplifying and separating variables yields

$$e^u u' = \frac{1}{x}.$$

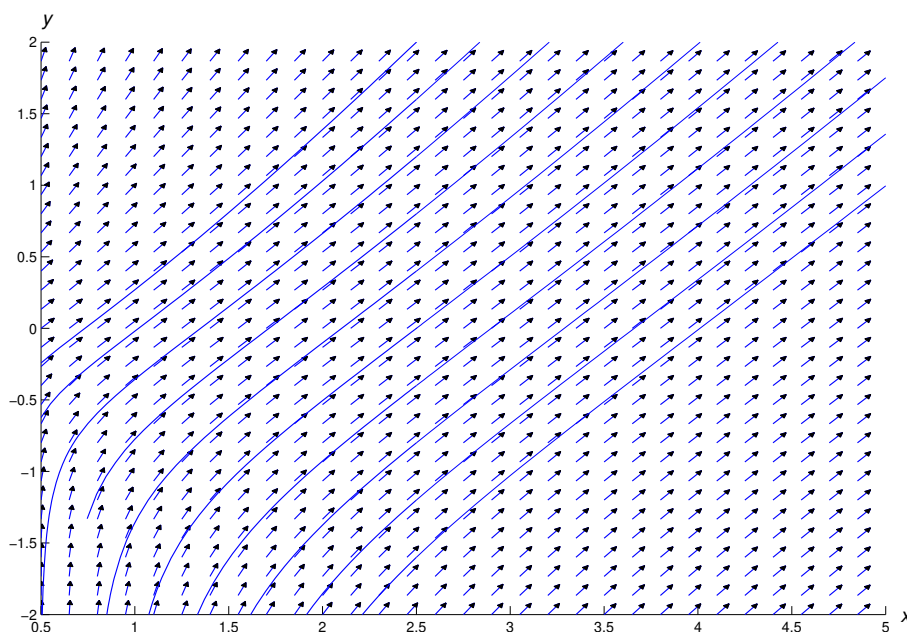


Figure 2.15: A direction field and some integral curves for $y' = \frac{y + xe^{-y/x}}{x}$

Integrating yields $e^u = \ln|x| + c$. Therefore $u = \ln(\ln|x| + c)$ and $y = ux = x \ln(\ln|x| + c)$.

Figure 2.15 shows a direction field and integral curves for (2.75).

Example 2.4.3:

(a) Solve

$$x^2 y' = y^2 + xy - x^2. \quad (2.76)$$

(b) Solve the initial value problem

$$x^2 y' = y^2 + xy - x^2, \quad y(1) = 2. \quad (2.77)$$

Solution (a): We first find solutions of (2.76) on open intervals that don't contain $x = 0$. We can rewrite (2.76) as

$$y' = \frac{y^2 + xy - x^2}{x^2}$$

for x in any such interval. Substituting $y = ux$ yields

$$u'x + u = \frac{(ux)^2 + x(ux) - x^2}{x^2} = u^2 + u - 1,$$

so

$$u'x = u^2 - 1. \quad (2.78)$$

By inspection this equation has the constant solutions $u \equiv 1$ and $u \equiv -1$. Therefore $y = x$ and $y = -x$ are solutions of (2.76). If u is a solution of (2.78) that doesn't assume the values ± 1 on some interval, separating variables yields

$$\frac{u'}{u^2 - 1} = \frac{1}{x},$$

or, after a partial fraction expansion,

$$\frac{1}{2} \left[\frac{1}{u-1} - \frac{1}{u+1} \right] u' = \frac{1}{x}.$$

Multiplying by 2 and integrating yields

$$\ln \left| \frac{u-1}{u+1} \right| = 2 \ln |x| + k,$$

or

$$\left| \frac{u-1}{u+1} \right| = e^k x^2,$$

which holds if

$$\frac{u-1}{u+1} = cx^2 \quad (2.79)$$

where c is an arbitrary constant. Solving for u yields

$$u = \frac{1 + cx^2}{1 - cx^2}.$$

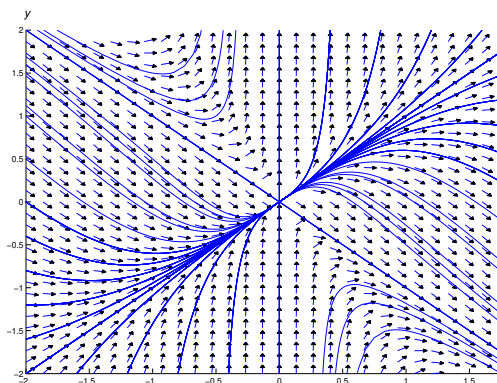


Figure 2.16: A direction field and integral curves for $x^2 y' = y^2 + xy - x^2$

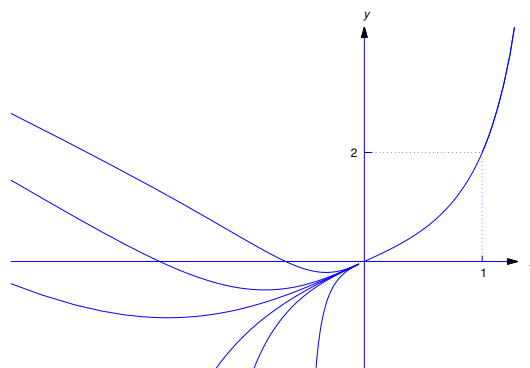


Figure 2.17: Solutions of $x^2 y' = y^2 + xy - x^2$, $y(1) = 2$

Therefore

$$y = ux = \frac{x(1 + cx^2)}{1 - cx^2} \quad (2.80)$$

is a solution of (2.77) for any choice of the constant c . Setting $c = 0$ in (2.80) yields the solution $y = x$. However, the solution $y = -x$ can't be obtained from (2.80). Thus, the solutions of (2.76) on intervals that don't contain $x = 0$ are $y = -x$ and functions of the form (2.80).

The situation is more complicated if $x = 0$ is the open interval. First, note that $y = -x$ satisfies (2.76) on $(-\infty, \infty)$. If c_1 and c_2 are arbitrary constants, the function

$$y = \begin{cases} \frac{x(1 + c_1 x^2)}{1 - c_1 x^2}, & a < x < 0, \\ \frac{x(1 + c_2 x^2)}{1 - c_2 x^2}, & 0 \leq x < b, \end{cases} \quad (2.81)$$

is a solution of (2.76) on (a, b) , where

$$a = \begin{cases} -\frac{1}{\sqrt{c_1}} & \text{if } c_1 > 0, \\ -\infty & \text{if } c_1 \leq 0, \end{cases} \quad \text{and} \quad b = \begin{cases} \frac{1}{\sqrt{c_2}} & \text{if } c_2 > 0, \\ \infty & \text{if } c_2 \leq 0. \end{cases}$$

We leave it to you to verify this. To do so, note that if y is any function of the form (2.80) then $y(0) = 0$ and $y'(0) = 1$.

Figure 2.16 shows a direction field and some integral curves for (2.76).

Solution (b): We could obtain c by imposing the initial condition $y(1) = 2$ in (2.80), and then solving for c . However, it's easier to use (2.79). Since $u = y/x$, the initial condition $y(1) = 2$ implies that $u(1) = 2$. Substituting this into (2.79) yields $c = 1/3$. Hence, the solution of (2.77) is

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}.$$

The interval of validity of this solution is $(-\sqrt{3}, \sqrt{3})$. However, the largest interval on which (2.77) has a unique solution is $(0, \sqrt{3})$. To see this, note from (2.81) that any function of the form

$$y = \begin{cases} \frac{x(1 + cx^2)}{1 - cx^2}, & a < x \leq 0, \\ \frac{x(1 + x^2/3)}{1 - x^2/3}, & 0 \leq x < \sqrt{3}, \end{cases} \quad (2.82)$$

is a solution of (2.77) on $(a, \sqrt{3})$, where $a = -1/\sqrt{c}$ if $c > 0$ or $a = -\infty$ if $c \leq 0$. (Why doesn't this contradict Theorem 2.3.1?)

Figure 2.17 shows several solutions of the initial value problem (2.77). Note that these solutions coincide on $(0, \sqrt{3})$.

In the last two examples we were able to solve the given equations explicitly. However, this isn't always possible, as you'll see in the exercises.

2.4.4 Exercises

In Exercises 2.4.1–2.4.4 solve the given Bernoulli equation.

1. $y' + y = y^2$
2. $7xy' - 2y = -\frac{x^2}{y^6}$
3. $x^2y' + 2y = 2e^{1/x}y^{1/2}$
4. $(1 + x^2)y' + 2xy = \frac{1}{(1 + x^2)y}$

In Exercises 2.4.5 and 2.4.6 find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

5. C/G $y' - xy = x^3y^3; \quad \{-3 \leq x \leq 3, 2 \leq y \leq 2\}$
6. C/G $y' - \frac{1+x}{3x}y = y^4; \quad \{-2 \leq x \leq 2, -2 \leq y \leq 2\}$

In Exercises 2.4.7–2.4.11 solve the initial value problem.

7. $y' - 2y = xy^3, \quad y(0) = 2\sqrt{2}$
8. $y' - xy = xy^{3/2}, \quad y(1) = 4$
9. $xy' + y = x^4y^4, \quad y(1) = 1/2$

10. $y' - 2y = 2y^{1/2}, \quad y(0) = 1$

11. $y' - 4y = \frac{48x}{y^2}, \quad y(0) = 1$

In Exercises 2.4.12 and 2.4.13 solve the initial value problem and graph the solution.

12. $\boxed{\text{C/G}} \quad x^2 y' + 2xy = y^3, \quad y(1) = 1/\sqrt{2}$

13. $\boxed{\text{C/G}} \quad y' - y = xy^{1/2}, \quad y(0) = 4$

14. You may have noticed that the logistic equation

$$P' = aP(1 - \alpha P)$$

from Verhulst's model for population growth can be written in Bernoulli form as

$$P' - aP = -a\alpha P^2.$$

This isn't particularly interesting, since the logistic equation is separable, and therefore solvable by the method studied in Section 2.2. So let's consider a more complicated model, where a is a positive constant and α is a positive continuous function of t on $[0, \infty)$. The equation for this model is

$$P' - aP = -a\alpha(t)P^2,$$

a non-separable Bernoulli equation.

- (a) Assuming that $P(0) = P_0 > 0$, find P for $t > 0$. HINT: Express your result in terms of the integral $\int_0^t \alpha(\tau)e^{a\tau} d\tau$.
- (b) Verify that your result reduces to the known results for the Malthusian model where $\alpha = 0$, and the Verhulst model where α is a nonzero constant.
- (c) Assuming that

$$\lim_{t \rightarrow \infty} e^{-at} \int_0^t \alpha(\tau)e^{a\tau} d\tau = L$$

exists (finite or infinite), find $\lim_{t \rightarrow \infty} P(t)$.

In Exercises 2.4.15–2.4.18 solve the equation explicitly.

15. $y' = \frac{y+x}{x}$

16. $y' = \frac{y^2 + 2xy}{x^2}$

17. $xy^3 y' = y^4 + x^4$

18. $y' = \frac{y}{x} + \sec \frac{y}{x}$

In Exercises 2.4.19–2.4.21 solve the equation explicitly. Also, plot a direction field and some integral curves on the indicated rectangular region.

19. $\boxed{\text{C/G}} \quad x^2 y' = xy + x^2 + y^2; \quad \{-8 \leq x \leq 8, -8 \leq y \leq 8\}$

20. $\boxed{\text{C/G}} \quad xy y' = x^2 + 2y^2; \quad \{-4 \leq x \leq 4, -4 \leq y \leq 4\}$

21. $\boxed{\text{C/G}} \quad y' = \frac{2y^2 + x^2 e^{-(y/x)^2}}{2xy}; \quad \{-8 \leq x \leq 8, -8 \leq y \leq 8\}$

In Exercises 2.4.22–2.4.27 solve the initial value problem.

22. $y' = \frac{xy + y^2}{x^2}, \quad y(-1) = 2$

23. $y' = \frac{x^3 + y^3}{xy^2}$, $y(1) = 3$
 24. $xyy' + x^2 + y^2 = 0$, $y(1) = 2$
 25. $y' = \frac{y^2 - 3xy - 5x^2}{x^2}$, $y(1) = -1$
 26. $x^2y' = 2x^2 + y^2 + 4xy$, $y(1) = 1$
 27. $xyy' = 3x^2 + 4y^2$, $y(1) = \sqrt{3}$

In Exercises 2.4.28–2.4.34 solve the given homogeneous equation implicitly.

28. $y' = \frac{x+y}{x-y}$ 29. $(y'x - y)(\ln|y| - \ln|x|) = x$
 30. $y' = \frac{y^3 + 2xy^2 + x^2y + x^3}{x(y+x)^2}$ 31. $y' = \frac{x+2y}{2x+y}$
 32. $y' = \frac{y}{y-2x}$ 33. $y' = \frac{xy^2 + 2y^3}{x^3 + x^2y + xy^2}$
 34. $y' = \frac{x^3 + x^2y + 3y^3}{x^3 + 3xy^2}$

35. L

- (a) Find a solution of the initial value problem

$$x^2y' = y^2 + xy - 4x^2, \quad y(-1) = 0 \tag{A}$$

on the interval $(-\infty, 0)$. Verify that this solution is actually valid on $(-\infty, \infty)$.

- (b) Use Theorem 2.3.1 to show that (A) has a unique solution on $(-\infty, 0)$.
 (c) Plot a direction field for the differential equation in (A) on a square

$$\{-r \leq x \leq r, -r \leq y \leq r\},$$

where r is any positive number. Graph the solution you obtained in (a) on this field.

- (d) Graph other solutions of (A) that are defined on $(-\infty, \infty)$.
 (e) Graph other solutions of (A) that are defined only on intervals of the form $(-\infty, a)$, where a is a finite positive number.

36. L

- (a) Solve the equation

$$xyy' = x^2 - xy + y^2 \tag{A}$$

implicitly.

- (b) Plot a direction field for (A) on a square

$$\{0 \leq x \leq r, 0 \leq y \leq r\}$$

where r is any positive number.

- (c) Let K be a positive integer. (You may have to try several choices for K .) Graph solutions of the initial value problems

$$xyy' = x^2 - xy + y^2, \quad y(r/2) = \frac{kr}{K},$$

for $k = 1, 2, \dots, K$. Based on your observations, find conditions on the positive numbers x_0 and y_0 such that the initial value problem

$$xyy' = x^2 - xy + y^2, \quad y(x_0) = y_0, \quad (\text{B})$$

has a unique solution (i) on $(0, \infty)$ or (ii) only on an interval (a, ∞) , where $a > 0$?

- (d) What can you say about the graph of the solution of (B) as $x \rightarrow \infty$? (Again, assume that $x_0 > 0$ and $y_0 > 0$.)

37. L

- (a) Solve the equation

$$y' = \frac{2y^2 - xy + 2x^2}{xy + 2x^2} \quad (\text{A})$$

implicitly.

- (b) Plot a direction field for (A) on a square

$$\{-r \leq x \leq r, -r \leq y \leq r\}$$

where r is any positive number. By graphing solutions of (A), determine necessary and sufficient conditions on (x_0, y_0) such that (A) has a solution on (i) $(-\infty, 0)$ or (ii) $(0, \infty)$ such that $y(x_0) = y_0$.

38. L Follow the instructions of Exercise 2.4.37 for the equation

$$y' = \frac{xy + x^2 + y^2}{xy}.$$

39. L Pick any nonlinear homogeneous equation $y' = q(y/x)$ you like, and plot direction fields on the square $\{-r \leq x \leq r, -r \leq y \leq r\}$, where $r > 0$. What happens to the direction field as you vary r ? Why?

40. Prove: If $ad - bc \neq 0$, the equation

$$y' = \frac{ax + by + \alpha}{cx + dy + \beta}$$

can be transformed into the homogeneous nonlinear equation

$$\frac{dY}{dX} = \frac{aX + bY}{cX + dY}$$

by the substitution $x = X - X_0$, $y = Y - Y_0$, where X_0 and Y_0 are suitably chosen constants.

In Exercises 2.4.41–2.4.43 use a method suggested by Exercise 2.4.40 to solve the given equation implicitly.

41. $y' = \frac{-6x + y - 3}{2x - y - 1}$

42. $y' = \frac{2x + y + 1}{x + 2y - 4}$

43. $y' = \frac{-x + 3y - 14}{x + y - 2}$

In Exercises 2.4.44–2.4.51 find a function y_1 such that the substitution $y = uy_1$ transforms the given equation into a separable equation of the form (2.73). Then solve the given equation explicitly.

44. $3xy^2y' = y^3 + x$

45. $xyy' = 3x^6 + 6y^2$

46. $x^3y' = 2(y^2 + x^2y - x^4)$

47. $y' = y^2e^{-x} + 4y + 2e^x$

$$48. \quad y' = \frac{y^2 + y \tan x + \tan^2 x}{\sin^2 x} \qquad 49. \quad x(\ln x)^2 y' = -4(\ln x)^2 + y \ln x + y^2$$

$$50. \quad 2x(y + 2\sqrt{x})y' = (y + \sqrt{x})^2 \qquad 51. \quad (y + e^{x^2})y' = 2x(y^2 + ye^{x^2} + e^{2x^2})$$

52. Solve the initial value problem

$$y' + \frac{2}{x}y = \frac{3x^2y^2 + 6xy + 2}{x^2(2xy + 3)}, \quad y(2) = 2.$$

53. Solve the initial value problem

$$y' + \frac{3}{x}y = \frac{3x^4y^2 + 10x^2y + 6}{x^3(2x^2y + 5)}, \quad y(1) = 1.$$

54. Prove: If y is a solution of a homogeneous nonlinear equation $y' = q(y/x)$, so is $y_1 = y(ax)/a$, where a is any nonzero constant.

55. A *generalized Riccati equation* is of the form

$$y' = P(x) + Q(x)y + R(x)y^2. \qquad (A)$$

(If $R \equiv -1$, (A) is a *Riccati equation*.) Let y_1 be a known solution and y an arbitrary solution of (A). Let $z = y - y_1$. Show that z is a solution of a Bernoulli equation with $n = 2$.

In Exercises 2.4.56–2.4.59, given that y_1 is a solution of the given equation, use the method suggested by Exercise 2.4.55 to find other solutions.

$$56. \quad y' = 1 + x - (1 + 2x)y + xy^2; \quad y_1 = 1$$

$$57. \quad y' = e^{2x} + (1 - 2e^x)y + y^2; \quad y_1 = e^x$$

$$58. \quad xy' = 2 - x + (2x - 2)y - xy^2; \quad y_1 = 1$$

$$59. \quad xy' = x^3 + (1 - 2x^2)y + xy^2; \quad y_1 = x$$

Chapter 3

Numerical Methods

3.1 Euler's Method

If an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (3.1)$$

can't be solved analytically, it's necessary to resort to numerical methods to obtain useful approximations to a solution of (3.1). We'll consider such methods in this chapter.

We're interested in computing approximate values of the solution of (3.1) at equally spaced points $x_0, x_1, \dots, x_n = b$ in an interval $[x_0, b]$. Thus,

$$x_i = x_0 + ih, \quad i = 0, 1, \dots, n,$$

where

$$h = \frac{b - x_0}{n}.$$

We'll denote the approximate values of the solution at these points by $y_0, y_1, \dots, y_n$; thus, y_i is an approximation to $y(x_i)$. We'll call

$$e_i = y(x_i) - y_i$$

the *error at the i th step*. Because of the initial condition $y(x_0) = y_0$, we'll always have $e_0 = 0$. However, in general $e_i \neq 0$ if $i > 0$.

We encounter two sources of error in applying a numerical method to solve an initial value problem:

- The formulas defining the method are based on some sort of approximation. Errors due to the inaccuracy of the approximation are called *truncation errors*.
- Computers do arithmetic with a fixed number of digits, and therefore make errors in evaluating the formulas defining the numerical methods. Errors due to the computer's inability to do exact arithmetic are called *roundoff errors*.

Since a careful analysis of roundoff error is beyond the scope of this book, we'll consider only truncation errors.

3.1.1 Euler's Method

The simplest numerical method for solving (3.1) is *Euler's method*. This method is so crude that it is seldom used in practice; however, its simplicity makes it useful for illustrative purposes.

Euler's method is based on the assumption that the tangent line to the integral curve of (3.1) at $(x_i, y(x_i))$ approximates the integral curve over the interval $[x_i, x_{i+1}]$. Since the slope of the integral curve of (3.1) at $(x_i, y(x_i))$ is $y'(x_i) = f(x_i, y(x_i))$, the equation of the tangent line to the integral curve at $(x_i, y(x_i))$ is

$$y = y(x_i) + f(x_i, y(x_i))(x - x_i). \quad (3.2)$$

Setting $x = x_{i+1} = x_i + h$ in (3.2) yields

$$y_{i+1} = y(x_i) + hf(x_i, y(x_i)) \quad (3.3)$$

as an approximation to $y(x_{i+1})$. Since $y(x_0) = y_0$ is known, we can use (3.3) with $i = 0$ to compute

$$y_1 = y_0 + hf(x_0, y_0).$$

However, setting $i = 1$ in (3.3) yields

$$y_2 = y(x_1) + hf(x_1, y(x_1)),$$

which isn't useful, since we *don't know* $y(x_1)$. Therefore we replace $y(x_1)$ by its approximate value y_1 and redefine

$$y_2 = y_1 + hf(x_1, y_1).$$

Having computed y_2 , we can compute

$$y_3 = y_2 + hf(x_2, y_2).$$

In general, Euler's method starts with the known value $y(x_0) = y_0$ and computes $y_1, y_2, \dots, y_n$ successively by with the formula

$$y_{i+1} = y_i + hf(x_i, y_i), \quad 0 \leq i \leq n-1. \quad (3.4)$$

The next example illustrates the computational procedure indicated in Euler's method.

Example 3.1.1: Use Euler's method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \quad (3.5)$$

at $x = 0.1, 0.2, 0.3$.

Solution: We rewrite (3.5) as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form (3.1), with

$$f(x, y) = -2y + x^3 e^{-2x}, \quad x_0 = 0, \text{ and } y_0 = 1.$$

Euler's method yields

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + (.1)f(0, 1) = 1 + (.1)(-2) = .8, \\ y_2 &= y_1 + hf(x_1, y_1) \\ &= .8 + (.1)f(.1, .8) = .8 + (.1)(-2(.8) + (.1)^3 e^{-.2}) = .640081873, \\ y_3 &= y_2 + hf(x_2, y_2) \\ &= .640081873 + (.1)(-2(.640081873) + (.2)^3 e^{-.4}) = .512601754. \blacksquare \end{aligned}$$

We've written the details of these computations to ensure that you understand the procedure. However, in the rest of the examples as well as the exercises in this chapter, we'll assume that you can use a programmable calculator or a computer to carry out the necessary computations.

3.1.2 Examples Illustrating The Error in Euler's Method

Example 3.1.2: Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4), \quad (3.6)$$

which can be obtained by the method of Section 2.1. (Verify.)

Solution: Table 3.1.2 shows the values of the exact solution (3.6) at the specified points, and the approximate values of the solution at these points obtained by Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$. In examining this table, keep in mind that the approximate values in the column corresponding to $h = .05$ are actually the results of 20 steps with Euler's method. We haven't listed the estimates of the solution obtained for $x = 0.05, 0.15, \dots$, since there's nothing to compare them with in the column corresponding to $h = 0.1$. Similarly, the approximate values in the column corresponding to $h = 0.025$ are actually the results of 40 steps with Euler's method.

Numerical solution of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.810005655	0.814518349	0.818751221
0.2	0.640081873	0.656266437	0.663635953	0.670588174
0.3	0.512601754	0.532290981	0.541339495	0.549922980
0.4	0.411563195	0.432887056	0.442774766	0.452204669
0.5	0.332126261	0.353785015	0.363915597	0.373627557
0.6	0.270299502	0.291404256	0.301359885	0.310952904
0.7	0.222745397	0.242707257	0.252202935	0.261398947
0.8	0.186654593	0.205105754	0.213956311	0.222570721
0.9	0.159660776	0.176396883	0.184492463	0.192412038
1.0	0.139778910	0.154715925	0.162003293	0.169169104

You can see from Table 3.1.2 that decreasing the step size improves the accuracy of Euler's method. For example,

$$y_{\text{exact}}(1) - y_{\text{approx}}(1) \approx \begin{cases} .0293 & \text{with } h = 0.1, \\ .0144 & \text{with } h = 0.05, \\ .0071 & \text{with } h = 0.025. \end{cases}$$

Based on this scanty evidence, you might guess that the error in approximating the exact solution at a *fixed value of* x by Euler's method is roughly halved when the step size is halved. You can find more evidence to support this conjecture by examining Table 3.1.2, which lists the approximate values of $y_{\text{exact}} - y_{\text{approx}}$ at $x = 0.1, 0.2, \dots, 1.0$.

Errors in approximate solutions of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, obtained by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.1	0.0187	0.0087	0.0042
0.2	0.0305	0.0143	0.0069
0.3	0.0373	0.0176	0.0085
0.4	0.0406	0.0193	0.0094
0.5	0.0415	0.0198	0.0097
0.6	0.0406	0.0195	0.0095
0.7	0.0386	0.0186	0.0091
0.8	0.0359	0.0174	0.0086
0.9	0.0327	0.0160	0.0079
1.0	0.0293	0.0144	0.0071

Example 3.1.3: Tables 3.1.2 and 3.1.2 show analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2, \quad y(0) = 1, \quad (3.7)$$

except in this case we can't solve (3.7) exactly. The results in the "Exact" column were obtained by using a more accurate numerical method known as the *Runge-Kutta* method with a small step size. They are exact to eight decimal places. ■

Since we think it's important in evaluating the accuracy of the numerical methods that we'll be studying in this chapter, we often include a column listing values of the exact solution of the initial value problem, even if the directions in the example or exercise don't specifically call for it. If quotation marks are included in the heading, the values were obtained by applying the Runge-Kutta method in a way that's explained in Section 3.3. If quotation marks are not included, the values were obtained from a known formula for the solution. In either case, the values are exact to eight places to the right of the decimal point.

Numerical solution of $y' = -2y^2 + xy + x^2$, $y(0) = 1$, by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.821375000	0.829977007	0.837584494
0.2	0.681000000	0.707795377	0.719226253	0.729641890
0.3	0.605867800	0.633776590	0.646115227	0.657580377
0.4	0.559628676	0.587454526	0.600045701	0.611901791
0.5	0.535376972	0.562906169	0.575556391	0.587575491
0.6	0.529820120	0.557143535	0.569824171	0.581942225
0.7	0.541467455	0.568716935	0.581435423	0.593629526
0.8	0.569732776	0.596951988	0.609684903	0.621907458
0.9	0.614392311	0.641457729	0.654110862	0.666250842
1.0	0.675192037	0.701764495	0.714151626	0.726015790

Errors in approximate solutions of $y' = -2y^2 + xy + x^2$, $y(0) = 1$, obtained by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.1	0.0376	0.0162	0.0076
0.2	0.0486	0.0218	0.0104
0.3	0.0517	0.0238	0.0115
0.4	0.0523	0.0244	0.0119
0.5	0.0522	0.0247	0.0121
0.6	0.0521	0.0248	0.0121
0.7	0.0522	0.0249	0.0122
0.8	0.0522	0.0250	0.0122
0.9	0.0519	0.0248	0.0121
1.0	0.0508	0.0243	0.0119

3.1.3 Truncation Error in Euler's Method

Consistent with the results indicated in Tables 3.1.2–3.1.2, we'll now show that under reasonable assumptions on f there's a constant K such that the error in approximating the solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0,$$

at a given point $b > x_0$ by Euler's method with step size $h = (b - x_0)/n$ satisfies the inequality

$$|y(b) - y_n| \leq Kh,$$

where K is a constant independent of n .

There are two sources of error (not counting roundoff) in Euler's method:

The error committed in approximating the integral curve by the tangent line (3.2) over the interval $[x_i, x_{i+1}]$.

The error committed in replacing $y(x_i)$ by y_i in (3.2) and using (3.4) rather than (3.2) to compute y_{i+1} .

Euler's method assumes that y_{i+1} defined in (3.2) is an approximation to $y(x_{i+1})$. We call the error in this approximation the *local truncation error at the i th step*, and denote it by T_i ; thus,

$$T_i = y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i)). \quad (3.8)$$

We'll now use *Taylor's theorem* to estimate T_i , assuming for simplicity that f , f_x , and f_y are continuous and bounded for all (x, y) . Then y'' exists and is bounded on $[x_0, b]$. To see this, we differentiate

$$y'(x) = f(x, y(x))$$

to obtain

$$\begin{aligned} y''(x) &= f_x(x, y(x)) + f_y(x, y(x))y'(x) \\ &= f_x(x, y(x)) + f_y(x, y(x))f(x, y(x)). \end{aligned}$$

Since we assumed that f , f_x and f_y are bounded, there's a constant M such that

$$|f_x(x, y(x)) + f_y(x, y(x))f(x, y(x))| \leq M, \quad x_0 < x < b,$$

which implies that

$$|y''(x)| \leq M, \quad x_0 < x < b. \quad (3.9)$$

Since $x_{i+1} = x_i + h$, Taylor's theorem implies that

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\tilde{x}_i),$$

where $\tilde{x}_i$ is some number between x_i and x_{i+1} . Since $y'(x_i) = f(x_i, y(x_i))$ this can be written as

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2}y''(\tilde{x}_i),$$

or, equivalently,

$$y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i)) = \frac{h^2}{2}y''(\tilde{x}_i).$$

Comparing this with (3.8) shows that

$$T_i = \frac{h^2}{2}y''(\tilde{x}_i).$$

Recalling (3.9), we can establish the bound

$$|T_i| \leq \frac{Mh^2}{2}, \quad 1 \leq i \leq n. \quad (3.10)$$

Although it may be difficult to determine the constant M , what is important is that there's an M such that (3.10) holds. We say that the local truncation error of Euler's method is *of order* h^2 , which we write as $O(h^2)$.

Note that the magnitude of the local truncation error in Euler's method is determined by the second derivative y'' of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y''|$ is large, or smaller where $|y''|$ is small.

Since the local truncation error for Euler's method is $O(h^2)$, it's reasonable to expect that halving h reduces the local truncation error by a factor of 4. This is true, but halving the step size also requires twice as many steps to approximate the solution at a given point. To analyze the overall effect of truncation error in Euler's method, it's useful to derive an equation relating the errors

$$e_{i+1} = y(x_{i+1}) - y_{i+1} \quad \text{and} \quad e_i = y(x_i) - y_i.$$

To this end, recall that

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + T_i \quad (3.11)$$

and

$$y_{i+1} = y_i + hf(x_i, y_i). \quad (3.12)$$

Subtracting (3.12) from (3.11) yields

$$e_{i+1} = e_i + h [f(x_i, y(x_i)) - f(x_i, y_i)] + T_i. \quad (3.13)$$

The last term on the right is the local truncation error at the i th step. The other terms reflect the way errors made at *previous steps* affect e_{i+1} . Since $|T_i| \leq Mh^2/2$, we see from (3.13) that

$$|e_{i+1}| \leq |e_i| + h|f(x_i, y(x_i)) - f(x_i, y_i)| + \frac{Mh^2}{2}. \quad (3.14)$$

Since we assumed that f_y is continuous and bounded, the mean value theorem implies that

$$f(x_i, y(x_i)) - f(x_i, y_i) = f_y(x_i, y_i^*)(y(x_i) - y_i) = f_y(x_i, y_i^*)e_i,$$

where y_i^* is between y_i and $y(x_i)$. Therefore

$$|f(x_i, y(x_i)) - f(x_i, y_i)| \leq R|e_i|$$

for some constant R . From this and (3.14),

$$|e_{i+1}| \leq (1 + Rh)|e_i| + \frac{Mh^2}{2}, \quad 0 \leq i \leq n-1. \quad (3.15)$$

For convenience, let $C = 1 + Rh$. Since $e_0 = y(x_0) - y_0 = 0$, applying (3.15) repeatedly yields

$$\begin{aligned} |e_1| &\leq \frac{Mh^2}{2} \\ |e_2| &\leq C|e_1| + \frac{Mh^2}{2} \leq (1 + C)\frac{Mh^2}{2} \\ |e_3| &\leq C|e_2| + \frac{Mh^2}{2} \leq (1 + C + C^2)\frac{Mh^2}{2} \\ &\vdots \\ |e_n| &\leq C|e_{n-1}| + \frac{Mh^2}{2} \leq (1 + C + \cdots + C^{n-1})\frac{Mh^2}{2}. \end{aligned} \quad (3.16)$$

Recalling the formula for the sum of a geometric series, we see that

$$1 + C + \cdots + C^{n-1} = \frac{1 - C^n}{1 - C} = \frac{(1 + Rh)^n - 1}{Rh}$$

(since $C = 1 + Rh$). From this and (3.16),

$$|y(b) - y_n| = |e_n| \leq \frac{(1 + Rh)^n - 1}{R} \frac{Mh}{2}. \quad (3.17)$$

Since Taylor's theorem implies that

$$1 + Rh < e^{Rh}$$

(verify),

$$(1 + Rh)^n < e^{nRh} = e^{R(b-x_0)} \quad (\text{since } nh = b - x_0).$$

This and (3.17) imply that

$$|y(b) - y_n| \leq Kh, \quad (3.18)$$

with

$$K = M \frac{e^{R(b-x_0)} - 1}{2R}.$$

Because of (3.18) we say that the *global truncation error of Euler's method is of order h* , which we write as $O(h)$.

3.1.4 Semilinear Equations and Variation of Parameters

An equation that can be written in the form

$$y' + p(x)y = h(x, y) \quad (3.19)$$

with $p \neq 0$ is said to be *semilinear*. (Of course, (3.19) is linear if h is independent of y .) One way to apply Euler's method to an initial value problem

$$y' + p(x)y = h(x, y), \quad y(x_0) = y_0 \quad (3.20)$$

for (3.19) is to think of it as

$$y' = f(x, y), \quad y(x_0) = y_0,$$

where

$$f(x, y) = -p(x)y + h(x, y).$$

However, we can also start by applying variation of parameters to (3.20), as in Sections 2.1 and 2.4; thus, we write the solution of (3.20) as $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation $y' + p(x)y = 0$. Then $y = uy_1$ is a solution of (3.20) if and only if u is a solution of the initial value problem

$$u' = h(x, uy_1(x))/y_1(x), \quad u(x_0) = y(x_0)/y_1(x_0). \quad (3.21)$$

We can apply Euler's method to obtain approximate values $u_0, u_1, \dots, u_n$ of this initial value problem, and then take

$$y_i = u_i y_1(x_i)$$

as approximate values of the solution of (3.20). We'll call this procedure the *Euler semilinear method*.

The next two examples show that the Euler and Euler semilinear methods may yield drastically different results.

Example 3.1.4: In Example 2.1.7 we had to leave the solution of the initial value problem

$$y' - 2xy = 1, \quad y(0) = 3 \quad (3.22)$$

in the form

$$y = e^{x^2} \left(3 + \int_0^x e^{-t^2} dt \right) \quad (3.23)$$

because it was impossible to evaluate this integral exactly in terms of elementary functions. Use step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$ to find approximate values of the solution of (3.22) at $x = 0, 0.2, 0.4, 0.6, \dots, 2.0$ by (a) Euler's method; (b) the Euler semilinear method.

Solution (a): Rewriting (3.22) as

$$y' = 1 + 2xy, \quad y(0) = 3 \quad (3.24)$$

and applying Euler's method with $f(x, y) = 1 + 2xy$ yields the results shown in Table 3.1.4. Because of the large differences between the estimates obtained for the three values of h , it would be clear that these results are useless even if the "exact" values were not included in the table.

Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, with Euler's method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.200000000	3.262000000	3.294348537	3.327851973
0.4	3.656000000	3.802028800	3.881421103	3.966059348
0.6	4.440960000	4.726810214	4.888870783	5.067039535
0.8	5.706790400	6.249191282	6.570796235	6.936700945
1.0	7.732963328	8.771893026	9.419105620	10.184923955
1.2	11.026148659	13.064051391	14.405772067	16.067111677
1.4	16.518700016	20.637273893	23.522935872	27.289392347
1.6	25.969172024	34.570423758	41.033441257	50.000377775
1.8	42.789442120	61.382165543	76.491018246	98.982969504
2.0	73.797840446	115.440048291	152.363866569	211.954462214

It's easy to see why Euler's method yields such poor results. Recall that the constant M in (3.10) – which plays an important role in determining the local truncation error in Euler's method – must be an upper bound for the values of the second derivative y'' of the solution of the initial value problem (3.22) on $(0, 2)$. The problem is that y'' assumes very large values on this interval. To see this, we differentiate (3.24) to obtain

$$y''(x) = 2y(x) + 2xy'(x) = 2y(x) + 2x(1 + 2xy(x)) = 2(1 + 2x^2)y(x) + 2x,$$

where the second equality follows again from (3.24). Since (3.23) implies that $y(x) > 3e^{x^2}$ if $x > 0$,

$$y''(x) > 6(1 + 2x^2)e^{x^2} + 2x, \quad x > 0.$$

For example, letting $x = 2$ shows that $y''(2) > 2952$.

Solution (b): Since $y_1 = e^{x^2}$ is a solution of the complementary equation $y' - 2xy = 0$, we can apply the Euler semilinear method to (3.22), with

$$y = ue^{x^2} \quad \text{and} \quad u' = e^{-x^2}, \quad u(0) = 3.$$

The results listed in Table 3.1.4 are clearly better than those obtained by Euler's method.

Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the Euler semilinear method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.330594477	3.329558853	3.328788889	3.327851973
0.4	3.980734157	3.974067628	3.970230415	3.966059348
0.6	5.106360231	5.087705244	5.077622723	5.067039535
0.8	7.021003417	6.980190891	6.958779586	6.936700945
1.0	10.350076600	10.269170824	10.227464299	10.184923955
1.2	16.381180092	16.226146390	16.147129067	16.067111677
1.4	27.890003380	27.592026085	27.441292235	27.289392347
1.6	51.183323262	50.594503863	50.298106659	50.000377775
1.8	101.424397595	100.206659076	99.595562766	98.982969504
2.0	217.301032800	214.631041938	213.293582978	211.954462214

We can't give a general procedure for determining in advance whether Euler's method or the semilinear Euler method will produce better results for a given semilinear initial value problem (3.19). As a rule of thumb, the Euler semilinear method will yield better results than Euler's method if $|u''|$ is small on $[x_0, b]$, while Euler's method yields better results if $|u''|$ is large on $[x_0, b]$. In many cases the results obtained by the two methods don't differ appreciably. However, we propose the an intuitive way to decide which is the better method: Try both methods with multiple step sizes, as we did in Example 3.1.4, and accept the results obtained by the method for which the approximations change less as the step size decreases.

Example 3.1.5: Applying Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to the initial value problem

$$y' - 2y = \frac{x}{1 + y^2}, \quad y(1) = 7 \quad (3.25)$$

on $[1, 2]$ yields the results in Table 3.1.4. Applying the Euler semilinear method with

$$y = ue^{2x} \quad \text{and} \quad u' = \frac{xe^{-2x}}{1 + ue^{4x}}, \quad u(1) = 7e^{-2}$$

yields the results in Table 3.1.4. Since the latter are clearly less dependent on step size than the former, we conclude that the Euler semilinear method is better than Euler's method for (3.25). This conclusion is supported by comparing the approximate results obtained by the two methods with the "exact" values of the solution.

Numerical solution of $y' - 2y = x/(1 + y^2)$, $y(1) = 7$, by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	7.000000000	7.000000000	7.000000000	7.000000000
1.1	8.402000000	8.471970569	8.510493955	8.551744786
1.2	10.083936450	10.252570169	10.346014101	10.446546230
1.3	12.101892354	12.406719381	12.576720827	12.760480158
1.4	14.523152445	15.012952416	15.287872104	15.586440425
1.5	17.428443554	18.166277405	18.583079406	19.037865752
1.6	20.914624471	21.981638487	22.588266217	23.253292359
1.7	25.097914310	26.598105180	27.456479695	28.401914416
1.8	30.117766627	32.183941340	33.373738944	34.690375086
1.9	36.141518172	38.942738252	40.566143158	42.371060528
2.0	43.369967155	47.120835251	49.308511126	51.752229656

Numerical solution of $y' - 2y = x/(1 + y^2)$, $y(1) = 7$, by the Euler semilinear method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	7.000000000	7.000000000	7.000000000	7.000000000
1.1	8.552262113	8.551993978	8.551867007	8.551744786
1.2	10.447568674	10.447038547	10.446787646	10.446546230
1.3	12.762019799	12.761221313	12.760843543	12.760480158
1.4	15.588535141	15.587448600	15.586934680	15.586440425
1.5	19.040580614	19.039172241	19.038506211	19.037865752
1.6	23.256721636	23.254942517	23.254101253	23.253292359
1.7	28.406184597	28.403969107	28.402921581	28.401914416
1.8	34.695649222	34.692912768	34.691618979	34.690375086
1.9	42.377544138	42.374180090	42.372589624	42.371060528
2.0	51.760178446	51.756054133	51.754104262	51.752229656

Example 3.1.6: Applying Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to the initial value problem

$$y' + 3x^2y = 1 + y^2, \quad y(2) = 2 \quad (3.26)$$

on $[2, 3]$ yields the results in Table 3.1.4. Applying the Euler semilinear method with

$$y = ue^{-x^3} \quad \text{and} \quad u' = e^{x^3}(1 + u^2e^{-2x^3}), \quad u(2) = 2e^8$$

yields the results in Table 3.1.4. Noting the close agreement among the three columns of Table 3.1.4 (at least for larger values of x) and the lack of any such agreement among the columns of Table 3.1.4, we conclude that Euler's method is better than the Euler semilinear method for (3.26). Comparing the results with the exact values supports this conclusion.

Numerical solution of $y' + 3x^2y = 1 + y^2$, $y(2) = 2$, by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	0.100000000	0.493231250	0.609611171	0.701162906
2.2	0.068700000	0.122879586	0.180113445	0.236986800
2.3	0.069419569	0.070670890	0.083934459	0.103815729
2.4	0.059732621	0.061338956	0.063337561	0.068390786
2.5	0.056871451	0.056002363	0.056249670	0.057281091
2.6	0.050560917	0.051465256	0.051517501	0.051711676
2.7	0.048279018	0.047484716	0.047514202	0.047564141
2.8	0.042925892	0.043967002	0.043989239	0.044014438
2.9	0.042148458	0.040839683	0.040857109	0.040875333
3.0	0.035985548	0.038044692	0.038058536	0.038072838

Numerical solution of $y' + 3x^2y = 1 + y^2$, $y(2) = 2$, by the Euler semilinear method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = .0125$
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	0.708426286	0.702568171	0.701214274	0.701162906
2.2	0.214501852	0.222599468	0.228942240	0.236986800
2.3	0.069861436	0.083620494	0.092852806	0.103815729
2.4	0.032487396	0.047079261	0.056825805	0.068390786
2.5	0.021895559	0.036030018	0.045683801	0.057281091
2.6	0.017332058	0.030750181	0.040189920	0.051711676
2.7	0.014271492	0.026931911	0.036134674	0.047564141
2.8	0.011819555	0.023720670	0.032679767	0.044014438
2.9	0.009776792	0.020925522	0.029636506	0.040875333
3.0	0.008065020	0.018472302	0.026931099	0.038072838

In the next two sections we'll study other numerical methods for solving initial value problems, called the *improved Euler method*, the *midpoint method*, *Heun's method* and the *Runge-Kutta method*. If the initial value problem is semilinear as in (3.19), we also have the option of using variation of parameters and then applying the given numerical method to the initial value problem (3.21) for u . By analogy with the terminology used here, we'll call the resulting procedure *the improved Euler semilinear method*, the *midpoint semilinear method*, *Heun's semilinear method* or *the Runge-Kutta semilinear method*, as the case may be.

3.1.5 Exercises

You may want to save the results of these exercises, since we'll revisit in the next two sections. In Exercises 3.1.1–3.1.5 use Euler's method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point where the initial condition is imposed and $i = 1, 2, 3$. The purpose of these exercises is to familiarize you with the computational procedure of Euler's method.

1. C $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$
2. C $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$
3. C $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$
4. C $y' = \frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$
5. C $y' + x^2y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$
6. C Use Euler's method with step sizes $h = 0.1, h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.1.2.

7. C Use Euler's method with step sizes $h = 0.1, h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \dots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9 \ln x + x^3 + 2),$$

which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.1.2.

8. C Use Euler's method with step sizes $h = 0.05, h = 0.025$, and $h = 0.0125$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2$$

at $x = 1.0, 1.05, 1.10, 1.15, \dots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}$$

obtained in Example 2.4.3. Present your results in a table like Table 3.1.2.

9. C In Example 2.2.3 it was shown that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1. \quad (\text{A})$$

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \dots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = y^5 + y - x^2 - x + 4$$

for each value of (x, y) appearing in the first table.

10. C You can see from Example ?? that

$$x^4 y^3 + x^2 y^5 + 2xy = 4$$

is an implicit solution of the initial value problem

$$y' = -\frac{4x^3 y^3 + 2x y^5 + 2y}{3x^4 y^2 + 5x^2 y^4 + 2x}, \quad y(1) = 1. \quad (\text{A})$$

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 1.0, 1.1, 1.2, 1.3, \dots, 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = x^4 y^3 + x^2 y^5 + 2xy - 4$$

for each value of (x, y) appearing in the first table.

11. C Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$(3y^2 + 4y)y' + 2x + \cos x = 0, \quad y(0) = 1; \quad (\text{Exercise 2.2.13})$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$.

12. C Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0 \quad (\text{Exercise 2.2.14})$$

at $x = 1.0, 1.1, 1.2, 1.3, \dots, 2.0$.

13. C Use Euler's method and the Euler semilinear method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{-3x}, \quad y(0) = 6$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{-3x}(7x + 6)$, which can be obtained by the method of Section 2.1. Do you notice anything special about the results? Explain.

The linear initial value problems in Exercises 3.1.14–3.1.19 can't be solved exactly in terms of known elementary functions. In each exercise, use Euler's method and the Euler semilinear methods with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. C $y' - 2y = \frac{1}{1+x^2}, \quad y(2) = 2; \quad h = 0.1, 0.05, 0.025 \text{ on } [2, 3]$
15. C $y' + 2xy = x^2, \quad y(0) = 3 \text{ (Exercise 2.1.38); } h = 0.2, 0.1, 0.05 \text{ on } [0, 2]$
16. C $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2; \text{ (Exercise 2.1.39); } h = 0.2, 0.1, 0.05 \text{ on } [1, 3]$
17. C $y' + y = \frac{e^{-x} \tan x}{x}, \quad y(1) = 0; \text{ (Exercise 2.1.40); } h = 0.05, 0.025, 0.0125 \text{ on } [1, 1.5]$
18. C $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1; \text{ (Exercise 2.1.41); } h = 0.2, 0.1, 0.05 \text{ on } [0, 2]$
19. C $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2; \text{ (Exercise 2.1.42); } h = 0.05, 0.025, 0.0125 \text{ on } [1, 1.5]$

In Exercises 3.1.20–3.1.22, use Euler's method and the Euler semilinear method with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

20. C $y' + 3y = xy^2(y+1), \quad y(0) = 1; \quad h = 0.1, 0.05, 0.025 \text{ on } [0, 1]$
21. C $y' - 4y = \frac{x}{y^2(y+1)}, \quad y(0) = 1; \quad h = 0.1, 0.05, 0.025 \text{ on } [0, 1]$
22. C $y' + 2y = \frac{x^2}{1+y^2}, \quad y(2) = 1; \quad h = 0.1, 0.05, 0.025 \text{ on } [2, 3]$
23. NUMERICAL QUADRATURE. The fundamental theorem of calculus says that if f is continuous on a closed interval $[a, b]$ then it has an antiderivative F such that $F'(x) = f(x)$ on $[a, b]$ and

$$\int_a^b f(x) dx = F(b) - F(a). \quad (\text{A})$$

This solves the problem of evaluating a definite integral if the integrand f has an antiderivative that can be found and evaluated easily. However, if f doesn't have this property, (A) doesn't provide a useful way to evaluate the definite integral. In this case we must resort to approximate methods. There's a class of such methods called *numerical quadrature*, where the approximation takes the form

$$\int_a^b f(x) dx \approx \sum_{i=0}^n c_i f(x_i), \quad (\text{B})$$

where $a = x_0 < x_1 < \cdots < x_n = b$ are suitably chosen points and $c_0, c_1, \dots, c_n$ are suitably chosen constants. We call (B) a *quadrature formula*.

(a) Derive the quadrature formula

$$\int_a^b f(x) dx \approx h \sum_{i=0}^{n-1} f(a + ih) \quad (\text{where } h = (b-a)/n) \quad (\text{C})$$

by applying Euler's method to the initial value problem

$$y' = f(x), \quad y(a) = 0.$$

(b) The quadrature formula (C) is sometimes called *the left rectangle rule*. Draw a figure that justifies this terminology.

- (c) L For several choices of a , b , and A , apply (C) to $f(x) = A$ with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.
- (d) L For several choices of a , b , A , and B , apply (C) to $f(x) = A + Bx$ with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.

3.2 The Improved Euler Method and Related Methods

In Section 3.1 we saw that the global truncation error of Euler's method is $O(h)$, which would seem to imply that we can achieve arbitrarily accurate results with Euler's method by simply choosing the step size sufficiently small. However, this isn't a good idea, for two reasons. First, after a certain point decreasing the step size will increase roundoff errors to the point where the accuracy will deteriorate rather than improve. The second and more important reason is that in most applications of numerical methods to an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \quad (3.27)$$

the expensive part of the computation is the evaluation of f . Therefore we want methods that give good results for a given number of such evaluations. This is what motivates us to look for numerical methods better than Euler's.

To clarify this point, suppose we want to approximate the value of e by applying Euler's method to the initial value problem

$$y' = y, \quad y(0) = 1, \quad (\text{with solution } y = e^x)$$

on $[0, 1]$, with $h = 1/12, 1/24$, and $1/48$, respectively. Since each step in Euler's method requires one evaluation of f , the number of evaluations of f in each of these attempts is $n = 12, 24$, and 48 , respectively. In each case we accept y_n as an approximation to e . The second column of Table 3.2 shows the results. The first column of the table indicates the number of evaluations of f required to obtain the approximation, and the last column contains the value of e rounded to ten significant figures.

In this section we'll study the *improved Euler method*, which requires two evaluations of f at each step. We've used this method with $h = 1/6, 1/12$, and $1/24$. The required number of evaluations of f were 12, 24, and 48, as in the three applications of Euler's method; however, you can see from the third column of Table 3.2 that the approximation to e obtained by the improved Euler method with only 12 evaluations of f is better than the approximation obtained by Euler's method with 48 evaluations.

In Section 3.1 we'll study the *Runge-Kutta method*, which requires four evaluations of f at each step. We've used this method with $h = 1/3, 1/6$, and $1/12$. The required number of evaluations of f were again 12, 24, and 48, as in the three applications of Euler's method and the improved Euler method; however, you can see from the fourth column of Table 3.2 that the approximation to e obtained by the Runge-Kutta method with only 12 evaluations of f is better than the approximation obtained by the improved Euler method with 48 evaluations.

Approximations to e obtained by three numerical methods.

n	Euler	Improved Euler	Runge-Kutta	Exact
12	2.613035290	2.707188994	2.718069764	2.718281828
24	2.663731258	2.715327371	2.718266612	2.718281828
48	2.690496599	2.717519565	2.718280809	2.718281828

3.2.1 The Improved Euler Method

The *improved Euler method* for solving the initial value problem (3.27) is based on approximating the integral curve of (3.27) at $(x_i, y(x_i))$ by the line through $(x_i, y(x_i))$ with slope

$$m_i = \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2};$$

that is, m_i is the average of the slopes of the tangents to the integral curve at the endpoints of $[x_i, x_{i+1}]$. The equation of the approximating line is therefore

$$y = y(x_i) + \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2}(x - x_i). \quad (3.28)$$

Setting $x = x_{i+1} = x_i + h$ in (3.28) yields

$$y_{i+1} = y(x_i) + \frac{h}{2} (f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))) \quad (3.29)$$

as an approximation to $y(x_{i+1})$. As in our derivation of Euler's method, we replace $y(x_i)$ (unknown if $i > 0$) by its approximate value y_i ; then (3.29) becomes

$$y_{i+1} = y_i + \frac{h}{2} (f(x_i, y_i) + f(x_{i+1}, y(x_{i+1}))).$$

However, this still won't work, because we don't know $y(x_{i+1})$, which appears on the right. We overcome this by replacing $y(x_{i+1})$ by $y_i + hf(x_i, y_i)$, the value that the Euler method would assign to y_{i+1} . Thus, the improved Euler method starts with the known value $y(x_0) = y_0$ and computes $y_1, y_2, \dots, y_n$ successively with the formula

$$y_{i+1} = y_i + \frac{h}{2} (f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))). \quad (3.30)$$

The computation indicated here can be conveniently organized as follows: given y_i , compute

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f(x_i + h, y_i + hk_{1i}), \\ y_{i+1} &= y_i + \frac{h}{2}(k_{1i} + k_{2i}). \end{aligned}$$

The improved Euler method requires two evaluations of $f(x, y)$ per step, while Euler's method requires only one. However, we'll see at the end of this section that if f satisfies appropriate assumptions, the local truncation error with the improved Euler method is $O(h^3)$, rather than $O(h^2)$ as with Euler's method. Therefore the global truncation error with the improved Euler method is $O(h^2)$; however, we won't prove this.

We note that the magnitude of the local truncation error in the improved Euler method and other methods discussed in this section is determined by the third derivative y''' of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y'''|$ is large, or smaller where $|y'''|$ is small.

The next example, which deals with the initial value problem considered in Example 3.1.1, illustrates the computational procedure indicated in the improved Euler method.

Example 3.2.1: Use the improved Euler method with $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \quad (3.31)$$

at $x = 0.1, 0.2, 0.3$.

Solution: As in Example 3.1.1, we rewrite (3.31) as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form (3.27), with

$$f(x, y) = -2y + x^3 e^{-2x}, \quad x_0 = 0, \quad \text{and } y_0 = 1.$$

The improved Euler method yields

$$\begin{aligned}
 k_{10} &= f(x_0, y_0) = f(0, 1) = -2, \\
 k_{20} &= f(x_1, y_0 + hk_{10}) = f(.1, 1 + (.1)(-2)) \\
 &= f(.1, .8) = -2(.8) + (.1)^3 e^{-.2} = -1.599181269, \\
 y_1 &= y_0 + \frac{h}{2}(k_{10} + k_{20}), \\
 &= 1 + (.05)(-2 - 1.599181269) = .820040937, \\
 k_{11} &= f(x_1, y_1) = f(.1, .820040937) = -2(.820040937) + (.1)^3 e^{-.2} = -1.639263142, \\
 k_{21} &= f(x_2, y_1 + hk_{11}) = f(.2, .820040937 + .1(-1.639263142)), \\
 &= f(.2, .656114622) = -2(.656114622) + (.2)^3 e^{-.4} = -1.306866684, \\
 y_2 &= y_1 + \frac{h}{2}(k_{11} + k_{21}), \\
 &= .820040937 + (.05)(-1.639263142 - 1.306866684) = .672734445, \\
 k_{12} &= f(x_2, y_2) = f(.2, .672734445) = -2(.672734445) + (.2)^3 e^{-.4} = -1.340106330, \\
 k_{22} &= f(x_3, y_2 + hk_{12}) = f(.3, .672734445 + .1(-1.340106330)), \\
 &= f(.3, .538723812) = -2(.538723812) + (.3)^3 e^{-.6} = -1.062629710, \\
 y_3 &= y_2 + \frac{h}{2}(k_{12} + k_{22}) \\
 &= .672734445 + (.05)(-1.340106330 - 1.062629710) = .552597643.
 \end{aligned}$$

Example 3.2.2: Table 3.2.1 shows results of using the improved Euler method with step sizes $h = 0.1$ and $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. For comparison, it also shows the corresponding approximate values obtained with Euler's method in 3.1.2, and the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4).$$

The results obtained by the improved Euler method with $h = 0.1$ are better than those obtained by Euler's method with $h = 0.05$.

Numerical solution of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, by Euler's method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.810005655	0.820040937	0.819050572	0.818751221
0.2	0.640081873	0.656266437	0.672734445	0.671086455	0.670588174
0.3	0.512601754	0.532290981	0.552597643	0.550543878	0.549922980
0.4	0.411563195	0.432887056	0.455160637	0.452890616	0.452204669
0.5	0.332126261	0.353785015	0.376681251	0.374335747	0.373627557
0.6	0.270299502	0.291404256	0.313970920	0.311652239	0.310952904
0.7	0.222745397	0.242707257	0.264287611	0.262067624	0.261398947
0.8	0.186654593	0.205105754	0.225267702	0.223194281	0.222570721
0.9	0.159660776	0.176396883	0.194879501	0.192981757	0.192412038
1.0	0.139778910	0.154715925	0.171388070	0.169680673	0.169169104
	Euler		Improved Euler		Exact

Example 3.2.3: Table 3.2.1 shows analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2, \quad y(0) = 1.$$

We applied Euler's method to this problem in Example 3.1.3.

Numerical solution of $y' = -2y^2 + xy + x^2$, $y(0) = 1$, by Euler's method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.821375000	0.840500000	0.838288371	0.837584494
0.2	0.681000000	0.707795377	0.733430846	0.730556677	0.729641890
0.3	0.605867800	0.633776590	0.661600806	0.658552190	0.657580377
0.4	0.559628676	0.587454526	0.615961841	0.612884493	0.611901791
0.5	0.535376972	0.562906169	0.591634742	0.588558952	0.587575491
0.6	0.529820120	0.557143535	0.586006935	0.582927224	0.581942225
0.7	0.541467455	0.568716935	0.597712120	0.594618012	0.593629526
0.8	0.569732776	0.596951988	0.626008824	0.622898279	0.621907458
0.9	0.614392311	0.641457729	0.670351225	0.667237617	0.666250842
1.0	0.675192037	0.701764495	0.730069610	0.726985837	0.726015790
	Euler		Improved Euler		"Exact"

Example 3.2.4: Use step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$ to find approximate values of the solution of

$$y' - 2xy = 1, \quad y(0) = 3 \quad (3.32)$$

at $x = 0, 0.2, 0.4, 0.6, \dots, 2.0$ by **(a)** the improved Euler method; **(b)** the improved Euler semilinear method. (We used Euler's method and the Euler semilinear method on this problem in 3.1.4.)

Solution (a): Rewriting (3.32) as

$$y' = 1 + 2xy, \quad y(0) = 3$$

and applying the improved Euler method with $f(x, y) = 1 + 2xy$ yields the results shown in Table 3.2.1.

Solution (b): Since $y_1 = e^{x^2}$ is a solution of the complementary equation $y' - 2xy = 0$, we can apply the improved Euler semilinear method to (3.32), with

$$y = ue^{x^2} \quad \text{and} \quad u' = e^{-x^2}, \quad u(0) = 3.$$

The results listed in Table 3.2.1 are clearly better than those obtained by the improved Euler method.

Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the improved Euler method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.328000000	3.328182400	3.327973600	3.327851973
0.4	3.964659200	3.966340117	3.966216690	3.966059348
0.6	5.057712497	5.065700515	5.066848381	5.067039535
0.8	6.900088156	6.928648973	6.934862367	6.936700945
1.0	10.065725534	10.154872547	10.177430736	10.184923955
1.2	15.708954420	15.970033261	16.041904862	16.067111677
1.4	26.244894192	26.991620960	27.210001715	27.289392347
1.6	46.958915746	49.096125524	49.754131060	50.000377775
1.8	89.982312641	96.200506218	98.210577385	98.982969504
2.0	184.563776288	203.151922739	209.464744495	211.954462214

Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the improved Euler semilinear method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.326513400	3.327518315	3.327768620	3.327851973
0.4	3.963383070	3.965392084	3.965892644	3.966059348
0.6	5.063027290	5.066038774	5.066789487	5.067039535
0.8	6.931355329	6.935366847	6.936367564	6.936700945
1.0	10.178248417	10.183256733	10.184507253	10.184923955
1.2	16.059110511	16.065111599	16.066611672	16.067111677
1.4	27.280070674	27.287059732	27.288809058	27.289392347
1.6	49.989741531	49.997712997	49.999711226	50.000377775
1.8	98.971025420	98.979972988	98.982219722	98.982969504
2.0	211.941217796	211.951134436	211.953629228	211.954462214

3.2.2 A Family of Methods with $O(h^3)$ Local Truncation Error

We'll now derive a class of methods with $O(h^3)$ local truncation error for solving (3.27). For simplicity, we assume that $f, f_x, f_y, f_{xx}, f_{yy},$ and f_{xy} are continuous and bounded for all (x, y) . This implies that if y is the solution of (3.27) then y'' and y''' are bounded (Exercise 3.2.31).

We begin by approximating the integral curve of (3.27) at $(x_i, y(x_i))$ by the line through $(x_i, y(x_i))$ with slope

$$m_i = \sigma y'(x_i) + \rho y'(x_i + \theta h),$$

where $\sigma, \rho,$ and θ are constants that we'll soon specify; however, we insist at the outset that $0 < \theta \leq 1$, so that

$$x_i < x_i + \theta h \leq x_{i+1}.$$

The equation of the approximating line is

$$\begin{aligned} y &= y(x_i) + m_i(x - x_i) \\ &= y(x_i) + [\sigma y'(x_i) + \rho y'(x_i + \theta h)](x - x_i). \end{aligned} \quad (3.33)$$

Setting $x = x_{i+1} = x_i + h$ in (3.33) yields

$$\hat{y}_{i+1} = y(x_i) + h [\sigma y'(x_i) + \rho y'(x_i + \theta h)]$$

as an approximation to $y(x_{i+1})$.

To determine $\sigma, \rho,$ and θ so that the error

$$\begin{aligned} E_i &= y(x_{i+1}) - \hat{y}_{i+1} \\ &= y(x_{i+1}) - y(x_i) - h [\sigma y'(x_i) + \rho y'(x_i + \theta h)] \end{aligned} \quad (3.34)$$

in this approximation is $O(h^3)$, we begin by recalling from Taylor's theorem that

$$y(x_{i+1}) = y(x_i) + h y'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{6} y'''(\hat{x}_i),$$

where $\hat{x}_i$ is in (x_i, x_{i+1}) . Since y''' is bounded this implies that

$$y(x_{i+1}) - y(x_i) - h y'(x_i) - \frac{h^2}{2} y''(x_i) = O(h^3).$$

Comparing this with (3.34) shows that $E_i = O(h^3)$ if

$$\sigma y'(x_i) + \rho y'(x_i + \theta h) = y'(x_i) + \frac{h}{2} y''(x_i) + O(h^2). \quad (3.35)$$

However, applying Taylor's theorem to y' shows that

$$y'(x_i + \theta h) = y'(x_i) + \theta h y''(x_i) + \frac{(\theta h)^2}{2} y'''(\bar{x}_i),$$

where $\bar{x}_i$ is in $(x_i, x_i + \theta h)$. Since y''' is bounded, this implies that

$$y'(x_i + \theta h) = y'(x_i) + \theta h y''(x_i) + O(h^2).$$

Substituting this into (3.35) and noting that the sum of two $O(h^2)$ terms is again $O(h^2)$ shows that $E_i = O(h^3)$ if

$$(\sigma + \rho) y'(x_i) + \rho \theta h y''(x_i) = y'(x_i) + \frac{h}{2} y''(x_i),$$

which is true if

$$\sigma + \rho = 1 \quad \text{and} \quad \rho \theta = \frac{1}{2}. \quad (3.36)$$

Since $y' = f(x, y)$, we can now conclude from (3.34) that

$$y(x_{i+1}) = y(x_i) + h [\sigma f(x_i, y_i) + \rho f(x_i + \theta h, y(x_i + \theta h))] + O(h^3) \quad (3.37)$$

if σ , ρ , and θ satisfy (3.36). However, this formula would not be useful even if we knew $y(x_i)$ exactly (as we would for $i = 0$), since we still wouldn't know $y(x_i + \theta h)$ exactly. To overcome this difficulty, we again use Taylor's theorem to write

$$y(x_i + \theta h) = y(x_i) + \theta h y'(x_i) + \frac{h^2}{2} y''(\tilde{x}_i),$$

where $\tilde{x}_i$ is in $(x_i, x_i + \theta h)$. Since $y'(x_i) = f(x_i, y(x_i))$ and y'' is bounded, this implies that

$$|y(x_i + \theta h) - y(x_i) - \theta h f(x_i, y(x_i))| \leq K h^2 \quad (3.38)$$

for some constant K . Since f_y is bounded, the mean value theorem implies that

$$|f(x_i + \theta h, u) - f(x_i + \theta h, v)| \leq M |u - v|$$

for some constant M . Letting

$$u = y(x_i + \theta h) \quad \text{and} \quad v = y(x_i) + \theta h f(x_i, y(x_i))$$

and recalling (3.38) shows that

$$f(x_i + \theta h, y(x_i + \theta h)) = f(x_i + \theta h, y(x_i) + \theta h f(x_i, y(x_i))) + O(h^2).$$

Substituting this into (3.37) yields

$$\begin{aligned} y(x_{i+1}) = & y(x_i) + h [\sigma f(x_i, y(x_i)) + \\ & \rho f(x_i + \theta h, y(x_i) + \theta h f(x_i, y(x_i)))] + O(h^3). \end{aligned}$$

This implies that the formula

$$y_{i+1} = y_i + h [\sigma f(x_i, y_i) + \rho f(x_i + \theta h, y_i + \theta h f(x_i, y_i))]$$

has $O(h^3)$ local truncation error if σ , ρ , and θ satisfy (3.36). Substituting $\sigma = 1 - \rho$ and $\theta = 1/2\rho$ here yields

$$y_{i+1} = y_i + h \left[(1 - \rho) f(x_i, y_i) + \rho f \left(x_i + \frac{h}{2\rho}, y_i + \frac{h}{2\rho} f(x_i, y_i) \right) \right]. \quad (3.39)$$

The computation indicated here can be conveniently organized as follows: given y_i , compute

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f \left(x_i + \frac{h}{2\rho}, y_i + \frac{h}{2\rho} k_{1i} \right), \\ y_{i+1} &= y_i + h[(1 - \rho)k_{1i} + \rho k_{2i}]. \end{aligned}$$

Consistent with our requirement that $0 < \theta < 1$, we require that $\rho \geq 1/2$. Letting $\rho = 1/2$ in (3.39) yields the improved Euler method (3.30). Letting $\rho = 3/4$ yields *Heun's method*,

$$y_{i+1} = y_i + h \left[\frac{1}{4} f(x_i, y_i) + \frac{3}{4} f \left(x_i + \frac{2}{3} h, y_i + \frac{2}{3} h f(x_i, y_i) \right) \right],$$

which can be organized as

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f \left(x_i + \frac{2h}{3}, y_i + \frac{2h}{3} k_{1i} \right), \\ y_{i+1} &= y_i + \frac{h}{4} (k_{1i} + 3k_{2i}). \end{aligned}$$

Letting $\rho = 1$ yields the *midpoint method*,

$$y_{i+1} = y_i + hf\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}f(x_i, y_i)\right),$$

which can be organized as

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}\right), \\ y_{i+1} &= y_i + hk_{2i}. \end{aligned}$$

Examples involving the midpoint method and Heun's method are given in Exercises 3.2.23–3.2.30.

3.2.3 Exercises

Most of the following numerical exercises involve initial value problems considered in the exercises in Section 3.1. You'll find it instructive to compare the results that you obtain here with the corresponding results that you obtained in Section 3.1.

In Exercises 3.2.1–3.2.5 use the improved Euler method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point where the initial condition is imposed and $i = 1, 2, 3$.

1. ☐ $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$
2. ☐ $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$
3. ☐ $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$
4. ☐ $y' = \frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$
5. ☐ $y' + x^2y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$
6. ☐ Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.2.1.

7. ☐ Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \dots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9 \ln x + x^3 + 2)$$

which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.2.1.

8. ☐ Use the improved Euler method with step sizes $h = 0.05$, $h = 0.025$, and $h = 0.0125$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2,$$

at $x = 1.0, 1.05, 1.10, 1.15, \dots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}$$

obtained in Example 2.4.3. Present your results in a table like Table 3.2.1.

9. C In Example 3.2.2 it was shown that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \quad (\text{A})$$

Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \dots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = y^5 + y - x^2 - x + 4$$

for each value of (x, y) appearing in the first table.

10. C You can see from Example ?? that

$$x^4 y^3 + x^2 y^5 + 2xy = 4$$

is an implicit solution of the initial value problem

$$y' = -\frac{4x^3 y^3 + 2x y^5 + 2y}{3x^4 y^2 + 5x^2 y^4 + 2x}, \quad y(1) = 1. \quad (\text{A})$$

Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 1.0, 1.14, 1.2, 1.3, \dots, 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = x^4 y^3 + x^2 y^5 + 2xy - 4$$

for each value of (x, y) appearing in the first table.

11. C Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$(3y^2 + 4y)y' + 2x + \cos x = 0, \quad y(0) = 1 \quad (\text{Exercise 2.2.13})$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$.

12. C Use the improved Euler method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0 \quad (\text{Exercise 2.2.14})$$

at $x = 1.0, 1.1, 1.2, 1.3, \dots, 2.0$.

13. C Use the improved Euler method and the improved Euler semilinear method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = e^{-3x}(1 - 2x), \quad y(0) = 2,$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{-3x}(2 + x - x^2)$, which can be obtained by the method of Section 2.1. Do you notice anything special about the results? Explain.

The linear initial value problems in Exercises 3.2.14–3.2.19 can't be solved exactly in terms of known elementary functions. In each exercise use the improved Euler and improved Euler semilinear methods with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14. C $y' - 2y = \frac{1}{1+x^2}, \quad y(2) = 2; \quad h = 0.1, 0.05, 0.025 \text{ on } [2, 3]$
15. C $y' + 2xy = x^2, \quad y(0) = 3; \quad h = 0.2, 0.1, 0.05 \text{ on } [0, 2] \quad (\text{Exercise 2.1.38})$
16. C $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2, \quad h = 0.2, 0.1, 0.05 \text{ on } [1, 3] \quad (\text{Exercise 2.1.39})$
17. C $y' + y = \frac{e^{-x} \tan x}{x}, \quad y(1) = 0; \quad h = 0.05, 0.025, 0.0125 \text{ on } [1, 1.5] \quad (\text{Exercise 2.1.40}),$
18. C $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1; \quad h = 0.2, 0.1, 0.05 \text{ on } [0, 2] \quad (\text{Exercise 2.1.41})$
19. C $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2; \quad h = 0.05, 0.025, 0.0125 \text{ on } [1, 1.5] \quad (\text{Exercise 2.1.42})$

In Exercises 3.2.20–3.2.22 use the improved Euler method and the improved Euler semilinear method with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

20. C $y' + 3y = xy^2(y+1), \quad y(0) = 1; \quad h = 0.1, 0.05, 0.025 \text{ on } [0, 1]$
21. C $y' - 4y = \frac{x}{y^2(y+1)}, \quad y(0) = 1; \quad h = 0.1, 0.05, 0.025 \text{ on } [0, 1]$
22. C $y' + 2y = \frac{x^2}{1+y^2}, \quad y(2) = 1; \quad h = 0.1, 0.05, 0.025 \text{ on } [2, 3]$
23. C Do Exercise 3.2.7 with “improved Euler method” replaced by “midpoint method.”
24. C Do Exercise 3.2.7 with “improved Euler method” replaced by “Heun’s method.”
25. C Do Exercise 3.2.8 with “improved Euler method” replaced by “midpoint method.”
26. C Do Exercise 3.2.8 with “improved Euler method” replaced by “Heun’s method.”
27. C Do Exercise 3.2.11 with “improved Euler method” replaced by “midpoint method.”
28. C Do Exercise 3.2.11 with “improved Euler method” replaced by “Heun’s method.”
29. C Do Exercise 3.2.12 with “improved Euler method” replaced by “midpoint method.”
30. C Do Exercise 3.2.12 with “improved Euler method” replaced by “Heun’s method.”
31. Show that if $f, f_x, f_y, f_{xx}, f_{yy},$ and f_{xy} are continuous and bounded for all (x, y) and y is the solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0,$$

then y'' and y''' are bounded.

32. NUMERICAL QUADRATURE (see Exercise 3.1.23).

(a) Derive the quadrature formula

$$\int_a^b f(x) dx \approx .5h(f(a) + f(b)) + h \sum_{i=1}^{n-1} f(a + ih) \quad (\text{where } h = (b - a)/n) \quad (\text{A})$$

by applying the improved Euler method to the initial value problem

$$y' = f(x), \quad y(a) = 0.$$

- (b) The quadrature formula (A) is called *the trapezoid rule*. Draw a figure that justifies this terminology.
- (c) L For several choices of a, b, A , and B , apply (A) to $f(x) = A+Bx$, with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.
- (d) L For several choices of a, b, A, B , and C , apply (A) to $f(x) = A + Bx + Cx^2$, with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.

3.3 The Runge-Kutta Method

In general, if k is any positive integer and f satisfies appropriate assumptions, there are numerical methods with local truncation error $O(h^{k+1})$ for solving an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0. \quad (3.40)$$

Moreover, it can be shown that a method with local truncation error $O(h^{k+1})$ has global truncation error $O(h^k)$. In Sections 3.1 and 3.2 we studied numerical methods where $k = 1$ and $k = 2$. We'll skip methods for which $k = 3$ and proceed to the *Runge-Kutta* method, the most widely used method, for which $k = 4$. The magnitude of the local truncation error is determined by the fifth derivative $y^{(5)}$ of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y^{(5)}|$ is large, or smaller where $|y^{(5)}|$ is small. The Runge-Kutta method computes approximate values $y_1, y_2, \dots, y_n$ of the solution of (3.40) at $x_0, x_0 + h, \dots, x_0 + nh$ as follows: Given y_i , compute

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}\right), \\ k_{3i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{2i}\right), \\ k_{4i} &= f(x_i + h, y_i + hk_{3i}), \end{aligned}$$

and

$$y_{i+1} = y_i + \frac{h}{6}(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i}).$$

The next example, which deals with the initial value problem considered in Examples 3.1.1 and 3.2.1, illustrates the computational procedure indicated in the Runge-Kutta method.

Example 3.3.1: Use the Runge-Kutta method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1, \quad (3.41)$$

at $x = 0.1, 0.2$.

Solution: Again we rewrite (3.41) as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form (3.40), with

$$f(x, y) = -2y + x^3 e^{-2x}, \quad x_0 = 0, \text{ and } y_0 = 1.$$

The Runge-Kutta method yields

$$\begin{aligned}
 k_{10} &= f(x_0, y_0) = f(0, 1) = -2, \\
 k_{20} &= f(x_0 + h/2, y_0 + hk_{10}/2) = f(.05, 1 + (.05)(-2)) \\
 &= f(.05, .9) = -2(.9) + (.05)^3 e^{-.1} = -1.799886895, \\
 k_{30} &= f(x_0 + h/2, y_0 + hk_{20}/2) = f(.05, 1 + (.05)(-1.799886895)) \\
 &= f(.05, .910005655) = -2(.910005655) + (.05)^3 e^{-.1} = -1.819898206, \\
 k_{40} &= f(x_0 + h, y_0 + hk_{30}) = f(.1, 1 + (.1)(-1.819898206)) \\
 &= f(.1, .818010179) = -2(.818010179) + (.1)^3 e^{-.2} = -1.635201628, \\
 y_1 &= y_0 + \frac{h}{6}(k_{10} + 2k_{20} + 2k_{30} + k_{40}), \\
 &= 1 + \frac{.1}{6}(-2 + 2(-1.799886895) + 2(-1.819898206) - 1.635201628) = .818753803, \\
 k_{11} &= f(x_1, y_1) = f(.1, .818753803) = -2(.818753803) + (.1)^3 e^{-.2} = -1.636688875, \\
 k_{21} &= f(x_1 + h/2, y_1 + hk_{11}/2) = f(.15, .818753803 + (.05)(-1.636688875)) \\
 &= f(.15, .736919359) = -2(.736919359) + (.15)^3 e^{-.3} = -1.471338457, \\
 k_{31} &= f(x_1 + h/2, y_1 + hk_{21}/2) = f(.15, .818753803 + (.05)(-1.471338457)) \\
 &= f(.15, .745186880) = -2(.745186880) + (.15)^3 e^{-.3} = -1.487873498, \\
 k_{41} &= f(x_1 + h, y_1 + hk_{31}) = f(.2, .818753803 + (.1)(-1.487873498)) \\
 &= f(.2, .669966453) = -2(.669966453) + (.2)^3 e^{-.4} = -1.334570346, \\
 y_2 &= y_1 + \frac{h}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}), \\
 &= .818753803 + \frac{.1}{6}(-1.636688875 + 2(-1.471338457) + 2(-1.487873498) - 1.334570346) \\
 &= .670592417.
 \end{aligned}$$

The Runge-Kutta method is sufficiently accurate for most applications.

Example 3.3.2: Table 3.3 shows results of using the Runge-Kutta method with step sizes $h = 0.1$ and $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. For comparison, it also shows the corresponding approximate values obtained with the improved Euler method in Example 3.2.2, and the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4).$$

The results obtained by the Runge-Kutta method are clearly better than those obtained by the improved Euler method in fact; the results obtained by the Runge-Kutta method with $h = 0.1$ are better than those obtained by the improved Euler method with $h = 0.05$.

Numerical solution of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, by the Runge-Kutta method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.820040937	0.819050572	0.818753803	0.818751370	0.818751221
0.2	0.672734445	0.671086455	0.670592417	0.670588418	0.670588174
0.3	0.552597643	0.550543878	0.549928221	0.549923281	0.549922980
0.4	0.455160637	0.452890616	0.452210430	0.452205001	0.452204669
0.5	0.376681251	0.374335747	0.373633492	0.373627899	0.373627557
0.6	0.313970920	0.311652239	0.310958768	0.310953242	0.310952904
0.7	0.264287611	0.262067624	0.261404568	0.261399270	0.261398947
0.8	0.225267702	0.223194281	0.222575989	0.222571024	0.222570721
0.9	0.194879501	0.192981757	0.192416882	0.192412317	0.192412038
1.0	0.171388070	0.169680673	0.169173489	0.169169356	0.169169104
	Improved Euler		Runge-Kutta		Exact

Example 3.3.3: Table 3.3 shows analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2, \quad y(0) = 1.$$

We applied the improved Euler method to this problem in Example 3.2.3.

Numerical solution of $y' = -2y^2 + xy + x^2$, $y(0) = 1$, by the Runge-Kutta method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.840500000	0.838288371	0.837587192	0.837584759	0.837584494
0.2	0.733430846	0.730556677	0.729644487	0.729642155	0.729641890
0.3	0.661600806	0.658552190	0.657582449	0.657580598	0.657580377
0.4	0.615961841	0.612884493	0.611903380	0.611901969	0.611901791
0.5	0.591634742	0.588558952	0.587576716	0.587575635	0.587575491
0.6	0.586006935	0.582927224	0.581943210	0.581942342	0.581942225
0.7	0.597712120	0.594618012	0.593630403	0.593629627	0.593629526
0.8	0.626008824	0.622898279	0.621908378	0.621907553	0.621907458
0.9	0.670351225	0.667237617	0.666251988	0.666250942	0.666250842
1.0	0.730069610	0.726985837	0.726017378	0.726015908	0.726015790
	Improved Euler		Runge-Kutta		"Exact"

Example 3.3.4: Tables 3.3 and 3.3 show results obtained by applying the Runge-Kutta and Runge-Kutta semi-linear methods to the initial value problem

$$y' - 2xy = 1, \quad y(0) = 3,$$

which we considered in Examples 3.1.4 and 3.2.4.

Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the Runge-Kutta method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.327846400	3.327851633	3.327851952	3.327851973
0.4	3.966044973	3.966058535	3.966059300	3.966059348
0.6	5.066996754	5.067037123	5.067039396	5.067039535
0.8	6.936534178	6.936690679	6.936700320	6.936700945
1.0	10.184232252	10.184877733	10.184920997	10.184923955
1.2	16.064344805	16.066915583	16.067098699	16.067111677
1.4	27.278771833	27.288605217	27.289338955	27.289392347
1.6	49.960553660	49.997313966	50.000165744	50.000377775
1.8	98.834337815	98.971146146	98.982136702	98.982969504
2.0	211.393800152	211.908445283	211.951167637	211.954462214

Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the Runge-Kutta semilinear method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.327853286	3.327852055	3.327851978	3.327851973
0.4	3.966061755	3.966059497	3.966059357	3.966059348
0.6	5.067042602	5.067039725	5.067039547	5.067039535
0.8	6.936704019	6.936701137	6.936700957	6.936700945
1.0	10.184926171	10.184924093	10.184923963	10.184923955
1.2	16.067111961	16.067111696	16.067111678	16.067111677
1.4	27.289389418	27.289392167	27.289392335	27.289392347
1.6	50.000370152	50.000377302	50.000377745	50.000377775
1.8	98.982955511	98.982968633	98.982969450	98.982969504
2.0	211.954439983	211.954460825	211.954462127	211.954462214

3.3.1 The Case Where x_0 Isn't The Left Endpoint

So far in this chapter we've considered numerical methods for solving an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (3.42)$$

on an interval $[x_0, b]$, for which x_0 is the left endpoint. We haven't discussed numerical methods for solving (3.42) on an interval $[a, x_0]$, for which x_0 is the right endpoint. To be specific, how can we obtain approximate values $y_{-1}, y_{-2}, \dots, y_{-n}$ of the solution of (3.42) at $x_0 - h, \dots, x_0 - nh$, where $h = (x_0 - a)/n$? Here's the answer to this question:

Consider the initial value problem

$$z' = -f(-x, z), \quad z(-x_0) = y_0, \quad (3.43)$$

on the interval $[-x_0, -a]$, for which $-x_0$ is the left endpoint. Use a numerical method to obtain approximate values $z_1, z_2, \dots, z_n$ of the solution of (3.43) at $-x_0 + h, -x_0 + 2h, \dots, -x_0 + nh = -a$. Then $y_{-1} = z_1, y_{-2} = z_2, \dots, y_{-n} = z_n$ are approximate values of the solution of (3.42) at $x_0 - h, x_0 - 2h, \dots, x_0 - nh = a$.

The justification for this answer is sketched in Exercise 3.3.23. Note how easy it is to make the change the given problem (3.42) to the modified problem (3.43): first replace f by $-f$ and then replace x, x_0 , and y by $-x, -x_0$, and z , respectively.

Example 3.3.5: Use the Runge-Kutta method with step size $h = 0.1$ to find approximate values of the solution of

$$(y - 1)^2 y' = 2x + 3, \quad y(1) = 4 \quad (3.44)$$

at $x = 0, 0.1, 0.2, \dots, 1$.

Solution: We first rewrite (3.44) in the form (3.42) as

$$y' = \frac{2x + 3}{(y - 1)^2}, \quad y(1) = 4. \quad (3.45)$$

Since the initial condition $y(1) = 4$ is imposed at the right endpoint of the interval $[0, 1]$, we apply the Runge-Kutta method to the initial value problem

$$z' = \frac{2x - 3}{(z - 1)^2}, \quad z(-1) = 4 \quad (3.46)$$

on the interval $[-1, 0]$. (You should verify that (3.46) is related to (3.45) as (3.43) is related to (3.42).) Table 3.3.1 shows the results. Reversing the order of the rows in Table 3.3.1 and changing the signs of the values of x yields the first two columns of Table 3.3.1. The last column of Table 3.3.1 shows the exact values of y , which are given by

$$y = 1 + (3x^2 + 9x + 15)^{1/3}.$$

(Since the differential equation in (3.45) is separable, this formula can be obtained by the method of Section 2.2.)

Numerical solution of $z' = \frac{2x-3}{(z-1)^2}$, $z(-1) = 4$, on $[-1, 0]$.

x	z
-1.0	4.000000000
-0.9	3.944536474
-0.8	3.889298649
-0.7	3.834355648
-0.6	3.779786399
-0.5	3.725680888
-0.4	3.672141529
-0.3	3.619284615
-0.2	3.567241862
-0.1	3.516161955
0.0	3.466212070

Numerical solution of $(y-1)^2 y' = 2x+3$, $y(1) = 4$, on $[0, 1]$.

x	y	Exact
0.00	3.466212070	3.466212074
0.10	3.516161955	3.516161958
0.20	3.567241862	3.567241864
0.30	3.619284615	3.619284617
0.40	3.672141529	3.672141530
0.50	3.725680888	3.725680889
0.60	3.779786399	3.779786399
0.70	3.834355648	3.834355648
0.80	3.889298649	3.889298649
0.90	3.944536474	3.944536474
1.00	4.000000000	4.000000000

We leave it to you to develop a procedure for handling the numerical solution of (3.42) on an interval $[a, b]$ such that $a < x_0 < b$ (Exercises 3.3.26 and 3.3.27).

3.3.2 Exercises

Most of the following numerical exercises involve initial value problems considered in the exercises in Sections 3.2. You'll find it instructive to compare the results that you obtain here with the corresponding results that you obtained in those sections.

In Exercises 3.3.1–3.3.5 use the Runge-Kutta method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point where the initial condition is imposed and $i = 1, 2$.

1. $\boxed{\text{C}}$ $y' = 2x^2 + 3y^2 - 2$, $y(2) = 1$; $h = 0.05$
2. $\boxed{\text{C}}$ $y' = y + \sqrt{x^2 + y^2}$, $y(0) = 1$; $h = 0.1$
3. $\boxed{\text{C}}$ $y' + 3y = x^2 - 3xy + y^2$, $y(0) = 2$; $h = 0.05$
4. $\boxed{\text{C}}$ $y' = \frac{1+x}{1-y^2}$, $y(2) = 3$; $h = 0.1$
5. $\boxed{\text{C}}$ $y' + x^2 y = \sin xy$, $y(1) = \pi$; $h = 0.2$

6. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2,$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.3.

7. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1$$

at $x = 1.0, 1.1, 1.2, 1.3, \dots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9 \ln x + x^3 + 2),$$

which can be obtained by the method of Section 2.1. Present your results in a table like Table 3.3.

8. C Use the Runge-Kutta method with step sizes $h = 0.05$, $h = 0.025$, and $h = 0.0125$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2$$

at $x = 1.0, 1.05, 1.10, 1.15, \dots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3},$$

which was obtained in Example 2.2.3. Present your results in a table like Table 3.3.

9. C In Example 2.2.3 it was shown that

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \tag{A}$$

Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 2.0, 2.1, 2.2, 2.3, \dots, 3.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = y^5 + y - x^2 - x + 4$$

for each value of (x, y) appearing in the first table.

10. C You can see from Example ?? that

$$x^4y^3 + x^2y^5 + 2xy = 4$$

is an implicit solution of the initial value problem

$$y' = -\frac{4x^3y^3 + 2xy^5 + 2y}{3x^4y^2 + 5x^2y^4 + 2x}, \quad y(1) = 1. \tag{A}$$

Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of (A) at $x = 1.0, 1.1, 1.2, 1.3, \dots, 2.0$. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x, y) = x^4 y^3 + x^2 y^5 + 2xy - 4$$

for each value of (x, y) appearing in the first table.

11.  Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$(3y^2 + 4y)y' + 2x + \cos x = 0, \quad y(0) = 1 \text{ (Exercise 2.2.13),}$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$.

12.  Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0 \text{ (Exercise 2.2.14),}$$

at $x = 1.0, 1.1, 1.2, 1.3, \dots, 2.0$.

13.  Use the Runge-Kutta method and the Runge-Kutta semilinear method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = e^{-3x}(1 - 4x + 3x^2 - 4x^3), \quad y(0) = -3$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. Compare these approximate values with the values of the exact solution $y = -e^{-3x}(3 - x + 2x^2 - x^3 + x^4)$, which can be obtained by the method of Section 2.1. Do you notice anything special about the results? Explain.

The linear initial value problems in Exercises 3.3.14–3.3.19 can't be solved exactly in terms of known elementary functions. In each exercise use the Runge-Kutta and the Runge-Kutta semilinear methods with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

14.  $y' - 2y = \frac{1}{1+x^2}$, $y(2) = 2$; $h = 0.1, 0.05, 0.025$ on $[2, 3]$
15.  $y' + 2xy = x^2$, $y(0) = 3$; $h = 0.2, 0.1, 0.05$ on $[0, 2]$ (Exercise 2.1.38)
16.  $y' + \frac{1}{x}y = \frac{\sin x}{x^2}$, $y(1) = 2$; $h = 0.2, 0.1, 0.05$ on $[1, 3]$ (Exercise 2.1.39)
17.  $y' + y = \frac{e^{-x} \tan x}{x}$, $y(1) = 0$; $h = 0.05, 0.025, 0.0125$ on $[1, 1.5]$ (Exercise 2.1.40)
18.  $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}$, $y(0) = 1$; $h = 0.2, 0.1, 0.05$ on $[0, 2]$ (Exercise 2.1.41)
19.  $xy' + (x+1)y = e^{x^2}$, $y(1) = 2$; $h = 0.05, 0.025, 0.0125$ on $[1, 1.5]$ (Exercise 2.1.42)

In Exercises 3.3.20–3.3.22 use the Runge-Kutta method and the Runge-Kutta semilinear method with the indicated step sizes to find approximate values of the solution of the given initial value problem at 11 equally spaced points (including the endpoints) in the interval.

20.  $y' + 3y = xy^2(y+1)$, $y(0) = 1$; $h = 0.1, 0.05, 0.025$ on $[0, 1]$
21.  $y' - 4y = \frac{x}{y^2(y+1)}$, $y(0) = 1$; $h = 0.1, 0.05, 0.025$ on $[0, 1]$

22. C $y' + 2y = \frac{x^2}{1 + y^2}, \quad y(2) = 1; \quad h = 0.1, 0.05, 0.025 \text{ on } [2, 3]$

23. C Suppose $a < x_0$, so that $-x_0 < -a$. Use the chain rule to show that if z is a solution of

$$z' = -f(-x, z), \quad z(-x_0) = y_0,$$

on $[-x_0, -a]$, then $y = z(-x)$ is a solution of

$$y' = f(x, y), \quad y(x_0) = y_0,$$

on $[a, x_0]$.

24. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(2) = -1$$

at $x = 1.1, 1.2, 1.3, \dots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(4 - 3x^2)}{4 + 3x^2},$$

which can be obtained by referring to Example 2.4.3.

25. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of

$$y' = -x^2y - xy^2, \quad y(1) = 1$$

at $x = 0, 0.1, 0.2, \dots, 1$.

26. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of

$$y' + \frac{1}{x}y = \frac{7}{x^2} + 3, \quad y(1) = \frac{3}{2}$$

at $x = 0.5, 0.6, \dots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{7 \ln x}{x} + \frac{3x}{2},$$

which can be obtained by the method discussed in Section 2.1.

27. C Use the Runge-Kutta method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of

$$xy' + 2y = 8x^2, \quad y(2) = 5$$

at $x = 1.0, 1.1, 1.2, \dots, 3.0$. Compare these approximate values with the values of the exact solution

$$y = 2x^2 - \frac{12}{x^2},$$

which can be obtained by the method discussed in Section 2.1.

28. NUMERICAL QUADRATURE (see Exercise 3.1.23).

(a) Derive the quadrature formula

$$\int_a^b f(x) dx \approx \frac{h}{6}(f(a) + f(b)) + \frac{h}{3} \sum_{i=1}^{n-1} f(a + ih) + \frac{2h}{3} \sum_{i=1}^n f(a + (2i - 1)h/2) \quad (\text{A})$$

(where $h = (b - a)/n$) by applying the Runge-Kutta method to the initial value problem

$$y' = f(x), \quad y(a) = 0.$$

This quadrature formula is called *Simpson's Rule*.

- (b) L For several choices of a, b, A, B, C , and D apply (A) to $f(x) = A + Bx + Cx + Dx^3$, with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.
- (c) L For several choices of a, b, A, B, C, D , and E apply (A) to $f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$, with $n = 10, 20, 40, 80, 160, 320$. Compare your results with the exact answers and explain what you find.

Chapter 4

More about First Order Equations

4.1 Integrals as Solutions

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y),$$

or just

$$y' = f(x, y).$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few lectures, we will look at cases where solutions are not difficult to obtain. In this section, we consider the case when f is a function of x alone, that is, the equation is

$$y' = f(x). \tag{4.1}$$

We can integrate (antidifferentiate) both sides of the equation with respect to x :

$$\int y'(x) dx = \int f(x) dx + C,$$

that is,

$$y(x) = \int f(x) dx + C.$$

This $y(x)$ is actually the general solution. So to solve (4.1), we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

Example 4.1.1: Find the general solution of $y' = 3x^2$.

Elementary calculus tells us that the general solution must be $y = x^3 + C$. Let us check by differentiating: $y' = 3x^2$. We got *precisely* our equation back.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called indefinite integral. The indefinite integral is really the *antiderivative* (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write $\int f(x) dx + C$ as

$$\int_{x_0}^x f(t) dt + C.$$

Hence the terminology *to integrate* when we may really mean *to antidifferentiate*. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should *always* think of the definite integral as a way to write it.

Normally, we also have an initial condition such as $y(x_0) = y_0$ for some two numbers x_0 and y_0 (x_0 is often 0, but not always). We can then write the solution as a definite integral in a nice way. Suppose our problem is $y' = f(x)$, $y(x_0) = y_0$. Then the solution is

$$y(x) = \int_{x_0}^x f(t) dt + y_0. \quad (4.2)$$

Let us check! We compute $y' = f(x)$ via the fundamental theorem of calculus, and by Jupiter, y is a solution. Is it the one satisfying the initial condition? Well, $y(x_0) = \int_{x_0}^{x_0} f(t) dt + y_0 = y_0$. It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, (4.2) is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

Example 4.1.2: Solve

$$y' = e^{-x^2}, \quad y(0) = 1.$$

By the preceding discussion, the solution must be

$$y(x) = \int_0^x e^{-s^2} ds + 1.$$

Here is a good way to make fun of your friends taking second semester calculus. Tell them to find the closed form solution. Ha ha ha (bad math joke). It is not possible (in closed form). There is absolutely nothing wrong with writing the solution as a definite integral. This particular integral is in fact very important in statistics.

Using this method, we can also solve equations of the form

$$y' = f(y).$$

We write the equation in Leibniz notation:

$$\frac{dy}{dx} = f(y).$$

Now we use the inverse function theorem from calculus to switch the roles of x and y to obtain

$$\frac{dx}{dy} = \frac{1}{f(y)}.$$

What we are doing seems like algebra with dx and dy . It is tempting to just do algebra with dx and dy as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point, we can simply integrate with respect to y ,

$$x(y) = \int \frac{1}{f(y)} dy + C.$$

Finally, we try to solve for y .

Example 4.1.3: Previously, we guessed $y' = ky$ (for some $k > 0$) has the solution $y = Ce^{kx}$. We could have found the solution by integrating. First we note that $y = 0$ is a solution. Henceforth, we assume $y \neq 0$. We write

$$\frac{dx}{dy} = \frac{1}{ky}.$$

We integrate in y to obtain

$$x(y) = x = \frac{1}{k} \ln|y| + D,$$

where D is an arbitrary constant. Now we solve for y (actually for $|y|$).

$$|y| = e^{kx-kD} = e^{-kD} e^{kx}.$$

If we replace e^{-kD} with an arbitrary constant C , we can get rid of the absolute value bars (which we can do as D was arbitrary). In this way, we also incorporate the solution $y = 0$. We get the same general solution as we guessed before, $y = Ce^{kx}$.

Example 4.1.4: Find the general solution of $y' = y^2$.

First we note that $y = 0$ is a solution. We can now assume that $y \neq 0$. Write

$$\frac{dx}{dy} = \frac{1}{y^2}.$$

We integrate to get

$$x = \frac{-1}{y} + C.$$

We solve for $y = \frac{1}{C-x}$. So the general solution is

$$y = \frac{1}{C-x} \quad \text{together with} \quad y = 0.$$

Note the singularities of the solution. If, for example, $C = 1$, then the solution “blows up” as we approach $x = 1$. See [Figure 4.1](#) on the following page. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation $y' = y^2$ is very nice and defined everywhere, but the solution is only defined on the interval $(-\infty, C)$ or (C, ∞) . Usually when this happens, we only consider the solution on one of these intervals and not both. For example, if we impose an initial condition $y(0) = 1$, then the solution is $y = \frac{1}{1-x}$, and we would consider this solution only for x on the interval $(-\infty, 1)$. In the figure, it is the left side of the graph.

Classical problems leading to differential equations solvable by integration are problems dealing with velocity, acceleration, and distance. You have surely seen these problems before in your calculus class.

Example 4.1.5: Suppose a car drives at a speed of $e^{t/2}$ meters per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

Let x denote the distance the car traveled. The equation is

$$x' = e^{t/2}.$$

We just integrate this equation to get that

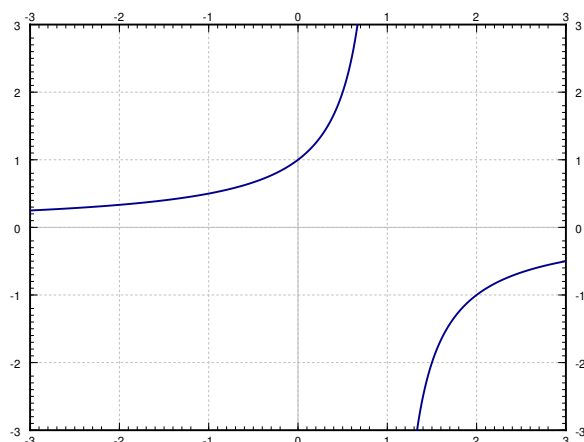
$$x(t) = 2e^{t/2} + C.$$

We still need to figure out C . We know that when $t = 0$, then $x = 0$. That is, $x(0) = 0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C.$$

Thus $C = -2$ and

$$x(t) = 2e^{t/2} - 2.$$

Figure 4.1: Plot of $y = \frac{1}{1-x}$.

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44 \text{ meters}, \quad x(10) = 2e^{10/2} - 2 \approx 294 \text{ meters}.$$

Example 4.1.6: Suppose that the car accelerates at $t^2 \text{ m/s}^2$. At time $t = 0$ the car is at the 1 meter mark and is traveling at 10 m/s . Where is the car at time $t = 10$?

Well this is actually a second order problem. If x is the distance traveled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is

$$x'' = t^2, \quad x(0) = 1, \quad x'(0) = 10.$$

What if we say $x' = v$. Then we have the problem

$$v' = t^2, \quad v(0) = 10.$$

Once we solve for v , we can integrate and find x .

Exercise 4.1.1: Solve for v , and then solve for x . Find $x(10)$ to answer the question.

4.1.1 Exercises

Exercise 4.1.2: Solve $\frac{dy}{dx} = x^2 + x$ for $y(1) = 3$.

Exercise 4.1.3: Solve $\frac{dy}{dx} = \sin(5x)$ for $y(0) = 2$.

Exercise 4.1.4: Solve $\frac{dy}{dx} = \frac{1}{x^2-1}$ for $y(0) = 0$.

Exercise 4.1.5: Solve $y' = y^3$ for $y(0) = 1$.

Exercise 4.1.6 (little harder): Solve $y' = (y-1)(y+1)$ for $y(0) = 3$.

Exercise 4.1.7: Solve $\frac{dy}{dx} = \frac{1}{y+1}$ for $y(0) = 0$.

Exercise 4.1.8 (harder): Solve $y'' = \sin x$ for $y(0) = 0$, $y'(0) = 2$.

Exercise 4.1.9: A spaceship is traveling at the speed $2t^2 + 1 \text{ km/s}$ (t is time in seconds). It is pointing directly away from earth and at time $t = 0$ it is 1000 kilometers from earth. How far from earth is it at one minute from time $t = 0$?

Exercise 4.1.10: Solve $\frac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. It is OK to leave your answer as a definite integral.

Exercise 4.1.11: A dropped ball accelerates downwards at a constant rate 9.8 meters per second squared. Set up the differential equation for the height above ground h in meters. Then supposing $h(0) = 100$ meters, how long does it take for the ball to hit the ground.

Exercise 4.1.12: Find the general solution of $y' = e^x$, and then $y' = e^y$.

Exercise 4.1.101: Solve $\frac{dy}{dx} = e^x + x$ and $y(0) = 10$.

Exercise 4.1.102: Solve $x' = \frac{1}{x^2}$, $x(1) = 1$.

Exercise 4.1.103: Solve $x' = \frac{1}{\cos(x)}$, $x(0) = \frac{\pi}{4}$.

Exercise 4.1.104: Sid is in a car traveling at speed $10t + 70$ miles per hour away from Las Vegas, where t is in hours. At $t = 0$, Sid is 10 miles away from Vegas. How far from Vegas is Sid 2 hours later?

Exercise 4.1.105: Solve $y' = y^n$, $y(0) = 1$, where n is a positive integer. Hint: You have to consider different cases.

Exercise 4.1.106: The rate of change of the volume of a snowball that is melting is proportional to the surface area of the snowball. Suppose the snowball is perfectly spherical. The volume (in centimeters cubed) of a ball of radius r centimeters is $(4/3)\pi r^3$. The surface area is $4\pi r^2$. Set up the differential equation for how the radius r is changing. Then, suppose that at time $t = 0$ minutes, the radius is 10 centimeters. After 5 minutes, the radius is 8 centimeters. At what time t will the snowball be completely melted?

Exercise 4.1.107: Find the general solution to $y'''' = 0$. How many distinct constants do you need?

4.2 Slope Fields

As we said, the general first order equation we are studying looks like

$$y' = f(x, y).$$

A lot of the time, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behavior of the solutions, or find approximate solutions.

4.2.1 Slope Fields

The equation $y' = f(x, y)$ gives you a slope at each point in the (x, y) -plane. And this is the slope a solution $y(x)$ would have at x if its value was y . In other words, $f(x, y)$ is the slope of a solution whose graph runs through the point (x, y) . At a point (x, y) , we draw a short line with the slope $f(x, y)$. For example, if $f(x, y) = xy$, then at point $(2, 1.5)$ we draw a short line of slope $xy = 2 \times 1.5 = 3$. If $y(x)$ is a solution and $y(2) = 1.5$, then the equation mandates that $y'(2) = 3$. See [Figure 4.2](#).

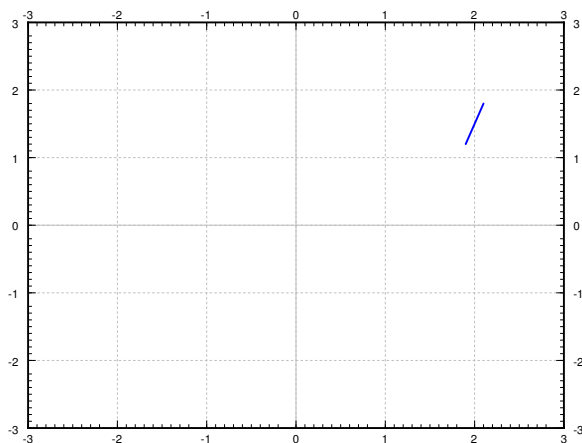


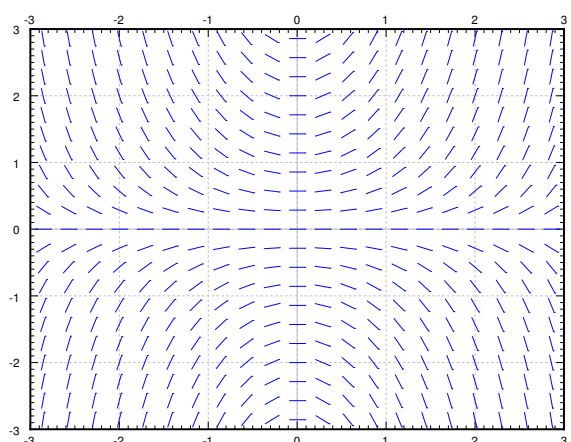
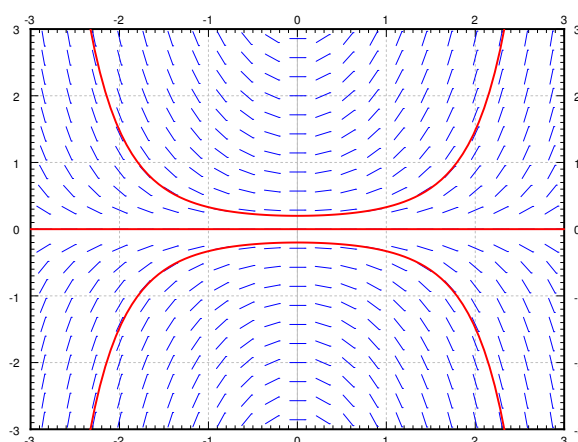
Figure 4.2: The slope $y' = xy$ at $(2, 1.5)$.

To get an idea of how solutions behave, we draw such lines at lots of points in the plane, not just the point $(2, 1.5)$. We would ideally want to see the slope at every point, but that is just not possible. Usually we pick a grid of points fine enough so that it shows the behavior, but not too fine so that we can still recognize the individual lines. We call this picture the *slope field* of the equation $y' = xy$. See [Figure 4.3](#) on the facing page for the slope field of the equation $y' = xy$. In practice, one does not do this by hand, a computer can do the drawing.

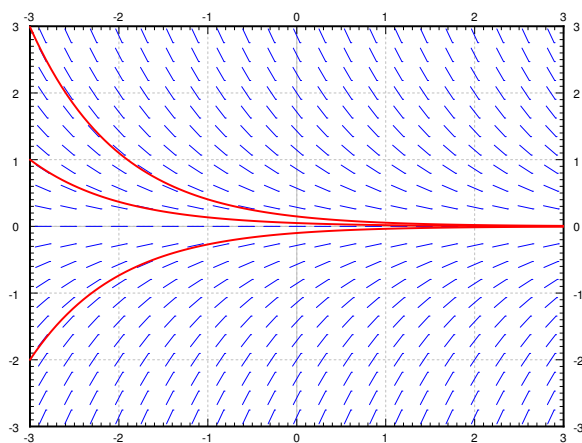
Suppose we are given a specific initial condition $y(x_0) = y_0$. A solution, that is, the graph of the solution, would be a curve that follows the slopes we drew. For a few sample solutions, see [Figure 4.4](#) on the next page. It is easy to roughly sketch (or at least imagine) possible solutions in the slope field, just from looking at the slope field itself. You simply sketch a line that roughly fits the little line segments and goes through your initial condition.

By looking at the slope field we get a lot of information about the behavior of solutions without having to solve the equation. For example, in [Figure 4.4](#) on the facing page we see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. A small change in the initial condition causes quite different behavior. We see this behavior just from the slope field and imagining what solutions ought to do.

We see a different behavior for the equation $y' = -y$. The slope field and a few solutions is in see [Figure 4.5](#) on the next page. If we think of moving from left to right (perhaps x is time and time is usually increasing),

Figure 4.3: Slope field of $y' = xy$.Figure 4.4: Slope field of $y' = xy$ with a graph of solutions satisfying $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

then we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. Again that behavior is clear from simply looking at the slope field itself.

Figure 4.5: Slope field of $y' = -y$ with a graph of a few solutions.

4.2.2 Existence and Uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

- (i) Does a solution *exist*?
- (ii) Is the solution *unique* (if it exists)?

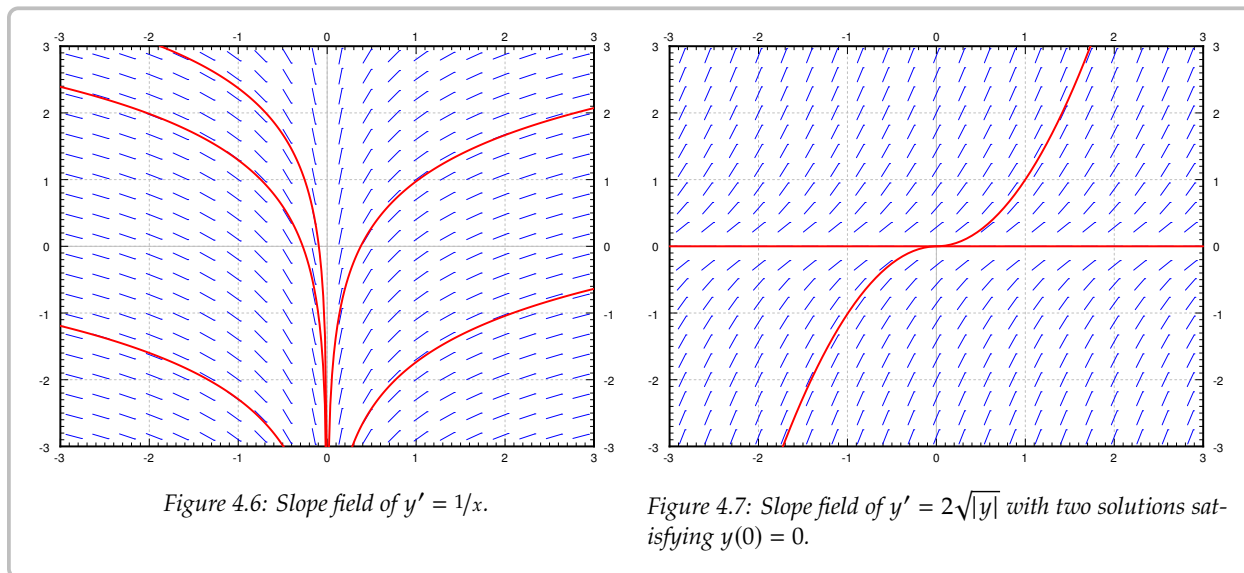
What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

Example 4.2.1: Attempt to solve:

$$y' = \frac{1}{x}, \quad y(0) = 0.$$

Integrate to find the general solution $y = \ln|x| + C$. The solution does not exist at $x = 0$. See Figure 4.6. Moreover, the equation may have been written as the seemingly harmless $xy' = 1$.



Example 4.2.2: Solve:

$$y' = 2\sqrt{|y|}, \quad y(0) = 0.$$

See Figure 4.7. Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as Picard's theorem*.

Theorem 4.2.1 (Picard's theorem on existence and uniqueness). *If $f(x, y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0) , then a solution to*

$$y' = f(x, y), \quad y(x_0) = y_0,$$

exists (at least for x in some small interval) and is unique.

Note that the problems $y' = 1/x$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

*Named after the French mathematician [Charles Émile Picard](#) (1856–1941)

Example 4.2.3: For some constant A , solve:

$$y' = y^2, \quad y(0) = A.$$

We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = 1/y^2$, so $x = -1/y + C$, so $y = \frac{1}{C-x}$. If $y(0) = A$, then $C = 1/A$ so

$$y = \frac{1}{1/A - x}.$$

If $A = 0$, then $y = 0$ is a solution.

For example, when $A = 1$ the solution “blows up” at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds “globally” unlike for the equation $y' = y^2$.

4.2.3 Exercises

Exercise 4.2.1: Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?

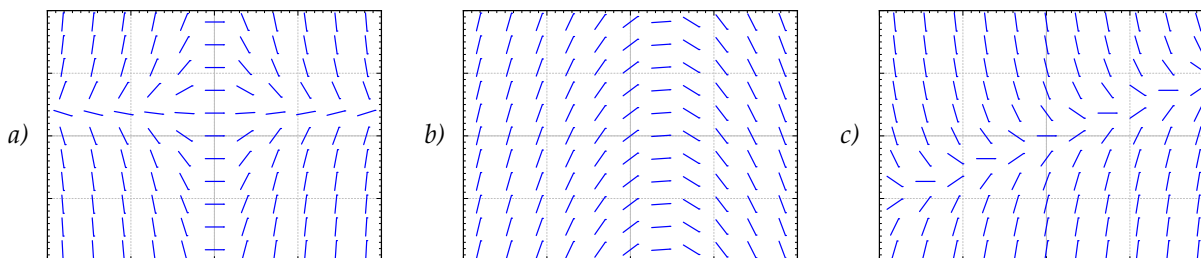
Exercise 4.2.2: Sketch slope field for $y' = x^2$.

Exercise 4.2.3: Sketch slope field for $y' = y^2$.

Exercise 4.2.4: Is it possible to solve the equation $y' = \frac{xy}{\cos x}$ for $y(0) = 1$? Justify.

Exercise 4.2.5: Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.

Exercise 4.2.6: Match equations $y' = 1 - x$, $y' = x - 2y$, $y' = x(1 - y)$ to slope fields. Justify.



Exercise 4.2.7 (challenging): Take $y' = f(x, y)$, $y(0) = 0$, where $f(x, y) > 1$ for all x and y . If the solution exists for all x , can you say what happens to $y(x)$ as x goes to positive infinity? Explain.

Exercise 4.2.8 (challenging): Take $(y - x)y' = 0$, $y(0) = 0$.

- Find two distinct solutions.
- Explain why this does not violate Picard's theorem.

Exercise 4.2.9: Suppose $y' = f(x, y)$. What will the slope field look like, explain and sketch an example, if you know the following about $f(x, y)$:

- f does not depend on y .
- f does not depend on x .
- $f(t, t) = 0$ for any number t .
- $f(x, 0) = 0$ and $f(x, 1) = 1$ for all x .

Exercise 4.2.10: Find a solution to $y' = |y|$, $y(0) = 0$. Does Picard's theorem apply?

Exercise 4.2.11: Take an equation $y' = (y - 2x)g(x, y) + 2$ for some function $g(x, y)$. Can you solve the problem for the initial condition $y(0) = 0$, and if so what is the solution?

Exercise 4.2.12 (challenging): Suppose $y' = f(x, y)$ is such that $f(x, 1) = 0$ for every x , f is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous for every x and y .

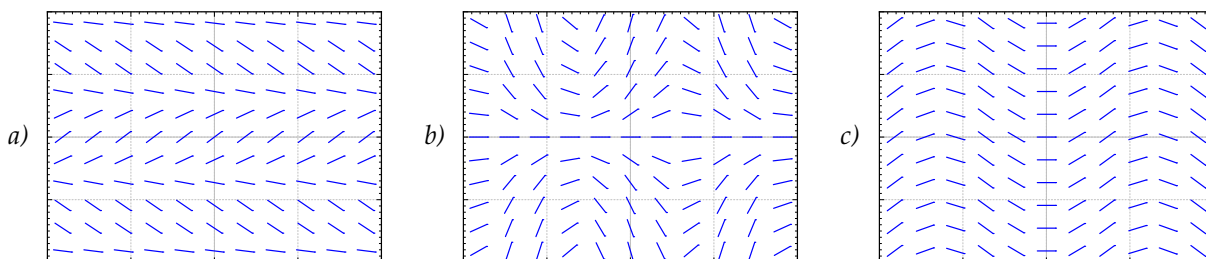
- Guess a solution given the initial condition $y(0) = 1$.
- Can graphs of two solutions of the equation for different initial conditions ever intersect?
- Given $y(0) = 0$, what can you say about the solution. In particular, can $y(x) > 1$ for any x ? Can $y(x) = 1$ for any x ? Why or why not?

Exercise 4.2.101: Sketch the slope field of $y' = y^3$. Can you visually find the solution that satisfies $y(0) = 0$?

Exercise 4.2.102: Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique?

Exercise 4.2.103: Is it possible to solve $y' = \frac{x}{x^2-1}$ for $y(1) = 0$?

Exercise 4.2.104: Match equations $y' = \sin x$, $y' = \cos y$, $y' = y \cos(x)$ to slope fields. Justify.



Exercise 4.2.105 (tricky): Suppose

$$f(y) = \begin{cases} 0 & \text{if } y > 0, \\ 1 & \text{if } y \leq 0. \end{cases}$$

Does $y' = f(y)$, $y(0) = 0$ have a continuously differentiable solution? Does Picard apply? Why, or why not?

Exercise 4.2.106: Consider an equation of the form $y' = f(x)$ for some continuous function f , and an initial condition $y(x_0) = y_0$. Does a solution exist for all x ? Why or why not?

4.3 Separable Equations

When a differential equation is of the form $y' = f(x)$, we integrate: $y = \int f(x) dx + C$. Unfortunately, simply integrating no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields the rather unhelpful expression

$$y = \int f(x, y) dx + C.$$

Notice the dependence on y in the integral.

4.3.1 Separable Equations

We say a differential equation is *separable* if we can write it as

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y).$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x) dx.$$

Both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x) dx + C.$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y .

Example 4.3.1: Take the equation

$$y' = xy.$$

Note that $y = 0$ is a solution. We will remember that fact and assume $y \neq 0$ from now on, so that we can divide by y . Write the equation as $\frac{dy}{dx} = xy$ or $\frac{dy}{y} = x dx$. Then

$$\int \frac{dy}{y} = \int x dx + C.$$

We compute the antiderivatives to get

$$\ln|y| = \frac{x^2}{2} + C,$$

or

$$|y| = e^{\frac{x^2}{2} + C} = e^{\frac{x^2}{2}} e^C = D e^{\frac{x^2}{2}},$$

where $D > 0$ is some constant. Because $y = 0$ is also a solution and because of the absolute value, we can write:

$$y = D e^{\frac{x^2}{2}},$$

for any number D (including zero or negative).

We check:

$$y' = D x e^{\frac{x^2}{2}} = x \left(D e^{\frac{x^2}{2}} \right) = xy.$$

Yay!

You may be worried that we integrated in two different variables. We seemingly did a different operation to each side. Perhaps we should be a little bit more careful and work through this method more rigorously. Consider

$$\frac{dy}{dx} = f(x)g(y).$$

We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x).$$

We integrate both sides with respect to x :

$$\int \frac{1}{g(y)} \frac{dy}{dx} dx = \int f(x) dx + C.$$

We use the change of variables formula (substitution) on the left hand side:

$$\int \frac{1}{g(y)} dy = \int f(x) dx + C.$$

And we are done.

4.3.2 Implicit Solutions

We sometimes get stuck even if we can do the integration. Consider the separable equation

$$y' = \frac{xy}{y^2 + 1}.$$

We separate variables,

$$\frac{y^2 + 1}{y} dy = \left(y + \frac{1}{y} \right) dy = x dx.$$

We integrate to get

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C,$$

or perhaps the less intimidating expression (where $D = 2C$)

$$y^2 + 2 \ln|y| = x^2 + D.$$

It is not easy to find the solution explicitly—it is hard to solve for y . We, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x , and remember that y is a function of x , to get

$$y' \left(2y + \frac{2}{y} \right) = 2x.$$

Multiply both sides by y and divide by $2(y^2 + 1)$ and you will get exactly the differential equation. We leave this computation to the reader.

If you have an implicit solution, and you want to compute values for y , you might have to be tricky. You might get multiple solutions y for each x , so you have to pick one. Sometimes you can graph x as a function of y , and then turn your paper to see a graph. Sometimes you have to do more.

Computers are also good at some of these tricks. More advanced mathematical software usually has some way of plotting solutions to implicit equations. For example, for $D = 0$, if you plot all the points (x, y) that are solutions to $y^2 + 2 \ln|y| = x^2$, you find the two curves in [Figure 4.8](#) on the facing page. This is not quite

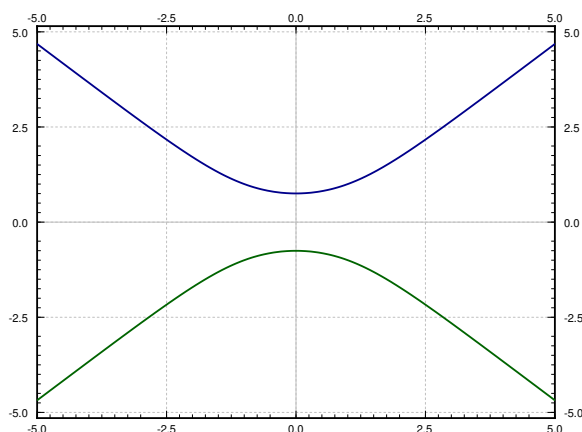


Figure 4.8: The implicit solution $y^2 + 2 \ln |y| = x^2$ to $y' = \frac{xy}{y^2+1}$.

a graph of a function. For each x there are two choices of y . To find a function, you have to pick one of these two curves. You pick the one that satisfies your initial condition if you have one. For instance, the top curve satisfies the condition $y(1) = 1$. So for each D , we really got two solutions. As you can see, computing values from an implicit solution can be somewhat tricky. But sometimes, an implicit solution is the best we can do.

The equation above also has the solution $y = 0$. So the general solution is

$$y^2 + 2 \ln |y| = x^2 + D, \quad \text{and} \quad y = 0.$$

Sometimes these extra solutions that came up due to division by zero such as $y = 0$ are called *singular solutions*.

4.3.3 Examples of Separable Equations

Example 4.3.2: Solve $x^2 y' = 1 - x^2 + y^2 - x^2 y^2$, $y(1) = 0$.

Factor the right-hand side

$$x^2 y' = (1 - x^2)(1 + y^2).$$

Separate variables, integrate, and solve for y :

$$\begin{aligned} \frac{y'}{1 + y^2} &= \frac{1 - x^2}{x^2}, \\ \frac{y'}{1 + y^2} &= \frac{1}{x^2} - 1, \\ \arctan(y) &= \frac{-1}{x} - x + C, \\ y &= \tan\left(\frac{-1}{x} - x + C\right). \end{aligned}$$

Solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $C = 2 + \pi$, or $C = 2 + 2\pi$, etc.). The particular solution we seek is, therefore,

$$y = \tan\left(\frac{-1}{x} - x + 2\right).$$

Example 4.3.3: Bob made a cup of coffee, and Bob likes to drink coffee only once reaches 60 degrees Celsius and will not burn him. Initially at time $t = 0$ minutes, Bob measured the temperature and the coffee was 89

degrees Celsius. One minute later, Bob measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Bob start drinking?

Let T be the temperature of the coffee in degrees Celsius, and let A be the ambient (room) temperature, also in degrees Celsius. Newton's law of cooling states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

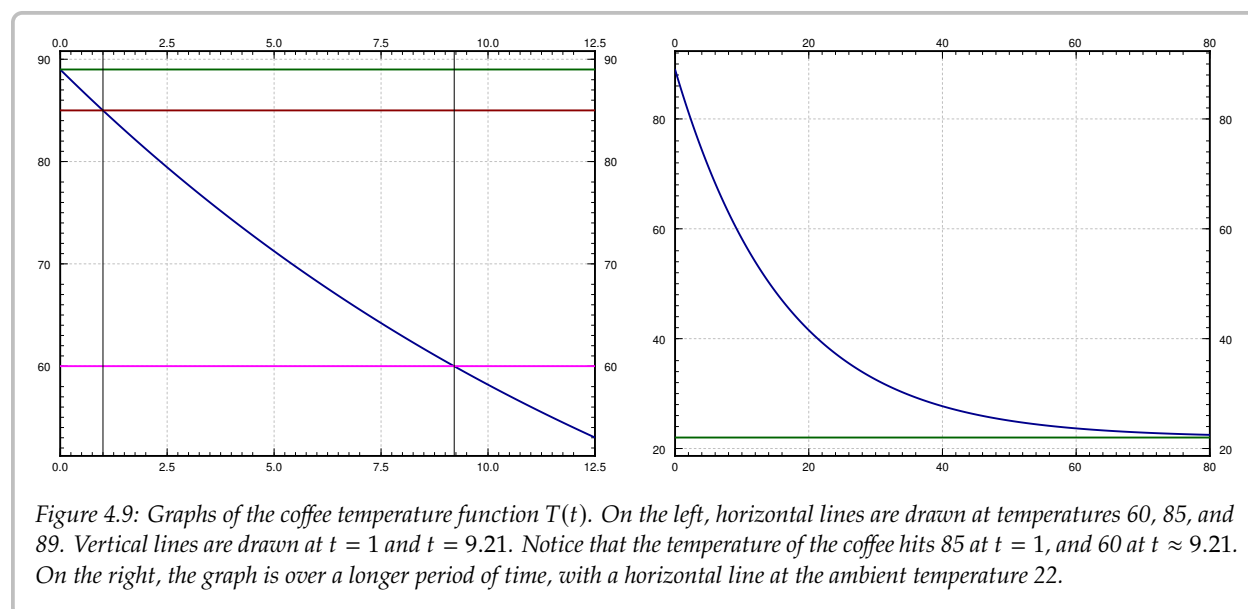
for some positive constant k . For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants):

$$\begin{aligned} \frac{1}{T - A} \frac{dT}{dt} &= -k, \\ \ln(T - A) &= -kt + C, \quad (\text{note that } T - A > 0) \\ T - A &= D e^{-kt}, \\ T &= A + D e^{-kt}. \end{aligned}$$

That is, $T = 22 + D e^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67 e^{-kt}$. The second condition says $85 = T(1) = 22 + 67 e^{-k}$. Solving for k , we get $k = -\ln \frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. Namely, we solve

$$60 = 22 + 67e^{-0.0616t}$$

to get $t = -\frac{\ln \frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Bob can begin to drink the coffee at just over 9 minutes from the time Bob made it. That is probably about the amount of time it took us to calculate how long it would take. See Figure 4.9.



Example 4.3.4: Find the general solution to $y' = \frac{-xy^2}{3}$ (including any singular solutions). First note that $y = 0$ is a solution (a singular solution). Now assume that $y \neq 0$.

$$\frac{-3}{y^2} y' = x,$$

$$\frac{3}{y} = \frac{x^2}{2} + C,$$

$$y = \frac{3}{x^2/2 + C} = \frac{6}{x^2 + 2C}.$$

So the general solution is

$$y = \frac{6}{x^2 + 2C} \quad \text{and} \quad y = 0.$$

4.3.4 Exercises

Exercise 4.3.1: Solve $y' = x/y$.

Exercise 4.3.2: Solve $y' = x^2 y$.

Exercise 4.3.3: Solve $\frac{dx}{dt} = (x^2 - 1)t$, for $x(0) = 0$.

Exercise 4.3.4: Solve $\frac{dx}{dt} = x \sin(t)$, for $x(0) = 1$.

Exercise 4.3.5: Solve $\frac{dy}{dx} = xy + x + y + 1$. Hint: Factor the right-hand side.

Exercise 4.3.6: Solve $xy' = y + 2x^2 y$, where $y(1) = 1$.

Exercise 4.3.7: Solve $\frac{dy}{dx} = \frac{y^2 + 1}{x^2 + 1}$, for $y(0) = 1$.

Exercise 4.3.8: Find an implicit solution for $\frac{dy}{dx} = \frac{x^2 + 1}{y^2 + 1}$, for $y(0) = 1$.

Exercise 4.3.9: Find an explicit solution for $y' = xe^{-y}$, $y(0) = 1$.

Exercise 4.3.10: Find an explicit solution for $xy' = e^{-y}$, for $y(1) = 1$.

Exercise 4.3.11: Find an explicit solution for $y' = ye^{-x^2}$, $y(0) = 1$. It is alright to leave a definite integral in your answer.

Exercise 4.3.12: Suppose a cup of coffee is at 100 degrees Celsius at time $t = 0$, it is at 70 degrees at $t = 10$ minutes, and it is at 50 degrees at $t = 20$ minutes. Compute the ambient temperature.

Exercise 4.3.101: Solve $y' = 2xy$.

Exercise 4.3.102: Solve $x' = 3xt^2 - 3t^2$, $x(0) = 2$.

Exercise 4.3.103: Find an implicit solution for $x' = \frac{1}{3x^2 + 1}$, $x(0) = 1$.

Exercise 4.3.104: Find an explicit solution to $xy' = y^2$, $y(1) = 1$.

Exercise 4.3.105: Find an implicit solution to $y' = \frac{\sin(x)}{\cos(y)}$.

Exercise 4.3.106: Take [Example 4.3.3](#) with the same numbers: 89 degrees at $t = 0$, 85 degrees at $t = 1$, and ambient temperature of 22 degrees. Suppose these temperatures were measured with precision of ± 0.5 degrees. Given this imprecision, the time it takes the coffee to cool to (exactly) 60 degrees is also only known in a certain range. Find this range. Hint: Think about what kind of error makes the cooling time longer and what shorter.

Exercise 4.3.107: A population x of rabbits on an island is modeled by $x' = x - (1/1000)x^2$, where the independent variable is time in months. At time $t = 0$, there are 40 rabbits on the island.

a) Find the solution to the equation with the initial condition.

b) How many rabbits are on the island in 1 month, 5 months, 10 months, 15 months (round to the nearest integer).

4.4 Linear Equations and the Integrating Factor

One of the most important types of equations we will learn to solve are the so-called *linear equations*. In fact, the majority of the course is about linear equations. In this section we focus on the *first order linear equation*. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \quad (4.3)$$

The word “linear” means linear in y and y' ; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has essentially the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations.

The trick is to rewrite the left-hand side of (4.3) as a derivative of a product of y with another function. To this end, we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx} [r(x)y].$$

This is the left-hand side of (4.3) multiplied by $r(x)$. If we multiply (4.3) by $r(x)$, we obtain

$$\frac{d}{dx} [r(x)y] = r(x)f(x).$$

We can now integrate both sides, which we can do as the right-hand side does not depend on y and the left-hand side is written as a derivative of a function. After the integration, we solve for y by dividing by $r(x)$. The function $r(x)$ is called the *integrating factor* and the method is called the *integrating factor method*.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x) dx}.$$

We compute:

$$\begin{aligned} y' + p(x)y &= f(x), \\ e^{\int p(x) dx} y' + e^{\int p(x) dx} p(x)y &= e^{\int p(x) dx} f(x), \\ \frac{d}{dx} [e^{\int p(x) dx} y] &= e^{\int p(x) dx} f(x), \\ e^{\int p(x) dx} y &= \int e^{\int p(x) dx} f(x) dx + C, \\ y &= e^{-\int p(x) dx} \left(\int e^{\int p(x) dx} f(x) dx + C \right). \end{aligned}$$

Of course, to get a closed form formula for y , we need to be able to find a closed form formula for the integrals appearing above.

Example 4.4.1: Solve

$$y' + 2xy = e^{x-x^2}, \quad y(0) = -1.$$

First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x) dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$\begin{aligned} e^{x^2} y' + 2xe^{x^2} y &= e^{x-x^2} e^{x^2}, \\ \frac{d}{dx} [e^{x^2} y] &= e^x. \end{aligned}$$

We integrate

$$\begin{aligned} e^{x^2} y &= e^x + C, \\ y &= e^{x-x^2} + C e^{-x^2}. \end{aligned}$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

Exercise 4.4.1: Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above.

Advice: Do not try to remember the formula for y itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \quad y(x_0) = y_0.$$

Look at the solution and write the integrals as definite integrals.

$$y(x) = e^{-\int_{x_0}^x p(s) ds} \left(\int_{x_0}^x e^{\int_{x_0}^t p(s) ds} f(t) dt + y_0 \right). \quad (4.4)$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

Exercise 4.4.2: Check that $y(x_0) = y_0$ in formula (4.4).

Exercise 4.4.3: Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.

$$y' + y = e^{x^2-x}, \quad y(0) = 10.$$

Remark 4.4.1: Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is an explicit formula (4.4) for the solution. Second, it follows from the formula (4.4) that if $p(x)$ and $f(x)$ are continuous on some interval (a, b) , then the solution $y(x)$ exists and is differentiable on (a, b) . Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to Theorem 4.2.1.

Example 4.4.2: Let us discuss a common simple application of linear equations. Real life applications of this type of problem include figuring out the concentration of chemicals in bodies of water (rivers and lakes).

A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per liter is flowing in at the rate of 5 liters a minute. The solution in the tank is well stirred and flows out at a rate of 3 liters a minute. How much salt is in the tank when the tank is full?

Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in} \times \text{concentration in})\Delta t - (\text{rate out} \times \text{concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \rightarrow 0$, we see that

$$\frac{dx}{dt} = (\text{rate in} \times \text{concentration in}) - (\text{rate out} \times \text{concentration out}).$$

In our example,

$$\begin{aligned}\text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5 - 3)t}.\end{aligned}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3 \frac{x}{60 + 2t}\right).$$

Or in the form (4.3),

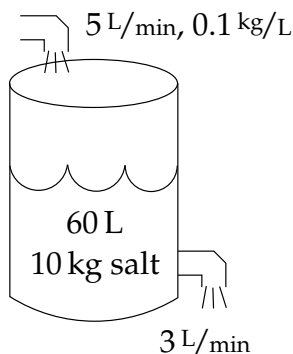
$$\frac{dx}{dt} + \frac{3}{60 + 2t}x = 0.5.$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60 + 2t} dt\right) = \exp\left(\frac{3}{2} \ln(60 + 2t)\right) = (60 + 2t)^{3/2}.$$

We multiply both sides of the equation to get

$$\begin{aligned}(60 + 2t)^{3/2} \frac{dx}{dt} + (60 + 2t)^{3/2} \frac{3}{60 + 2t} x &= 0.5(60 + 2t)^{3/2}, \\ \frac{d}{dt} \left[(60 + 2t)^{3/2} x \right] &= 0.5(60 + 2t)^{3/2}, \\ (60 + 2t)^{3/2} x &= \int 0.5(60 + 2t)^{3/2} dt + C, \\ x &= (60 + 2t)^{-3/2} \int \frac{(60 + 2t)^{3/2}}{2} dt + C(60 + 2t)^{-3/2}, \\ x &= (60 + 2t)^{-3/2} \frac{1}{10} (60 + 2t)^{5/2} + C(60 + 2t)^{-3/2}, \\ x &= \frac{60 + 2t}{10} + C(60 + 2t)^{-3/2}.\end{aligned}$$



To find C , note that at $t = 0$, we have $x = 10$. That is,

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2},$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We know 5 liters per minute are flowing in and 3 liters per minute are flowing out, so the volume is increasing by 2 liters a minute. So the tank is full when $60 + 2t = 100$, or when $t = 20$. We are interested in

the value of x when the tank is full, that is we want to compute $x(20)$:

$$\begin{aligned} x(20) &= \frac{60 + 40}{10} + C(60 + 40)^{-3/2} \\ &\approx 10 + 1859.03(100)^{-3/2} \approx 11.86. \end{aligned}$$

There are 11.86 kg of salt in the tank when it is full. See [Figure 4.10](#) on the following page for the graph of x over t .

The concentration when the tank is full is approximately $11.86/100 = 0.1186$ kg/liter, and we started with $1/6$ or approximately 0.1667 kg/liter.

4.4.1 Exercises

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

Exercise 4.4.4: Solve $y' + xy = x$.

Exercise 4.4.5: Solve $y' + 6y = e^x$.

Exercise 4.4.6: Solve $y' + 3x^2y = \sin(x)e^{-x^3}$, with $y(0) = 1$.

Exercise 4.4.7: Solve $y' + \cos(x)y = \cos(x)$.

Exercise 4.4.8: Solve $\frac{1}{x^2+1}y' + xy = 3$, with $y(0) = 0$.

Exercise 4.4.9: Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 liters per hour. The first lake contains 100 thousand liters of water and the second lake contains 200 thousand liters of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream.

- Find the concentration of toxic substance as a function of time in both lakes.
- When will the concentration in the first lake be below 0.001 kg per liter?
- When will the concentration in the second lake be maximal?

Exercise 4.4.10: Newton's law of cooling states that $\frac{dx}{dt} = -k(x - A)$ where x is the temperature, t is time, A is the ambient temperature, and $k > 0$ is a constant. Suppose that $A = A_0 \cos(\omega t)$ for some constants A_0 and ω . That is, the ambient temperature oscillates (for example night and day temperatures).

- Find the general solution.
- In the long term, will the initial conditions make much of a difference? Why or why not?

Exercise 4.4.11: Initially 5 grams of salt are dissolved in 20 liters of water. Brine with concentration of salt 2 grams of salt per liter is added at a rate of 3 liters a minute. The tank is mixed well and is drained at 3 liters a minute. How long does the process have to continue until there are 20 grams of salt in the tank?

Exercise 4.4.12: Initially a tank contains 10 liters of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 liter per minute. The water is mixed well and drained at 1 liter per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?

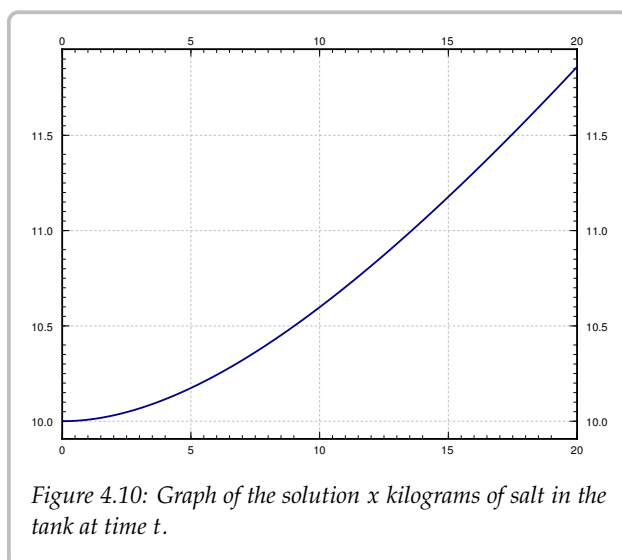
Exercise 4.4.101: Solve $y' + 3x^2y = x^2$.

Exercise 4.4.102: Solve $y' + 2 \sin(2x)y = 2 \sin(2x)$, $y(\pi/2) = 3$.

Exercise 4.4.103: Suppose a water tank is being pumped out at 3 L/min . The water tank starts at 10 L of clean water. Water with toxic substance is flowing into the tank at 2 L/min , with concentration $20t \text{ g/L}$ at time t . When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?

Exercise 4.4.104: There is bacteria on a plate and a toxic substance is being added that slows down the rate of growth of the bacteria. That is, suppose that $\frac{dP}{dt} = (2 - 0.1t)P$. If $P(0) = 1000$, find the population at $t = 5$.

Exercise 4.4.105: A cylindrical water tank has water flowing in at 1 cubic meters per second. Let A be the area of the cross section of the tank in square meters. Suppose water is flowing out from the bottom of the tank at a rate proportional to the height of the water level. Set up the differential equation for h , the height of the water, introducing and naming constants that you need. You should also give the units for your constants.



4.5 Substitution

Just as when solving integrals, one method to try is to change variables to end up with a simpler equation to solve.

4.5.1 Substitution

The equation

$$y' = (x - y + 1)^2$$

is neither separable nor linear. What can we do? How about trying to change variables, so that in the new variables the equation is simpler. We use another variable v , which we treat as a function of x . We try

$$v = x - y + 1.$$

We need to figure out y' in terms of v' , v and x . We differentiate (in x) to obtain $v' = 1 - y'$. So $y' = 1 - v'$. We plug this into the equation to get

$$1 - v' = v^2.$$

In other words, $v' = 1 - v^2$. Such an equation we know how to solve by separating variables:

$$\frac{1}{1 - v^2} dv = dx.$$

So

$$\frac{1}{2} \ln \left| \frac{v+1}{v-1} \right| = x + C, \quad \text{or} \quad \left| \frac{v+1}{v-1} \right| = e^{2x+2C}, \quad \text{or} \quad \frac{v+1}{v-1} = De^{2x},$$

for some constant D . Note that $v = 1$ and $v = -1$ are also solutions.

Now we need to “unsubstitute” to obtain

$$\frac{x - y + 2}{x - y} = De^{2x},$$

and also the two solutions $x - y + 1 = 1$ or $y = x$, and $x - y + 1 = -1$ or $y = x + 2$. We solve the first equation for y :

$$\begin{aligned} x - y + 2 &= (x - y)De^{2x}, \\ x - y + 2 &= Dxe^{2x} - yDe^{2x}, \\ -y + yDe^{2x} &= Dxe^{2x} - x - 2, \\ y(-1 + De^{2x}) &= Dxe^{2x} - x - 2, \\ y &= \frac{Dxe^{2x} - x - 2}{De^{2x} - 1}. \end{aligned}$$

Note that $D = 0$ gives $y = x + 2$, but no value of D gives the solution $y = x$.

Substitution in differential equations is applied in much the same way that it is applied in calculus. You guess. Several different substitutions might work. There are some general patterns to look for. We summarize a few of these in a table.

When you see	Try substituting
yy'	$v = y^2$
y^2y'	$v = y^3$
$(\cos y)y'$	$v = \sin y$
$(\sin y)y'$	$v = \cos y$
$e^y y'$	$v = e^y$

Usually you try to substitute in the “most complicated” part of the equation with the hopes of simplifying it. The table above is just a rule of thumb. You might have to modify your guesses. If a substitution does not work (it does not make the equation any simpler), try a different one.

4.5.2 Bernoulli Equations

There are some forms of equations where there is a general rule for substitution that always works. One such example is the so-called *Bernoulli equation**:

$$y' + p(x)y = q(x)y^n.$$

This equation looks a lot like a linear equation except for the y^n . If $n = 0$ or $n = 1$, then the equation is linear and we can solve it. Otherwise, the substitution $v = y^{1-n}$ transforms the Bernoulli equation into a linear equation. Note that n need not be an integer.

Example 4.5.1: Solve

$$xy' + y(x+1) + xy^5 = 0, \quad y(1) = 1.$$

The equation is a Bernoulli equation, $p(x) = (x+1)/x$ and $q(x) = -1$. We substitute

$$v = y^{1-5} = y^{-4}, \quad v' = -4y^{-5}y'.$$

In other words, $(-1/4)y^5v' = y'$. So

$$\begin{aligned} xy' + y(x+1) + xy^5 &= 0, \\ \frac{-xy^5}{4}v' + y(x+1) + xy^5 &= 0, \\ \frac{-x}{4}v' + y^{-4}(x+1) + x &= 0, \\ \frac{-x}{4}v' + v(x+1) + x &= 0, \end{aligned}$$

and finally

$$v' - \frac{4(x+1)}{x}v = 4.$$

The equation is now linear. We can use the integrating factor method. In particular, we use formula (4.4). We assume that $x > 0$ so $|x| = x$. This assumption is OK, as our initial condition is at $x = 1 > 0$. Let us compute the integrating factor. Here $p(s)$ from formula (4.4) is $\frac{-4(s+1)}{s}$.

$$\begin{aligned} e^{\int_1^x p(s) ds} &= \exp\left(\int_1^x \frac{-4(s+1)}{s} ds\right) = e^{-4x-4\ln(x)+4} = e^{-4x+4}x^{-4} = \frac{e^{-4x+4}}{x^4}, \\ e^{-\int_1^x p(s) ds} &= e^{4x+4\ln(x)-4} = e^{4x-4}x^4. \end{aligned}$$

We now plug in to (4.4)

$$\begin{aligned} v(x) &= e^{-\int_1^x p(s) ds} \left(\int_1^x e^{\int_1^t p(s) ds} 4 dt + 1 \right) \\ &= e^{4x-4}x^4 \left(\int_1^x 4 \frac{e^{-4t+4}}{t^4} dt + 1 \right). \end{aligned}$$

*There are several things called Bernoulli equations, this is just one of them. The Bernoullis were a prominent Swiss family of mathematicians. These particular equations are named for [Jacob Bernoulli](#) (1654–1705).

The integral in this expression is not possible to find in closed form. As we said before, it is perfectly fine to have a definite integral in our solution. Now “unsubstitute”

$$y^{-4} = e^{4x-4} x^4 \left(4 \int_1^x \frac{e^{-4t+4}}{t^4} dt + 1 \right),$$

$$y = \frac{e^{-x+1}}{x \left(4 \int_1^x \frac{e^{-4t+4}}{t^4} dt + 1 \right)^{1/4}}.$$

4.5.3 Homogeneous Equations

Another type of equations we can solve by substitution are the so-called *homogeneous equations*. Suppose that we can write the differential equation as

$$y' = F\left(\frac{y}{x}\right).$$

Here we try the substitutions

$$v = \frac{y}{x} \quad \text{and therefore} \quad y' = v + xv'.$$

We note that the equation is transformed into

$$v + xv' = F(v) \quad \text{or} \quad xv' = F(v) - v \quad \text{or} \quad \frac{v'}{F(v) - v} = \frac{1}{x}.$$

Hence an implicit solution is

$$\int \frac{1}{F(v) - v} dv = \ln|x| + C.$$

Clearly this solution does not work when $x = 0$ (we would, afterall, divide by zero in y/x). So we will either assume $x > 0$ or $x < 0$ depending on the initial condition.

Example 4.5.2: Solve

$$x^2 y' = y^2 + xy, \quad y(1) = 1.$$

We put the equation into the form $y' = (y/x)^2 + y/x$, that is, $F(v) = v^2 + v$. As the initial condition is for a positive x value, we will assume $x > 0$. We substitute $v = y/x$ to get the separable equation

$$xv' = v^2 + v - v = v^2,$$

which has a solution

$$\int \frac{1}{v^2} dv = \ln|x| + C,$$

$$\frac{-1}{v} = \ln x + C,$$

$$v = \frac{-1}{\ln x + C}.$$

We unsubstitute

$$\frac{y}{x} = \frac{-1}{\ln x + C}, \quad \text{or} \quad y = \frac{-x}{\ln x + C}.$$

We want $y(1) = 1$, so

$$1 = y(1) = \frac{-1}{\ln 1 + C} = \frac{-1}{C}.$$

Thus $C = -1$ and the solution we are looking for is

$$y = \frac{-x}{\ln x - 1}.$$

4.5.4 Exercises

Hint: Answers need not always be in closed form.

Exercise 4.5.1: Solve $y' + y(x^2 - 1) + xy^6 = 0$, with $y(1) = 1$.

Exercise 4.5.2: Solve $2yy' + 1 = y^2 + x$, with $y(0) = 1$.

Exercise 4.5.3: Solve $y' + xy = y^4$, with $y(0) = 1$.

Exercise 4.5.4: Solve $yy' + x = \sqrt{x^2 + y^2}$.

Exercise 4.5.5: Solve $y' = (x + y - 1)^2$.

Exercise 4.5.6: Solve $y' = \frac{x^2 - y^2}{xy}$, with $y(1) = 2$.

Exercise 4.5.101: Solve $xy' + y + y^2 = 0$, $y(1) = 2$.

Exercise 4.5.102: Solve $xy' + y + x = 0$, $y(1) = 1$.

Exercise 4.5.103: Solve $y^2y' = y^3 - 3x$, $y(0) = 2$.

Exercise 4.5.104: Solve $2yy' = e^{y^2 - x^2} + 2x$.

4.6 Autonomous Equations

Consider problems of the form

$$\frac{dx}{dt} = f(x),$$

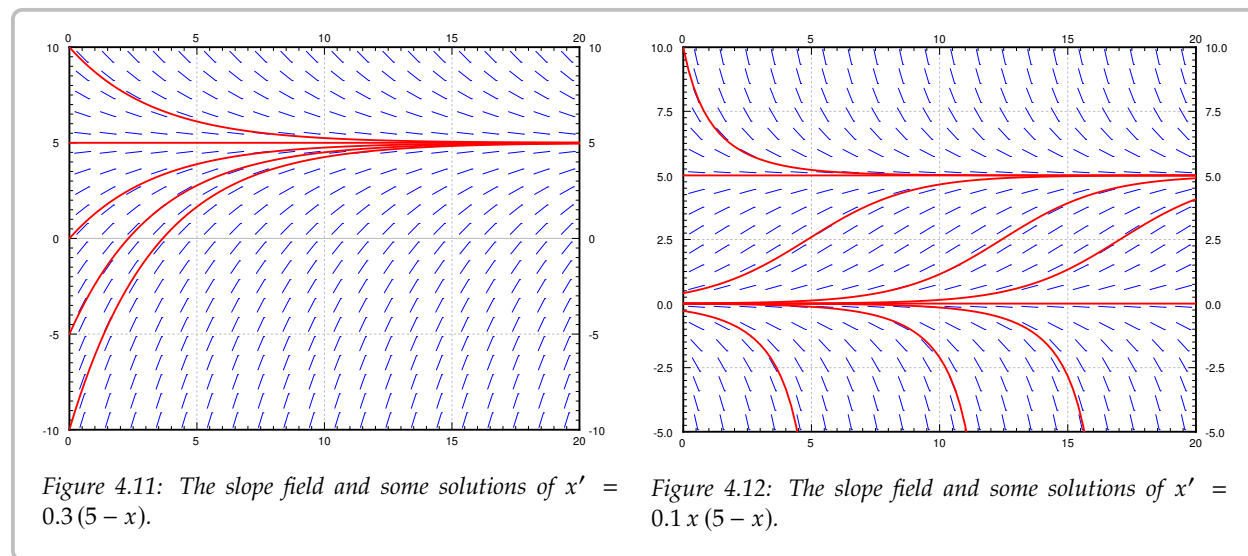
where the derivative of solutions depends only on x (the dependent variable). Such equations are called *autonomous equations*. If we think of t as time, the naming comes from the fact that the equation is independent of time.

We return to the cooling coffee problem (Example 4.3.3). Newton's law of cooling says

$$\frac{dx}{dt} = k(A - x),$$

where x is the temperature, t is time, k is some positive constant, and A is the ambient temperature. See Figure 4.11 for an example with $k = 0.3$ and $A = 5$.

Note the solution $x = A$ (in the figure $x = 5$). We call these constant solutions the *equilibrium solutions*. The points on the x -axis where $f(x) = 0$ are called *critical points*. The point $x = A$ is a critical point. In fact, each critical point corresponds to an equilibrium solution. Note also, by looking at the graph, that the solution $x = A$ is “stable” in that small perturbations in x do not lead to substantially different solutions as t grows. If we change the initial condition a little bit, then as $t \rightarrow \infty$ we get $x(t) \rightarrow A$. We call such a critical point *stable*. In this simple example it turns out that all solutions in fact go to A as $t \rightarrow \infty$. If a critical point is not stable, we say it is *unstable*.



Consider now the *logistic equation*

$$\frac{dx}{dt} = kx(M - x),$$

for some positive k and M . This equation is commonly used to model population if we know the limiting population M , that is the maximum sustainable population. The logistic equation leads to less catastrophic predictions on world population than $x' = kx$. In the real world there is no such thing as negative population, but we will still consider negative x for the purposes of the math.

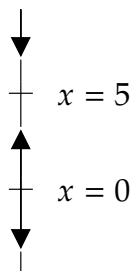
See Figure 4.12 for an example, $x' = 0.1x(5 - x)$. There are two critical points, $x = 0$ and $x = 5$. The critical point at $x = 5$ is stable, while the critical point at $x = 0$ is unstable. It is not necessary to find the exact

solutions to understand their long term behavior, that is, behavior as time goes to infinity. From the slope field above of $x' = 0.1x(5 - x)$, we see that

$$\lim_{t \rightarrow \infty} x(t) = \begin{cases} 5 & \text{if } x(0) > 0, \\ 0 & \text{if } x(0) = 0, \\ \text{DNE or } -\infty & \text{if } x(0) < 0. \end{cases}$$

Here DNE means “does not exist.” From just looking at the slope field we cannot quite decide what happens if $x(0) < 0$. It could be that the solution does not exist for t all the way to ∞ . Think of the equation $x' = x^2$; we have seen that solutions only exist for some finite period of time. Same can happen here. In our example equation above it turns out that the solution does not exist for all time, but to see that we would have to solve the equation. In any case, the solution does go to $-\infty$, but it may get there rather quickly.

If we are interested only in the long term behavior of the solution, we would be doing unnecessary work if we solved the equation exactly. We could draw the slope field, but it is easier to just look at the *phase diagram* or *phase portrait*, which is a simple way to visualize the behavior of autonomous equations. In this case there is one dependent variable x . We draw the x -axis, we mark all the critical points, and then we draw arrows in between. Since x is the dependent variable we draw the axis vertically, as it appears in the slope field diagrams above. If $f(x) > 0$, we draw an up arrow. If $f(x) < 0$, we draw a down arrow. To figure this out, we could just plug in some x between the critical points, $f(x)$ will have the same sign at all x between two critical points as long $f(x)$ is continuous. For example, $f(6) = -0.6 < 0$, so $f(x) < 0$ for $x > 5$, and the arrow above $x = 5$ is a down arrow. Next, $f(1) = 0.4 > 0$, so $f(x) > 0$ whenever $0 < x < 5$, and the arrow points up. Finally, $f(-1) = -0.6 < 0$ so $f(x) < 0$ when $x < 0$, and the arrow points down.



Armed with the phase diagram, it is easy to sketch the solutions approximately: As time t moves from left to right, the graph of a solution goes up if the arrow is up, and it goes down if the arrow is down.

Exercise 4.6.1: Try sketching a few solutions simply from looking at the phase diagram. Check with the preceding graphs if you are getting the same type of curves.

Once we draw the phase diagram, we classify critical points as stable or unstable*. Since any mathematical model we cook up will only be an approximation to the real world, unstable points are generally bad news.



We remark that you can figure out the arrows by plotting the graph $y = f(x)$. However, in that case note that x is then the dependent variable and will be on the horizontal axis.

Let us think about the logistic equation with harvesting. Suppose an alien race really likes to eat humans. They keep a planet with humans and harvest the humans at a rate of h million humans per year. Suppose x

*Unstable points with one of the arrows pointing towards the critical point are sometimes called *semistable*.

is the number of humans in millions on the planet and t is time in years. Let M be the limiting population when no harvesting is done. The number $k > 0$ is a constant depending on how fast humans multiply. Our equation becomes

$$\frac{dx}{dt} = kx(M - x) - h.$$

We expand the right-hand side and set it to zero.

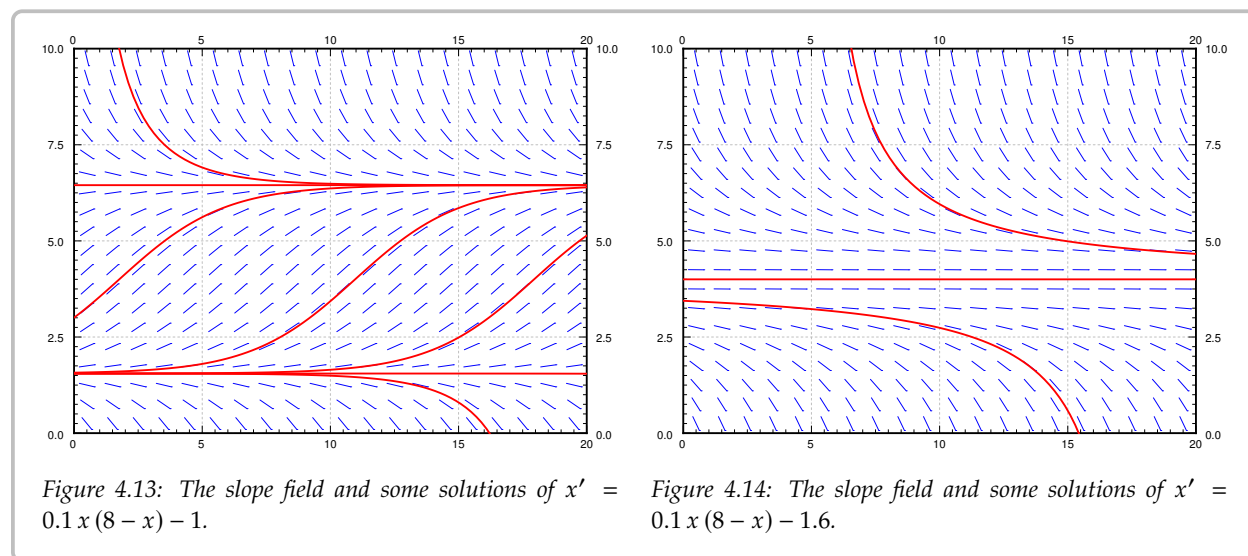
$$kx(M - x) - h = -kx^2 + kMx - h = 0.$$

Solving for the critical points, let us call them A and B , we get

$$A = \frac{kM + \sqrt{(kM)^2 - 4hk}}{2k}, \quad B = \frac{kM - \sqrt{(kM)^2 - 4hk}}{2k}.$$

Exercise 4.6.2: Sketch a phase diagram for different possibilities. Note that these possibilities are $A > B$, or $A = B$, or A and B both complex (i.e. no real solutions). Hint: Fix some simple k and M and then vary h .

For example, let $M = 8$ and $k = 0.1$. When $h = 1$, then A and B are distinct and positive. See Figure 4.13 for the slope field. As long as the population starts above B , which is approximately 1.55 million, then the population will not die out, it will tend towards $A \approx 6.45$ million. If ever a catastrophe happens and the population drops below B , humans will die out, and the fast food restaurant serving them will go out of business.



When $h = 1.6$, then $A = B = 4$. There is only one critical point and it is unstable. When the population starts above 4 million, it will tend towards 4 million. However, if it ever drops below 4 million, perhaps a worse than normal hurricane season one year, then humans will die out on the planet. This scenario is not one that we (as the human fast food proprietor) want to be in. A small perturbation of the equilibrium state and we are out of business. There is no room for error. See Figure 4.14.

Finally, if we are harvesting at 2 million humans per year, there are no critical points. The population will always plummet towards zero, no matter how well stocked the planet starts. See Figure 4.15 on the next page.

4.6.1 Exercises

Exercise 4.6.3: Consider $x' = x^2$.

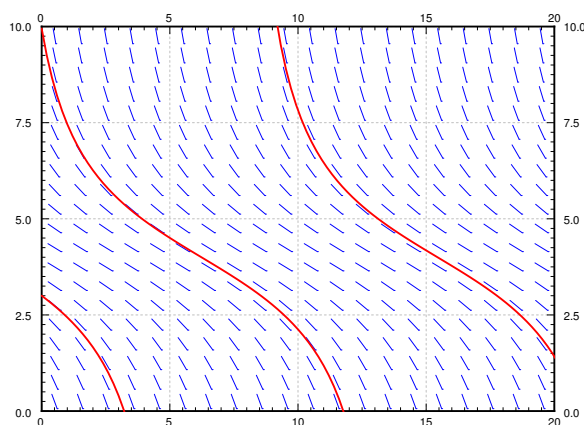


Figure 4.15: The slope field and some solutions of $x' = 0.1x(8 - x) - 2$.

- Draw the phase diagram, find the critical points, and mark them stable or unstable.
- Sketch typical solutions of the equation.
- Find $\lim_{t \rightarrow \infty} x(t)$ for the solution with the initial condition $x(0) = -1$.

Exercise 4.6.4: Consider $x' = \sin x$.

- Draw the phase diagram for $-4\pi \leq x \leq 4\pi$. On this interval mark the critical points stable or unstable.
- Sketch typical solutions of the equation.
- Find $\lim_{t \rightarrow \infty} x(t)$ for the solution with the initial condition $x(0) = 1$.

Exercise 4.6.5: Suppose $f(x)$ is positive for $0 < x < 1$, it is zero when $x = 0$ and $x = 1$, and it is negative for all other x .

- Draw the phase diagram for $x' = f(x)$, find the critical points, and mark them stable or unstable.
- Sketch typical solutions of the equation.
- Find $\lim_{t \rightarrow \infty} x(t)$ for the solution with the initial condition $x(0) = 0.5$.

Exercise 4.6.6: Start with the logistic equation $\frac{dx}{dt} = kx(M - x)$. Suppose we modify our harvesting. That is we will only harvest an amount proportional to current population. In other words, we harvest hx per unit of time for some $h > 0$ (similar to earlier example with h replaced with hx).

- Construct the differential equation.
- Show that if $kM > h$, then the equation is still logistic.
- What happens when $kM < h$?

Exercise 4.6.7: A disease is spreading through the country. Let x be the number of people infected. Let the constant S be the number of people susceptible to infection. The infection rate $\frac{dx}{dt}$ is proportional to the product of already infected people, x , and the number of susceptible but uninfected people, $S - x$.

- Write down the differential equation.
- Supposing $x(0) > 0$, that is, some people are infected at time $t = 0$, what is $\lim_{t \rightarrow \infty} x(t)$.
- Does the solution to part b) agree with your intuition? Why or why not?

Exercise 4.6.101: Let $x' = (x - 1)(x - 2)x^2$.

- a) Sketch the phase diagram and find critical points.
- b) Classify the critical points.
- c) If $x(0) = 0.5$, then find $\lim_{t \rightarrow \infty} x(t)$.

Exercise 4.6.102: Let $x' = e^{-x}$.

- a) Find and classify all critical points.
- b) Find $\lim_{t \rightarrow \infty} x(t)$ given any initial condition.

Exercise 4.6.103: Assume that a population of fish in a lake satisfies $\frac{dx}{dt} = kx(M - x)$. Now suppose that fish are continually added at A fish per unit of time.

- a) Find the differential equation for x .
- b) What is the new limiting population?

Exercise 4.6.104: Suppose $\frac{dx}{dt} = (x - \alpha)(x - \beta)$ for two numbers $\alpha < \beta$.

- a) Find the critical points, and classify them.

For b), c), d), find $\lim_{t \rightarrow \infty} x(t)$ based on the phase diagram.

- b) $x(0) < \alpha$,
- c) $\alpha < x(0) < \beta$,
- d) $\beta < x(0)$.

4.7 Numerical Methods: Euler's Method

Unless $f(x, y)$ is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

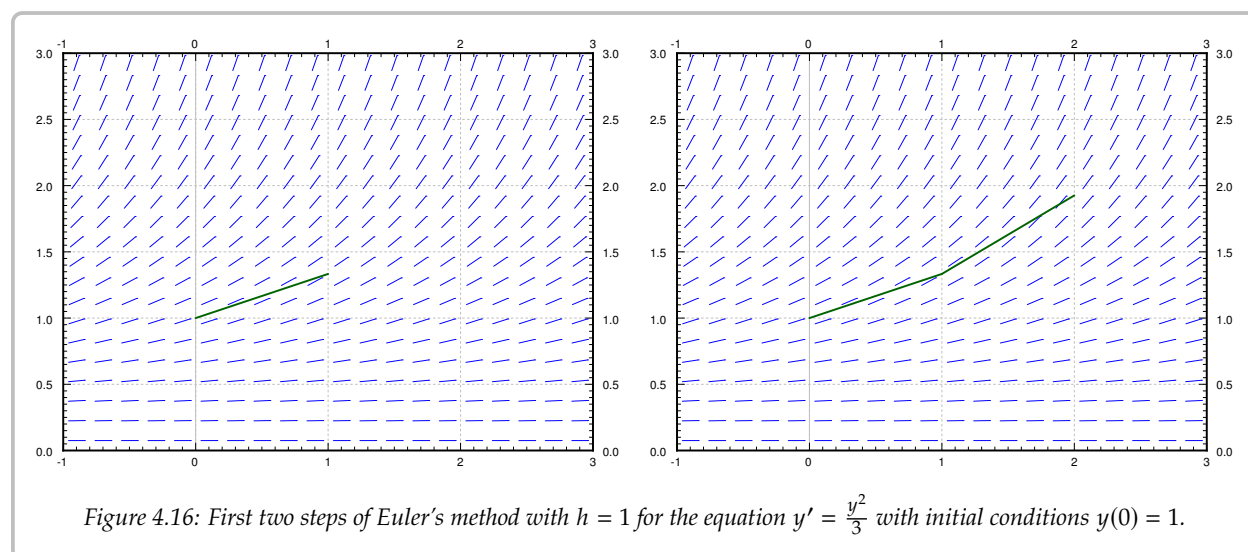
$$y' = f(x, y), \quad y(x_0) = y_0.$$

If the equation can be solved in closed form, we should do that. But what if we have an equation that cannot be solved in closed form? What if we want to find the value of the solution at some particular x ? Or perhaps we want to produce a graph of the solution to inspect the behavior. In this section we will learn about the basics of numerical approximation of solutions.

The simplest method for approximating a solution is *Euler's method**. It works as follows: Take x_0 and y_0 and compute the slope $k = f(x_0, y_0)$. The slope is the change in y per unit change in x . Follow the line for an interval of length h on the x -axis. Hence if $y = y_0$ at x_0 , then we say that y_1 , the approximate value of y at $x_1 = x_0 + h$, is $y_1 = y_0 + hk$. Rinse, repeat! Let $k = f(x_1, y_1)$, and then compute $x_2 = x_1 + h$, and $y_2 = y_1 + hk$. Now compute x_3 and y_3 using x_2 and y_2 , etc. Consider the equation $y' = y^2/3$, $y(0) = 1$, and $h = 1$. Then $x_0 = 0$ and $y_0 = 1$. We compute

$$\begin{aligned} x_1 &= x_0 + h = 0 + 1 = 1, & y_1 &= y_0 + h f(x_0, y_0) = 1 + 1 \cdot 1/3 = 4/3 \approx 1.333, \\ x_2 &= x_1 + h = 1 + 1 = 2, & y_2 &= y_1 + h f(x_1, y_1) = 4/3 + 1 \cdot \frac{(4/3)^2}{3} = 52/27 \approx 1.926. \end{aligned}$$

We then draw an approximate graph of the solution by connecting the points (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , ... See Figure 4.16 for the first two steps of the method.



More abstractly, for any $i = 0, 1, 2, 3, \dots$, we compute

$$x_{i+1} = x_i + h, \quad y_{i+1} = y_i + h f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure 4.17 on the next page for the plot of the real solution and the approximation.

We continue with the equation $y' = y^2/3$, $y(0) = 1$. Let us try to approximate $y(2)$ using Euler's method. In Figures 4.16 and 4.17 we have graphically approximated $y(2)$ with step size 1. With step size 1, we have

*Named after the Swiss mathematician [Leonhard Paul Euler](#) (1707–1783). The correct pronunciation of the name sounds more like “oiler.”

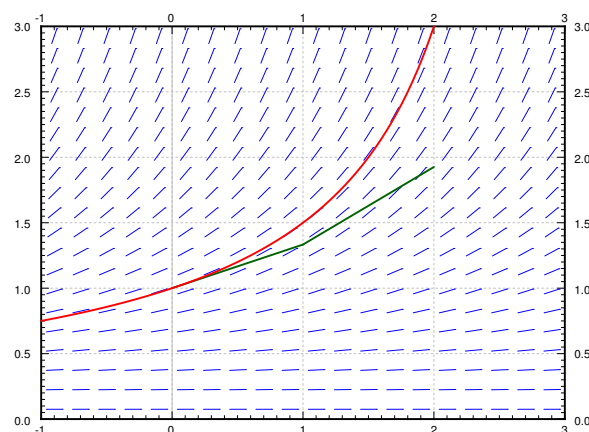


Figure 4.17: Two steps of Euler's method (step size 1) and the exact solution for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

$y(2) \approx 1.926$. The real answer is 3. We are approximately 1.074 off. Let us halve the step size. Computing y_4 with $h = 0.5$, we find that $y(2) \approx 2.209$, so an error of about 0.791. Table 4.1 gives the values computed for various parameters.

Exercise 4.7.1: Solve this equation exactly and show that $y(2) = 3$.

The difference between the actual solution and the approximate solution is called the error. We usually talk about the size of the error and we do not care much about its sign.

$$\text{Error} = |\text{Actual } y - \text{Approximate } y|.$$

The point is, we do not know the real solution. If we knew the error exactly, we would know the actual solution ...so what is the point of doing the approximation?

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

Table 4.1: Euler's method approximation of $y(2)$ where of $y' = y^2/3$, $y(0) = 1$.

Note that except for the first few times, each time we halve the h , the error approximately halves. Halving of the error is a general feature of Euler's method as it is a *first order method*. A simple improvement of the Euler method, see the exercises, produces a second order method. A second order method reduces the error to approximately one quarter every time we halve the interval. The order being "second" means the squaring in $1/4 = 1/2 \times 1/2 = (1/2)^2$.

To get the error to be within 0.1 of the answer, we had to do 64 steps. To get it to within 0.01, we would have to halve another three or four times, meaning doing 512 to 1024 steps. The improved Euler method from the exercises should quarter the error every time we halve the interval, so we would have to do (approximately) half as many “halvings” to get the same error. This reduction can be a big deal. With 10 halvings (starting at $h = 1$) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more difficult to compute than $y^2/3$). Then the difference is 32 seconds versus about 17 minutes. We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

In practice, we do not know how large the error is! How do we know what is the right step size? Well, essentially, we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

Exercise 4.7.2: In the table above, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?

Let us talk a little bit more about the example $y' = y^2/3$, $y(0) = 1$. Suppose that instead of $y(2)$ we wish to find $y(3)$. Table 4.2 lists the results of this effort for successive halvings of h . What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at $x = 3$. In fact, the solution goes to infinity when you approach $x = 3$.

h	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

Table 4.2: Attempts to use Euler's to approximate $y(3)$ where of $y' = y^2/3$, $y(0) = 1$.

Another case where things go bad is if the solution oscillates wildly near some point. The solution may exist at all points, but even a much better numerical method than Euler would need an insanely small step size to approximate the solution with reasonable precision. And computers might not be able to easily handle such a small step size.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge–Kutta method (see exercises). That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16 (it is fourth order as $1/16 = 1/2 \times 1/2 \times 1/2 \times 1/2$).

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

- Computational time: Each step takes computer time. Even if the function f is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.

- **Roundoff errors:** Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse. There is a certain optimum step size such that the precision increases as we approach it, but then starts getting worse as we make our step size smaller still. Trouble is: this optimum may be hard to find.
- **Stability:** Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called *stiff*. In the worst case, the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers seem to have stabilized after successive halving, does not mean that we must have the right answer.

We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area for engineers and mathematicians. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

4.7.1 Exercises

Exercise 4.7.3: Consider $\frac{dx}{dt} = (2t - x)^2$, $x(0) = 2$. Use Euler's method with step size $h = 0.5$ to approximate $x(1)$.

Exercise 4.7.4: Consider $\frac{dx}{dt} = t - x$, $x(0) = 1$.

- Use Euler's method with step sizes $h = 1, 1/2, 1/4, 1/8$ to approximate $x(1)$.
- Solve the equation exactly.
- Describe what happens to the errors for each h you used. That is, find the factor by which the error changed each time you halved the interval.

Exercise 4.7.5: Approximate the value of e by looking at the initial value problem $y' = y$ with $y(0) = 1$ and approximating $y(1)$ using Euler's method with a step size of 0.2.

Exercise 4.7.6: Example of numerical instability: Take $y' = -5y$, $y(0) = 1$. We know that the solution should decay to zero as x grows. Using Euler's method, start with $h = 1$ and compute y_1, y_2, y_3, y_4 to try to approximate $y(4)$. What happened? Now halve the interval. Keep halving the interval and approximating $y(4)$ until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.

The simplest method used in practice is the *Runge–Kutta method*. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h . Everything is the same as in Euler's method, except the computation of y_{i+1} and x_{i+1} .

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f(x_i + h/2, y_i + k_1(h/2)), & x_{i+1} &= x_i + h, \\ k_3 &= f(x_i + h/2, y_i + k_2(h/2)), & y_{i+1} &= y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} h, \\ k_4 &= f(x_i + h, y_i + k_3h). \end{aligned}$$

Exercise 4.7.7: Consider $\frac{dy}{dx} = yx^2$, $y(0) = 1$.

- Use Runge–Kutta (see above) with step sizes $h = 1$ and $h = 1/2$ to approximate $y(1)$.
- Use Euler's method with $h = 1$ and $h = 1/2$.
- Solve exactly, find the exact value of $y(1)$, and compare.

Exercise 4.7.101: Let $x' = \sin(xt)$, and $x(0) = 1$. Approximate $x(1)$ using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.

Exercise 4.7.102: Let $x' = 2t$, and $x(0) = 0$.

- Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
- Solve exactly, and compute the errors.
- Compute the factor by which the errors changed.

Exercise 4.7.103: Let $x' = xe^{xt+1}$, and $x(0) = 0$.

- Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
- Guess an exact solution based on part a) and compute the errors.

There is a simple way to improve Euler's method to make it a second order method by doing just one extra step. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h . What we do is to pretend we compute the next step as in Euler, that is, we start with (x_i, y_i) , we compute a slope $k_1 = f(x_i, y_i)$, and then look at the point $(x_i + h, y_i + k_1h)$. Instead of letting our new point be $(x_i + h, y_i + k_1h)$, we compute the slope at that point, call it k_2 , and then take the average of k_1 and k_2 , hoping that the average is going to be closer to the actual slope on the interval from x_i to $x_i + h$. And we are correct, if we halve the step, the error should go down by a factor of $2^2 = 4$. To summarize, the setup is the same as for regular Euler, except the computation of y_{i+1} and x_{i+1} .

$$\begin{aligned} k_1 &= f(x_i, y_i), & x_{i+1} &= x_i + h, \\ k_2 &= f(x_i + h, y_i + k_1h), & y_{i+1} &= y_i + \frac{k_1 + k_2}{2} h. \end{aligned}$$

Exercise 4.7.104: Consider $\frac{dy}{dx} = x + y$, $y(0) = 1$.

- Use the improved Euler's method (see above) with step sizes $h = 1/4$ and $h = 1/8$ to approximate $y(1)$.
- Use Euler's method with $h = 1/4$ and $h = 1/8$.
- Solve exactly, find the exact value of $y(1)$.
- Compute the errors, and the factors by which the errors changed.

Chapter 5

Applications of First Order Equations

5.1 Growth and Decay

Since the applications in this section deal with functions of time, we'll denote the independent variable by t . If Q is a function of t , Q' will denote the derivative of Q with respect to t ; thus,

$$Q' = \frac{dQ}{dt}.$$

5.1.1 Exponential Growth and Decay

One of the most common mathematical models for a physical process is the *exponential model*, where it's assumed that the rate of change of a quantity Q is proportional to Q ; thus

$$Q' = aQ, \tag{5.1}$$

where a is the constant of proportionality.

From Example 2.1.3, the general solution of (5.1) is

$$Q = ce^{at}$$

and the solution of the initial value problem

$$Q' = aQ, \quad Q(t_0) = Q_0$$

is

$$Q = Q_0 e^{a(t-t_0)}. \tag{5.2}$$

Since the solutions of $Q' = aQ$ are exponential functions, we say that a quantity Q that satisfies this equation *grows exponentially* if $a > 0$, or *decays exponentially* if $a < 0$ (Figure 5.1).

5.1.2 Radioactive Decay

Experimental evidence shows that radioactive material decays at a rate proportional to the mass of the material present. According to this model the mass $Q(t)$ of a radioactive material present at time t satisfies (5.1), where a is a negative constant whose value for any given material must be determined by experimental observation. For simplicity, we'll replace the negative constant a by $-k$, where k is a positive number that we'll call the *decay constant* of the material. Thus, (5.1) becomes

$$Q' = -kQ.$$

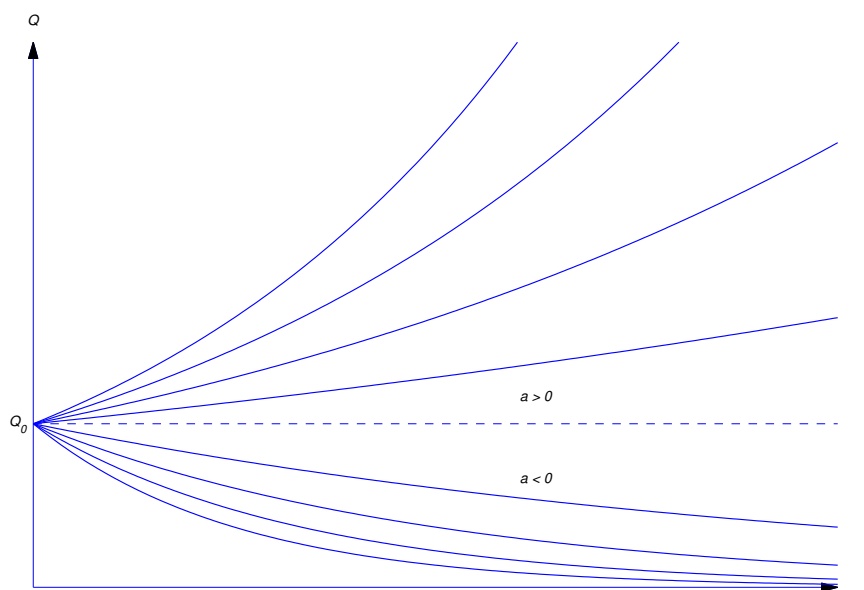


Figure 5.1: Exponential growth and decay

If the mass of the material present at $t = t_0$ is Q_0 , the mass present at time t is the solution of

$$Q' = -kQ, \quad Q(t_0) = Q_0.$$

From (5.2) with $a = -k$, the solution of this initial value problem is

$$Q = Q_0 e^{-k(t-t_0)}. \quad (5.3)$$

The *half-life* τ of a radioactive material is defined to be the time required for half of its mass to decay; that is, if $Q(t_0) = Q_0$, then

$$Q(\tau + t_0) = \frac{Q_0}{2}. \quad (5.4)$$

From (5.3) with $t = \tau + t_0$, (5.4) is equivalent to

$$Q_0 e^{-k\tau} = \frac{Q_0}{2},$$

so

$$e^{-k\tau} = \frac{1}{2}.$$

Taking logarithms yields

$$-k\tau = \ln \frac{1}{2} = -\ln 2,$$

so the half-life is

$$\tau = \frac{1}{k} \ln 2. \quad (5.5)$$

(Figure 5.2). The half-life is independent of t_0 and Q_0 , since it's determined by the properties of material, not by the amount of the material present at any particular time.

Example 5.1.1: A radioactive substance has a half-life of 1620 years.

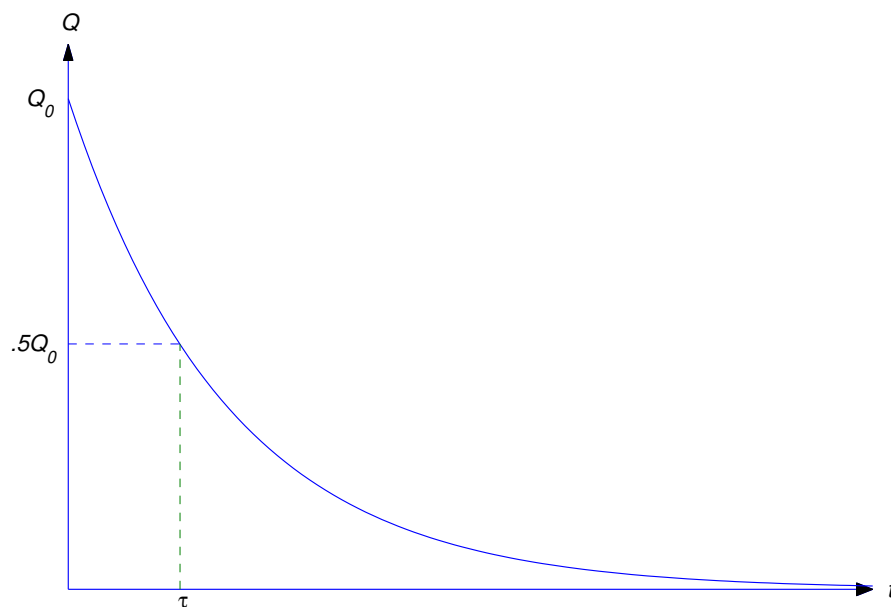


Figure 5.2: Half-life of a radioactive substance

- (a) If its mass is now 4 g (grams), how much will be left 810 years from now?
 (b) Find the time t_1 when 1.5 g of the substance remain.

Solution (a): From (5.3) with $t_0 = 0$ and $Q_0 = 4$,

$$Q = 4e^{-kt}, \quad (5.6)$$

where we determine k from (5.5), with $\tau = 1620$ years:

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}.$$

Substituting this in (5.6) yields

$$Q = 4e^{-(t \ln 2)/1620}. \quad (5.7)$$

Therefore the mass left after 810 years will be

$$\begin{aligned} Q(810) &= 4e^{-(810 \ln 2)/1620} = 4e^{-(\ln 2)/2} \\ &= 2\sqrt{2} \text{ g.} \end{aligned}$$

Solution (b): Setting $t = t_1$ in (5.7) and requiring that $Q(t_1) = 1.5$ yields

$$\frac{3}{2} = 4e^{(-t_1 \ln 2)/1620}.$$

Dividing by 4 and taking logarithms yields

$$\ln \frac{3}{8} = -\frac{t_1 \ln 2}{1620}.$$

Since $\ln 3/8 = -\ln 8/3$,

$$t_1 = 1620 \frac{\ln 8/3}{\ln 2} \approx 2292.4 \text{ years.}$$

5.1.3 Interest Compounded Continuously

Suppose we deposit an amount of money Q_0 in an interest-bearing account and make no further deposits or withdrawals for t years, during which the account bears interest at a constant annual rate r . To calculate the value of the account at the end of t years, we need one more piece of information: how the interest is added to the account, or—as the bankers say—how it is *compounded*. If the interest is compounded annually, the value of the account is multiplied by $1 + r$ at the end of each year. This means that after t years the value of the account is

$$Q(t) = Q_0(1 + r)^t.$$

If interest is compounded semiannually, the value of the account is multiplied by $(1 + r/2)$ every 6 months. Since this occurs twice annually, the value of the account after t years is

$$Q(t) = Q_0 \left(1 + \frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded n times per year, the value of the account is multiplied n times per year by $(1 + r/n)$; therefore, the value of the account after t years is

$$Q(t) = Q_0 \left(1 + \frac{r}{n}\right)^{nt}. \quad (5.8)$$

Thus, increasing the frequency of compounding increases the value of the account after a fixed period of time. Table 5.1.3 shows the effect of increasing the number of compoundings over $t = 5$ years on an initial deposit of $Q_0 = 100$ (dollars), at an annual interest rate of 6%.

Table The effect of compound interest

n (number of compoundings per year)	$\$100 \left(1 + \frac{.06}{n}\right)^{5n}$ (value in dollars after 5 years)
1	\$133.82
2	\$134.39
4	\$134.68
8	\$134.83
364	\$134.98

You can see from Table 5.1.3 that the value of the account after 5 years is an increasing function of n . Now suppose the maximum allowable rate of interest on savings accounts is restricted by law, but the time intervals between successive compoundings isn't; then competing banks can attract savers by compounding often. The ultimate step in this direction is to *compound continuously*, by which we mean that $n \rightarrow \infty$ in (5.8). Since we know from calculus that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = e^r,$$

this yields

$$\begin{aligned} Q(t) &= \lim_{n \rightarrow \infty} Q_0 \left(1 + \frac{r}{n}\right)^{nt} = Q_0 \left[\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n \right]^t \\ &= Q_0 e^{rt}. \end{aligned}$$

Observe that $Q = Q_0 e^{rt}$ is the solution of the initial value problem

$$Q' = rQ, \quad Q(0) = Q_0;$$

that is, with continuous compounding the value of the account grows exponentially.

Example 5.1.2: If \$150 is deposited in a bank that pays $5\frac{1}{2}\%$ annual interest compounded continuously, the value of the account after t years is

$$Q(t) = 150e^{.055t}$$

dollars. (Note that it's necessary to write the interest rate as a decimal; thus, $r = .055$.) Therefore, after $t = 10$ years the value of the account is

$$Q(10) = 150e^{.55} \approx \$259.99.$$

Example 5.1.3: We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}\%$ annual interest compounded continuously. How much must we deposit in the account?

Solution: The value of the account at time t is

$$Q(t) = Q_0e^{.055t}. \quad (5.9)$$

Since we want $Q(10)$ to be \$10,000, the initial deposit Q_0 must satisfy the equation

$$10000 = Q_0e^{.55}, \quad (5.10)$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in (5.9). Solving (5.10) for Q_0 yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

5.1.4 Mixed Growth and Decay

Example 5.1.4: A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time.

(a) Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time t .

(b) Find $\lim_{t \rightarrow \infty} Q(t)$.

Solution (a): Here

$$Q' = \text{rate of increase of } Q - \text{rate of decrease of } Q.$$

The rate of increase is the constant a . Since Q is radioactive with decay constant k , the rate of decrease is kQ . Therefore

$$Q' = a - kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \quad (5.11)$$

Since e^{-kt} is a solution of the complementary equation, the solutions of (5.11) are of the form $Q = ue^{-kt}$, where $u'e^{-kt} = a$, so $u' = ae^{kt}$. Hence,

$$u = \frac{a}{k}e^{kt} + c$$

and

$$Q = ue^{-kt} = \frac{a}{k} + ce^{-kt}.$$

Since $Q(0) = Q_0$, setting $t = 0$ here yields

$$Q_0 = \frac{a}{k} + c \quad \text{or} \quad c = Q_0 - \frac{a}{k}.$$

Therefore

$$Q = \frac{a}{k} + \left(Q_0 - \frac{a}{k}\right)e^{-kt}. \quad (5.12)$$

Solution (b): Since $k > 0$, $\lim_{t \rightarrow \infty} e^{-kt} = 0$, so from (5.12)

$$\lim_{t \rightarrow \infty} Q(t) = \frac{a}{k}.$$

This limit depends only on a and k , and not on Q_0 . We say that a/k is the *steady state* value of Q . From (5.12) we also see that Q approaches its steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. If $Q_0 = a/k$, then Q remains constant (Figure 5.3).

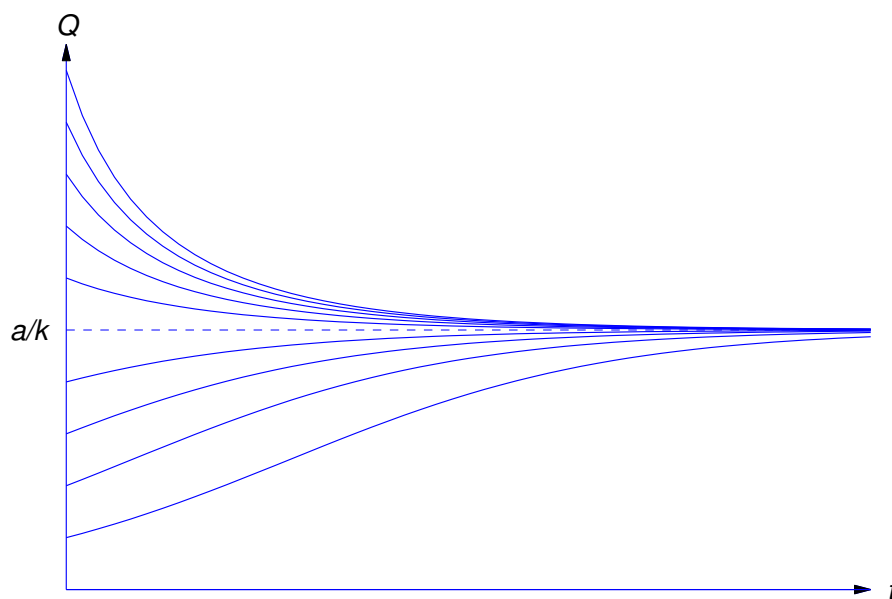


Figure 5.3: $Q(t)$ approaches the steady state value $\frac{a}{k}$ as $t \rightarrow \infty$

5.1.5 Carbon Dating

The fact that Q approaches a steady state value in the situation discussed in Example 4 underlies the method of *carbon dating*, devised by the American chemist and Nobel Prize Winner *W.S. Libby*.

Carbon 12 is stable, but carbon-14, which is produced by cosmic bombardment of nitrogen in the upper atmosphere, is radioactive with a half-life of about 5570 years. Libby assumed that the quantity of carbon-12 in the atmosphere has been constant throughout time, and that the quantity of radioactive carbon-14 achieved its steady state value long ago as a result of its creation and decomposition over millions of years. These assumptions led Libby to conclude that the ratio of carbon-14 to carbon-12 has been nearly constant for a long time. This constant, which we denote by R , has been determined experimentally.

Living cells absorb both carbon-12 and carbon-14 in the proportion in which they are present in the environment. Therefore the ratio of carbon-14 to carbon-12 in a living cell is always R . However, when the cell dies it ceases to absorb carbon, and the ratio of carbon-14 to carbon-12 decreases exponentially as the radioactive carbon-14 decays. This is the basis for the method of carbon dating, as illustrated in the next example.

Example 5.1.5: An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

Solution: Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, its decay constant is

$$k = \frac{\ln 2}{5570}.$$

Therefore

$$Q = Q_0 e^{-t(\ln 2)/5570}$$

if we choose our time scale so that $t_0 = 0$ is the time of death. If we know the present value of Q we can solve this equation for t , the number of years since death occurred. This yields

$$t = -5570 \frac{\ln Q/Q_0}{\ln 2}.$$

It is given that $Q = .42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = -5570 \frac{\ln .42}{\ln 2} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = -5570 \frac{\ln .44}{\ln 2} \approx 6597$$

years ago. Therefore it's reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

5.1.6 A Savings Program

Example 5.1.6: A person opens a savings account with an initial deposit of \$1000 and subsequently deposits \$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

Solution: Observe that Q isn't continuous, since there are 52 discrete deposits per year of \$50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate of \$2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, Q increases continuously at the rate

$$Q' = 2600 + .06Q$$

and therefore Q satisfies the differential equation

$$Q' - .06Q = 2600. \quad (5.13)$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of Q at any given time. We'll discuss this further below.) Since $e^{.06t}$ is a solution of the complementary equation, the solutions of (5.13) are of the form $Q = ue^{.06t}$, where $u'e^{.06t} = 2600$. Hence, $u' = 2600e^{-.06t}$,

$$u = -\frac{2600}{.06} e^{-.06t} + c$$

and

$$Q = ue^{.06t} = -\frac{2600}{.06} + ce^{.06t}. \quad (5.14)$$

Setting $t = 0$ and $Q = 1000$ here yields

$$c = 1000 + \frac{2600}{.06},$$

and substituting this into (5.14) yields

$$Q = 1000e^{.06t} + \frac{2600}{.06}(e^{.06t} - 1), \quad (5.15)$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits. ■

Mathematical models must be tested for validity by comparing predictions based on them with the actual outcome of experiments. Example 6 is unusual in that we can compute the exact value of the account at any specified time and compare it with the approximate value predicted by (5.15) (See Exercise 5.1.21.). The following table gives a comparison for a ten year period. Each exact answer corresponds to the time of the year-end deposit, and each year is assumed to have exactly 52 weeks.

Year	Approximate Value of Q (Example 5.1.6)	Exact Value of P (Exercise 5.1.21)	Error $Q - P$	Percentage Error $(Q - P)/P$
1	\$ 3741.42	\$ 3739.87	\$ 1.55	.0413%
2	6652.36	6649.17	3.19	.0479
3	9743.30	9738.37	4.93	.0506
4	13,025.38	13,018.60	6.78	.0521
5	16,510.41	16,501.66	8.75	.0530
6	20,210.94	20,200.11	10.83	.0536
7	24,140.30	24,127.25	13.05	.0541
8	28,312.63	28,297.23	15.40	.0544
9	32,742.97	32,725.07	17.90	.0547
10	37,447.27	37,426.72	20.55	.0549

5.1.7 Exercises

- The half-life of a radioactive substance is 3200 years. Find the quantity $Q(t)$ of the substance left at time $t > 0$ if $Q(0) = 20$ g.
- The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to $1/10$ of its original mass.
- A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?
- A tree contains a known percentage p_0 of a radioactive substance with half-life τ . When the tree dies the substance decays and isn't replaced. If the percentage of the substance in the fossilized remains of such a tree is found to be p_1 , how long has the tree been dead?
- If t_p and t_q are the times required for a radioactive material to decay to $1/p$ and $1/q$ times its original mass (respectively), how are t_p and t_q related?
- Find the decay constant k for a radioactive substance, given that the mass of the substance is Q_1 at time t_1 and Q_2 at time t_2 .
- A process creates a radioactive substance at the rate of 2 g/hr and the substance decays at a rate proportional to its mass, with constant of proportionality $k = .1(\text{hr})^{-1}$. If $Q(t)$ is the mass of the substance at time t , find $\lim_{t \rightarrow \infty} Q(t)$.
- A bank pays interest continuously at the rate of 6%. How long does it take for a deposit of Q_0 to grow in value to $2Q_0$?
- At what rate of interest, compounded continuously, will a bank deposit double in value in 8 years?
- A savings account pays 5% per annum interest compounded continuously. The initial deposit is Q_0 dollars. Assume that there are no subsequent withdrawals or deposits.
 - How long will it take for the value of the account to triple?
 - What is Q_0 if the value of the account after 10 years is \$100,000 dollars?
- A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time t .
 - Find $Q(t)$ for $t > 0$ if the candymaker has 250 pounds of candy at $t = 0$.
 - Find $\lim_{t \rightarrow \infty} Q(t)$.
- Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50 g of the substance, how long will it be until only 25 g remain?
- A super bread dough increases in volume at a rate proportional to the volume V present. If V increases by a factor of 10 in 2 hours and $V(0) = V_0$, find V at any time t . How long will it take for V to increase to $100V_0$?
- A radioactive substance decays at a rate proportional to the amount present, and half the original quantity Q_0 is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?
- A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of however much is there per hour. Let $W(t)$ be the number of ounces that the wizard has at time t . Find $W(t)$ and $\lim_{t \rightarrow \infty} W(t)$ if $W(0) = 1$.
- A process creates a radioactive substance at the rate of 1 g/hr, and the substance decays at an hourly rate equal to $1/10$ of the mass present (expressed in grams). Assuming that there are initially 20 g, find the mass $S(t)$ of the substance present at time t , and find $\lim_{t \rightarrow \infty} S(t)$.
- A tank is empty at $t = 0$. Water is added to the tank at the rate of 10 gal/min, but it leaks out at a rate (in gallons per minute) equal to the number of gallons in the tank. What is the smallest capacity the tank can have if this process is to continue forever?

18. A person deposits \$25,000 in a bank that pays 5% per year interest, compounded continuously. The person continuously withdraws from the account at the rate of \$750 per year. Find $V(t)$, the value of the account at time t after the initial deposit.
19. A person has a fortune that grows at rate proportional to the square root of its worth. Find the worth W of the fortune as a function of t if it was \$1 million 6 months ago and is \$4 million today.
20. Let $p = p(t)$ be the quantity of a product present at time t . The product is manufactured continuously at a rate proportional to p , with proportionality constant $1/2$, and it's consumed continuously at a rate proportional to p^2 , with proportionality constant $1/8$. Find $p(t)$ if $p(0) = 100$.

21. (a) In the situation of Example 5.1.6 find the exact value $P(t)$ of the person's account after t years, where t is an integer. Assume that each year has exactly 52 weeks, and include the year-end deposit in the computation.

HINT: At time t the initial \$1000 has been on deposit for t years. There have been $52t$ deposits of \$50 each. The first \$50 has been on deposit for $t - 1/52$ years, the second for $t - 2/52$ years $\cdots$ in general, the j th \$50 has been on deposit for $t - j/52$ years ($1 \leq j \leq 52t$). Find the present value of each \$50 deposit assuming 6% interest compounded continuously, and use the formula

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad (x \neq 1)$$

to find their total value.

- (b) Let

$$p(t) = \frac{Q(t) - P(t)}{P(t)}$$

be the relative error after t years. Find

$$p(\infty) = \lim_{t \rightarrow \infty} p(t).$$

22. A homebuyer borrows P_0 dollars at an annual interest rate r , agreeing to repay the loan with equal monthly payments of M dollars per month over N years.
- (a) Derive a differential equation for the loan principal (amount that the homebuyer owes) $P(t)$ at time $t > 0$, making the simplifying assumption that the homebuyer repays the loan continuously rather than in discrete steps. (See Example 5.1.6.)
- (b) Solve the equation derived in (a).
- (c) Use the result of (b) to determine an approximate value for M assuming that each year has exactly 12 months of equal length.
- (d) It can be shown that the exact value of M is given by

$$M = \frac{rP_0}{12} (1 - (1 + r/12)^{-12N})^{-1}.$$

Compare the value of M obtained from the answer in (c) to the exact value if (i) $P_0 = \$50,000$, $r = 7\frac{1}{2}\%$, $N = 20$ (ii) $P_0 = \$150,000$, $r = 9.0\%$, $N = 30$.

23. Assume that the homebuyer of Exercise 5.1.22 elects to repay the loan continuously at the rate of αM dollars per month, where α is a constant greater than 1. (This is called *accelerated payment*.)
- (a) Determine the time $T(\alpha)$ when the loan will be paid off and the amount $S(\alpha)$ that the homebuyer will save.
- (b) Suppose $P_0 = \$50,000$, $r = 8\%$, and $N = 15$. Compute the savings realized by accelerated payments with $\alpha = 1.05, 1.10$, and 1.15 .

24. A benefactor wishes to establish a trust fund to pay a researcher's salary for T years. The salary is to start at S_0 dollars per year and increase at a fractional rate of a per year. Find the amount of money P_0 that the benefactor must deposit in a trust fund paying interest at a rate r per year. Assume that the researcher's salary is paid continuously, the interest is compounded continuously, and the salary increases are granted continuously.

25. L A radioactive substance with decay constant k is produced at the rate of

$$\frac{at}{1 + btQ(t)}$$

units of mass per unit time, where a and b are positive constants and $Q(t)$ is the mass of the substance present at time t ; thus, the rate of production is small at the start and tends to slow when Q is large.

- (a) Set up a differential equation for Q .
- (b) Choose your own positive values for a , b , k , and $Q_0 = Q(0)$. Use a numerical method to discover what happens to $Q(t)$ as $t \rightarrow \infty$. (Be precise, expressing your conclusions in terms of a , b , k . However, no proof is required.)
26. L Follow the instructions of Exercise 5.1.25, assuming that the substance is produced at the rate of $at/(1 + bt(Q(t))^2)$ units of mass per unit of time.
27. L Follow the instructions of Exercise 5.1.25, assuming that the substance is produced at the rate of $at/(1 + bt)$ units of mass per unit of time.

5.2 Cooling and Mixing

5.2.1 Newton's Law of Cooling

Newton's law of cooling states that if an object with temperature $T(t)$ at time t is in a medium with temperature $T_m(t)$, the rate of change of T at time t is proportional to $T(t) - T_m(t)$; thus, T satisfies a differential equation of the form

$$T' = -k(T - T_m). \quad (5.16)$$

Here $k > 0$, since the temperature of the object must decrease if $T > T_m$, or increase if $T < T_m$. We'll call k the *temperature decay constant of the medium*.

For simplicity, in this section we'll assume that the medium is maintained at a constant temperature T_m . This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it's reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it's a huge cauldron of molten metal. (For more on this see Exercise 5.2.17.)

To solve (5.16), we rewrite it as

$$T' + kT = kT_m.$$

Since e^{-kt} is a solution of the complementary equation, the solutions of this equation are of the form $T = ue^{-kt}$, where $u'e^{-kt} = kT_m$, so $u' = kT_me^{kt}$. Hence,

$$u = T_me^{kt} + c,$$

so

$$T = ue^{-kt} = T_m + ce^{-kt}.$$

If $T(0) = T_0$, setting $t = 0$ here yields $c = T_0 - T_m$, so

$$T = T_m + (T_0 - T_m)e^{-kt}. \quad (5.17)$$

Note that $T - T_m$ decays exponentially, with decay constant k .

Example 5.2.1: A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C . After 4 minutes the temperature of the insulator is 200°C . What is its temperature after 8 minutes?

Solution: Here $T_0 = 400$ and $T_m = 25$, so (5.17) becomes

$$T = 25 + 375e^{-kt}. \quad (5.18)$$

We determine k from the stated condition that $T(4) = 200$; that is,

$$200 = 25 + 375e^{-4k};$$

hence,

$$e^{-4k} = \frac{175}{375} = \frac{7}{15}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{4} \ln \frac{7}{15} = \frac{1}{4} \ln \frac{15}{7}.$$

Substituting this into (5.18) yields

$$T = 25 + 375e^{-\frac{1}{4} \ln \frac{15}{7}}$$

(Figure 5.4). Therefore the temperature of the insulator after 8 minutes is

$$\begin{aligned} T(8) &= 25 + 375e^{-2\ln \frac{15}{7}} \\ &= 25 + 375\left(\frac{7}{15}\right)^2 \approx 107^\circ\text{C}. \end{aligned}$$

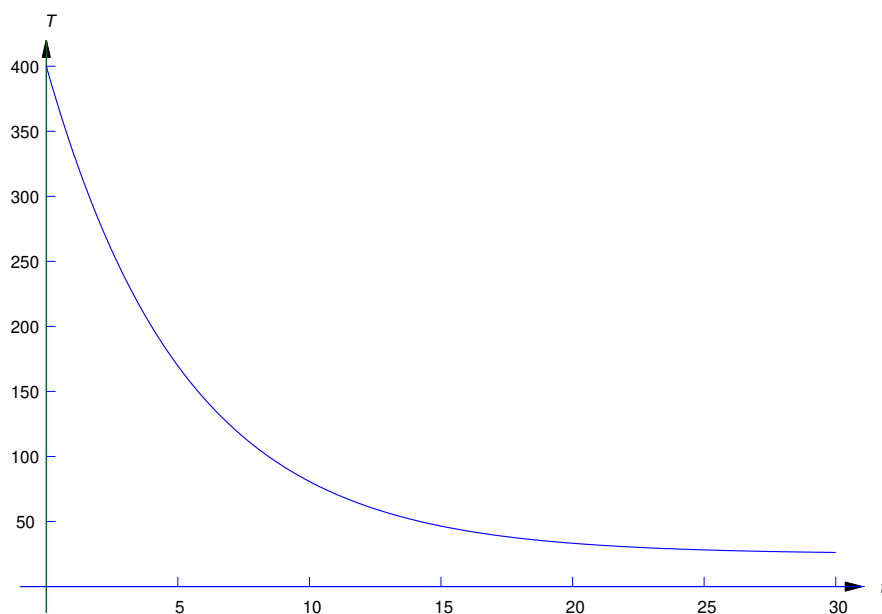


Figure 5.4: $T = 25 + 375e^{-(t/4)\ln 15/7}$

Example 5.2.2: An object with temperature 72°F is placed outside, where the temperature is -20°F . At 11:05 the temperature of the object is 60°F and at 11:07 its temperature is 50°F . At what time was the object placed outside?

Solution: Let $T(t)$ be the temperature of the object at time t . For convenience, we choose the origin $t_0 = 0$ of the time scale to be 11:05 so that $T_0 = 60$. We must determine the time τ when $T(\tau) = 72$. Substituting $T_0 = 60$ and $T_m = -20$ into (5.17) yields

$$T = -20 + (60 - (-20))e^{-kt}$$

or

$$T = -20 + 80e^{-kt}. \quad (5.19)$$

We obtain k from the stated condition that the temperature of the object is 50°F at 11:07. Since 11:07 is $t = 2$ on our time scale, we can determine k by substituting $T = 50$ and $t = 2$ into (5.19) to obtain

$$50 = -20 + 80e^{-2k}$$

(Figure 5.5); hence,

$$e^{-2k} = \frac{70}{80} = \frac{7}{8}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{2} \ln \frac{7}{8} = \frac{1}{2} \ln \frac{8}{7}.$$

Substituting this into (5.19) yields

$$T = -20 + 80e^{-\frac{t}{2} \ln \frac{8}{7}},$$

and the condition $T(\tau) = 72$ implies that

$$72 = -20 + 80e^{-\frac{\tau}{2} \ln \frac{8}{7}};$$

hence,

$$e^{-\frac{\tau}{2} \ln \frac{8}{7}} = \frac{92}{80} = \frac{23}{20}.$$

Taking logarithms and solving for τ yields

$$\tau = -\frac{2 \ln \frac{23}{20}}{\ln \frac{8}{7}} \approx -2.09 \text{ min.}$$

Therefore the object was placed outside about 2 minutes and 5 seconds before 11:05; that is, at 11:02:55.

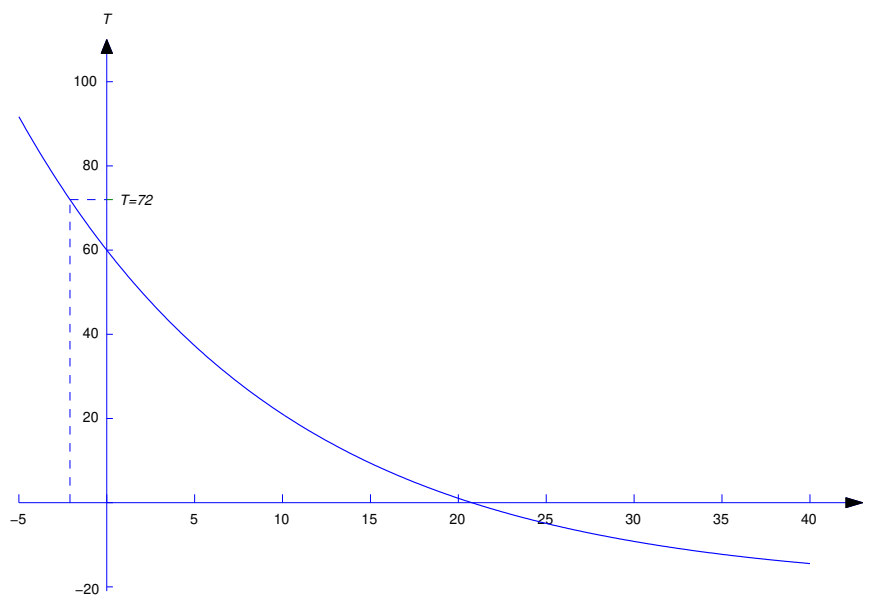


Figure 5.5: $T = -20 + 80e^{-\frac{t}{2} \ln \frac{8}{7}}$

5.2.2 Mixing Problems

In the next two examples a saltwater solution with a given concentration (weight of salt per unit volume of solution) is added at a specified rate to a tank that initially contains saltwater with a different concentration. The problem is to determine the quantity of salt in the tank as a function of time. This is an example of a *mixing problem*. To construct a tractable mathematical model for mixing problems we assume in our

examples (and most exercises) that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Exercises 5.2.22 and 5.2.23 deal with situations where this isn't so, but the distribution of salt becomes approximately uniform as $t \rightarrow \infty$.

Example 5.2.3: A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate (Figure 5.6).

- (a) Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.
- (b) Find $\lim_{t \rightarrow \infty} Q(t)$.

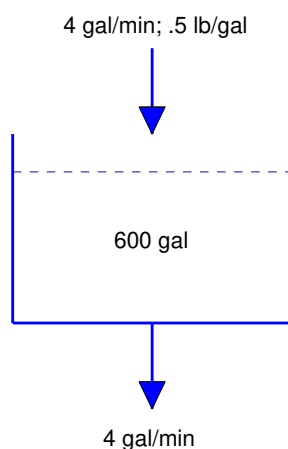


Figure 5.6: A mixing problem

Solution (a): To find a differential equation for Q , we must use the given information to derive an expression for Q' . But Q' is the rate of change of the quantity of salt in the tank changes with respect to time; thus, if *rate in* denotes the rate at which salt enters the tank and *rate out* denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out}. \quad (5.20)$$

The rate in is

$$\left(\frac{1}{2} \text{ lb/gal}\right) \times (4 \text{ gal/min}) = 2 \text{ lb/min}.$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we're removing $1/150$ of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time t , the rate out at any time t is $Q(t)/150$. Alternatively, we

can arrive at this conclusion by arguing that

$$\begin{aligned}\text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) \\ &= (\text{lb/gal}) \times (\text{gal/min}) \\ &= \frac{Q(t)}{600} \times 4 = \frac{Q(t)}{150}.\end{aligned}$$

We can now write (5.20) as

$$Q' = 2 - \frac{Q}{150}.$$

This first order equation can be rewritten as

$$Q' + \frac{Q}{150} = 2.$$

Since $e^{-t/150}$ is a solution of the complementary equation, the solutions of this equation are of the form $Q = ue^{-t/150}$, where $u'e^{-t/150} = 2$, so $u' = 2e^{t/150}$. Hence,

$$u = 300e^{t/150} + c,$$

so

$$Q = ue^{-t/150} = 300 + ce^{-t/150} \quad (5.21)$$

(Figure 5.7). Since $Q(0) = 40$, $c = -260$; therefore,

$$Q = 300 - 260e^{-t/150}.$$

Solution (b): From (5.21), we see that $\lim_{t \rightarrow \infty} Q(t) = 300$ for any value of $Q(0)$. This is intuitively reasonable, since the incoming solution contains 1/2 pound of salt per gallon and there are always 600 gallons of water in the tank.

Example 5.2.4: A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min (Figure 5.8). Find a differential equation for the quantity $Q(t)$ of salt in the tank at time t prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

Solution: We first determine the amount $W(t)$ of solution in the tank at any time t prior to overflow. Since $W(0) = 200$ and we're adding 4 liters/min while removing only 2 liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ liters (capacity of the tank), this formula is valid for $0 \leq t \leq 150$.

Now let $Q(t)$ be the number of grams of salt in the tank at time t , where $0 \leq t \leq 150$. As in Example 5.2.3,

$$Q' = \text{rate in} - \text{rate out}. \quad (5.22)$$

The rate in is

$$\left(\frac{1}{4} \text{ g/liter}\right) \times (4 \text{ liters/min}) = 1 \text{ g/min}. \quad (5.23)$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t + 200$ liters in the tank at time t , the fraction of the mixture being removed per minute at time t is

$$\frac{2}{2t + 200} = \frac{1}{t + 100}.$$

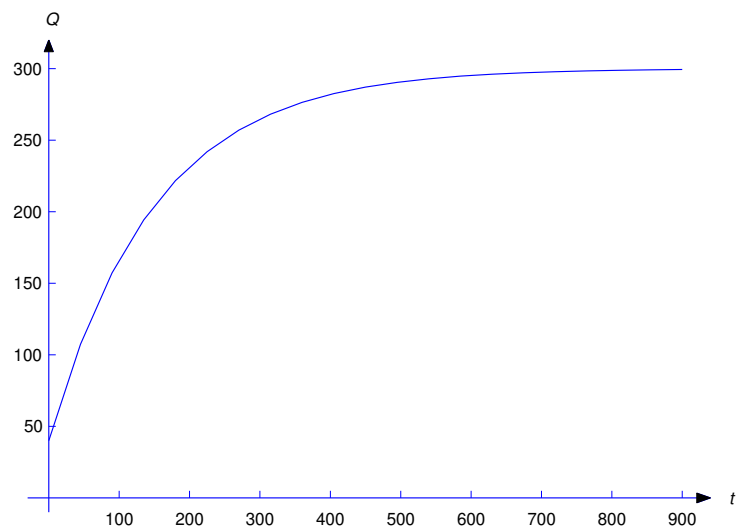


Figure 5.7: $Q = 300 - 260e^{-t/150}$

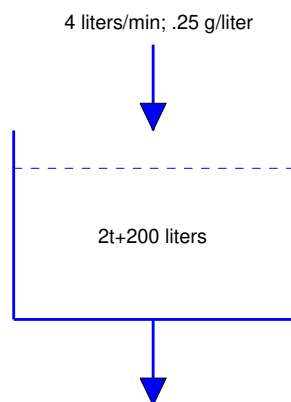


Figure 5.8: Another mixing problem

We're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ grams of salt in the tank at time t ,

$$\text{rate out} = \frac{Q(t)}{t + 100}. \quad (5.24)$$

Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) = (\text{g/liter}) \times (\text{liters/min}) \\ &= \frac{Q(t)}{2t + 200} \times 2 = \frac{Q(t)}{t + 100}. \end{aligned}$$

Substituting (5.23) and (5.24) into (5.22) yields

$$Q' = 1 - \frac{Q}{t + 100}, \quad \text{so} \quad Q' + \frac{1}{t + 100}Q = 1. \quad (5.25)$$

By separation of variables, $1/(t + 100)$ is a solution of the complementary equation, so the solutions of (5.25) are of the form

$$Q = \frac{u}{t + 100}, \quad \text{where} \quad \frac{u'}{t + 100} = 1, \quad \text{so} \quad u' = t + 100.$$

Hence,

$$u = \frac{(t + 100)^2}{2} + c. \quad (5.26)$$

Since $Q(0) = 10$ and $u = (t + 100)Q$, (5.26) implies that

$$(100)(10) = \frac{(100)^2}{2} + c,$$

so

$$c = 100(10) - \frac{(100)^2}{2} = -4000$$

and therefore

$$u = \frac{(t + 100)^2}{2} - 4000.$$

Hence,

$$Q = \frac{u}{t + 100} = \frac{t + 100}{2} - \frac{4000}{t + 100}.$$

Now let $K(t)$ be the concentration of salt at time t . Then

$$K(t) = \frac{1}{4} - \frac{2000}{(t + 100)^2}$$

(Figure 5.9).

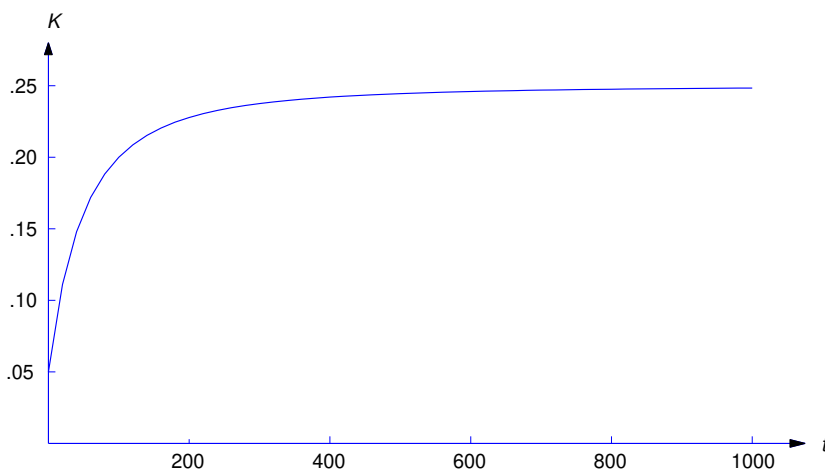


Figure 5.9: $K(t) = \frac{1}{4} - \frac{2000}{(t + 100)^2}$

5.2.3 Exercises

1. A thermometer is moved from a room where the temperature is 70°F to a freezer where the temperature is 12°F . After 30 seconds the thermometer reads 40°F . What does it read after 2 minutes?
2. A fluid initially at 100°C is placed outside on a day when the temperature is -10°C , and the temperature of the fluid drops 20°C in one minute. Find the temperature $T(t)$ of the fluid for $t > 0$.
3. At 12:00 PM a thermometer reading 10°F is placed in a room where the temperature is 70°F . It reads 56° when it's placed outside, where the temperature is 5°F , at 12:03. What does it read at 12:05 PM?
4. A thermometer initially reading 212°F is placed in a room where the temperature is 70°F . After 2 minutes the thermometer reads 125°F .
 - (a) What does the thermometer read after 4 minutes?
 - (b) When will the thermometer read 72°F ?
 - (c) When will the thermometer read 69°F ?
5. An object with initial temperature 150°C is placed outside, where the temperature is 35°C . Its temperatures at 12:15 and 12:20 are 120°C and 90°C , respectively.
 - (a) At what time was the object placed outside?
 - (b) When will its temperature be 40°C ?
6. An object is placed in a room where the temperature is 20°C . The temperature of the object drops by 5°C in 4 minutes and by 7°C in 8 minutes. What was the temperature of the object when it was initially placed in the room?
7. A cup of boiling water is placed outside at 1:00 PM. One minute later the temperature of the water is 152°F . After another minute its temperature is 112°F . What is the outside temperature?
8. A tank initially contains 40 gallons of pure water. A solution with 1 gram of salt per gallon of water is added to the tank at 3 gal/min, and the resulting solution drains out at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

9. A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with $1/2$ pound of salt per gallon is added to the tank at 6 gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.
10. A tank initially contains 100 liters of a salt solution with a concentration of .1 g/liter. A solution with a salt concentration of .3 g/liter is added to the tank at 5 liters/min, and the resulting mixture is drained out at the same rate. Find the concentration $K(t)$ of salt in the tank as a function of t .
11. A 200 gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with $1/4$ pound of salt per gallon is added to the tank at 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find the quantity of salt in the tank as it's about to overflow.
12. Suppose water is added to a tank at 10 gal/min, but leaks out at the rate of $1/5$ gal/min for each gallon in the tank. What is the smallest capacity the tank can have if the process is to continue indefinitely?
13. A chemical reaction in a laboratory with volume V (in ft^3) produces $q_1 \text{ ft}^3/\text{min}$ of a noxious gas as a byproduct. The gas is dangerous at concentrations greater than $\bar{c}$, but harmless at concentrations $\leq \bar{c}$. Intake fans at one end of the laboratory pull in fresh air at the rate of $q_2 \text{ ft}^3/\text{min}$ and exhaust fans at the other end exhaust the mixture of gas and air from the laboratory at the same rate. Assuming that the gas is always uniformly distributed in the room and its initial concentration c_0 is at a safe level, find the smallest value of q_2 required to maintain safe conditions in the laboratory for all time.
14. A 1200-gallon tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is added to the tank at the rate of 6 gal/min and the resulting mixture is drained from the tank at 4 gal/min. Find the quantity $Q(t)$ of salt in the tank at any time $t > 0$ prior to overflow.
15. Tank T_1 initially contain 50 gallons of pure water. Starting at $t_0 = 0$, water that contains 1 pound of salt per gallon is poured into T_1 at the rate of 2 gal/min. The mixture is drained from T_1 at the same rate into a second tank T_2 , which initially contains 50 gallons of pure water. Also starting at $t_0 = 0$, a mixture from another source that contains 2 pounds of salt per gallon is poured into T_2 at the rate of 2 gal/min. The mixture is drained from T_2 at the rate of 4 gal/min.
 - (a) Find a differential equation for the quantity $Q(t)$ of salt in tank T_2 at time $t > 0$.
 - (b) Solve the equation derived in (a) to determine $Q(t)$.
 - (c) Find $\lim_{t \rightarrow \infty} Q(t)$.
16. Suppose an object with initial temperature T_0 is placed in a sealed container, which is in turn placed in a medium with temperature T_m . Let the initial temperature of the container be S_0 . Assume that the temperature of the object does not affect the temperature of the container, which in turn does not affect the temperature of the medium. (These assumptions are reasonable, for example, if the object is a cup of coffee, the container is a house, and the medium is the atmosphere.)
 - (a) Assuming that the container and the medium have distinct temperature decay constants k and k_m respectively, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t .
 - (b) Assuming that the container and the medium have the same temperature decay constant k , use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t .
 - (c) Find $\lim_{t \rightarrow \infty} S(t)$ and $\lim_{t \rightarrow \infty} T(t)$.
17. In our previous examples and exercises concerning Newton's law of cooling we assumed that the temperature of the medium remains constant. This model is adequate if the heat lost or gained by the object is insignificant compared to the heat required to cause an appreciable change in the temperature of the medium. If this isn't so, we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium, respectively, and let T_0 and T_{m0} be their initial values. Again, we assume that T and T_m are related by Newton's law of cooling,

$$T' = -k(T - T_m). \quad (\text{A})$$

We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T - T_0)$ and that the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m - T_{m0})$, where a and a_m are positive constants depending upon the masses and thermal properties of the object and medium, respectively. If we assume that the total heat of the system consisting of the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0. \quad (\text{B})$$

- (a) Equation (A) involves two unknown functions T and T_m . Use (A) and (B) to derive a differential equation involving only T .
- (b) Find $T(t)$ and $T_m(t)$ for $t > 0$.
- (c) Find $\lim_{t \rightarrow \infty} T(t)$ and $\lim_{t \rightarrow \infty} T_m(t)$.
18. Control mechanisms allow fluid to flow into a tank at a rate proportional to the volume V of fluid in the tank, and to flow out at a rate proportional to V^2 . Suppose $V(0) = V_0$ and the constants of proportionality are a and b , respectively. Find $V(t)$ for $t > 0$ and find $\lim_{t \rightarrow \infty} V(t)$.
19. Identical tanks T_1 and T_2 initially contain W gallons each of pure water. Starting at $t_0 = 0$, a salt solution with constant concentration c is pumped into T_1 at r gal/min and drained from T_1 into T_2 at the same rate. The resulting mixture in T_2 is also drained at the same rate. Find the concentrations $c_1(t)$ and $c_2(t)$ in tanks T_1 and T_2 for $t > 0$.
20. An infinite sequence of identical tanks $T_1, T_2, \dots, T_n, \dots$, initially contain W gallons each of pure water. They are hooked together so that fluid drains from T_n into T_{n+1} ($n = 1, 2, \dots$). A salt solution is circulated through the tanks so that it enters and leaves each tank at the constant rate of r gal/min. The solution has a concentration of c pounds of salt per gallon when it enters T_1 .
- (a) Find the concentration $c_n(t)$ in tank T_n for $t > 0$.
- (b) Find $\lim_{t \rightarrow \infty} c_n(t)$ for each n .
21. Tanks T_1 and T_2 have capacities W_1 and W_2 liters, respectively. Initially they are both full of dye solutions with concentrations c_1 and c_2 grams per liter. Starting at $t_0 = 0$, the solution from T_1 is pumped into T_2 at a rate of r liters per minute, and the solution from T_2 is pumped into T_1 at the same rate.
- (a) Find the concentrations $c_1(t)$ and $c_2(t)$ of the dye in T_1 and T_2 for $t > 0$.
- (b) Find $\lim_{t \rightarrow \infty} c_1(t)$ and $\lim_{t \rightarrow \infty} c_2(t)$.
22. **L** Consider the mixing problem of Example 5.2.3, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \rightarrow \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{150}Q = 2$$

where $\lim_{t \rightarrow \infty} a(t) = 1$.

- (a) Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t \rightarrow \infty} Q(t)$?
- (b) Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 - \sin(e^{-t}).$$

23. **L** Consider the mixing problem of Example 5.2.4 in a tank with infinite capacity, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \rightarrow \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{t+100}Q = 1$$

where $\lim_{t \rightarrow \infty} a(t) = 1$.

- (a) Let $K(t)$ be the concentration of salt at time t . Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t \rightarrow \infty} K(t)$?
- (b) Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 + \sin(e^{-t}).$$

5.3 Elementary Mechanics

5.3.1 Newton's Second Law of Motion

In this section we consider an object with constant mass m moving along a line under a force F . Let $y = y(t)$ be the displacement of the object from a reference point on the line at time t , and let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time t . Thus, $v = y'$ and $a = v' = y''$, where the prime denotes differentiation with respect to t . Newton's second law of motion asserts that the force F and the acceleration a are related by the equation

$$F = ma. \quad (5.27)$$

5.3.2 Units

In applications there are three main sets of units in use for length, mass, force, and time: the cgs, mks, and British systems. All three use the second as the unit of time. Table 1 shows the other units. Consistent with (5.27), the unit of force in each system is defined to be the force required to impart an acceleration of (one unit of length)/ s^2 to one unit of mass.

	Length	Force	Mass
cgs	centimeter (cm)	dyne (d)	gram (g)
mks	meter (m)	newton (N)	kilogram (kg)
British	foot (ft)	pound (lb)	slug (sl)

Table 1.

If we assume that Earth is a perfect sphere with constant mass density, Newton's law of gravitation (discussed later in this section) asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth. However, if the object remains sufficiently close to Earth's surface, we may assume that the gravitational force is constant and equal to its value at the surface. The magnitude of this force is mg , where g is called the *acceleration due to gravity*. (To be completely accurate, g should be called the *magnitude of the acceleration due to gravity at Earth's surface*.) This quantity has been determined experimentally. Approximate values of g are

$$\begin{aligned} g &= 980 \text{ cm/s}^2 && \text{(cgs)} \\ g &= 9.8 \text{ m/s}^2 && \text{(mks)} \\ g &= 32 \text{ ft/s}^2 && \text{(British).} \end{aligned}$$

In general, the force F in (5.27) may depend upon t , y , and y' . Since $a = y''$, (5.27) can be written in the form

$$my'' = F(t, y, y'), \quad (5.28)$$

which is a second order equation. We'll consider this equation with restrictions on F later; however, since Chapter 2 dealt only with first order equations, we consider here only problems in which (5.28) can be recast as a first order equation. This is possible if F does not depend on y , so (5.28) is of the form

$$my'' = F(t, y').$$

Letting $v = y'$ and $v' = y''$ yields a first order equation for v :

$$mv' = F(t, v). \quad (5.29)$$

Solving this equation yields v as a function of t . If we know $y(t_0)$ for some time t_0 , we can integrate v to obtain y as a function of t .

Equations of the form (5.29) occur in problems involving motion through a resisting medium.

5.3.3 Motion Through a Resisting Medium Under Constant Gravitational Force

Now we consider an object moving vertically in some medium. We assume that the only forces acting on the object are gravity and resistance from the medium. We also assume that the motion takes place close to Earth's surface and take the upward direction to be positive, so the gravitational force can be assumed to have the constant value $-mg$. We'll see that, under reasonable assumptions on the resisting force, the velocity approaches a limit as $t \rightarrow \infty$. We call this limit the *terminal velocity*.

Example 5.3.1: An object with mass m moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object. (Recall that the speed of an object is $|v|$, the absolute value of its velocity v .) Find the velocity of the object as a function of t , and find the terminal velocity. Assume that the initial velocity is v_0 .

Solution: The total force acting on the object is

$$F = -mg + F_1, \quad (5.30)$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium, which has magnitude $k|v|$, where k is a positive constant. If the object is moving downward ($v \leq 0$), the resisting force is upward (Figure 5.10(a)), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \geq 0$), the resisting force is downward (Figure 5.10(b)), so

$$F_1 = -k|v| = -kv.$$

Thus, (5.30) can be written as

$$F = -mg - kv, \quad (5.31)$$

regardless of the sign of the velocity.

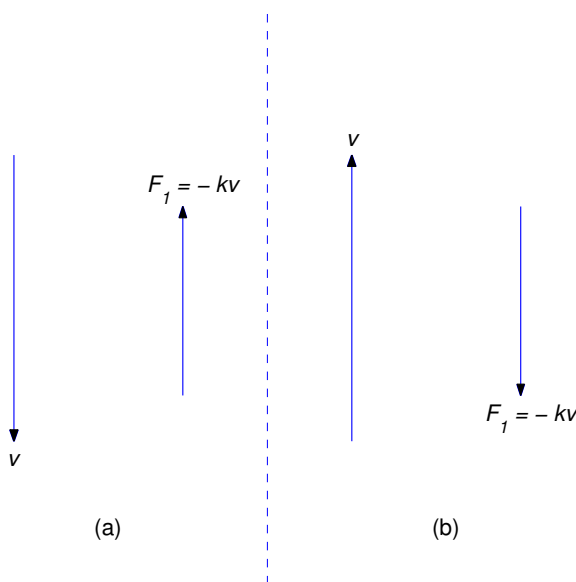


Figure 5.10: Resistive forces

From Newton's second law of motion,

$$F = ma = mv',$$

so (5.31) yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g. \quad (5.32)$$

Since $e^{-kt/m}$ is a solution of the complementary equation, the solutions of (5.32) are of the form $v = ue^{-kt/m}$, where $u'e^{-kt/m} = -g$, so $u' = -ge^{kt/m}$. Hence,

$$u = -\frac{mg}{k}e^{kt/m} + c,$$

so

$$v = ue^{-kt/m} = -\frac{mg}{k} + ce^{-kt/m}. \quad (5.33)$$

Since $v(0) = v_0$,

$$v_0 = -\frac{mg}{k} + c,$$

so

$$c = v_0 + \frac{mg}{k}$$

and (5.33) becomes

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Letting $t \rightarrow \infty$ here shows that the terminal velocity is

$$\lim_{t \rightarrow \infty} v(t) = -\frac{mg}{k},$$

which is independent of the initial velocity v_0 (Figure 5.11).

Example 5.3.2: A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity v as a function of t , and find its terminal velocity.

Solution: Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if v is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32.$$

Since $e^{-t/10}$ is a solution of the complementary equation, the solutions of this equation are of the form $v = ue^{-t/10}$, where $u'e^{-t/10} = -32$, so $u' = -32e^{t/10}$. Hence,

$$u = -320e^{t/10} + c,$$

so

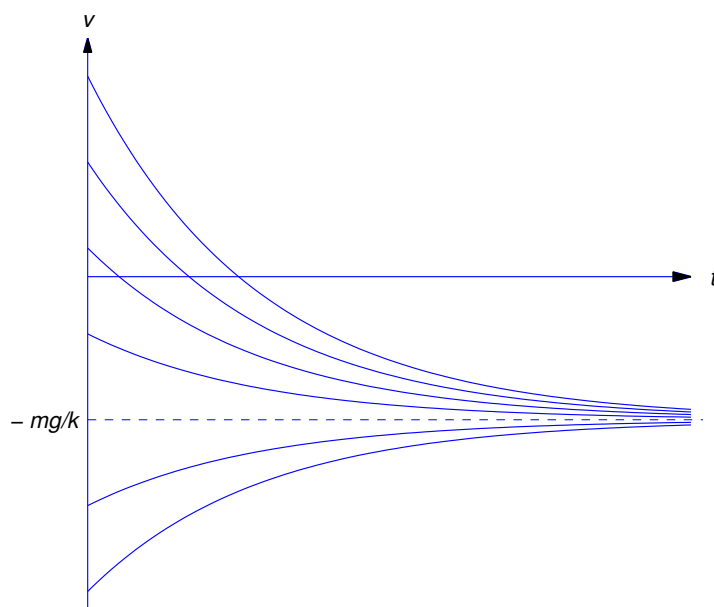
$$v = ue^{-t/10} = -320 + ce^{-t/10}. \quad (5.34)$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v_0 = 60$. Substituting $t = 0$ and $v = 60$ in (5.34) yields

$$60 = -320 + c,$$

so $c = 380$, and (5.34) becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

Figure 5.11: Solutions of $mv' = -mg - kv$

The terminal velocity is

$$\lim_{t \rightarrow \infty} v(t) = -320 \text{ ft/s.}$$

Example 5.3.3: A 10 kg mass is given an initial velocity $v_0 \leq 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of t , and find the terminal velocity.

Solution: Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \quad (5.35)$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2 \text{ N}\cdot\text{s}^2/\text{m}^2$. Since $m = 10$ and $g = 9.8$, (5.35) becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \quad (5.36)$$

If $v_0 = -7$, then $v \equiv -7$ for all $t \geq 0$. If $v_0 \neq -7$, we separate variables to obtain

$$\frac{1}{v^2 - 49} v' = \frac{1}{5}, \quad (5.37)$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2 - 49} = \frac{1}{(v - 7)(v + 7)} = \frac{1}{14} \left[\frac{1}{v - 7} - \frac{1}{v + 7} \right]. \quad (5.38)$$

Substituting (5.38) into (5.37) yields

$$\frac{1}{14} \left[\frac{1}{v-7} - \frac{1}{v+7} \right] v' = \frac{1}{5},$$

so

$$\left[\frac{1}{v-7} - \frac{1}{v+7} \right] v' = \frac{14}{5}.$$

Integrating this yields

$$\ln|v-7| - \ln|v+7| = 14t/5 + k.$$

Therefore

$$\left| \frac{v-7}{v+7} \right| = e^k e^{14t/5}.$$

Since Theorem 2.3.1 implies that $(v-7)/(v+7)$ can't change sign (why?), we can rewrite the last equation as

$$\frac{v-7}{v+7} = ce^{14t/5}, \quad (5.39)$$

which is an implicit solution of (5.36). Solving this for v yields

$$v = -7 \frac{c + e^{-14t/5}}{c - e^{-14t/5}}. \quad (5.40)$$

Since $v(0) = v_0$, it (5.39) implies that

$$c = \frac{v_0 - 7}{v_0 + 7}.$$

Substituting this into (5.40) and simplifying yields

$$v = -7 \frac{v_0(1 + e^{-14t/5}) - 7(1 - e^{-14t/5})}{v_0(1 - e^{-14t/5}) - 7(1 + e^{-14t/5})}.$$

Since $v_0 \leq 0$, v is defined and negative for all $t > 0$. The terminal velocity is

$$\lim_{t \rightarrow \infty} v(t) = -7 \text{ m/s},$$

independent of v_0 . More generally, it can be shown (Exercise 5.3.11) that if v is any solution of (5.35) such that $v_0 \leq 0$ then

$$\lim_{t \rightarrow \infty} v(t) = -\sqrt{\frac{mg}{k}}$$

(Figure 5.12).

Example 5.3.4: A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of v_0 m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time T required for the mass to reach maximum altitude.

Solution: The mass will climb while $v > 0$ and reach its maximum altitude when $v = 0$. Therefore $v > 0$ for $0 \leq t < T$ and $v(T) = 0$. Although the mass of the object and our assumptions concerning the forces acting on it are the same as those in Example 3, (5.36) does not apply here, since the resisting force is negative if $v > 0$; therefore, we replace (5.36) by

$$10v' = -98 - 2v^2. \quad (5.41)$$

Separating variables yields

$$\frac{5}{v^2 + 49} v' = -1,$$

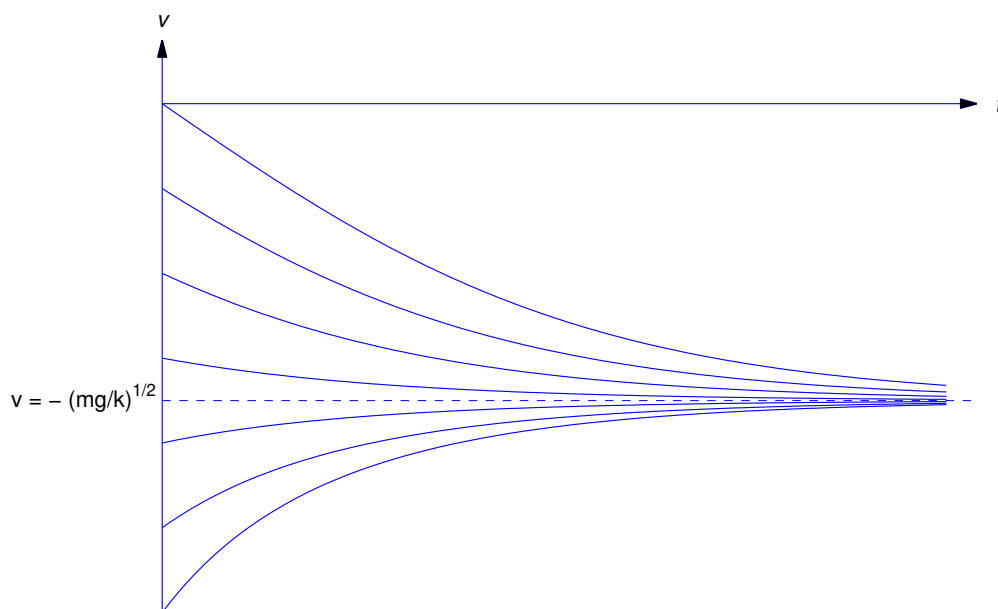


Figure 5.12: Solutions of $mv' = -mg + kv^2$, $v(0) = v_0 \leq 0$

and integrating this yields

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + c.$$

(Recall that $\tan^{-1} u$ is the number θ such that $-\pi/2 < \theta < \pi/2$ and $\tan \theta = u$.) Since $v(0) = v_0$,

$$c = \frac{5}{7} \tan^{-1} \frac{v_0}{7},$$

so v is defined implicitly by

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + \frac{5}{7} \tan^{-1} \frac{v_0}{7}, \quad 0 \leq t \leq T. \quad (5.42)$$

Solving this for v yields

$$v = 7 \tan \left(-\frac{7t}{5} + \tan^{-1} \frac{v_0}{7} \right). \quad (5.43)$$

Using the identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

with $A = \tan^{-1}(v_0/7)$ and $B = 7t/5$, and noting that $\tan(\tan^{-1} \theta) = \theta$, we can simplify (5.43) to

$$v = 7 \frac{v_0 - 7 \tan(7t/5)}{7 + v_0 \tan(7t/5)}.$$

Since $v(T) = 0$ and $\tan^{-1}(0) = 0$, (5.42) implies that

$$-T + \frac{5}{7} \tan^{-1} \frac{v_0}{7} = 0.$$

Therefore

$$T = \frac{5}{7} \tan^{-1} \frac{v_0}{7}.$$

Since $\tan^{-1}(v_0/7) < \pi/2$ for all v_0 , the time required for the mass to reach its maximum altitude is less than

$$\frac{5\pi}{14} \approx 1.122 \text{ s}$$

regardless of the initial velocity. Figure 5.13 shows graphs of v over $[0, T]$ for various values of v_0 .

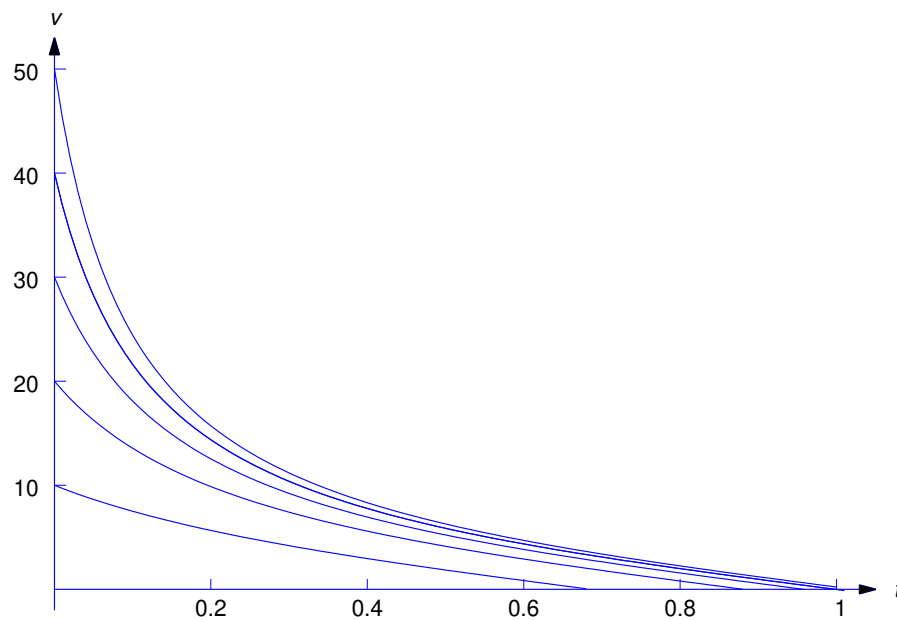


Figure 5.13: Solutions of (5.41) for various $v_0 > 0$

5.3.4 Escape Velocity

Suppose a space vehicle is launched vertically and its fuel is exhausted when the vehicle reaches an altitude h above Earth, where h is sufficiently large so that resistance due to Earth's atmosphere can be neglected. Let $t = 0$ be the time when burnout occurs. Assuming that the gravitational forces of all other celestial bodies can be neglected, the motion of the vehicle for $t > 0$ is that of an object with constant mass m under the influence of Earth's gravitational force, which we now assume to vary inversely with the square of the distance from Earth's center; thus, if we take the upward direction to be positive then gravitational force on the vehicle at an altitude y above Earth is

$$F = -\frac{K}{(y + R)^2}, \quad (5.44)$$

where R is Earth's radius (Figure 5.14).

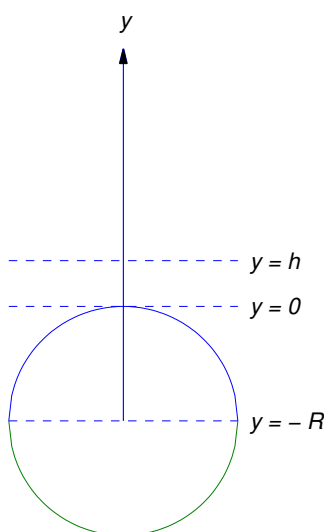


Figure 5.14: Escape velocity

Since $F = -mg$ when $y = 0$, setting $y = 0$ in (5.44) yields

$$-mg = -\frac{K}{R^2};$$

therefore $K = mgR^2$ and (5.44) can be written more specifically as

$$F = -\frac{mgR^2}{(y + R)^2}. \quad (5.45)$$

From Newton's second law of motion,

$$F = m \frac{d^2y}{dt^2},$$

so (5.45) implies that

$$\frac{d^2y}{dt^2} = -\frac{gR^2}{(y + R)^2}. \quad (5.46)$$

We'll show that there's a number v_e , called the *escape velocity*, with these properties:

1. If $v_0 \geq v_e$ then $v(t) > 0$ for all $t > 0$, and the vehicle continues to climb for all $t > 0$; that is, it “escapes” Earth. (Is it really so obvious that $\lim_{t \rightarrow \infty} y(t) = \infty$ in this case? For a proof, see Exercise 5.3.20.)
2. If $v_0 < v_e$ then $v(t)$ decreases to zero and becomes negative. Therefore the vehicle attains a maximum altitude y_m and falls back to Earth.

Since (5.46) is second order, we can’t solve it by methods discussed so far. However, we’re concerned with v rather than y , and v is easier to find. Since $v = y'$ the chain rule implies that

$$\frac{d^2 y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy}.$$

Substituting this into (5.46) yields the first order separable equation

$$v \frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}. \quad (5.47)$$

When $t = 0$, the velocity is v_0 and the altitude is h . Therefore we can obtain v as a function of y by solving the initial value problem

$$v \frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}, \quad v(h) = v_0.$$

Integrating (5.47) with respect to y yields

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + c. \quad (5.48)$$

Since $v(h) = v_0$,

$$c = \frac{v_0^2}{2} - \frac{gR^2}{h+R},$$

so (5.48) becomes

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R} \right). \quad (5.49)$$

If

$$v_0 \geq \left(\frac{2gR^2}{h+R} \right)^{1/2},$$

the parenthetical expression in (5.49) is nonnegative, so $v(y) > 0$ for $y > h$. This proves that there’s an escape velocity v_e . We’ll now prove that

$$v_e = \left(\frac{2gR^2}{h+R} \right)^{1/2}$$

by showing that the vehicle falls back to Earth if

$$v_0 < \left(\frac{2gR^2}{h+R} \right)^{1/2}. \quad (5.50)$$

If (5.50) holds then the parenthetical expression in (5.49) is negative and the vehicle will attain a maximum altitude $y_m > h$ that satisfies the equation

$$0 = \frac{gR^2}{y_m+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R} \right).$$

The velocity will be zero at the maximum altitude, and the object will then fall to Earth under the influence of gravity.

5.3.5 Exercises

Except where directed otherwise, assume that the magnitude of the gravitational force on an object with mass m is constant and equal to mg . In exercises involving vertical motion take the upward direction to be positive.

1. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to his speed, with $k = 2.5$ lb-s/ft. Assuming that he starts from rest, find his velocity as a function of time and find his terminal velocity.
2. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to her speed, with constant of proportionality k . Find k , given that her terminal velocity is -16 ft/s, and then find her velocity v as a function of t . Assume that she starts from rest.
3. A boat weighs 64,000 lb. Its propellor produces a constant thrust of 50,000 lb and the water exerts a resistive force with magnitude proportional to the speed, with $k = 2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.
4. A constant horizontal force of 10 N pushes a 20 kg-mass through a medium that resists its motion with .5 N for every m/s of speed. The initial velocity of the mass is 7 m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t > 0$.
5. A stone weighing $1/2$ lb is thrown upward from an initial height of 5 ft with an initial speed of 32 ft/s. Air resistance is proportional to speed, with $k = 1/128$ lb-s/ft. Find the maximum height attained by the stone.
6. A 3200-lb car is moving at 64 ft/s down a 30-degree grade when it runs out of fuel. Find its velocity after that if friction exerts a resistive force with magnitude proportional to the square of the speed, with $k = 1$ lb-s²/ft². Also find its terminal velocity.
7. A 96 lb weight is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the speed. Find its velocity as a function of time if its terminal velocity is -128 ft/s.
8. An object with mass m moves vertically through a medium that exerts a resistive force with magnitude proportional to the speed. Let $y = y(t)$ be the altitude of the object at time t , with $y(0) = y_0$. Use the results of Example 5.3.1 to show that

$$y(t) = y_0 + \frac{m}{k}(v_0 - v - gt).$$

9. An object with mass m is launched vertically upward with initial velocity v_0 from Earth's surface ($y_0 = 0$) in a medium that exerts a resistive force with magnitude proportional to the speed. Find the time T when the object attains its maximum altitude y_m . Then use the result of Exercise 5.3.8 to find y_m .
10. An object weighing 256 lb is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the square of the speed. The magnitude of the resisting force is 1 lb when $|v| = 4$ ft/s. Find v for $t > 0$, and find its terminal velocity.
11. An object with mass m is given an initial velocity $v_0 \leq 0$ in a medium that exerts a resistive force with magnitude proportional to the square of the speed. Find the velocity of the object for $t > 0$, and find its terminal velocity.
12. An object with mass m is launched vertically upward with initial velocity v_0 in a medium that exerts a resistive force with magnitude proportional to the square of the speed.
 - (a) Find the time T when the object reaches its maximum altitude.
 - (b) Use the result of Exercise 5.3.11 to find the velocity of the object for $t > T$.
13. L An object with mass m is given an initial velocity $v_0 \leq 0$ in a medium that exerts a resistive force of the form $a|v|/(1 + |v|)$, where a is positive constant.

- (a) Set up a differential equation for the speed of the object.
- (b) Use your favorite numerical method to solve the equation you found in (a), to convince yourself that there's a unique number a_0 such that $\lim_{t \rightarrow \infty} s(t) = \infty$ if $a \leq a_0$ and $\lim_{t \rightarrow \infty} s(t)$ exists (finite) if $a > a_0$. (We say that a_0 is the *bifurcation value* of a .) Try to find a_0 and $\lim_{t \rightarrow \infty} s(t)$ in the case where $a > a_0$. HINT: See Exercise 5.3.14.
14. An object of mass m falls in a medium that exerts a resistive force $f = f(s)$, where $s = |v|$ is the speed of the object. Assume that $f(0) = 0$ and f is strictly increasing and differentiable on $(0, \infty)$.
- (a) Write a differential equation for the speed $s = s(t)$ of the object. Take it as given that all solutions of this equation with $s(0) \geq 0$ are defined for all $t > 0$ (which makes good sense on physical grounds).
- (b) Show that if $\lim_{s \rightarrow \infty} f(s) \leq mg$ then $\lim_{t \rightarrow \infty} s(t) = \infty$.
- (c) Show that if $\lim_{s \rightarrow \infty} f(s) > mg$ then $\lim_{t \rightarrow \infty} s(t) = s_T$ (terminal speed), where $f(s_T) = mg$. HINT: Use Theorem 2.3.1.
15. A 100-g mass with initial velocity $v_0 \leq 0$ falls in a medium that exerts a resistive force proportional to the fourth power of the speed. The resistance is .1 N if the speed is 3 m/s.
- (a) Set up the initial value problem for the velocity v of the mass for $t > 0$.
- (b) Use Exercise 5.3.14(c) to determine the terminal velocity of the object.
- (c) C To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 1]$ (seconds) of the initial value problem of (a), with initial values $v_0 = 0, -2, -4, \dots, -12$. Present your results in graphical form similar to Figure 5.12.
16. A 64-lb object with initial velocity $v_0 \leq 0$ falls through a dense fluid that exerts a resistive force proportional to the square root of the speed. The resistance is 64 lb if the speed is 16 ft/s.
- (a) Set up the initial value problem for the velocity v of the mass for $t > 0$.
- (b) Use Exercise 5.3.14(c) to determine the terminal velocity of the object.
- (c) C To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 4]$ (seconds) of the initial value problem of (a), with initial values $v_0 = 0, -5, -10, \dots, -30$. Present your results in graphical form similar to Figure 5.12.

In Exercises 5.3.17–5.3.20, assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

17. A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.
18. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s². Find the escape velocity in miles/s.
19. (a) Show that Eqn. (5.49) can be rewritten as

$$v^2 = \frac{h-y}{y+R} v_e^2 + v_0^2.$$

- (b) Show that if $v_0 = \rho v_e$ with $0 \leq \rho < 1$, then the maximum altitude y_m attained by the space vehicle is

$$y_m = \frac{h + R\rho^2}{1 - \rho^2}.$$

- (c) By requiring that $v(y_m) = 0$, use Eqn. (5.48) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[\frac{(1 - \rho^2)(y_m - y)}{y + R} \right]^{1/2},$$

where y_m and ρ are as defined in (b) and $y \geq h$.

- (d) Deduce from (c) that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.
20. In the situation considered in the discussion of escape velocity, show that $\lim_{t \rightarrow \infty} y(t) = \infty$ if $v(t) > 0$ for all $t > 0$.

HINT: Use a proof by contradiction. Assume that there's a number y_m such that $y(t) \leq y_m$ for all $t > 0$. Deduce from this that there's positive number α such that $y''(t) \leq -\alpha$ for all $t \geq 0$. Show that this contradicts the assumption that $v(t) > 0$ for all $t > 0$.

5.4 Autonomous Second Order Equations

A second order differential equation that can be written as

$$y'' = F(y, y') \quad (5.51)$$

where F is independent of t , is said to be *autonomous*. An autonomous second order equation can be converted into a first order equation relating $v = y'$ and y . If we let $v = y'$, (5.51) becomes

$$v' = F(y, v). \quad (5.52)$$

Since

$$v' = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy}, \quad (5.53)$$

(5.52) can be rewritten as

$$v \frac{dv}{dy} = F(y, v). \quad (5.54)$$

The integral curves of (5.54) can be plotted in the (y, v) plane, which is called the *Poincaré phase plane* of (5.51). If y is a solution of (5.51) then $y = y(t)$, $v = y'(t)$ is a parametric equation for an integral curve of (5.54). We'll call these integral curves *trajectories* of (5.51), and we'll call (5.54) the *phase plane equivalent* of (5.51).

In this section we'll consider autonomous equations that can be written as

$$y'' + q(y, y')y' + p(y) = 0. \quad (5.55)$$

Equations of this form often arise in applications of Newton's second law of motion. For example, suppose y is the displacement of a moving object with mass m . It's reasonable to think of two types of time-independent forces acting on the object. One type - such as gravity - depends only on position. We could write such a force as $-mp(y)$. The second type - such as atmospheric resistance or friction - may depend on position and velocity. (Forces that depend on velocity are called *damping* forces.) We write this force as $-mq(y, y')y'$, where $q(y, y')$ is usually a positive function and we've put the factor y' outside to make it explicit that the force is in the direction opposing the motion. In this case Newton's, second law of motion leads to (5.55).

The phase plane equivalent of (5.55) is

$$v \frac{dv}{dy} + q(y, v)v + p(y) = 0. \quad (5.56)$$

Some statements that we'll be making about the properties of (5.55) and (5.56) are intuitively reasonable, but difficult to prove. Therefore our presentation in this section will be informal: we'll just say things without proof, all of which are true if we assume that $p = p(y)$ is continuously differentiable for all y and $q = q(y, v)$ is continuously differentiable for all (y, v) . We begin with the following statements:

- **Statement 1.** If y_0 and v_0 are arbitrary real numbers then (5.55) has a unique solution on $(-\infty, \infty)$ such that $y(0) = y_0$ and $y'(0) = v_0$.
- **Statement 2.)** If $y = y(t)$ is a solution of (5.55) and τ is any constant then $y_1 = y(t - \tau)$ is also a solution of (5.55), and y and y_1 have the same trajectory.
- **Statement 3.** If two solutions y and y_1 of (5.55) have the same trajectory then $y_1(t) = y(t - \tau)$ for some constant τ .
- **Statement 4.** Distinct trajectories of (5.55) can't intersect; that is, if two trajectories of (5.55) intersect, they are identical.

- **Statement 5.** If the trajectory of a solution of (5.55) is a closed curve then $(y(t), v(t))$ traverses the trajectory in a finite time T , and the solution is periodic with period T ; that is, $y(t + T) = y(t)$ for all t in $(-\infty, \infty)$.

If $\bar{y}$ is a constant such that $p(\bar{y}) = 0$ then $y \equiv \bar{y}$ is a constant solution of (5.55). We say that $\bar{y}$ is an *equilibrium* of (5.55) and $(\bar{y}, 0)$ is a *critical point* of the phase plane equivalent equation (5.56). We say that the equilibrium and the critical point are *stable* if, for any given $\epsilon > 0$ *no matter how small*, there's a $\delta > 0$, *sufficiently small*, such that if

$$\sqrt{(y_0 - \bar{y})^2 + v_0^2} < \delta$$

then the solution of the initial value problem

$$y'' + q(y, y')y' + p(y) = 0, \quad y(0) = y_0, \quad y'(0) = v_0$$

satisfies the inequality

$$\sqrt{(y(t) - \bar{y})^2 + (v(t))^2} < \epsilon$$

for all $t > 0$. Figure 5.15 illustrates the geometrical interpretation of this definition in the Poincaré phase plane: if (y_0, v_0) is in the smaller shaded circle (with radius δ), then $(y(t), v(t))$ must be in the larger circle (with radius ϵ) for all $t > 0$.

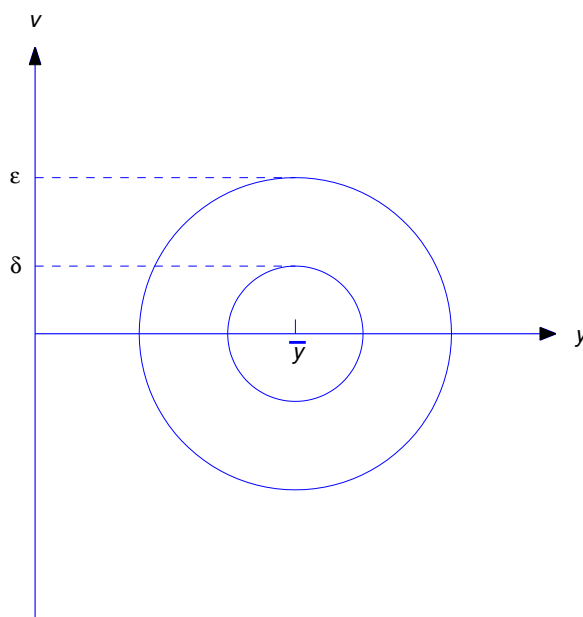


Figure 5.15: Stability: if (y_0, v_0) is in the smaller circle then $(y(t), v(t))$ is in the larger circle for all $t > 0$

If an equilibrium and the associated critical point are not stable, we say they are *unstable*. To see if you really understand what *stable* means, try to give a direct definition of *unstable* (Exercise 5.4.22). We'll illustrate both definitions in the following examples.

5.4.1 The Undamped Case

We'll begin with the case where $q \equiv 0$, so (5.55) reduces to

$$y'' + p(y) = 0. \tag{5.57}$$

We say that this equation - as well as any physical situation that it may model - is *undamped*. The phase plane equivalent of (5.57) is the separable equation

$$v \frac{dv}{dy} + p(y) = 0.$$

Integrating this yields

$$\frac{v^2}{2} + P(y) = c, \quad (5.58)$$

where c is a constant of integration and $P(y) = \int p(y) dy$ is an antiderivative of p .

If (5.57) is the equation of motion of an object of mass m , then $mv^2/2$ is the kinetic energy and $mP(y)$ is the potential energy of the object; thus, (5.58) says that the total energy of the object remains constant, or is *conserved*. In particular, if a trajectory passes through a given point (y_0, v_0) then

$$c = \frac{v_0^2}{2} + P(y_0).$$

Example 5.4.1: [The Undamped Spring - Mass System] Consider an object with mass m suspended from a spring and moving vertically. Let y be the displacement of the object from the position it occupies when suspended at rest from the spring (Figure 5.16).

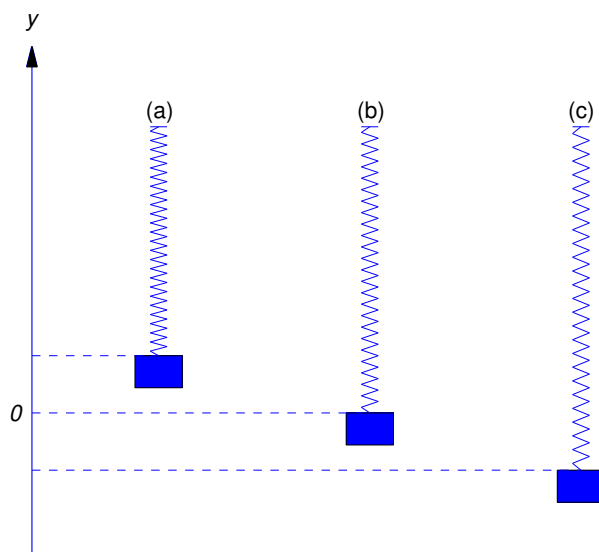


Figure 5.16: (a) $y > 0$ (b) $y = 0$ (c) $y < 0$

Assume that if the length of the spring is changed by an amount ΔL (positive or negative), then the spring exerts an opposing force with magnitude $k|\Delta L|$, where k is a positive constant. In Section 6.1 it will be shown that if the mass of the spring is negligible compared to m and no other forces act on the object then Newton's second law of motion implies that

$$my'' = -ky, \quad (5.59)$$

which can be written in the form (5.57) with $p(y) = ky/m$. This equation can be solved easily by a method that we'll study in Section 5.2, but that method isn't available here. Instead, we'll consider the phase plane equivalent of (5.59).

From (5.53), we can rewrite (5.59) as the separable equation

$$mv \frac{dv}{dy} = -ky.$$

Integrating this yields

$$\frac{mv^2}{2} = -\frac{ky^2}{2} + c,$$

which implies that

$$mv^2 + ky^2 = \rho \tag{5.60}$$

($\rho = 2c$). This defines an ellipse in the Poincaré phase plane (Figure 5.17).

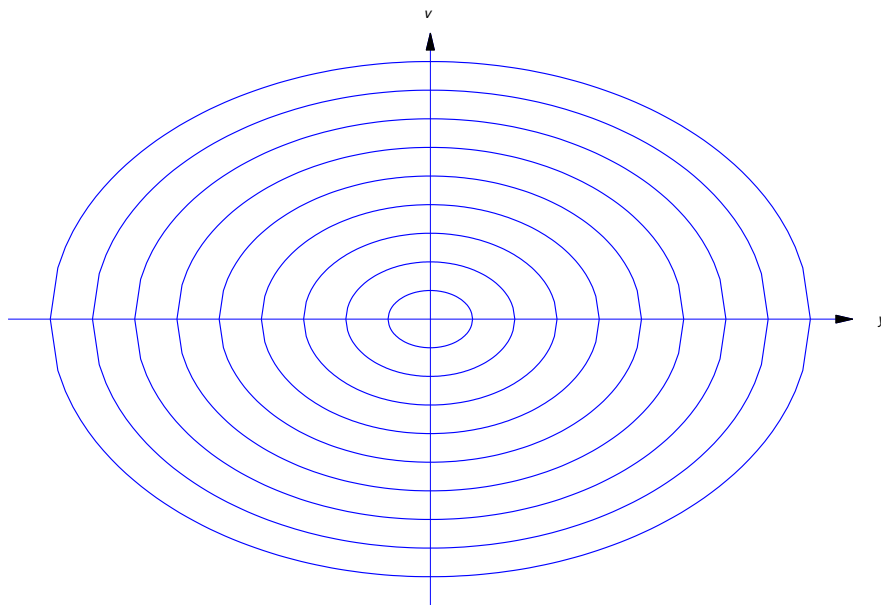


Figure 5.17: Trajectories of $my'' + ky = 0$

We can identify ρ by setting $t = 0$ in (5.60); thus, $\rho = mv_0^2 + ky_0^2$, where $y_0 = y(0)$ and $v_0 = v(0)$. To determine the maximum and minimum values of y we set $v = 0$ in (5.60); thus,

$$y_{\max} = R \quad \text{and} \quad y_{\min} = -R, \quad \text{with } R = \sqrt{\frac{\rho}{k}}. \tag{5.61}$$

Equation (5.59) has exactly one equilibrium, $\bar{y} = 0$, and it's stable. You can see intuitively why this is so: if the object is displaced in either direction from equilibrium, the spring tries to bring it back.

In this case we can find y explicitly as a function of t . (Don't expect this to happen in more complicated problems!) If $v > 0$ on an interval I , (5.60) implies that

$$\frac{dy}{dt} = v = \sqrt{\frac{\rho - ky^2}{m}}$$

on I . This is equivalent to

$$\frac{\sqrt{k}}{\sqrt{\rho - ky^2}} \frac{dy}{dt} = \omega_0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{5.62}$$

Since

$$\int \frac{\sqrt{k} dy}{\sqrt{\rho - ky^2}} = \sin^{-1} \left(\sqrt{\frac{k}{\rho}} y \right) + c = \sin^{-1} \left(\frac{y}{R} \right) + c$$

(see (5.61)), (5.62) implies that there's a constant ϕ such that

$$\sin^{-1} \left(\frac{y}{R} \right) = \omega_0 t + \phi$$

or

$$y = R \sin(\omega_0 t + \phi)$$

for all t in I . Although we obtained this function by assuming that $v > 0$, you can easily verify that y satisfies (5.59) for all values of t . Thus, the displacement varies periodically between $-R$ and R , with period $T = 2\pi/\omega_0$ (Figure 5.18). (If you've taken a course in elementary mechanics you may recognize this as *simple harmonic motion*.)

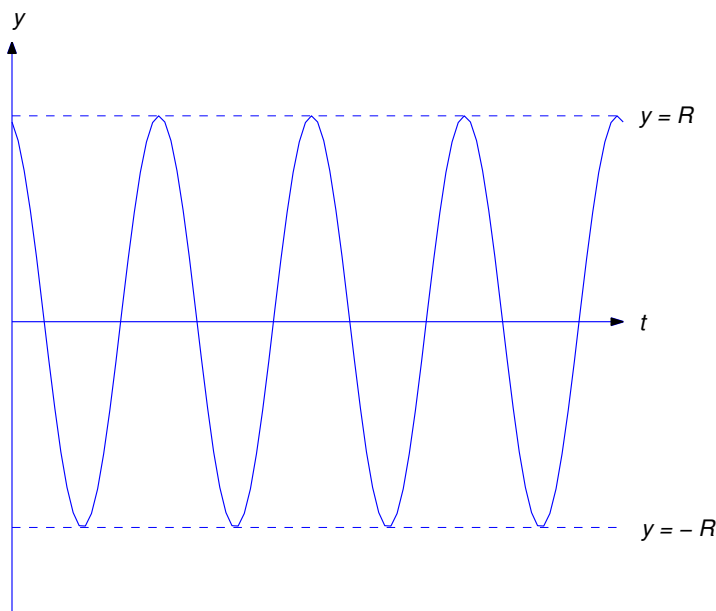


Figure 5.18: $y = R \sin(\omega_0 t + \phi)$

Example 5.4.2: [The Undamped Pendulum] Now we consider the motion of a pendulum with mass m , attached to the end of a weightless rod with length L that rotates on a frictionless axle (Figure 5.19). We assume that there's no air resistance.

Let y be the angle measured from the rest position (vertically downward) of the pendulum, as shown in Figure 5.19. Newton's second law of motion says that the product of m and the tangential acceleration equals the tangential component of the gravitational force; therefore, from Figure 5.19,

$$mLy'' = -mg \sin y,$$

or

$$y'' = -\frac{g}{L} \sin y. \quad (5.63)$$

Since $\sin n\pi = 0$ if n is any integer, (5.63) has infinitely many equilibria $\bar{y}_n = n\pi$. If n is even, the mass is directly below the axle (Figure 5.20 (a)) and gravity opposes any deviation from the equilibrium. However,

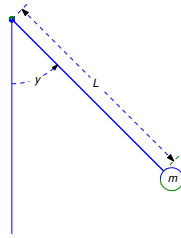


Figure 5.19: The undamped pendulum

if n is odd, the mass is directly above the axle (Figure 5.20 (b)) and gravity increases any deviation from the equilibrium. Therefore we conclude on physical grounds that $\bar{y}_{2m} = 2m\pi$ is stable and $\bar{y}_{2m+1} = (2m+1)\pi$ is unstable.

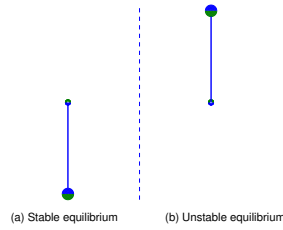


Figure 5.20: (a) Stable equilibrium (b) Unstable equilibrium

The phase plane equivalent of (5.63) is

$$v \frac{dv}{dy} = -\frac{g}{L} \sin y,$$

where $v = y'$ is the angular velocity of the pendulum. Integrating this yields

$$\frac{v^2}{2} = \frac{g}{L} \cos y + c. \quad (5.64)$$

If $v = v_0$ when $y = 0$, then

$$c = \frac{v_0^2}{2} - \frac{g}{L},$$

so (5.64) becomes

$$\frac{v^2}{2} = \frac{v_0^2}{2} - \frac{g}{L}(1 - \cos y) = \frac{v_0^2}{2} - \frac{2g}{L} \sin^2 \frac{y}{2},$$

which is equivalent to

$$v^2 = v_0^2 - v_c^2 \sin^2 \frac{y}{2}, \quad (5.65)$$

where

$$v_c = 2\sqrt{\frac{g}{L}}.$$

The curves defined by (5.65) are the trajectories of (5.63). They are periodic with period 2π in y , which isn't surprising, since if $y = y(t)$ is a solution of (5.63) then so is $y_n = y(t) + 2n\pi$ for any integer n . Figure 5.21 shows trajectories over the interval $[-\pi, \pi]$. From (5.65), you can see that if $|v_0| > v_c$ then v is nonzero for all t , which means that the object whirls in the same direction forever, as in Figure 5.22. The trajectories associated with this whirling motion are above the upper dashed curve and below the lower dashed curve in Figure 5.21. You can also see from (5.65) that if $0 < |v_0| < v_c$, then $v = 0$ when $y = \pm y_{\max}$, where

$$y_{\max} = 2 \sin^{-1}(|v_0|/v_c).$$

In this case the pendulum oscillates periodically between $-y_{\max}$ and $y_{\max}$, as shown in Figure 5.23. The trajectories associated with this kind of motion are the ovals between the dashed curves in Figure 5.21. It can be shown (see Exercise 5.4.21 for a partial proof) that the period of the oscillation is

$$T = 8 \int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2 \theta}}. \quad (5.66)$$

Although this integral can't be evaluated in terms of familiar elementary functions, you can see that it's finite if $|v_0| < v_c$.

The dashed curves in Figure 5.21 contain four trajectories. The critical points $(\pi, 0)$ and $(-\pi, 0)$ are the trajectories of the unstable equilibrium solutions $\bar{y} = \pm\pi$. The upper dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0$, $v(t_0) = v_c$. If y is any solution with this trajectory then

$$\lim_{t \rightarrow \infty} y(t) = \pi \quad \text{and} \quad \lim_{t \rightarrow -\infty} y(t) = -\pi.$$

The lower dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0$, $v(t_0) = -v_c$. If y is any solution with this trajectory then

$$\lim_{t \rightarrow \infty} y(t) = -\pi \quad \text{and} \quad \lim_{t \rightarrow -\infty} y(t) = \pi.$$

Consistent with this, the integral (5.66) diverges to ∞ if $v_0 = \pm v_c$. (Exercise 5.4.21).

Since the dashed curves separate trajectories of whirling solutions from trajectories of oscillating solutions, each of these curves is called a *separatrix*.

In general, if (5.57) has both stable and unstable equilibria then the separatrices are the curves given by (5.58) that pass through unstable critical points. Thus, if $(\bar{y}, 0)$ is an unstable critical point, then

$$\frac{v^2}{2} + P(y) = P(\bar{y}) \quad (5.67)$$

defines a separatrix passing through $(\bar{y}, 0)$.

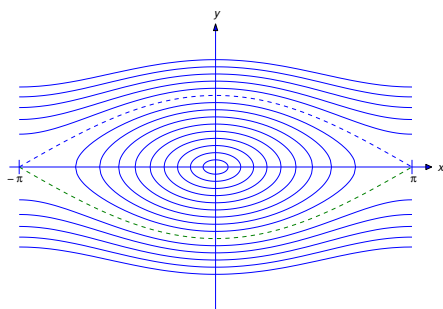


Figure 5.21: Trajectories of the undamped pendulum

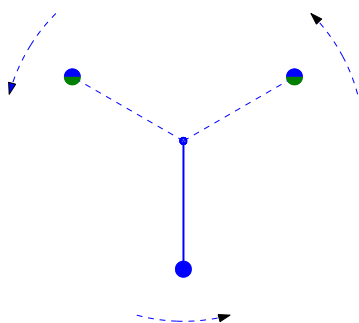


Figure 5.22: The whirling undamped pendulum

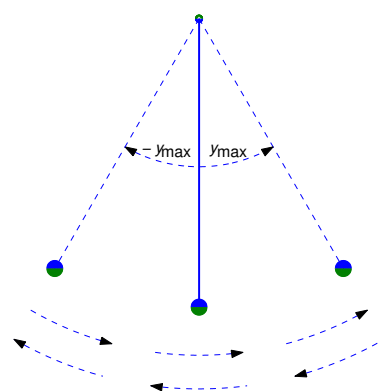


Figure 5.23: The oscillating undamped pendulum

5.4.2 Stability and Instability Conditions for $y'' + p(y) = 0$

It can be shown (Exercise 5.4.23) that an equilibrium $\bar{y}$ of an undamped equation

$$y'' + p(y) = 0 \quad (5.68)$$

is stable if there's an open interval (a, b) containing $\bar{y}$ such that

$$p(y) < 0 \text{ if } a < y < \bar{y} \text{ and } p(y) > 0 \text{ if } \bar{y} < y < b. \quad (5.69)$$

If we regard $p(y)$ as a force acting on a unit mass, (5.69) means that the force resists all sufficiently small displacements from $\bar{y}$.

We've already seen examples illustrating this principle. The equation (5.59) for the undamped spring-mass system is of the form (5.68) with $p(y) = ky/m$, which has only the stable equilibrium $\bar{y} = 0$. In this case (5.69) holds with $a = -\infty$ and $b = \infty$. The equation (5.63) for the undamped pendulum is of the form (5.68) with $p(y) = (g/L) \sin y$. We've seen that $\bar{y} = 2m\pi$ is a stable equilibrium if m is an integer. In this case

$$p(y) = \sin y < 0 \text{ if } (2m - 1)\pi < y < 2m\pi$$

and

$$p(y) > 0 \text{ if } 2m\pi < y < (2m + 1)\pi.$$

It can also be shown (Exercise 5.4.24) that $\bar{y}$ is unstable if there's a $b > \bar{y}$ such that

$$p(y) < 0 \text{ if } \bar{y} < y < b \quad (5.70)$$

or an $a < \bar{y}$ such that

$$p(y) > 0 \text{ if } a < y < \bar{y}. \quad (5.71)$$

If we regard $p(y)$ as a force acting on a unit mass, (5.70) means that the force tends to increase all sufficiently small positive displacements from $\bar{y}$, while (5.71) means that the force tends to increase the magnitude of all sufficiently small negative displacements from $\bar{y}$.

The undamped pendulum also illustrates this principle. We've seen that $\bar{y} = (2m + 1)\pi$ is an unstable equilibrium if m is an integer. In this case

$$\sin y < 0 \text{ if } (2m + 1)\pi < y < (2m + 2)\pi,$$

so (5.70) holds with $b = (2m + 2)\pi$, and

$$\sin y > 0 \text{ if } 2m\pi < y < (2m + 1)\pi,$$

so (5.71) holds with $a = 2m\pi$.

Example 5.4.3: The equation

$$y'' + y(y - 1) = 0 \quad (5.72)$$

is of the form (5.68) with $p(y) = y(y - 1)$. Therefore $\bar{y} = 0$ and $\bar{y} = 1$ are the equilibria of (5.72). Since

$$\begin{aligned} y(y - 1) &> 0 && \text{if } y < 0 \text{ or } y > 1, \\ &< 0 && \text{if } 0 < y < 1, \end{aligned}$$

$\bar{y} = 0$ is unstable and $\bar{y} = 1$ is stable.

The phase plane equivalent of (5.72) is the separable equation

$$v \frac{dv}{dy} + y(y - 1) = 0.$$

Integrating yields

$$\frac{v^2}{2} + \frac{y^3}{3} - \frac{y^2}{2} = C,$$

which we rewrite as

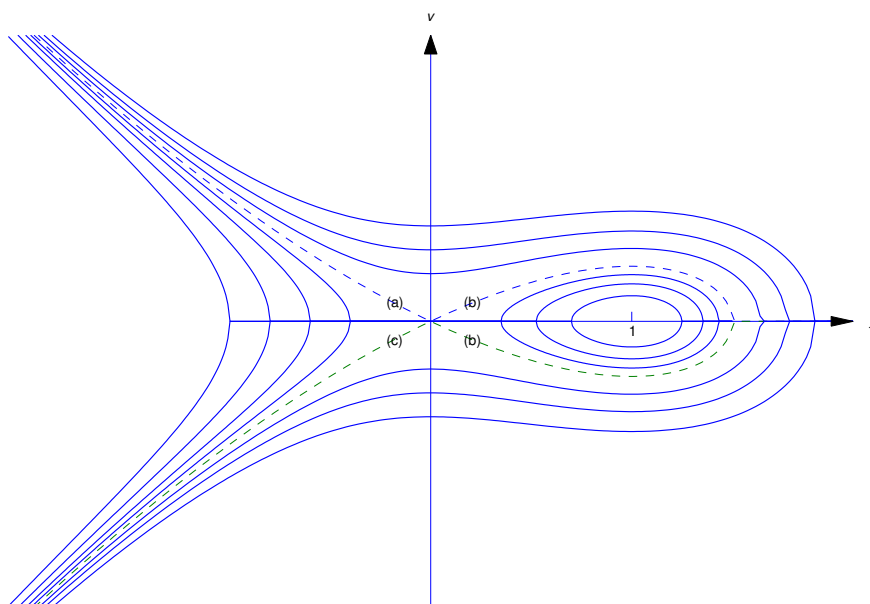
$$v^2 + \frac{1}{3}y^2(2y - 3) = c \quad (5.73)$$

after renaming the constant of integration. These are the trajectories of (5.72). If y is any solution of (5.72), the point $(y(t), v(t))$ moves along the trajectory of y in the direction of increasing y in the upper half plane ($v = y' > 0$), or in the direction of decreasing y in the lower half plane ($v = y' < 0$).

Figure 5.24 shows typical trajectories. The dashed curve through the critical point $(0, 0)$, obtained by setting $c = 0$ in (5.73), separates the y - v plane into regions that contain different kinds of trajectories; again, we call this curve a *separatrix*. Trajectories in the region bounded by the closed loop **(b)** are closed curves, so solutions associated with them are periodic. Solutions associated with other trajectories are not periodic. If y is any such solution with trajectory not on the separatrix, then

$$\begin{aligned} \lim_{t \rightarrow \infty} y(t) &= -\infty, & \lim_{t \rightarrow -\infty} y(t) &= -\infty, \\ \lim_{t \rightarrow \infty} v(t) &= -\infty, & \lim_{t \rightarrow -\infty} v(t) &= \infty. \end{aligned}$$

The separatrix contains four trajectories of (5.72). One is the point $(0, 0)$, the trajectory of the equilibrium $\bar{y} = 0$. Since distinct trajectories can't intersect, the segments of the separatrix marked **(a)**, **(b)**, and **(c)** – which don't include $(0, 0)$ – are distinct trajectories, none of which can be traversed in finite time. Solutions

Figure 5.24: Trajectories of $y'' + y(y - 1) = 0$

with these trajectories have the following asymptotic behavior:

$$\begin{array}{ll}
 \lim_{t \rightarrow \infty} y(t) = 0, & \lim_{t \rightarrow -\infty} y(t) = -\infty, \\
 \lim_{t \rightarrow \infty} v(t) = 0, & \lim_{t \rightarrow -\infty} v(t) = \infty \quad (\text{on (a)}) \\
 \lim_{t \rightarrow \infty} y(t) = 0, & \lim_{t \rightarrow -\infty} y(t) = 0, \\
 \lim_{t \rightarrow \infty} v(t) = 0, & \lim_{t \rightarrow -\infty} v(t) = 0 \quad (\text{on (b)}) \\
 \lim_{t \rightarrow \infty} y(t) = -\infty, & \lim_{t \rightarrow -\infty} y(t) = 0, \\
 \lim_{t \rightarrow \infty} v(t) = -\infty, & \lim_{t \rightarrow -\infty} v(t) = 0 \quad (\text{on (c)}).
 \end{array}$$

5.4.3 The Damped Case

The phase plane equivalent of the damped autonomous equation

$$y'' + q(y, y')y' + p(y) = 0 \quad (5.74)$$

is

$$v \frac{dv}{dy} + q(y, v)v + p(y) = 0.$$

This equation isn't separable, so we can't solve it for v in terms of y , as we did in the undamped case, and conservation of energy doesn't hold. (For example, energy expended in overcoming friction is lost.) However, we can study the qualitative behavior of its solutions by rewriting it as

$$\frac{dv}{dy} = -q(y, v) - \frac{p(y)}{v} \quad (5.75)$$

and considering the direction fields for this equation. In the following examples we'll also be showing computer generated trajectories of this equation, obtained by numerical methods. The exercises call for

similar computations. The methods discussed in Chapter 3 are not suitable for this task, since $p(y)/v$ in (5.75) is undefined on the y axis of the Poincaré phase plane. Therefore we're forced to apply numerical methods briefly discussed in Section 10.1 to the system

$$\begin{aligned}y' &= v \\v' &= -q(y, v)v - p(y),\end{aligned}$$

which is equivalent to (5.74) in the sense defined in Section 10.1. Fortunately, most differential equation software packages enable you to do this painlessly.

In the text we'll confine ourselves to the case where q is constant, so (5.74) and (5.75) reduce to

$$y'' + cy' + p(y) = 0 \tag{5.76}$$

and

$$\frac{dv}{dy} = -c - \frac{p(y)}{v}.$$

(We'll consider more general equations in the exercises.) The constant c is called the *damping constant*. In situations where (5.76) is the equation of motion of an object, c is positive; however, there are situations where c may be negative.

5.4.4 The Damped Spring-Mass System

Earlier we considered the spring - mass system under the assumption that the only forces acting on the object were gravity and the spring's resistance to changes in its length. Now we'll assume that some mechanism (for example, friction in the spring or atmospheric resistance) opposes the motion of the object with a force proportional to its velocity. In Section 6.1 it will be shown that in this case Newton's second law of motion implies that

$$my'' + cy' + ky = 0, \tag{5.77}$$

where $c > 0$ is the *damping constant*. Again, this equation can be solved easily by a method that we'll study in Section 5.2, but that method isn't available here. Instead, we'll consider its phase plane equivalent, which can be written in the form (5.75) as

$$\frac{dv}{dy} = -\frac{c}{m} - \frac{ky}{mv}. \tag{5.78}$$

(A minor note: the c in (5.76) actually corresponds to c/m in this equation.) Figure 5.25 shows a typical direction field for an equation of this form. Recalling that motion along a trajectory must be in the direction of increasing y in the upper half plane ($v > 0$) and in the direction of decreasing y in the lower half plane ($v < 0$), you can infer that all trajectories approach the origin in clockwise fashion. To confirm this, Figure 5.26 shows the same direction field with some trajectories filled in. All the trajectories shown there correspond to solutions of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

where

$$mv_0^2 + ky_0^2 = \rho \quad (\text{a positive constant});$$

thus, if there were no damping ($c = 0$), all the solutions would have the same dashed elliptic trajectory, shown in Figure 5.28.

Solutions corresponding to the trajectories in Figure 5.26 cross the y -axis infinitely many times. The corresponding solutions are said to be *oscillatory* (Figure 5.27). It is shown in Section 6.2 that there's a number c_1 such that if $0 \leq c < c_1$ then all solutions of (5.77) are oscillatory, while if $c \geq c_1$, no solutions of (5.77) have this property. (In fact, no solution not identically zero can have more than two zeros in this case.) Figure 5.28 shows a direction field and some integral curves for (5.78) in this case.

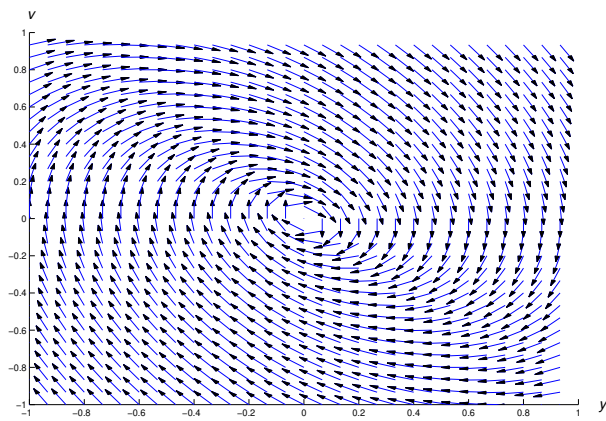


Figure 5.25: A typical direction field for $my'' + cy' + ky = 0$ with $0 < c < c_1$

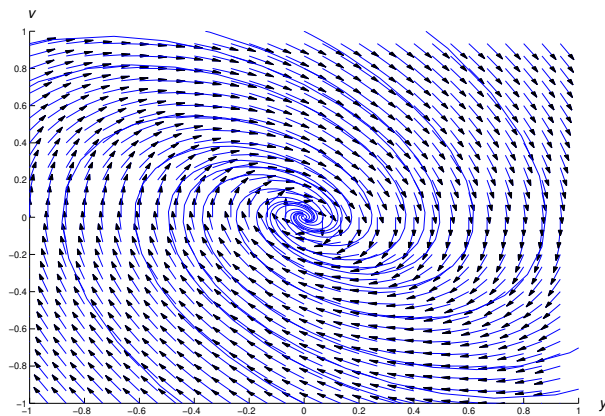


Figure 5.26: Figure 5.25 with some trajectories added

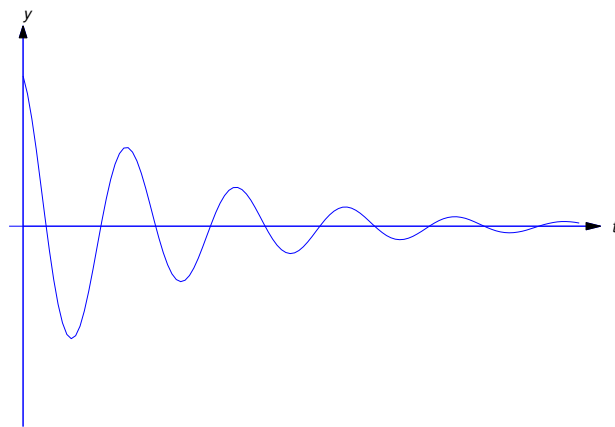


Figure 5.27: An oscillatory solution of $my'' + cy' + ky = 0$

Example 5.4.4 (The Damped Pendulum): Now we return to the pendulum. If we assume that some mechanism (for example, friction in the axle or atmospheric resistance) opposes the motion of the pendulum with a force proportional to its angular velocity, Newton's second law of motion implies that

$$mLy'' = -cy' - mg \sin y, \quad (5.79)$$

where $c > 0$ is the damping constant. (Again, a minor note: the c in (5.76) actually corresponds to c/mL in this equation.) To plot a direction field for (5.79) we write its phase plane equivalent as

$$\frac{dv}{dy} = -\frac{c}{mL} - \frac{g}{Lv} \sin y.$$

Figure 5.29 shows trajectories of four solutions of (5.79), all satisfying $y(0) = 0$. For each $m = 0, 1, 2, 3$, imparting the initial velocity $v(0) = v_m$ causes the pendulum to make m complete revolutions and then settle into decaying oscillation about the stable equilibrium $\bar{y} = 2m\pi$.

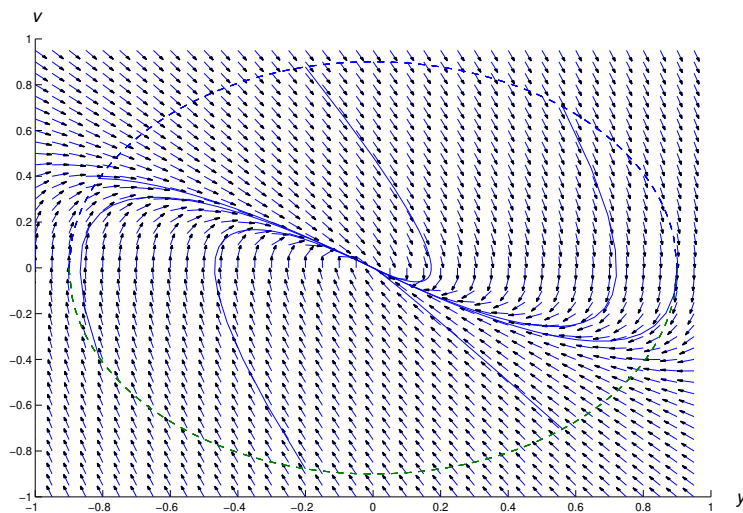


Figure 5.28: A typical direction field for $my'' + cy' + ky = 0$ with $c > c_1$

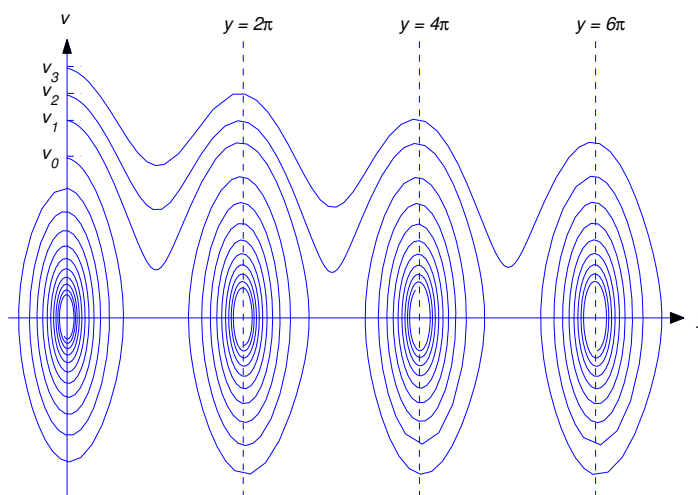


Figure 5.29: Four trajectories of the damped pendulum

5.4.5 Exercises

In Exercises 5.4.1–5.4.4 find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and plot some trajectories. HINT: Use Eqn. (5.58) to obtain the equations of the trajectories.

1. $\boxed{\text{C/G}}$ $y'' + y^3 = 0$
2. $\boxed{\text{C/G}}$ $y'' + y^2 = 0$
3. $\boxed{\text{C/G}}$ $y'' + y|y| = 0$
4. $\boxed{\text{C/G}}$ $y'' + ye^{-y} = 0$

In Exercises 5.4.5–5.4.8 find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and find the equations of the separatrices (that is, the curves through the unstable equilibria). Plot the separatrices and some trajectories in each of the regions of Poincaré plane determined by them. HINT: Use Eqn. (5.67) to determine the separatrices.

5. $\boxed{\text{C/G}}$ $y'' - y^3 + 4y = 0$
6. $\boxed{\text{C/G}}$ $y'' + y^3 - 4y = 0$
7. $\boxed{\text{C/G}}$ $y'' + y(y^2 - 1)(y^2 - 4) = 0$
8. $\boxed{\text{C/G}}$ $y'' + y(y - 2)(y - 1)(y + 2) = 0$

In Exercises 5.4.9–5.4.12 plot some trajectories of the given equation for various values (positive, negative, zero) of the parameter a . Find the equilibria of the equation and classify them as stable or unstable. Explain why the phase plane plots corresponding to positive and negative values of a differ so markedly. Can you think of a reason why zero deserves to be called the *critical value* of a ?

9. $\boxed{\text{L}}$ $y'' + y^2 - a = 0$
10. $\boxed{\text{L}}$ $y'' + y^3 - ay = 0$
11. $\boxed{\text{L}}$ $y'' - y^3 + ay = 0$
12. $\boxed{\text{L}}$ $y'' + y - ay^3 = 0$

In Exercises 5.4.13–5.4.18 plot trajectories of the given equation for $c = 0$ and small nonzero (positive and negative)

values of c to observe the effects of damping.

13. $\boxed{\text{L}}$ $y'' + cy' + y^3 = 0$

14. $\boxed{\text{L}}$ $y'' + cy' - y = 0$

15. $\boxed{\text{L}}$ $y'' + cy' + y^3 = 0$

16. $\boxed{\text{L}}$ $y'' + cy' + y^2 = 0$

17. $\boxed{\text{L}}$ $y'' + cy' + y|y| = 0$

18. $\boxed{\text{L}}$ $y'' + y(y - 1) + cy = 0$

19. $\boxed{\text{L}}$ The *van der Pol equation*

$$y'' - \mu(1 - y^2)y' + y = 0, \quad (\text{A})$$

where μ is a positive constant and y is electrical current (Section 6.3), arises in the study of an electrical circuit whose resistive properties depend upon the current. The damping term $-\mu(1 - y^2)y'$ works to reduce $|y|$ if $|y| < 1$ or to increase $|y|$ if $|y| > 1$. It can be shown that van der Pol's equation has exactly one closed trajectory, which is called a *limit cycle*. Trajectories inside the limit cycle spiral outward to it, while trajectories outside the limit cycle spiral inward to it (Figure 5.30). Use your favorite differential equations software to verify this for $\mu = .5, 1.1.5, 2$. Use a grid with $-4 < y < 4$ and $-4 < v < 4$.

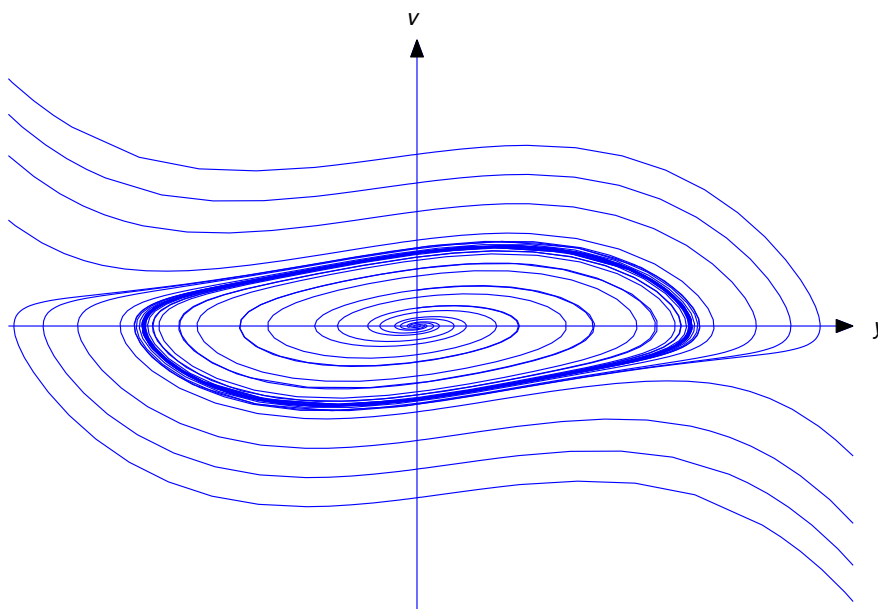


Figure 5.30: Trajectories of van der Pol's equation

20. $\boxed{\text{L}}$ *Rayleigh's equation*,

$$y'' - \mu(1 - (y')^2/3)y' + y = 0$$

also has a limit cycle. Follow the directions of Exercise 5.4.19 for this equation.

21. In connection with Eqn (5.65), suppose $y(0) = 0$ and $y'(0) = v_0$, where $0 < v_0 < v_c$.

(a) Let T_1 be the time required for y to increase from zero to $y_{\max} = 2 \sin^{-1}(v_0/v_c)$. Show that

$$\frac{dy}{dt} = \sqrt{v_0^2 - v_c^2 \sin^2 y/2}, \quad 0 \leq t < T_1. \quad (\text{A})$$

- (b) Separate variables in (A) and show that

$$T_1 = \int_0^{y_{\max}} \frac{du}{\sqrt{v_0^2 - v_c^2 \sin^2 u/2}} \quad (\text{B})$$

- (c) Substitute $\sin u/2 = (v_0/v_c) \sin \theta$ in (B) to obtain

$$T_1 = 2 \int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2 \theta}}. \quad (\text{C})$$

- (d) Conclude from symmetry that the time required for $(y(t), v(t))$ to traverse the trajectory

$$v^2 = v_0^2 - v_c^2 \sin^2 y/2$$

is $T = 4T_1$, and that consequently $y(t + T) = y(t)$ and $v(t + T) = v(t)$; that is, the oscillation is periodic with period T .

- (e) Show that if $v_0 = v_c$, the integral in (C) is improper and diverges to ∞ . Conclude from this that $y(t) < \pi$ for all t and $\lim_{t \rightarrow \infty} y(t) = \pi$.

22. Give a direct definition of an unstable equilibrium of $y'' + p(y) = 0$.
23. Let p be continuous for all y and $p(0) = 0$. Suppose there's a positive number ρ such that $p(y) > 0$ if $0 < y \leq \rho$ and $p(y) < 0$ if $-\rho \leq y < 0$. For $0 < r \leq \rho$ let

$$\alpha(r) = \min \left\{ \int_0^r p(x) dx, \int_{-r}^0 |p(x)| dx \right\} \quad \text{and} \quad \beta(r) = \max \left\{ \int_0^r p(x) dx, \int_{-r}^0 |p(x)| dx \right\}.$$

Let y be the solution of the initial value problem

$$y'' + p(y) = 0, \quad y(0) = v_0, \quad y'(0) = v_0,$$

and define $c(y_0, v_0) = v_0^2 + 2 \int_0^{y_0} p(x) dx$.

- (a) Show that

$$0 < c(y_0, v_0) < v_0^2 + 2\beta(|y_0|) \quad \text{if} \quad 0 < |y_0| \leq \rho.$$

- (b) Show that

$$v^2 + 2 \int_0^y p(x) dx = c(y_0, v_0), \quad t > 0.$$

- (c) Conclude from (b) that if $c(y_0, v_0) < 2\alpha(r)$ then $|y| < r$, $t > 0$.

- (d) Given $\epsilon > 0$, let $\delta > 0$ be chosen so that

$$\delta^2 + 2\beta(\delta) < \max \left\{ \epsilon^2/2, 2\alpha(\epsilon/\sqrt{2}) \right\}.$$

Show that if $\sqrt{y_0^2 + v_0^2} < \delta$ then $\sqrt{y^2 + v^2} < \epsilon$ for $t > 0$, which implies that $\bar{y} = 0$ is a stable equilibrium of $y'' + p(y) = 0$.

- (e) Now let p be continuous for all y and $p(\bar{y}) = 0$, where $\bar{y}$ is not necessarily zero. Suppose there's a positive number ρ such that $p(y) > 0$ if $\bar{y} < y \leq \bar{y} + \rho$ and $p(y) < 0$ if $\bar{y} - \rho \leq y < \bar{y}$. Show that $\bar{y}$ is a stable equilibrium of $y'' + p(y) = 0$.

24. Let p be continuous for all y .

- (a) Suppose $p(0) = 0$ and there's a positive number ρ such that $p(y) < 0$ if $0 < y \leq \rho$. Let ϵ be any number such that $0 < \epsilon < \rho$. Show that if y is the solution of the initial value problem

$$y'' + p(y) = 0, \quad y(0) = y_0, \quad y'(0) = 0$$

with $0 < y_0 < \epsilon$, then $y(t) \geq \epsilon$ for some $t > 0$. Conclude that $\bar{y} = 0$ is an unstable equilibrium of $y'' + p(y) = 0$. HINT: Let $k = \min_{y_0 \leq x \leq \epsilon} (-p(x))$, which is positive. Show that if $y(t) < \epsilon$ for $0 \leq t < T$ then $kT^2 < 2(\epsilon - y_0)$.

- (b) Now let $p(\bar{y}) = 0$, where $\bar{y}$ isn't necessarily zero. Suppose there's a positive number ρ such that $p(y) < 0$ if $\bar{y} < y \leq \bar{y} + \rho$. Show that $\bar{y}$ is an unstable equilibrium of $y'' + p(y) = 0$.
- (c) Modify your proofs of (a) and (b) to show that if there's a positive number ρ such that $p(y) > 0$ if $\bar{y} - \rho \leq y < \bar{y}$, then $\bar{y}$ is an unstable equilibrium of $y'' + p(y) = 0$.

5.5 Applications to Curves

5.5.1 One-Parameter Families of Curves

We begin with two examples of families of curves generated by varying a parameter over a set of real numbers.

Example 5.5.1: For each value of the parameter c , the equation

$$y - cx^2 = 0 \tag{5.80}$$

defines a curve in the xy -plane. If $c \neq 0$, the curve is a parabola through the origin, opening upward if $c > 0$ or downward if $c < 0$. If $c = 0$, the curve is the x axis (Figure 5.31).

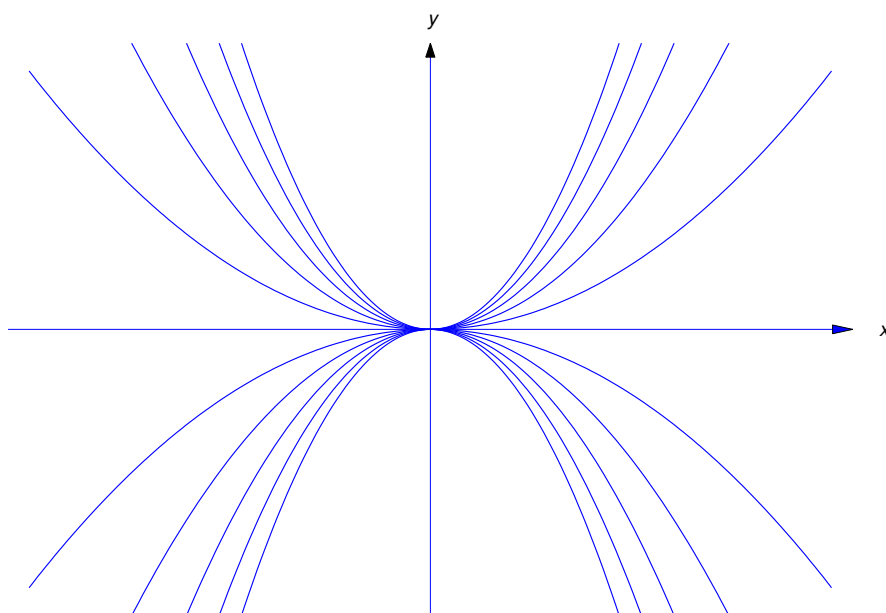


Figure 5.31: A family of curves defined by $y - cx^2 = 0$

Example 5.5.2: For each value of the parameter c the equation

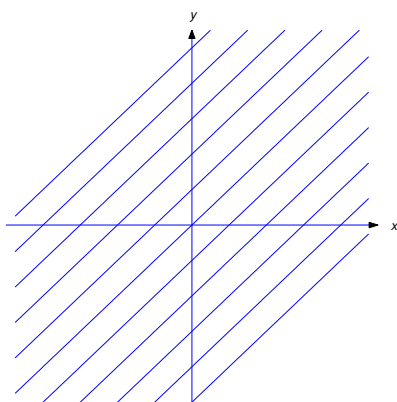
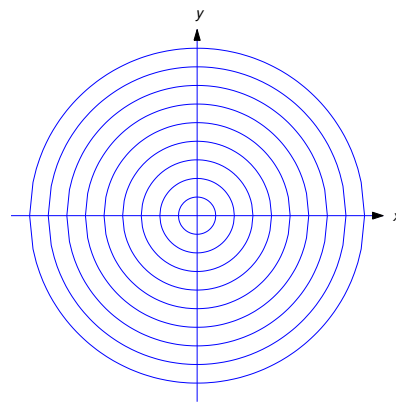
$$y = x + c \tag{5.81}$$

defines a line with slope 1 (Figure 5.32).

Definition 5.5.1 An equation that can be written in the form

$$H(x, y, c) = 0 \tag{5.82}$$

is said to define a *one-parameter family of curves* if, for each value of c in in some nonempty set of real numbers, the set of points (x, y) that satisfy (5.82) forms a curve in the xy -plane.

Figure 5.32: A family of lines defined by $y = x + c$ Figure 5.33: A family of circles defined by $x^2 + y^2 - c^2 = 0$

Equations (5.80) and (5.81) define one-parameter families of curves. (Although (5.81) isn't in the form (5.82), it can be written in this form as $y - x - c = 0$.)

Example 5.5.3: If $c > 0$, the graph of the equation

$$x^2 + y^2 - c = 0 \quad (5.83)$$

is a circle with center at $(0, 0)$ and radius $\sqrt{c}$. If $c = 0$, the graph is the single point $(0, 0)$. (We don't regard a single point as a curve.) If $c < 0$, the equation has no graph. Hence, (5.83) defines a one-parameter family of curves for positive values of c . This family consists of all circles centered at $(0, 0)$ (Figure 5.33).

Example 5.5.4: The equation

$$x^2 + y^2 + c^2 = 0$$

does not define a one-parameter family of curves, since no (x, y) satisfies the equation if $c \neq 0$, and only the single point $(0, 0)$ satisfies it if $c = 0$. ■

Recall from Section 1.2 that the graph of a solution of a differential equation is called an *integral curve* of the equation. Solving a first order differential equation usually produces a one-parameter family of integral curves of the equation. Here we are interested in the converse problem: given a one-parameter family of curves, is there a first order differential equation for which every member of the family is an integral curve. This suggests the next definition.

Definition 5.5.2 If every curve in a one-parameter family defined by the equation

$$H(x, y, c) = 0 \quad (5.84)$$

is an integral curve of the first order differential equation

$$F(x, y, y') = 0, \quad (5.85)$$

then (5.85) is said to be a *differential equation for the family*.

To find a differential equation for a one-parameter family we differentiate its defining equation (5.84) implicitly with respect to x , to obtain

$$H_x(x, y, c) + H_y(x, y, c)y' = 0. \quad (5.86)$$

If this equation doesn't, then it's a differential equation for the family. If it does contain c , it may be possible to obtain a differential equation for the family by eliminating c between (5.84) and (5.86).

Example 5.5.5: Find a differential equation for the family of curves defined by

$$y = cx^2. \quad (5.87)$$

Solution: Differentiating (5.87) with respect to x yields

$$y' = 2cx.$$

Therefore $c = y'/2x$, and substituting this into (5.87) yields

$$y = \frac{xy'}{2}$$

as a differential equation for the family of curves defined by (5.87). The graph of any function of the form $y = cx^2$ is an integral curve of this equation. ■

The next example shows that members of a given family of curves may be obtained by joining integral curves for more than one differential equation.

Example 5.5.6:

- (a) Try to find a differential equation for the family of lines tangent to the parabola $y = x^2$.
- (b) Find two tangent lines to the parabola $y = x^2$ that pass through $(2, 3)$, and find the points of tangency.

Solution (a): The equation of the line through a given point (x_0, y_0) with slope m is

$$y = y_0 + m(x - x_0). \quad (5.88)$$

If (x_0, y_0) is on the parabola, then $y_0 = x_0^2$ and the slope of the tangent line through (x_0, x_0^2) is $m = 2x_0$; hence, (5.88) becomes

$$y = x_0^2 + 2x_0(x - x_0),$$

or, equivalently,

$$y = -x_0^2 + 2x_0x. \quad (5.89)$$

Here x_0 plays the role of the constant c in Definition 5.5.1; that is, varying x_0 over $(-\infty, \infty)$ produces the family of tangent lines to the parabola $y = x^2$.

Differentiating (5.89) with respect to x yields $y' = 2x_0$. We can express x_0 in terms of x and y by rewriting (5.89) as

$$x_0^2 - 2x_0x + y = 0$$

and using the quadratic formula to obtain

$$x_0 = x \pm \sqrt{x^2 - y}. \quad (5.90)$$

We must choose the plus sign in (5.90) if $x < x_0$ and the minus sign if $x > x_0$; thus,

$$x_0 = \left(x + \sqrt{x^2 - y} \right) \text{ if } x < x_0$$

and

$$x_0 = \left(x - \sqrt{x^2 - y} \right) \text{ if } x > x_0.$$

Since $y' = 2x_0$, this implies that

$$y' = 2 \left(x + \sqrt{x^2 - y} \right), \text{ if } x < x_0 \quad (5.91)$$

and

$$y' = 2 \left(x - \sqrt{x^2 - y} \right), \quad \text{if } x > x_0. \quad (5.92)$$

Neither (5.91) nor (5.92) is a differential equation for the family of tangent lines to the parabola $y = x^2$. However, if each tangent line is regarded as consisting of two *tangent half lines* joined at the point of tangency, (5.91) is a differential equation for the family of tangent half lines on which x is less than the abscissa of the point of tangency (Figure 5.34(a)), while (5.92) is a differential equation for the family of tangent half lines on which x is greater than this abscissa (Figure 5.34(b)). The parabola $y = x^2$ is also an integral curve of both (5.91) and (5.92).

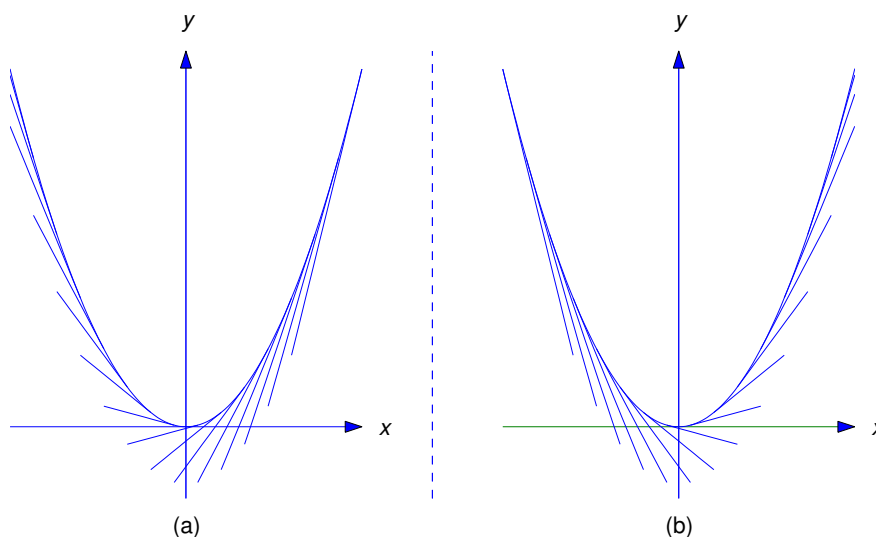


Figure 5.34

Solution (b): From (5.89) the point $(x, y) = (2, 3)$ is on the tangent line through (x_0, x_0^2) if and only if

$$3 = -x_0^2 + 4x_0,$$

which is equivalent to

$$x_0^2 - 4x_0 + 3 = (x_0 - 3)(x_0 - 1) = 0.$$

Letting $x_0 = 3$ in (5.89) shows that $(2, 3)$ is on the line

$$y = -9 + 6x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (3, 9)$, as shown in Figure 5.35

Letting $x_0 = 1$ in (5.89) shows that $(2, 3)$ is on the line

$$y = -1 + 2x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (1, 1)$, as shown in Figure 5.35.

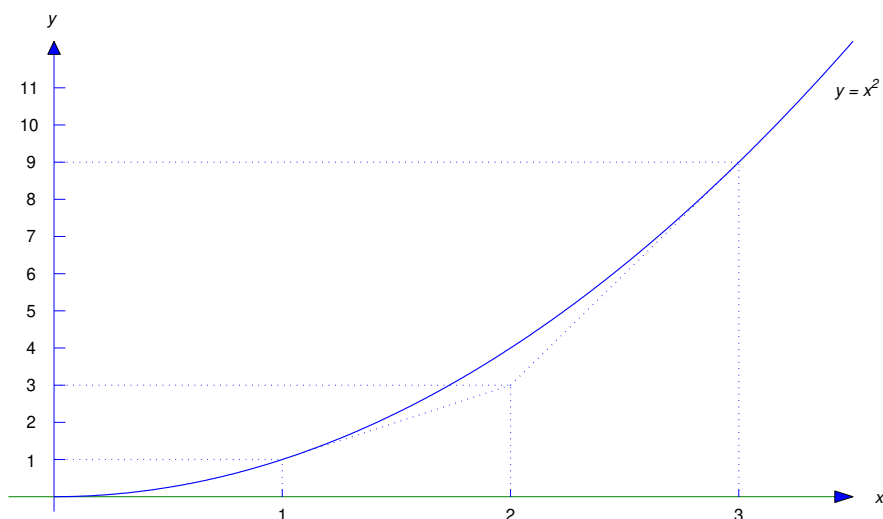


Figure 5.35

5.5.2 Geometric Problems

We now consider some geometric problems that can be solved by means of differential equations.

Example 5.5.7: Find curves $y = y(x)$ such that every point $(x_0, y(x_0))$ on the curve is the midpoint of the line segment with endpoints on the coordinate axes and tangent to the curve at $(x_0, y(x_0))$ (Figure 5.36).

Solution: The equation of the line tangent to the curve at $P = (x_0, y(x_0))$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

If we denote the x and y intercepts of the tangent line by x_I and y_I (Figure 5.36), then

$$0 = y(x_0) + y'(x_0)(x_I - x_0) \quad (5.93)$$

and

$$y_I = y(x_0) - y'(x_0)x_0. \quad (5.94)$$

From Figure 5.36, P is the midpoint of the line segment connecting $(x_I, 0)$ and $(0, y_I)$ if and only if $x_I = 2x_0$ and $y_I = 2y(x_0)$. Substituting the first of these conditions into (5.93) or the second into (5.94) yields

$$y(x_0) + y'(x_0)x_0 = 0.$$

Since x_0 is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + xy' = 0,$$

which can be rewritten as

$$(xy)' = 0.$$

Integrating yields $xy = c$, or

$$y = \frac{c}{x}.$$

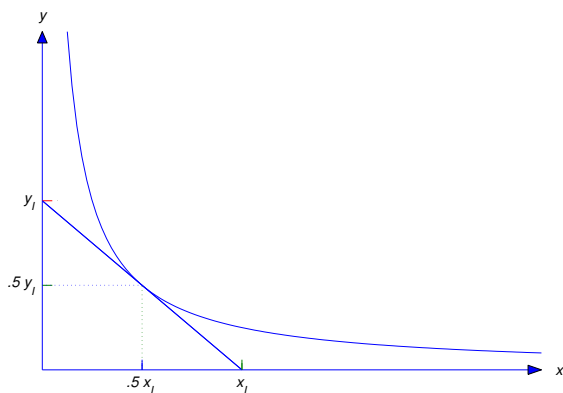


Figure 5.36

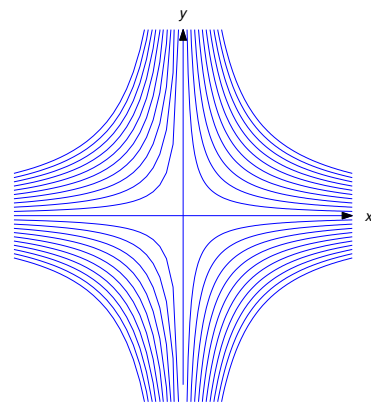


Figure 5.37

If $c = 0$ this curve is the line $y = 0$, which does not satisfy the geometric requirements imposed by the problem; thus, $c \neq 0$, and the solutions define a family of hyperbolas (Figure 5.37).

Example 5.5.8: Find curves $y = y(x)$ such that the tangent line to the curve at any point $(x_0, y(x_0))$ intersects the x -axis at $(x_0^2, 0)$. Figure 5.38 illustrates the situation in the case where the curve is in the first quadrant and $0 < x < 1$.

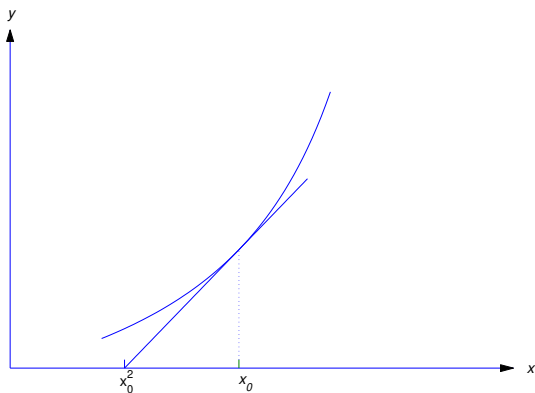


Figure 5.38

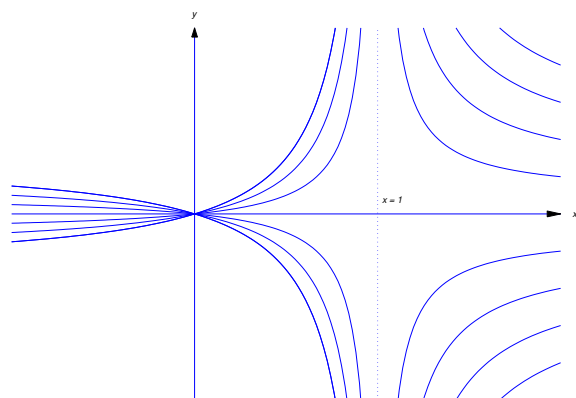


Figure 5.39

Solution: The equation of the line tangent to the curve at $(x_0, y(x_0))$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

Since $(x_0^2, 0)$ is on the tangent line,

$$0 = y(x_0) + y'(x_0)(x_0^2 - x_0).$$

Since x_0 is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + y'(x^2 - x) = 0.$$

Therefore

$$\frac{y'}{y} = -\frac{1}{x^2 - x} = -\frac{1}{x(x - 1)} = \frac{1}{x} - \frac{1}{x - 1},$$

so

$$\ln|y| = \ln|x| - \ln|x-1| + k = \ln\left|\frac{x}{x-1}\right| + k,$$

and

$$y = \frac{cx}{x-1}.$$

If $c = 0$, the graph of this function is the x -axis. If $c \neq 0$, it's a hyperbola with vertical asymptote $x = 1$ and horizontal asymptote $y = c$. Figure 5.39 shows the graphs for $c \neq 0$.

5.5.3 Orthogonal Trajectories

Two curves C_1 and C_2 are said to be *orthogonal* at a point of intersection (x_0, y_0) if they have perpendicular tangents at (x_0, y_0) . (Figure 5.40). A curve is said to be an *orthogonal trajectory* of a given family of curves if it's orthogonal to every curve in the family. For example, every line through the origin is an orthogonal trajectory of the family of circles centered at the origin. Conversely, any such circle is an orthogonal trajectory of the family of lines through the origin (Figure 5.41).

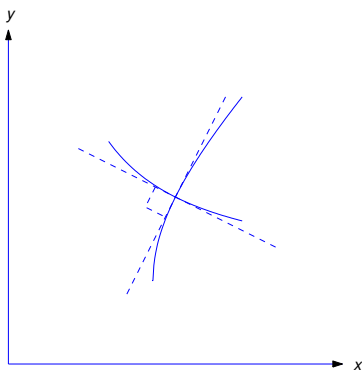


Figure 5.40: Curves orthogonal at a point of intersection

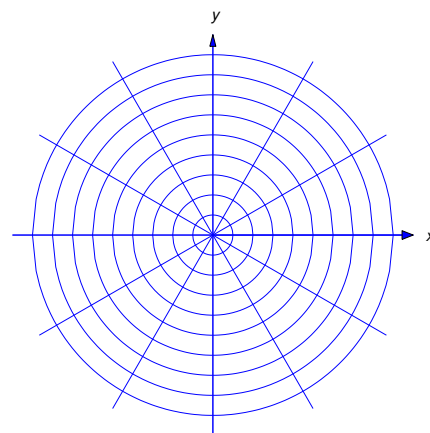


Figure 5.41: Orthogonal families of circles and lines

Orthogonal trajectories occur in many physical applications. For example, if $u = u(x, y)$ is the temperature at a point (x, y) , the curves defined by

$$u(x, y) = c \tag{5.95}$$

are called *isothermal* curves. The orthogonal trajectories of this family are called *heat-flow* lines, because at any given point the direction of maximum heat flow is perpendicular to the isothermal through the point. If u represents the potential energy of an object moving under a force that depends upon (x, y) , the curves (5.95) are called *equipotentials*, and the orthogonal trajectories are called *lines of force*.

From analytic geometry we know that two nonvertical lines L_1 and L_2 with slopes m_1 and m_2 , respectively, are perpendicular if and only if $m_2 = -1/m_1$; therefore, the integral curves of the differential equation

$$y' = -\frac{1}{f(x, y)}$$

are orthogonal trajectories of the integral curves of the differential equation

$$y' = f(x, y),$$

because at any point (x_0, y_0) where curves from the two families intersect the slopes of the respective tangent lines are

$$m_1 = f(x_0, y_0) \quad \text{and} \quad m_2 = -\frac{1}{f(x_0, y_0)}.$$

This suggests a method for finding orthogonal trajectories of a family of integral curves of a first order equation.

Finding Orthogonal Trajectories

Step 1. Find a differential equation

$$y' = f(x, y)$$

for the given family.

Step 2. Solve the differential equation

$$y' = -\frac{1}{f(x, y)}$$

to find the orthogonal trajectories.

Example 5.5.9: Find the orthogonal trajectories of the family of circles

$$x^2 + y^2 = c^2 \quad (c > 0). \quad (5.96)$$

Solution: To find a differential equation for the family of circles we differentiate (5.96) implicitly with respect to x to obtain

$$2x + 2yy' = 0,$$

or

$$y' = -\frac{x}{y}.$$

Therefore the integral curves of

$$y' = \frac{y}{x}$$

are orthogonal trajectories of the given family. We leave it to you to verify that the general solution of this equation is

$$y = kx,$$

where k is an arbitrary constant. This is the equation of a nonvertical line through $(0, 0)$. The y axis is also an orthogonal trajectory of the given family. Therefore every line through the origin is an orthogonal trajectory of the given family (5.96) (Figure 5.41). This is consistent with the theorem of plane geometry which states that a diameter of a circle and a tangent line to the circle at the end of the diameter are perpendicular.

Example 5.5.10: Find the orthogonal trajectories of the family of hyperbolas

$$xy = c \quad (c \neq 0) \quad (5.97)$$

(Figure 5.37).

Solution: Differentiating (5.97) implicitly with respect to x yields

$$y + xy' = 0,$$

or

$$y' = -\frac{y}{x};$$

thus, the integral curves of

$$y' = \frac{x}{y}$$

are orthogonal trajectories of the given family. Separating variables yields

$$y'y = x$$

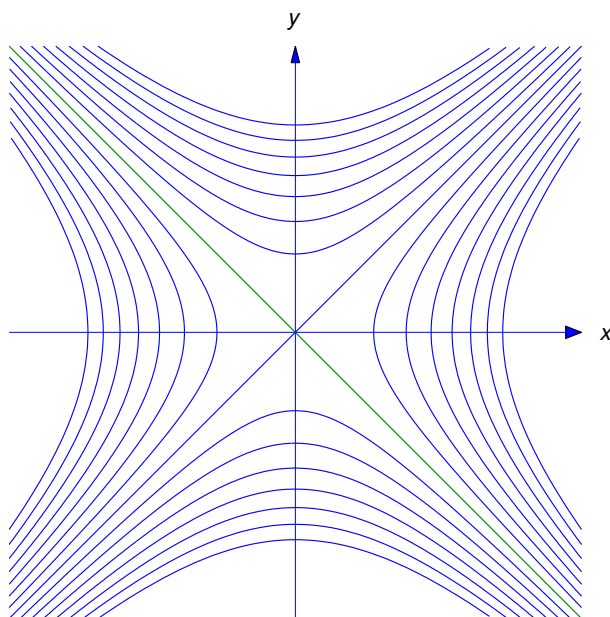


Figure 5.42: Orthogonal trajectories of the hyperbolas $xy = c$

and integrating yields

$$y^2 - x^2 = k,$$

which is the equation of a hyperbola if $k \neq 0$, or of the lines $y = x$ and $y = -x$ if $k = 0$ (Figure 5.42).

Example 5.5.11: Find the orthogonal trajectories of the family of circles defined by

$$(x - c)^2 + y^2 = c^2 \quad (c \neq 0). \quad (5.98)$$

These circles are centered on the x -axis and tangent to the y -axis (Figure 5.43(a)).

Solution: Multiplying out the left side of (5.98) yields

$$x^2 - 2cx + y^2 = 0, \quad (5.99)$$

and differentiating this implicitly with respect to x yields

$$2(x - c) + 2yy' = 0. \quad (5.100)$$

From (5.99),

$$c = \frac{x^2 + y^2}{2x},$$

so

$$x - c = x - \frac{x^2 + y^2}{2x} = \frac{x^2 - y^2}{2x}.$$

Substituting this into (5.100) and solving for y' yields

$$y' = \frac{y^2 - x^2}{2xy}. \quad (5.101)$$

The curves defined by (5.98) are integral curves of (5.101), and the integral curves of

$$y' = \frac{2xy}{x^2 - y^2}$$

are orthogonal trajectories of the family (5.98). This is a homogeneous nonlinear equation, which we studied in Section 2.4. Substituting $y = ux$ yields

$$u'x + u = \frac{2x(ux)}{x^2 - (ux)^2} = \frac{2u}{1 - u^2},$$

so

$$u'x = \frac{2u}{1 - u^2} - u = \frac{u(u^2 + 1)}{1 - u^2},$$

Separating variables yields

$$\frac{1 - u^2}{u(u^2 + 1)} u' = \frac{1}{x},$$

or, equivalently,

$$\left[\frac{1}{u} - \frac{2u}{u^2 + 1} \right] u' = \frac{1}{x}.$$

Therefore

$$\ln|u| - \ln(u^2 + 1) = \ln|x| + k.$$

By substituting $u = y/x$, we see that

$$\ln|y| - \ln|x| - \ln(x^2 + y^2) + \ln(x^2) = \ln|x| + k,$$

which, since $\ln(x^2) = 2\ln|x|$, is equivalent to

$$\ln|y| - \ln(x^2 + y^2) = k,$$

or

$$|y| = e^k(x^2 + y^2).$$

To see what these curves are we rewrite this equation as

$$x^2 + |y|^2 - e^{-k}|y| = 0$$

and complete the square to obtain

$$x^2 + (|y| - e^{-k}/2)^2 = (e^{-k}/2)^2.$$

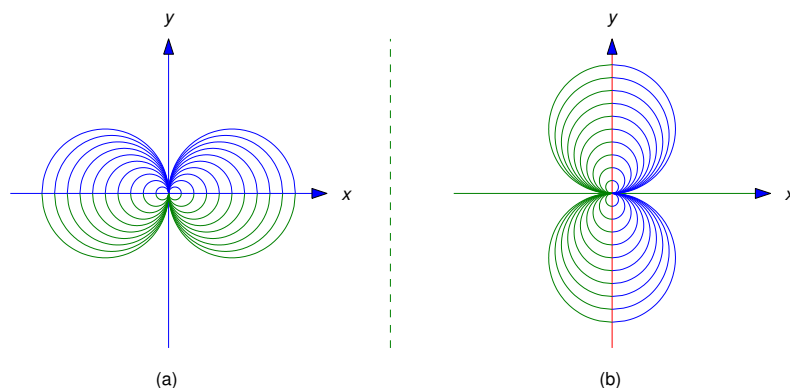
This can be rewritten as

$$x^2 + (y - h)^2 = h^2,$$

where

$$h = \begin{cases} \frac{e^{-k}}{2} & \text{if } y \geq 0, \\ -\frac{e^{-k}}{2} & \text{if } y \leq 0. \end{cases}$$

Thus, the orthogonal trajectories are circles centered on the y axis and tangent to the x axis (Figure 5.43(b)). The circles for which $h > 0$ are above the x -axis, while those for which $h < 0$ are below.



5.5.4 Exercises

Figure 5.43: **(a)** The circles $(x - c)^2 + y^2 = c^2$ **(b)** The circles $x^2 + (y - h)^2 = h^2$

In Exercises 5.5.1–5.5.8 find a first order differential equation for the given family of curves.

1. $y(x^2 + y^2) = c$
2. $e^{xy} = cy$
3. $\ln|xy| = c(x^2 + y^2)$
4. $y = x^{1/2} + cx$
5. $y = e^{x^2} + ce^{-x^2}$
6. $y = x^3 + \frac{c}{x}$
7. $y = \sin x + ce^x$
8. $y = e^x + c(1 + x^2)$
9. Show that the family of circles

$$(x - x_0)^2 + y^2 = 1, \quad -\infty < x_0 < \infty,$$

can be obtained by joining integral curves of two first order differential equations. More specifically, find differential equations for the families of semicircles

$$(x - x_0)^2 + y^2 = 1, \quad x_0 < x < x_0 + 1, \quad -\infty < x_0 < \infty,$$

$$(x - x_0)^2 + y^2 = 1, \quad x_0 - 1 < x < x_0, \quad -\infty < x_0 < \infty.$$

10. Suppose f and g are differentiable for all x . Find a differential equation for the family of functions $y = f + cg$ ($c = \text{constant}$).

In Exercises 5.5.11–5.5.13 find a first order differential equation for the given family of curves.

11. Lines through a given point (x_0, y_0) .
12. Circles through $(-1, 0)$ and $(1, 0)$.
13. Circles through $(0, 0)$ and $(0, 2)$.
14. Use the method Example 5.5.6(a) to find the equations of lines through the given points tangent to the parabola $y = x^2$. Also, find the points of tangency.

(a) $(5, 9)$	(b) $(6, 11)$	(c) $(-6, 20)$	(d) $(-3, 5)$

15. (a) Show that the equation of the line tangent to the circle

$$x^2 + y^2 = 1 \quad (\text{A})$$

at a point (x_0, y_0) on the circle is

$$y = \frac{1 - x_0 x}{y_0} \quad \text{if } x_0 \neq \pm 1. \quad (\text{B})$$

- (b) Show that if y' is the slope of a nonvertical tangent line to the circle (A) and (x, y) is a point on the tangent line then

$$(y')^2(x^2 - 1) - 2xyy' + y^2 - 1 = 0. \quad (\text{C})$$

- (c) Show that the segment of the tangent line (B) on which $(x - x_0)/y_0 > 0$ is an integral curve of the differential equation

$$y' = \frac{xy - \sqrt{x^2 + y^2 - 1}}{x^2 - 1}, \quad (\text{D})$$

while the segment on which $(x - x_0)/y_0 < 0$ is an integral curve of the differential equation

$$y' = \frac{xy + \sqrt{x^2 + y^2 - 1}}{x^2 - 1}. \quad (\text{E})$$

HINT: Use the quadratic formula to solve (C) for y' . Then substitute (B) for y and choose the $\pm$ sign in the quadratic formula so that the resulting expression for y' reduces to the known slope $y' = -x_0/y_0$.

- (d) Show that the upper and lower semicircles of (A) are also integral curves of (D) and (E).
 (e) Find the equations of two lines through (5,5) tangent to the circle (A), and find the points of tangency.
16. (a) Show that the equation of the line tangent to the parabola

$$x = y^2 \quad (\text{A})$$

at a point $(x_0, y_0) \neq (0, 0)$ on the parabola is

$$y = \frac{y_0}{2} + \frac{x}{2y_0}. \quad (\text{B})$$

- (b) Show that if y' is the slope of a nonvertical tangent line to the parabola (A) and (x, y) is a point on the tangent line then

$$4x^2(y')^2 - 4xyy' + x = 0. \quad (\text{C})$$

- (c) Show that the segment of the tangent line defined in (a) on which $x > x_0$ is an integral curve of the differential equation

$$y' = \frac{y + \sqrt{y^2 - x}}{2x}, \quad (\text{D})$$

while the segment on which $x < x_0$ is an integral curve of the differential equation

$$y' = \frac{y - \sqrt{y^2 - x}}{2x}, \quad (\text{E})$$

HINT: Use the quadratic formula to solve (C) for y' . Then substitute (B) for y and choose the $\pm$ sign in the quadratic formula so that the resulting expression for y' reduces to the known slope $y' = \frac{1}{2y_0}$.

- (d) Show that the upper and lower halves of the parabola (A), given by $y = \sqrt{x}$ and $y = -\sqrt{x}$ for $x > 0$, are also integral curves of (D) and (E).

17. Use the results of Exercise 5.5.16 to find the equations of two lines tangent to the parabola $x = y^2$ and passing through the given point. Also find the points of tangency.
 (a) $(-5, 2)$ (b) $(-4, 0)$ (c) $(7, 4)$ (d) $(5, -3)$
18. Find a curve $y = y(x)$ through $(1, 2)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the x axis at $x_I = x_0/2$.
19. Find all curves $y = y(x)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the x axis at $x_I = x_0^3$.
20. Find all curves $y = y(x)$ such that the tangent to the curve at any point passes through a given point (x_1, y_1) .
21. Find a curve $y = y(x)$ through $(1, -1)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the y axis at $y_I = x_0^3$.
22. Find all curves $y = y(x)$ such that the tangent to the curve at any point $(x_0, y(x_0))$ intersects the y axis at $y_I = x_0$.
23. Find a curve $y = y(x)$ through $(0, 2)$ such that the normal to the curve at any point $(x_0, y(x_0))$ intersects the x axis at $x_I = x_0 + 1$.
24. Find a curve $y = y(x)$ through $(2, 1)$ such that the normal to the curve at any point $(x_0, y(x_0))$ intersects the y axis at $y_I = 2y(x_0)$.

In Exercises 5.5.25–5.5.29 find the orthogonal trajectories of the given family of curves.

25. $x^2 + 2y^2 = c^2$ 26. $x^2 + 4xy + y^2 = c$
27. $y = ce^{2x}$ 28. $xye^{x^2} = c$
29. $y = \frac{ce^x}{x}$
30. Find a curve through $(-1, 3)$ orthogonal to every parabola of the form

$$y = 1 + cx^2$$

that it intersects. Which of these parabolas does the desired curve intersect?

31. Show that the orthogonal trajectories of

$$x^2 + 2axy + y^2 = c$$

satisfy

$$|y - x|^{a+1}|y + x|^{a-1} = k.$$

32. If lines L and L_1 intersect at (x_0, y_0) and α is the smallest angle through which L must be rotated counterclockwise about (x_0, y_0) to bring it into coincidence with L_1 , we say that α is the *angle from L to L_1* ; thus, $0 \leq \alpha < \pi$. If L and L_1 are tangents to curves C and C_1 , respectively, that intersect at (x_0, y_0) , we say that C_1 intersects C at the angle α . Use the identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

to show that if C and C_1 are intersecting integral curves of

$$y' = f(x, y) \quad \text{and} \quad y' = \frac{f(x, y) + \tan \alpha}{1 - f(x, y) \tan \alpha} \quad \left(\alpha \neq \frac{\pi}{2} \right),$$

respectively, then C_1 intersects C at the angle α .

33. Use the result of Exercise 5.5.32 to find a family of curves that intersect every nonvertical line through the origin at the angle $\alpha = \pi/4$.
34. Use the result of Exercise 5.5.32 to find a family of curves that intersect every circle centered at the origin at a given angle $\alpha \neq \pi/2$.

Chapter 6

Linear Second Order Equations

6.1 Homogeneous Linear Equations

A second order differential equation is said to be *linear* if it can be written as

$$y'' + p(x)y' + q(x)y = f(x). \quad (6.1)$$

We call the function f on the right a *forcing function*, since in physical applications it's often related to a force acting on some system modeled by the differential equation. We say that (6.1) is *homogeneous* if $f \equiv 0$ or *nonhomogeneous* if $f \not\equiv 0$. Since these definitions are like the corresponding definitions in Section 2.1 for the linear first order equation

$$y' + p(x)y = f(x), \quad (6.2)$$

it's natural to expect similarities between methods of solving (6.1) and (6.2). However, solving (6.1) is more difficult than solving (6.2). For example, while Theorem 2.1.1 gives a formula for the general solution of (6.2) in the case where $f \equiv 0$ and Theorem 2.1.2 gives a formula for the case where $f \not\equiv 0$, there are no formulas for the general solution of (6.1) in either case. Therefore we must be content to solve linear second order equations of special forms.

In Section 2.1 we considered the homogeneous equation $y' + p(x)y = 0$ first, and then used a nontrivial solution of this equation to find the general solution of the nonhomogeneous equation $y' + p(x)y = f(x)$. Although the progression from the homogeneous to the nonhomogeneous case isn't that simple for the linear second order equation, it's still necessary to solve the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \quad (6.3)$$

in order to solve the nonhomogeneous equation (6.1). This section is devoted to (6.3).

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for (6.3). We omit the proof.

Theorem 6.1.1. *Suppose p and q are continuous on an open interval (a, b) , let x_0 be any point in (a, b) , and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem*

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on (a, b) .

Since $y \equiv 0$ is obviously a solution of (6.3) we call it the *trivial* solution. Any other solution is *nontrivial*. Under the assumptions of Theorem 6.1.1, the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

on (a, b) is the trivial solution (Exercise 6.1.24).

The next three examples illustrate concepts that we'll develop later in this section. You shouldn't be concerned with how to *find* the given solutions of the equations in these examples. This will be explained in later sections.

Example 6.1.1: The coefficients of y' and y in

$$y'' - y = 0 \quad (6.4)$$

are the constant functions $p \equiv 0$ and $q \equiv -1$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 6.1.1 implies that every initial value problem for (6.4) has a unique solution on $(-\infty, \infty)$.

- (a) Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of (6.4) on $(-\infty, \infty)$.
- (b) Verify that if c_1 and c_2 are arbitrary constants, $y = c_1 e^x + c_2 e^{-x}$ is a solution of (6.4) on $(-\infty, \infty)$.
- (c) Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 3. \quad (6.5)$$

Solution (a): If $y_1 = e^x$ then $y'_1 = e^x$ and $y''_1 = e^x = y_1$, so $y''_1 - y_1 = 0$. If $y_2 = e^{-x}$, then $y'_2 = -e^{-x}$ and $y''_2 = e^{-x} = y_2$, so $y''_2 - y_2 = 0$.

Solution (b): If

$$y = c_1 e^x + c_2 e^{-x} \quad (6.6)$$

then

$$y' = c_1 e^x - c_2 e^{-x} \quad (6.7)$$

and

$$y'' = c_1 e^x + c_2 e^{-x},$$

so

$$\begin{aligned} y'' - y &= (c_1 e^x + c_2 e^{-x}) - (c_1 e^x + c_2 e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

for all x . Therefore $y = c_1 e^x + c_2 e^{-x}$ is a solution of (6.4) on $(-\infty, \infty)$.

Solution (c): We can solve (6.5) by choosing c_1 and c_2 in (6.6) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (6.6) and (6.7) shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving these equations yields $c_1 = 2$ and $c_2 = -1$. Therefore $y = 2e^x - e^{-x}$ is the unique solution of (6.5) on $(-\infty, \infty)$.

Example 6.1.2: Let ω be a positive constant. The coefficients of y' and y in

$$y'' + \omega^2 y = 0 \quad (6.8)$$

are the constant functions $p \equiv 0$ and $q \equiv \omega^2$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 6.1.1 implies that every initial value problem for (6.8) has a unique solution on $(-\infty, \infty)$.

- (a) Verify that $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of (6.8) on $(-\infty, \infty)$.
- (b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution of (6.8) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 3. \quad (6.9)$$

Solution (a): If $y_1 = \cos \omega x$ then $y_1' = -\omega \sin \omega x$ and $y_1'' = -\omega^2 \cos \omega x = -\omega^2 y_1$, so $y_1'' + \omega^2 y_1 = 0$. If $y_2 = \sin \omega x$ then, $y_2' = \omega \cos \omega x$ and $y_2'' = -\omega^2 \sin \omega x = -\omega^2 y_2$, so $y_2'' + \omega^2 y_2 = 0$.

Solution (b): If

$$y = c_1 \cos \omega x + c_2 \sin \omega x \quad (6.10)$$

then

$$y' = \omega(-c_1 \sin \omega x + c_2 \cos \omega x) \quad (6.11)$$

and

$$y'' = -\omega^2(c_1 \cos \omega x + c_2 \sin \omega x),$$

so

$$\begin{aligned} y'' + \omega^2 y &= -\omega^2(c_1 \cos \omega x + c_2 \sin \omega x) + \omega^2(c_1 \cos \omega x + c_2 \sin \omega x) \\ &= c_1 \omega^2(-\cos \omega x + \cos \omega x) + c_2 \omega^2(-\sin \omega x + \sin \omega x) = 0 \end{aligned}$$

for all x . Therefore $y = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution of (6.8) on $(-\infty, \infty)$.

Solution (c): To solve (6.9), we must choose c_1 and c_2 in (6.10) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (6.10) and (6.11) shows that $c_1 = 1$ and $c_2 = 3/\omega$. Therefore

$$y = \cos \omega x + \frac{3}{\omega} \sin \omega x$$

is the unique solution of (6.9) on $(-\infty, \infty)$. ■

Theorem 6.1.1 implies that if k_0 and k_1 are arbitrary real numbers then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \quad (6.12)$$

has a unique solution on an interval (a, b) that contains x_0 , provided that P_0, P_1 , and P_2 are continuous and P_0 has no zeros on (a, b) . To see this, we rewrite the differential equation in (6.12) as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 6.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$.

Example 6.1.3: The equation

$$x^2 y'' + x y' - 4y = 0 \quad (6.13)$$

has the form of the differential equation in (6.12), with $P_0(x) = x^2$, $P_1(x) = x$, and $P_2(x) = -4$, which are all continuous on $(-\infty, \infty)$. However, since $P_0(0) = 0$ we must consider solutions of (6.13) on $(-\infty, 0)$ and $(0, \infty)$. Since P_0 has no zeros on these intervals, Theorem 6.1.1 implies that the initial value problem

$$x^2 y'' + x y' - 4y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(0, \infty)$ if $x_0 > 0$, or on $(-\infty, 0)$ if $x_0 < 0$.

- (a) Verify that $y_1 = x^2$ is a solution of (6.13) on $(-\infty, \infty)$ and $y_2 = 1/x^2$ is a solution of (6.13) on $(-\infty, 0)$ and $(0, \infty)$.
- (b) Verify that if c_1 and c_2 are any constants then $y = c_1 x^2 + c_2/x^2$ is a solution of (6.13) on $(-\infty, 0)$ and $(0, \infty)$.

(c) Solve the initial value problem

$$x^2 y'' + x y' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 0. \quad (6.14)$$

(d) Solve the initial value problem

$$x^2 y'' + x y' - 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 0. \quad (6.15)$$

Solution (a): If $y_1 = x^2$ then $y'_1 = 2x$ and $y''_1 = 2$, so

$$x^2 y''_1 + x y'_1 - 4y_1 = x^2(2) + x(2x) - 4x^2 = 0$$

for x in $(-\infty, \infty)$. If $y_2 = 1/x^2$, then $y'_2 = -2/x^3$ and $y''_2 = 6/x^4$, so

$$x^2 y''_2 + x y'_2 - 4y_2 = x^2 \left(\frac{6}{x^4} \right) - x \left(\frac{2}{x^3} \right) - \frac{4}{x^2} = 0$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

Solution (b): If

$$y = c_1 x^2 + \frac{c_2}{x^2} \quad (6.16)$$

then

$$y' = 2c_1 x - \frac{2c_2}{x^3} \quad (6.17)$$

and

$$y'' = 2c_1 + \frac{6c_2}{x^4},$$

so

$$\begin{aligned} x^2 y'' + x y' - 4y &= x^2 \left(2c_1 + \frac{6c_2}{x^4} \right) + x \left(2c_1 x - \frac{2c_2}{x^3} \right) - 4 \left(c_1 x^2 + \frac{c_2}{x^2} \right) \\ &= c_1 (2x^2 + 2x^2 - 4x^2) + c_2 \left(\frac{6}{x^2} - \frac{2}{x^2} - \frac{4}{x^2} \right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

Solution (c): To solve (6.14), we choose c_1 and c_2 in (6.16) so that $y(1) = 2$ and $y'(1) = 0$. Setting $x = 1$ in (6.16) and (6.17) shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 2 \\ 2c_1 - 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of (6.14) on $(0, \infty)$.

Solution (d): We can solve (6.15) by choosing c_1 and c_2 in (6.16) so that $y(-1) = 2$ and $y'(-1) = 0$. Setting $x = -1$ in (6.16) and (6.17) shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 2 \\ -2c_1 + 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of (6.15) on $(-\infty, 0)$. ■

Although the *formulas* for the solutions of (6.14) and (6.15) are both $y = x^2 + 1/x^2$, you should not conclude that these two initial value problems have the same solution. Remember that a solution of an initial value problem is defined *on an interval that contains the initial point*; therefore, the solution of (6.14) is $y = x^2 + 1/x^2$ *on the interval* $(0, \infty)$, which contains the initial point $x_0 = 1$, while the solution of (6.15) is $y = x^2 + 1/x^2$ *on the interval* $(-\infty, 0)$, which contains the initial point $x_0 = -1$.

6.1.1 The General Solution of a Homogeneous Linear Second Order Equation

If y_1 and y_2 are defined on an interval (a, b) and c_1 and c_2 are constants, then

$$y = c_1 y_1 + c_2 y_2$$

is a *linear combination of y_1 and y_2* . For example, $y = 2 \cos x + 7 \sin x$ is a linear combination of $y_1 = \cos x$ and $y_2 = \sin x$, with $c_1 = 2$ and $c_2 = 7$.

The next theorem states a fact that we've already verified in Examples 6.1.1, 6.1.2, and 6.1.3.

Theorem 6.1.2. *If y_1 and y_2 are solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{6.18}$$

on (a, b) , then any linear combination

$$y = c_1 y_1 + c_2 y_2 \tag{6.19}$$

of y_1 and y_2 is also a solution of (6.18) on (a, b) .

Proof. If

$$y = c_1 y_1 + c_2 y_2$$

then

$$y' = c_1 y_1' + c_2 y_2' \quad \text{and} \quad y'' = c_1 y_1'' + c_2 y_2''.$$

Therefore

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 (y_1'' + p(x)y_1' + q(x)y_1) + c_2 (y_2'' + p(x)y_2' + q(x)y_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0, \end{aligned}$$

since y_1 and y_2 are solutions of (6.18). ■

We say that $\{y_1, y_2\}$ is a *fundamental set of solutions of (6.18) on (a, b)* if every solution of (6.18) on (a, b) can be written as a linear combination of y_1 and y_2 as in (6.19). In this case we say that (6.19) is *general solution of (6.18) on (a, b)* .

6.1.2 Linear Independence

We need a way to determine whether a given set $\{y_1, y_2\}$ of solutions of (6.18) is a fundamental set. The next definition will enable us to state necessary and sufficient conditions for this.

We say that two functions y_1 and y_2 defined on an interval (a, b) are *linearly independent on (a, b)* if neither is a constant multiple of the other on (a, b) . (In particular, this means that neither can be the trivial solution of (6.18), since, for example, if $y_1 \equiv 0$ we could write $y_1 = 0y_2$.) We'll also say that the set $\{y_1, y_2\}$ *is linearly independent on (a, b)* .

Theorem 6.1.3. *Suppose p and q are continuous on (a, b) . Then a set $\{y_1, y_2\}$ of solutions of*

$$y'' + p(x)y' + q(x)y = 0 \quad (6.20)$$

on (a, b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a, b) .

We'll present the proof of Theorem 6.1.3 in steps worth regarding as theorems in their own right. However, let's first interpret Theorem 6.1.3 in terms of Examples 6.1.1, 6.1.2, and 6.1.3.

Example 6.1.4:

- (a) Since $e^x/e^{-x} = e^{2x}$ is nonconstant, Theorem 6.1.3 implies that $y = c_1e^x + c_2e^{-x}$ is the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.
- (b) Since $\cos \omega x / \sin \omega x = \cot \omega x$ is nonconstant, Theorem 6.1.3 implies that $y = c_1 \cos \omega x + c_2 \sin \omega x$ is the general solution of $y'' + \omega^2 y = 0$ on $(-\infty, \infty)$.
- (c) Since $x^2/x^{-2} = x^4$ is nonconstant, Theorem 6.1.3 implies that $y = c_1x^2 + c_2/x^2$ is the general solution of $x^2y'' + xy' - 4y = 0$ on $(-\infty, 0)$ and $(0, \infty)$.

6.1.3 The Wronskian and Abel's Formula

To motivate a result that we need in order to prove Theorem 6.1.3, let's see what is required to prove that $\{y_1, y_2\}$ is a fundamental set of solutions of (6.20) on (a, b) . Let x_0 be an arbitrary point in (a, b) , and suppose y is an arbitrary solution of (6.20) on (a, b) . Then y is the unique solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1; \quad (6.21)$$

that is, k_0 and k_1 are the numbers obtained by evaluating y and y' at x_0 . Moreover, k_0 and k_1 can be any real numbers, since Theorem 6.1.1 implies that (6.21) has a solution no matter how k_0 and k_1 are chosen. Therefore $\{y_1, y_2\}$ is a fundamental set of solutions of (6.20) on (a, b) if and only if it's possible to write the solution of an arbitrary initial value problem (6.21) as $y = c_1y_1 + c_2y_2$. This is equivalent to requiring that the system

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= k_0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) &= k_1 \end{aligned} \quad (6.22)$$

has a solution (c_1, c_2) for every choice of (k_0, k_1) . Let's try to solve (6.22).

Multiplying the first equation in (6.22) by $y_2'(x_0)$ and the second by $y_2(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_2'(x_0) + c_2y_2(x_0)y_2'(x_0) &= y_2'(x_0)k_0 \\ c_1y_1'(x_0)y_2(x_0) + c_2y_2'(x_0)y_2(x_0) &= y_2(x_0)k_1, \end{aligned}$$

and subtracting the second equation here from the first yields

$$(y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0))c_1 = y_2'(x_0)k_0 - y_2(x_0)k_1. \quad (6.23)$$

Multiplying the first equation in (6.22) by $y_1'(x_0)$ and the second by $y_1(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_1'(x_0) + c_2y_2(x_0)y_1'(x_0) &= y_1'(x_0)k_0 \\ c_1y_1'(x_0)y_1(x_0) + c_2y_2'(x_0)y_1(x_0) &= y_1(x_0)k_1, \end{aligned}$$

and subtracting the first equation here from the second yields

$$(y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0))c_2 = y_1(x_0)k_1 - y_1'(x_0)k_0. \quad (6.24)$$

If

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) = 0,$$

it's impossible to satisfy (6.23) and (6.24) (and therefore (6.22)) unless k_0 and k_1 happen to satisfy

$$\begin{aligned} y_1(x_0)k_1 - y_1'(x_0)k_0 &= 0 \\ y_2'(x_0)k_0 - y_2(x_0)k_1 &= 0. \end{aligned}$$

On the other hand, if

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0 \quad (6.25)$$

we can divide (6.23) and (6.24) through by the quantity on the left to obtain

$$\begin{aligned} c_1 &= \frac{y_2'(x_0)k_0 - y_2(x_0)k_1}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)} \\ c_2 &= \frac{y_1(x_0)k_1 - y_1'(x_0)k_0}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}, \end{aligned} \quad (6.26)$$

no matter how k_0 and k_1 are chosen. This motivates us to consider conditions on y_1 and y_2 that imply (6.25).

Theorem 6.1.4. *Suppose p and q are continuous on (a, b) , let y_1 and y_2 be solutions of*

$$y'' + p(x)y' + q(x)y = 0 \quad (6.27)$$

on (a, b) , and define

$$W = y_1y_2' - y_1'y_2. \quad (6.28)$$

Let x_0 be any point in (a, b) . Then

$$W(x) = W(x_0)e^{-\int_{x_0}^x p(t)dt}, \quad a < x < b. \quad (6.29)$$

Therefore either W has no zeros in (a, b) or $W \equiv 0$ on (a, b) .

Proof. Differentiating (6.28) yields

$$W' = y_1'y_2' + y_1y_2'' - y_1'y_2' - y_1''y_2 = y_1y_2'' - y_1''y_2. \quad (6.30)$$

Since y_1 and y_2 both satisfy (6.27),

$$y_1'' = -py_1' - qy_1 \quad \text{and} \quad y_2'' = -py_2' - qy_2.$$

Substituting these into (6.30) yields

$$\begin{aligned} W' &= -y_1(py_2' + qy_2) + y_2(py_1' + qy_1) \\ &= -p(y_1y_2' - y_2y_1') - q(y_1y_2 - y_2y_1) \\ &= -p(y_1y_2' - y_2y_1') = -pW. \end{aligned}$$

Therefore $W' + p(x)W = 0$; that is, W is the solution of the initial value problem

$$y' + p(x)y = 0, \quad y(x_0) = W(x_0).$$

We leave it to you to verify by separation of variables that this implies (6.29). If $W(x_0) \neq 0$, (6.29) implies that W has no zeros in (a, b) , since an exponential is never zero. On the other hand, if $W(x_0) = 0$, (6.29) implies that $W(x) = 0$ for all x in (a, b) . ■

The function W defined in (6.28) is the *Wronskian of $\{y_1, y_2\}$* . Formula (6.29) is *Abel's formula*.

The Wronskian of $\{y_1, y_2\}$ is usually written as the determinant

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

The expressions in (6.26) for c_1 and c_2 can be written in terms of determinants as

$$c_1 = \frac{1}{W(x_0)} \begin{vmatrix} k_0 & y_2(x_0) \\ k_1 & y_2'(x_0) \end{vmatrix} \quad \text{and} \quad c_2 = \frac{1}{W(x_0)} \begin{vmatrix} y_1(x_0) & k_0 \\ y_1'(x_0) & k_1 \end{vmatrix}.$$

If you've taken linear algebra you may recognize this as *Cramer's rule*.

Example 6.1.5: Verify Abel's formula for the following differential equations and the corresponding solutions, from Examples 6.1.1, 6.1.2, and 6.1.3:

- (a) $y'' - y = 0$; $y_1 = e^x$, $y_2 = e^{-x}$
- (b) $y'' + \omega^2 y = 0$; $y_1 = \cos \omega x$, $y_2 = \sin \omega x$
- (c) $x^2 y'' + x y' - 4y = 0$; $y_1 = x^2$, $y_2 = 1/x^2$

Solution (a): Since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x e^{-x} = -2$$

for all x .

Solution (b): Again, since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} \\ &= \cos \omega x (\omega \cos \omega x) - (-\omega \sin \omega x) \sin \omega x \\ &= \omega (\cos^2 \omega x + \sin^2 \omega x) = \omega \end{aligned}$$

for all x .

Solution (c): Computing the Wronskian of $y_1 = x^2$ and $y_2 = 1/x^2$ directly yields

$$W = \begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix} = x^2 \left(-\frac{2}{x^3} \right) - 2x \left(\frac{1}{x^2} \right) = -\frac{4}{x}. \quad (6.31)$$

To verify Abel's formula we rewrite the differential equation as

$$y'' + \frac{1}{x} y' - \frac{4}{x^2} y = 0$$

to see that $p(x) = 1/x$. If x_0 and x are either both in $(-\infty, 0)$ or both in $(0, \infty)$ then

$$\int_{x_0}^x p(t) dt = \int_{x_0}^x \frac{dt}{t} = \ln \left(\frac{x}{x_0} \right),$$

so Abel's formula becomes

$$\begin{aligned} W(x) &= W(x_0) e^{-\ln(x/x_0)} = W(x_0) \frac{x_0}{x} \\ &= -\left(\frac{4}{x_0} \right) \left(\frac{x_0}{x} \right) \quad \text{from (6.31)} \\ &= -\frac{4}{x}, \end{aligned}$$

which is consistent with (6.31). ■

The next theorem will enable us to complete the proof of Theorem 6.1.3.

Theorem 6.1.5. Suppose p and q are continuous on an open interval (a, b) , let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \quad (6.32)$$

on (a, b) , and let $W = y_1y_2' - y_1'y_2$. Then y_1 and y_2 are linearly independent on (a, b) if and only if W has no zeros on (a, b) .

Proof. We first show that if $W(x_0) = 0$ for some x_0 in (a, b) , then y_1 and y_2 are linearly dependent on (a, b) . Let I be a subinterval of (a, b) on which y_1 has no zeros. (If there's no such subinterval, $y_1 \equiv 0$ on (a, b) , so y_1 and y_2 are linearly dependent, and we're finished with this part of the proof.) Then y_2/y_1 is defined on I , and

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W}{y_1^2}. \quad (6.33)$$

However, if $W(x_0) = 0$, Theorem 6.1.4 implies that $W \equiv 0$ on (a, b) . Therefore (6.33) implies that $(y_2/y_1)' \equiv 0$, so $y_2/y_1 = c$ (constant) on I . This shows that $y_2(x) = cy_1(x)$ for all x in I . However, we want to show that $y_2 = cy_1(x)$ for all x in (a, b) . Let $Y = y_2 - cy_1$. Then Y is a solution of (6.32) on (a, b) such that $Y \equiv 0$ on I , and therefore $Y' \equiv 0$ on I . Consequently, if x_0 is chosen arbitrarily in I then Y is a solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0,$$

which implies that $Y \equiv 0$ on (a, b) , by the paragraph following Theorem 6.1.1. (See also Exercise 6.1.24.) Hence, $y_2 - cy_1 \equiv 0$ on (a, b) , which implies that y_1 and y_2 are not linearly independent on (a, b) .

Now suppose W has no zeros on (a, b) . Then y_1 can't be identically zero on (a, b) (why not?), and therefore there is a subinterval I of (a, b) on which y_1 has no zeros. Since (6.33) implies that y_2/y_1 is nonconstant on I , y_2 isn't a constant multiple of y_1 on (a, b) . A similar argument shows that y_1 isn't a constant multiple of y_2 on (a, b) , since

$$\left(\frac{y_1}{y_2}\right)' = \frac{y_1'y_2 - y_1y_2'}{y_2^2} = -\frac{W}{y_2^2}$$

on any subinterval of (a, b) where y_2 has no zeros. ■

We can now complete the proof of Theorem 6.1.3. From Theorem 6.1.5, two solutions y_1 and y_2 of (6.32) are linearly independent on (a, b) if and only if W has no zeros on (a, b) . From Theorem 6.1.4 and the motivating comments preceding it, $\{y_1, y_2\}$ is a fundamental set of solutions of (6.32) if and only if W has no zeros on (a, b) . Therefore $\{y_1, y_2\}$ is a fundamental set for (6.32) on (a, b) if and only if $\{y_1, y_2\}$ is linearly independent on (a, b) . ■

The next theorem summarizes the relationships among the concepts discussed in this section.

Theorem 6.1.6. Suppose p and q are continuous on an open interval (a, b) and let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \quad (6.34)$$

on (a, b) . Then the following statements are equivalent; that is, they are either all true or all false.

- (a) The general solution of (6.34) on (a, b) is $y = c_1y_1 + c_2y_2$.
- (b) $\{y_1, y_2\}$ is a fundamental set of solutions of (6.34) on (a, b) .
- (c) $\{y_1, y_2\}$ is linearly independent on (a, b) .
- (d) The Wronskian of $\{y_1, y_2\}$ is nonzero at some point in (a, b) .
- (e) The Wronskian of $\{y_1, y_2\}$ is nonzero at all points in (a, b) .

We can apply this theorem to an equation written as

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

on an interval (a, b) where P_0, P_1 , and P_2 are continuous and P_0 has no zeros.

Theorem 6.1.7. Suppose c is in (a, b) and α and β are real numbers, not both zero. Under the assumptions of Theorem 6.1.7, suppose y_1 and y_2 are solutions of (6.34) such that

$$\alpha y_1(c) + \beta y_1'(c) = 0 \text{ and } \alpha y_2(c) + \beta y_2'(c) = 0. \quad (6.35)$$

Then $\{y_1, y_2\}$ isn't linearly independent on (a, b) .

Proof. Since α and β are not both zero, (6.35) implies that

$$\begin{vmatrix} y_1(c) & y_1'(c) \\ y_2(c) & y_2'(c) \end{vmatrix} = 0, \text{ so } \begin{vmatrix} y_1(c) & y_2(c) \\ y_1'(c) & y_2'(c) \end{vmatrix} = 0$$

and Theorem 6.1.6 implies the stated conclusion.

6.1.4 Exercises

1. (a) Verify that $y_1 = e^{2x}$ and $y_2 = e^{5x}$ are solutions of

$$y'' - 7y' + 10y = 0 \quad (A)$$

on $(-\infty, \infty)$.

- (b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 e^{2x} + c_2 e^{5x}$ is a solution of (A) on $(-\infty, \infty)$.

- (c) Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 1.$$

- (d) Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

2. (a) Verify that $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$ are solutions of

$$y'' - 2y' + 2y = 0 \quad (A)$$

on $(-\infty, \infty)$.

- (b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 e^x \cos x + c_2 e^x \sin x$ is a solution of (A) on $(-\infty, \infty)$.

- (c) Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2.$$

- (d) Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

3. (a) Verify that $y_1 = e^x$ and $y_2 = x e^x$ are solutions of

$$y'' - 2y' + y = 0 \quad (A)$$

on $(-\infty, \infty)$.

- (b) Verify that if c_1 and c_2 are arbitrary constants then $y = e^x(c_1 + c_2 x)$ is a solution of (A) on $(-\infty, \infty)$.

- (c) Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 7, \quad y'(0) = 4.$$

(d) Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

4. (a) Verify that $y_1 = 1/(x-1)$ and $y_2 = 1/(x+1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \quad (\text{A})$$

on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. What is the general solution of (A) on each of these intervals?

(b) Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = -5, \quad y'(0) = 1.$$

What is the interval of validity of the solution?

(c) C/G Graph the solution of the initial value problem.

(d) Verify Abel's formula for y_1 and y_2 , with $x_0 = 0$.

5. Compute the Wronskians of the given sets of functions.

(a) $\{1, e^x\}$

(b) $\{e^x, e^x \sin x\}$

(c) $\{x+1, x^2+2\}$

(d) $\{x^{1/2}, x^{-1/3}\}$

(e) $\left\{\frac{\sin x}{x}, \frac{\cos x}{x}\right\}$

(f) $\{x \ln |x|, x^2 \ln |x|\}$

(g) $\{e^x \cos \sqrt{x}, e^x \sin \sqrt{x}\}$

6. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$y'' + 3(x^2 + 1)y' - 2y = 0,$$

given that $W(\pi) = 0$.

7. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

given that $W(0) = 1$. (This is *Legendre's equation*.)

8. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$x^2 y'' + xy' + (x^2 - v^2)y = 0,$$

given that $W(1) = 1$. (This is *Bessel's equation*.)

9. (This exercise shows that if you know one nontrivial solution of $y'' + p(x)y' + q(x)y = 0$, you can use Abel's formula to find another.)

Suppose p and q are continuous and y_1 is a solution of

$$y'' + p(x)y' + q(x)y = 0 \quad (\text{A})$$

that has no zeros on (a, b) . Let $P(x) = \int p(x) dx$ be any antiderivative of p on (a, b) .

(a) Show that if K is an arbitrary nonzero constant and y_2 satisfies

$$y_1 y_2' - y_1' y_2 = K e^{-P(x)} \quad (\text{B})$$

on (a, b) , then y_2 also satisfies (A) on (a, b) , and $\{y_1, y_2\}$ is a fundamental set of solutions on (a, b) .

- (b) Conclude from (a) that if $y_2 = u y_1$ where $u' = K \frac{e^{-P(x)}}{y_1^2(x)}$, then $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a, b) .

In Exercises 6.1.10–6.1.23 use the method suggested by Exercise 6.1.9 to find a second solution y_2 that isn't a constant multiple of the solution y_1 . Choose K conveniently to simplify y_2 .

10. $y'' - 2y' - 3y = 0$; $y_1 = e^{3x}$
11. $y'' - 6y' + 9y = 0$; $y_1 = e^{3x}$
12. $y'' - 2ay' + a^2y = 0$ ($a = \text{constant}$); $y_1 = e^{ax}$
13. $x^2y'' + xy' - y = 0$; $y_1 = x$
14. $x^2y'' - xy' + y = 0$; $y_1 = x$
15. $x^2y'' - (2a - 1)xy' + a^2y = 0$ ($a = \text{nonzero constant}$); $x > 0$; $y_1 = x^a$
16. $4x^2y'' - 4xy' + (3 - 16x^2)y = 0$; $y_1 = x^{1/2}e^{2x}$
17. $(x - 1)y'' - xy' + y = 0$; $y_1 = e^x$
18. $x^2y'' - 2xy' + (x^2 + 2)y = 0$; $y_1 = x \cos x$
19. $4x^2(\sin x)y'' - 4x(x \cos x + \sin x)y' + (2x \cos x + 3 \sin x)y = 0$; $y_1 = x^{1/2}$
20. $(3x - 1)y'' - (3x + 2)y' - (6x - 8)y = 0$; $y_1 = e^{2x}$
21. $(x^2 - 4)y'' + 4xy' + 2y = 0$; $y_1 = \frac{1}{x - 2}$
22. $(2x + 1)xy'' - 2(2x^2 - 1)y' - 4(x + 1)y = 0$; $y_1 = \frac{1}{x}$
23. $(x^2 - 2x)y'' + (2 - x^2)y' + (2x - 2)y = 0$; $y_1 = e^x$
24. Suppose p and q are continuous on an open interval (a, b) and let x_0 be in (a, b) . Use Theorem 6.1.1 to show that the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

on (a, b) is the trivial solution $y \equiv 0$.

25. Suppose P_0 , P_1 , and P_2 are continuous on (a, b) and let x_0 be in (a, b) . Show that if either of the following statements is true then $P_0(x) = 0$ for some x in (a, b) .

(a) The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has more than one solution on (a, b) .

(b) The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

has a nontrivial solution on (a, b) .

26. Suppose p and q are continuous on (a, b) and y_1 and y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

on (a, b) . Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where α , β , γ , and δ are constants. Show that if $\{z_1, z_2\}$ is a fundamental set of solutions of (A) on (a, b) then so is $\{y_1, y_2\}$.

27. Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0 \quad (\text{A})$$

on (a, b) . Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where α, β, γ , and δ are constants. Show that $\{z_1, z_2\}$ is a fundamental set of solutions of (A) on (a, b) if and only if $\alpha\gamma - \beta\delta \neq 0$.

28. Suppose y_1 is differentiable on an interval (a, b) and $y_2 = ky_1$, where k is a constant. Show that the Wronskian of $\{y_1, y_2\}$ is identically zero on (a, b) .

29. Let

$$y_1 = x^3 \quad \text{and} \quad y_2 = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0. \end{cases}$$

- (a) Show that the Wronskian of $\{y_1, y_2\}$ is defined and identically zero on $(-\infty, \infty)$.
 (b) Suppose $a < 0 < b$. Show that $\{y_1, y_2\}$ is linearly independent on (a, b) .
 (c) Use Exercise 6.1.25(b) to show that these results don't contradict Theorem 6.1.5, because neither y_1 nor y_2 can be a solution of an equation

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) if p and q are continuous on (a, b) .

30. Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) such that either $y_1(x_0) = y_2(x_0) = 0$ or $y_1'(x_0) = y_2'(x_0) = 0$ for some x_0 in (a, b) . Show that $\{y_1, y_2\}$ is linearly dependent on (a, b) .

31. Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) . Show that if $y_1(x_1) = y_1(x_2) = 0$, where $a < x_1 < x_2 < b$, then $y_2(x) = 0$ for some x in (x_1, x_2) .
 HINT: Show that if y_2 has no zeros in (x_1, x_2) , then y_1/y_2 is either strictly increasing or strictly decreasing on (x_1, x_2) , and deduce a contradiction.

32. Suppose p and q are continuous on (a, b) and every solution of

$$y'' + p(x)y' + q(x)y = 0 \quad (\text{A})$$

on (a, b) can be written as a linear combination of the twice differentiable functions $\{y_1, y_2\}$. Use Theorem 6.1.1 to show that y_1 and y_2 are themselves solutions of (A) on (a, b) .

33. Suppose p_1, p_2, q_1 , and q_2 are continuous on (a, b) and the equations

$$y'' + p_1(x)y' + q_1(x)y = 0 \quad \text{and} \quad y'' + p_2(x)y' + q_2(x)y = 0$$

have the same solutions on (a, b) . Show that $p_1 = p_2$ and $q_1 = q_2$ on (a, b) . HINT: Use Abel's formula.

34. (For this exercise you have to know about 3×3 determinants.) Show that if y_1 and y_2 are twice continuously differentiable on (a, b) and the Wronskian W of $\{y_1, y_2\}$ has no zeros in (a, b) then the equation

$$\frac{1}{W} \begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0$$

can be written as

$$y'' + p(x)y' + q(x)y = 0, \quad (\text{A})$$

where p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a, b) .
 HINT: Expand the determinant by cofactors of its first column.

35. Use the method suggested by Exercise 6.1.34 to find a linear homogeneous equation for which the given functions form a fundamental set of solutions on some interval.

(a) $e^x \cos 2x, \quad e^x \sin 2x$

(b) $x, \quad e^{2x}$

(c) $x, \quad x \ln x$

(d) $\cos(\ln x), \quad \sin(\ln x)$

(e) $\cosh x, \quad \sinh x$

(f) $x^2 - 1, \quad x^2 + 1$

36. Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0 \quad (\text{A})$$

on (a, b) . Show that if y is a solution of (A) on (a, b) , there's exactly one way to choose c_1 and c_2 so that $y = c_1 y_1 + c_2 y_2$ on (a, b) .

37. Suppose p and q are continuous on (a, b) and x_0 is in (a, b) . Let y_1 and y_2 be the solutions of

$$y'' + p(x)y' + q(x)y = 0 \quad (\text{A})$$

such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

(Theorem 6.1.1 implies that each of these initial value problems has a unique solution on (a, b) .)

(a) Show that $\{y_1, y_2\}$ is linearly independent on (a, b) .

(b) Show that an arbitrary solution y of (A) on (a, b) can be written as $y = y(x_0)y_1 + y'(x_0)y_2$.

(c) Express the solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2 .

38. Find solutions y_1 and y_2 of the equation $y'' = 0$ that satisfy the initial conditions

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Then use Exercise 6.1.37 (c) to write the solution of the initial value problem

$$y'' = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

as a linear combination of y_1 and y_2 .

39. Let x_0 be an arbitrary real number. Given (Example 6.1.1) that e^x and e^{-x} are solutions of $y'' - y = 0$, find solutions y_1 and y_2 of $y'' - y = 0$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Then use Exercise 6.1.37 (c) to write the solution of the initial value problem

$$y'' - y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2 .

40. Let x_0 be an arbitrary real number. Given (Example 6.1.2) that $\cos \omega x$ and $\sin \omega x$ are solutions of $y'' + \omega^2 y = 0$, find solutions of $y'' + \omega^2 y = 0$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Then use Exercise 6.1.37 (c) to write the solution of the initial value problem

$$y'' + \omega^2 y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2 . Use the identities

$$\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B \end{aligned}$$

to simplify your expressions for y_1 , y_2 , and y .

41. Recall from Exercise 6.1.4 that $1/(x-1)$ and $1/(x+1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \tag{A}$$

on $(-1, 1)$. Find solutions of (A) such that

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

Then use Exercise 6.1.37 (c) to write the solution of initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

as a linear combination of y_1 and y_2 .

42. (a) Verify that $y_1 = x^2$ and $y_2 = x^3$ satisfy

$$x^2 y'' - 4xy' + 6y = 0 \tag{A}$$

on $(-\infty, \infty)$ and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.

- (b) Let a_1, a_2, b_1 , and b_2 be constants. Show that

$$y = \begin{cases} a_1 x^2 + a_2 x^3, & x \geq 0, \\ b_1 x^2 + b_2 x^3, & x < 0 \end{cases}$$

is a solution of (A) on $(-\infty, \infty)$ if and only if $a_1 = b_1$. From this, justify the statement that y is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} c_1 x^2 + c_2 x^3, & x \geq 0, \\ c_1 x^2 + c_3 x^3, & x < 0, \end{cases}$$

where c_1, c_2 , and c_3 are arbitrary constants.

- (c) For what values of k_0 and k_1 does the initial value problem

$$x^2 y'' - 4xy' + 6y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

- (d) Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants, the initial value problem

$$x^2 y'' - 4xy' + 6y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{B}$$

has infinitely many solutions on $(-\infty, \infty)$. On what interval does (B) have a unique solution?

43. (a) Verify that $y_1 = x$ and $y_2 = x^2$ satisfy

$$x^2 y'' - 2xy' + 2y = 0 \quad (\text{A})$$

on $(-\infty, \infty)$ and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.

- (b) Let a_1, a_2, b_1 , and b_2 be constants. Show that

$$y = \begin{cases} a_1 x + a_2 x^2, & x \geq 0, \\ b_1 x + b_2 x^2, & x < 0 \end{cases}$$

is a solution of (A) on $(-\infty, \infty)$ if and only if $a_1 = b_1$ and $a_2 = b_2$. From this, justify the statement that the general solution of (A) on $(-\infty, \infty)$ is $y = c_1 x + c_2 x^2$, where c_1 and c_2 are arbitrary constants.

- (c) For what values of k_0 and k_1 does the initial value problem

$$x^2 y'' - 2xy' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

- (d) Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants then the initial value problem

$$x^2 y'' - 2xy' + 2y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(-\infty, \infty)$.

44. (a) Verify that $y_1 = x^3$ and $y_2 = x^4$ satisfy

$$x^2 y'' - 6xy' + 12y = 0 \quad (\text{A})$$

on $(-\infty, \infty)$, and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.

- (b) Show that y is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} a_1 x^3 + a_2 x^4, & x \geq 0, \\ b_1 x^3 + b_2 x^4, & x < 0, \end{cases}$$

where a_1, a_2, b_1 , and b_2 are arbitrary constants.

- (c) For what values of k_0 and k_1 does the initial value problem

$$x^2 y'' - 6xy' + 12y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

- (d) Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants then the initial value problem

$$x^2 y'' - 6xy' + 12y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \quad (\text{B})$$

has infinitely many solutions on $(-\infty, \infty)$. On what interval does (B) have a unique solution?

6.2 Constant Coefficient Homogeneous Equations

If a, b , and c are real constants and $a \neq 0$, then

$$ay'' + by' + cy = F(x)$$

is said to be a *constant coefficient equation*. In this section we consider the homogeneous constant coefficient equation

$$ay'' + by' + cy = 0. \quad (6.36)$$

As we'll see, all solutions of (6.36) are defined on $(-\infty, \infty)$. This being the case, we'll omit references to the interval on which solutions are defined, or on which a given set of solutions is a fundamental set, etc., since the interval will always be $(-\infty, \infty)$.

The key to solving (6.36) is that if $y = e^{rx}$ where r is a constant then the left side of (6.36) is a multiple of e^{rx} ; thus, if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$, so

$$ay'' + by' + cy = ar^2e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx}. \quad (6.37)$$

The quadratic polynomial

$$p(r) = ar^2 + br + c$$

is the *characteristic polynomial* of (6.36), and $p(r) = 0$ is the *characteristic equation*. From (6.37) we can see that $y = e^{rx}$ is a solution of (6.36) if and only if $p(r) = 0$.

The roots of the characteristic equation are given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (6.38)$$

We consider three cases:

CASE 1. $b^2 - 4ac > 0$, so the characteristic equation has two distinct real roots.

CASE 2. $b^2 - 4ac = 0$, so the characteristic equation has a repeated real root.

CASE 3. $b^2 - 4ac < 0$, so the characteristic equation has complex roots.

In each case we'll start with an example.

6.2.1 Case 1: Distinct Real Roots

Example 6.2.1:

(a) Find the general solution of

$$y'' + 6y' + 5y = 0. \quad (6.39)$$

(b) Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3, \quad y'(0) = -1. \quad (6.40)$$

Solution (a): The characteristic polynomial of (6.39) is

$$p(r) = r^2 + 6r + 5 = (r + 1)(r + 5).$$

Since $p(-1) = p(-5) = 0$, $y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are solutions of (6.39). Since $y_2/y_1 = e^{-4x}$ is nonconstant, 6.1.6 implies that the general solution of (6.39) is

$$y = c_1 e^{-x} + c_2 e^{-5x}. \quad (6.41)$$

Solution (b): We must determine c_1 and c_2 in (6.41) so that y satisfies the initial conditions in (6.40). Differentiating (6.41) yields

$$y' = -c_1 e^{-x} - 5c_2 e^{-5x}. \quad (6.42)$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in (6.41) and (6.42) yields

$$\begin{aligned} c_1 + c_2 &= 3 \\ -c_1 - 5c_2 &= -1. \end{aligned}$$

The solution of this system is $c_1 = 7/2$, $c_2 = -1/2$. Therefore the solution of (6.40) is

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}.$$

Figure 6.1 is a graph of this solution.

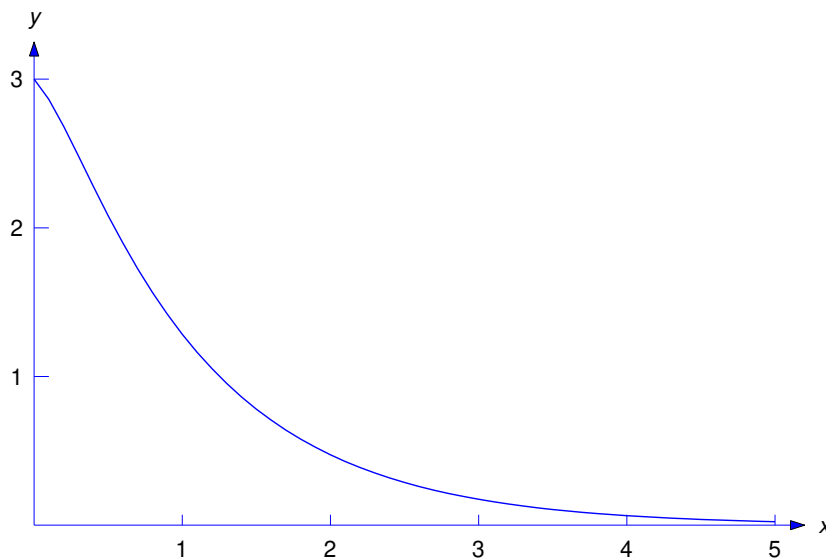


Figure 6.1: $y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}$

If the characteristic equation has arbitrary distinct real roots r_1 and r_2 , then $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions of $ay'' + by' + cy = 0$. Since $y_2/y_1 = e^{(r_2-r_1)x}$ is nonconstant, Theorem 6.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of $ay'' + by' + cy = 0$.

6.2.2 Case 2: A Repeated Real Root

Example 6.2.2:

(a) Find the general solution of

$$y'' + 6y' + 9y = 0. \quad (6.43)$$

(b) Solve the initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 3, \quad y'(0) = -1. \quad (6.44)$$

Solution (a): The characteristic polynomial of (6.43) is

$$p(r) = r^2 + 6r + 9 = (r + 3)^2,$$

so the characteristic equation has the repeated real root $r_1 = -3$. Therefore $y_1 = e^{-3x}$ is a solution of (6.43). Since the characteristic equation has no other roots, (6.43) has no other solutions of the form e^{rx} . We look for solutions of the form $y = uy_1 = ue^{-3x}$, where u is a function that we'll now determine. (This should remind you of the method of variation of parameters used in Section 2.1 to solve the nonhomogeneous equation $y' + p(x)y = f(x)$, given a solution y_1 of the complementary equation $y' + p(x)y = 0$. It's also a special case of a method called *reduction of order* that we'll study in 6.6. For other ways to obtain a second solution of (6.43) that's not a multiple of e^{-3x} , see Exercises 6.1.9, 6.1.12, and 6.2.33.

If $y = ue^{-3x}$, then

$$y' = u'e^{-3x} - 3ue^{-3x} \quad \text{and} \quad y'' = u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x},$$

so

$$\begin{aligned} y'' + 6y' + 9y &= e^{-3x} [(u'' - 6u' + 9u) + 6(u' - 3u) + 9u] \\ &= e^{-3x} [u'' - (6 - 6)u' + (9 - 18 + 9)u] = u''e^{-3x}. \end{aligned}$$

Therefore $y = ue^{-3x}$ is a solution of (6.43) if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2x$, where c_1 and c_2 are constants. Therefore any function of the form

$$y = e^{-3x}(c_1 + c_2x) \quad (6.45)$$

is a solution of (6.43). Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-3x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{-3x}$. Since $y_2/y_1 = x$ is nonconstant, 6.1.6 implies that $\{y_1, y_2\}$ is fundamental set of solutions of (6.43), and (6.45) is the general solution.

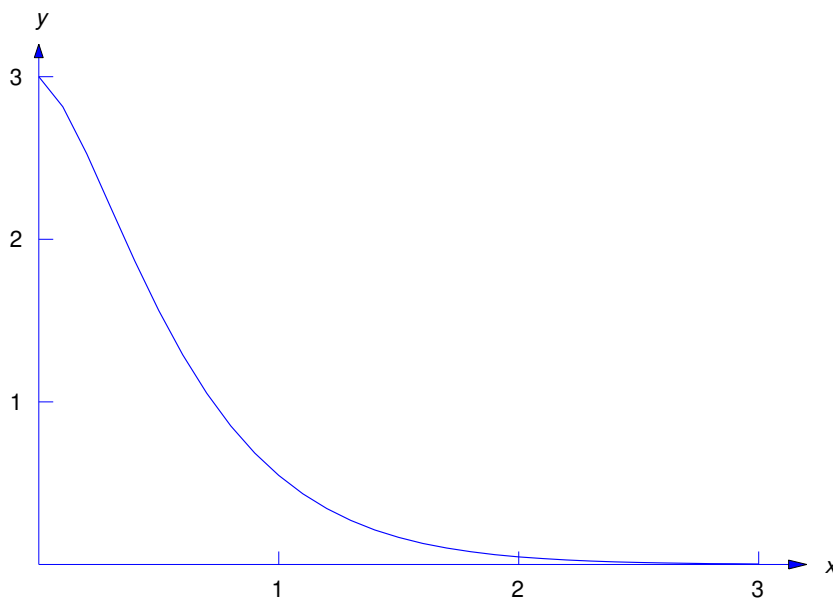
Solution (b): Differentiating (6.45) yields

$$y' = -3e^{-3x}(c_1 + c_2x) + c_2e^{-3x}. \quad (6.46)$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in (6.45) and (6.46) yields $c_1 = 3$ and $-3c_1 + c_2 = -1$, so $c_2 = 8$. Therefore the solution of (6.44) is

$$y = e^{-3x}(3 + 8x).$$

Figure 6.2 is a graph of this solution.

Figure 6.2: $y = e^{-3x}(3 + 8x)$

If the characteristic equation of $ay'' + by' + cy = 0$ has an arbitrary repeated root r_1 , the characteristic polynomial must be

$$p(r) = a(r - r_1)^2 = a(r^2 - 2r_1r + r_1^2).$$

Therefore

$$ar^2 + br + c = ar^2 - (2ar_1)r + ar_1^2,$$

which implies that $b = -2ar_1$ and $c = ar_1^2$. Therefore $ay'' + by' + cy = 0$ can be written as $a(y'' - 2r_1y' + r_1^2y) = 0$. Since $a \neq 0$ this equation has the same solutions as

$$y'' - 2r_1y' + r_1^2y = 0. \quad (6.47)$$

Since $p(r_1) = 0$, $y_1 = e^{r_1x}$ is a solution of $ay'' + by' + cy = 0$, and therefore of (6.47). Proceeding as in Example 6.2.2, we look for other solutions of (6.47) of the form $y = ue^{r_1x}$; then

$$y' = u'e^{r_1x} + r_1ue^{r_1x} \quad \text{and} \quad y'' = u''e^{r_1x} + 2r_1u'e^{r_1x} + r_1^2ue^{r_1x},$$

so

$$\begin{aligned} y'' - 2r_1y' + r_1^2y &= e^{r_1x} [(u'' + 2r_1u' + r_1^2u) - 2r_1(u' + r_1u) + r_1^2u] \\ &= e^{r_1x} [u'' + (2r_1 - 2r_1)u' + (r_1^2 - 2r_1^2 + r_1^2)u] = u''e^{r_1x}. \end{aligned}$$

Therefore $y = ue^{r_1x}$ is a solution of (6.47) if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2x$, where c_1 and c_2 are constants. Hence, any function of the form

$$y = e^{r_1x}(c_1 + c_2x) \quad (6.48)$$

is a solution of (6.47). Letting $c_1 = 1$ and $c_2 = 0$ here yields the solution $y_1 = e^{r_1x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{r_1x}$. Since $y_2/y_1 = x$ is nonconstant, 6.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (6.47), and (6.48) is the general solution.

6.2.3 Case 3: Complex Conjugate Roots

Example 6.2.3:

(a) Find the general solution of

$$y'' + 4y' + 13y = 0. \quad (6.49)$$

(b) Solve the initial value problem

$$y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = -3. \quad (6.50)$$

Solution (a): The characteristic polynomial of (6.49) is

$$p(r) = r^2 + 4r + 13 = r^2 + 4r + 4 + 9 = (r + 2)^2 + 9.$$

The roots of the characteristic equation are $r_1 = -2 + 3i$ and $r_2 = -2 - 3i$. By analogy with Case 1, it's reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of (6.49). This is true (see Exercise 6.2.34); however, there are difficulties here, since you are probably not familiar with exponential functions with complex arguments, and even if you are, it's inconvenient to work with them, since they are complex-valued. We'll take a simpler approach, which we motivate as follows: the exponential notation suggests that

$$e^{(-2+3i)x} = e^{-2x} e^{3ix} \quad \text{and} \quad e^{(-2-3i)x} = e^{-2x} e^{-3ix},$$

so even though we haven't defined e^{3ix} and e^{-3ix} , it's reasonable to expect that every linear combination of $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ can be written as $y = ue^{-2x}$, where u depends upon x . To determine u , we note that if $y = ue^{-2x}$ then

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y'' = u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x},$$

so

$$\begin{aligned} y'' + 4y' + 13y &= e^{-2x} [(u'' - 4u' + 4u) + 4(u' - 2u) + 13u] \\ &= e^{-2x} [u'' - (4 - 4)u' + (4 - 8 + 13)u] = e^{-2x}(u'' + 9u). \end{aligned}$$

Therefore $y = ue^{-2x}$ is a solution of (6.49) if and only if

$$u'' + 9u = 0.$$

From Example 6.1.2, the general solution of this equation is

$$u = c_1 \cos 3x + c_2 \sin 3x.$$

Therefore any function of the form

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) \quad (6.51)$$

is a solution of (6.49). Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-2x} \cos 3x$. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = e^{-2x} \sin 3x$. Since $y_2/y_1 = \tan 3x$ is nonconstant, 6.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (6.49), and (6.51) is the general solution.

Solution (b): Imposing the condition $y(0) = 2$ in (6.51) shows that $c_1 = 2$. Differentiating (6.51) yields

$$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 3e^{-2x}(-c_1 \sin 3x + c_2 \cos 3x),$$

and imposing the initial condition $y'(0) = -3$ here yields $-3 = -2c_1 + 3c_2 = -4 + 3c_2$, so $c_2 = 1/3$. Therefore the solution of (6.50) is

$$y = e^{-2x}\left(2 \cos 3x + \frac{1}{3} \sin 3x\right).$$

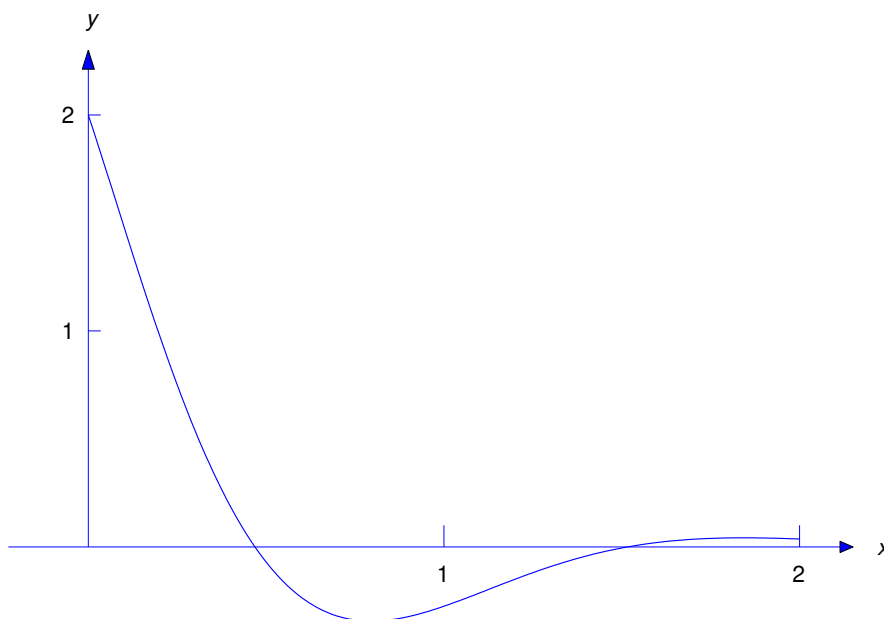


Figure 6.3: $y = e^{-2x}(2 \cos 3x + \frac{1}{3} \sin 3x)$

Figure 6.3 is a graph of this function. ■

Now suppose the characteristic equation of $ay'' + by' + cy = 0$ has arbitrary complex roots; thus, $b^2 - 4ac < 0$ and, from (6.38), the roots are

$$r_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}, \quad r_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a},$$

which we rewrite as

$$r_1 = \lambda + i\omega, \quad r_2 = \lambda - i\omega, \tag{6.52}$$

with

$$\lambda = -\frac{b}{2a}, \quad \omega = \frac{\sqrt{4ac - b^2}}{2a}.$$

Don't memorize these formulas. Just remember that r_1 and r_2 are of the form (6.52), where λ is an arbitrary real number and ω is positive; λ and ω are the *real* and *imaginary parts*, respectively, of r_1 . Similarly, λ and $-\omega$ are the real and imaginary parts of r_2 . We say that r_1 and r_2 are *complex conjugates*, which means that they have the same real part and their imaginary parts have the same absolute values, but opposite signs.

As in Example 6.2.3, it's reasonable to expect that the solutions of $ay'' + by' + cy = 0$ are linear combinations of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$. Again, the exponential notation suggests that

$$e^{(\lambda+i\omega)x} = e^{\lambda x} e^{i\omega x} \quad \text{and} \quad e^{(\lambda-i\omega)x} = e^{\lambda x} e^{-i\omega x},$$

so even though we haven't defined $e^{i\omega x}$ and $e^{-i\omega x}$, it's reasonable to expect that every linear combination of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$ can be written as $y = ue^{\lambda x}$, where u depends upon x . To determine u we first observe that since $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ are the roots of the characteristic equation, p must be of the form

$$\begin{aligned} p(r) &= a(r - r_1)(r - r_2) \\ &= a(r - \lambda - i\omega)(r - \lambda + i\omega) \\ &= a[(r - \lambda)^2 + \omega^2] \\ &= a(r^2 - 2\lambda r + \lambda^2 + \omega^2). \end{aligned}$$

Therefore $ay'' + by' + cy = 0$ can be written as

$$a [y'' - 2\lambda y' + (\lambda^2 + \omega^2)y] = 0.$$

Since $a \neq 0$ this equation has the same solutions as

$$y'' - 2\lambda y' + (\lambda^2 + \omega^2)y = 0. \quad (6.53)$$

To determine u we note that if $y = ue^{\lambda x}$ then

$$y' = u'e^{\lambda x} + \lambda ue^{\lambda x} \quad \text{and} \quad y'' = u''e^{\lambda x} + 2\lambda u'e^{\lambda x} + \lambda^2 ue^{\lambda x}.$$

Substituting these expressions into (6.53) and dropping the common factor $e^{\lambda x}$ yields

$$(u'' + 2\lambda u' + \lambda^2 u) - 2\lambda(u' + \lambda u) + (\lambda^2 + \omega^2)u = 0,$$

which simplifies to

$$u'' + \omega^2 u = 0.$$

From Example 6.1.2, the general solution of this equation is

$$u = c_1 \cos \omega x + c_2 \sin \omega x.$$

Therefore any function of the form

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x) \quad (6.54)$$

is a solution of (6.53). Letting $c_1 = 1$ and $c_2 = 0$ here yields the solution $y_1 = e^{\lambda x} \cos \omega x$. Letting $c_1 = 0$ and $c_2 = 1$ yields a second solution $y_2 = e^{\lambda x} \sin \omega x$. Since $y_2/y_1 = \tan \omega x$ is nonconstant, so Theorem 6.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (6.53), and (6.54) is the general solution.

6.2.4 Summary

The next theorem summarizes the results of this section.

Theorem 6.2.1. *Let $p(r) = ar^2 + br + c$ be the characteristic polynomial of*

$$ay'' + by' + cy = 0. \quad (6.55)$$

Then:

(a) *If $p(r) = 0$ has distinct real roots r_1 and r_2 , then the general solution of (6.55) is*

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

(b) *If $p(r) = 0$ has a repeated root r_1 , then the general solution of (6.55) is*

$$y = e^{r_1 x}(c_1 + c_2 x).$$

(c) *If $p(r) = 0$ has complex conjugate roots $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ (where $\omega > 0$), then the general solution of (6.55) is*

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x).$$

6.2.5 Exercises

In Exercises 6.2.1–6.2.12 find the general solution.

1. $y'' + 5y' - 6y = 0$
2. $y'' - 4y' + 5y = 0$
3. $y'' + 8y' + 7y = 0$
4. $y'' - 4y' + 4y = 0$
5. $y'' + 2y' + 10y = 0$
6. $y'' + 6y' + 10y = 0$
7. $y'' - 8y' + 16y = 0$
8. $y'' + y' = 0$
9. $y'' - 2y' + 3y = 0$
10. $y'' + 6y' + 13y = 0$
11. $4y'' + 4y' + 10y = 0$
12. $10y'' - 3y' - y = 0$

In Exercises 6.2.13–6.2.17 solve the initial value problem.

13. $y'' + 14y' + 50y = 0$, $y(0) = 2$, $y'(0) = -17$
14. $6y'' - y' - y = 0$, $y(0) = 10$, $y'(0) = 0$
15. $6y'' + y' - y = 0$, $y(0) = -1$, $y'(0) = 3$
16. $4y'' - 4y' - 3y = 0$, $y(0) = \frac{13}{12}$, $y'(0) = \frac{23}{24}$
17. $4y'' - 12y' + 9y = 0$, $y(0) = 3$, $y'(0) = \frac{5}{2}$

In Exercises 6.2.18–6.2.21 solve the initial value problem and graph the solution.

18. C/G $y'' + 7y' + 12y = 0$, $y(0) = -1$, $y'(0) = 0$
19. C/G $y'' - 6y' + 9y = 0$, $y(0) = 0$, $y'(0) = 2$
20. C/G $36y'' - 12y' + y = 0$, $y(0) = 3$, $y'(0) = \frac{5}{2}$
21. C/G $y'' + 4y' + 10y = 0$, $y(0) = 3$, $y'(0) = -2$
22. (a) Suppose y is a solution of the constant coefficient homogeneous equation

$$ay'' + by' + cy = 0. \quad (\text{A})$$

Let $z(x) = y(x - x_0)$, where x_0 is an arbitrary real number. Show that

$$az'' + bz' + cz = 0.$$

- (b) Let $z_1(x) = y_1(x - x_0)$ and $z_2(x) = y_2(x - x_0)$, where $\{y_1, y_2\}$ is a fundamental set of solutions of (A). Show that $\{z_1, z_2\}$ is also a fundamental set of solutions of (A).
- (c) The statement of Theorem 6.2.1 is convenient for solving an initial value problem

$$ay'' + by' + cy = 0, \quad y(0) = k_0, \quad y'(0) = k_1,$$

where the initial conditions are imposed at $x_0 = 0$. However, if the initial value problem is

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \quad (\text{B})$$

where $x_0 \neq 0$, then determining the constants in

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad y = e^{r_1 x}(c_1 + c_2 x), \quad \text{or} \quad y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x)$$

(whichever is applicable) is more complicated. Use **(b)** to restate Theorem 6.2.1 in a form more convenient for solving (B).

In Exercises 6.2.23–6.2.28 use a method suggested by Exercise 6.2.22 to solve the initial value problem.

23. $y'' + 3y' + 2y = 0, \quad y(1) = -1, \quad y'(1) = 4$

24. $y'' - 6y' - 7y = 0, \quad y(2) = -\frac{1}{3}, \quad y'(2) = -5$

25. $y'' - 14y' + 49y = 0, \quad y(1) = 2, \quad y'(1) = 11$

26. $9y'' + 6y' + y = 0, \quad y(2) = 2, \quad y'(2) = -\frac{14}{3}$

27. $9y'' + 4y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

28. $y'' + 3y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -1$

29. Prove: If the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated negative root or two roots with negative real parts, then every solution of (A) approaches zero as $x \rightarrow \infty$.

30. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has distinct real roots r_1 and r_2 . Use a method suggested by Exercise 6.2.22 to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

31. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has a repeated real root r_1 . Use a method suggested by Exercise 6.2.22 to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

32. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has complex conjugate roots $\lambda \pm i\omega$. Use a method suggested by Exercise 6.2.22 to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

33. Suppose the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated real root r_1 . Temporarily, think of e^{rx} as a function of two real variables x and r .

(a) Show that

$$a \frac{\partial^2}{\partial^2 x} (e^{rx}) + b \frac{\partial}{\partial x} (e^{rx}) + c e^{rx} = a(r - r_1)^2 e^{rx}. \tag{B}$$

(b) Differentiate (B) with respect to r to obtain

$$a \frac{\partial}{\partial r} \left(\frac{\partial^2}{\partial^2 x} (e^{rx}) \right) + b \frac{\partial}{\partial r} \left(\frac{\partial}{\partial x} (e^{rx}) \right) + c (x e^{rx}) = [2 + (r - r_1)x] a(r - r_1) e^{rx}. \tag{C}$$

(c) Reverse the orders of the partial differentiations in the first two terms on the left side of (C) to obtain

$$a \frac{\partial^2}{\partial x^2} (x e^{rx}) + b \frac{\partial}{\partial x} (x e^{rx}) + c (x e^{rx}) = [2 + (r - r_1)x] a(r - r_1) e^{rx}. \tag{D}$$

(d) Set $r = r_1$ in (B) and (D) to see that $y_1 = e^{r_1 x}$ and $y_2 = x e^{r_1 x}$ are solutions of (A)

34. In calculus you learned that e^u , $\cos u$, and $\sin u$ can be represented by the infinite series

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + \cdots \quad (\text{A})$$

$$\cos u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots + (-1)^n \frac{u^{2n}}{(2n)!} + \cdots, \quad (\text{B})$$

and

$$\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots + (-1)^n \frac{u^{2n+1}}{(2n+1)!} + \cdots \quad (\text{C})$$

for all real values of u . Even though you have previously considered (A) only for real values of u , we can set $u = i\theta$, where θ is real, to obtain

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}. \quad (\text{D})$$

Given the proper background in the theory of infinite series with complex terms, it can be shown that the series in (D) converges for all real θ .

- (a) Recalling that $i^2 = -1$, write enough terms of the sequence $\{i^n\}$ to convince yourself that the sequence is repetitive:

$$1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \dots$$

Use this to group the terms in (D) as

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}. \end{aligned}$$

By comparing this result with (B) and (C), conclude that

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (\text{E})$$

This is *Euler's identity*.

- (b) Starting from

$$e^{i\theta_1} e^{i\theta_2} = (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2),$$

collect the real part (the terms not multiplied by i) and the imaginary part (the terms multiplied by i) on the right, and use the trigonometric identities

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \\ \sin(\theta_1 + \theta_2) &= \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \end{aligned}$$

to verify that

$$e^{i(\theta_1 + \theta_2)} = e^{i\theta_1} e^{i\theta_2},$$

as you would expect from the use of the exponential notation $e^{i\theta}$.

- (c) If α and β are real numbers, define

$$e^{\alpha+i\beta} = e^{\alpha} e^{i\beta} = e^{\alpha} (\cos \beta + i \sin \beta). \quad (\text{F})$$

Show that if $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$ then

$$e^{z_1+z_2} = e^{z_1} e^{z_2}.$$

- (d) Let a, b , and c be real numbers, with $a \neq 0$. Let $z = u + iv$ where u and v are real-valued functions of x . Then we say that z is a solution of

$$ay'' + by' + cy = 0 \tag{G}$$

if u and v are both solutions of (G). Use Theorem 6.2.1(c) to verify that if the characteristic equation of (G) has complex conjugate roots $\lambda \pm i\omega$ then $z_1 = e^{(\lambda+i\omega)x}$ and $z_2 = e^{(\lambda-i\omega)x}$ are both solutions of (G).

6.3 Nonhomogeneous Linear Equations

We'll now consider the nonhomogeneous linear second order equation

$$y'' + p(x)y' + q(x)y = f(x), \quad (6.56)$$

where the forcing function f isn't identically zero. The next theorem, an extension of Theorem 6.1.1, gives sufficient conditions for existence and uniqueness of solutions of initial value problems for (6.56). We omit the proof, which is beyond the scope of this book.

Theorem 6.3.1. *Suppose p , q and f are continuous on an open interval (a, b) , let x_0 be any point in (a, b) , and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem*

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on (a, b) .

To find the general solution of (6.56) on an interval (a, b) where p , q , and f are continuous, it's necessary to find the general solution of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \quad (6.57)$$

on (a, b) . We call (6.57) the *complementary equation* for (6.56).

The next theorem shows how to find the general solution of (6.56) if we know one solution y_p of (6.56) and a fundamental set of solutions of (6.57). We call y_p a *particular solution* of (6.56); it can be any solution that we can find, one way or another.

Theorem 6.3.2. *Suppose p , q , and f are continuous on (a, b) . Let y_p be a particular solution of*

$$y'' + p(x)y' + q(x)y = f(x) \quad (6.58)$$

on (a, b) , and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation

$$y'' + p(x)y' + q(x)y = 0 \quad (6.59)$$

on (a, b) . Then y is a solution of (6.58) on (a, b) if and only if

$$y = y_p + c_1y_1 + c_2y_2, \quad (6.60)$$

where c_1 and c_2 are constants.

Proof. We first show that y in (6.60) is a solution of (6.58) for any choice of the constants c_1 and c_2 . Differentiating (6.60) twice yields

$$y' = y'_p + c_1y'_1 + c_2y'_2 \quad \text{and} \quad y'' = y''_p + c_1y''_1 + c_2y''_2,$$

so

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (y''_p + c_1y''_1 + c_2y''_2) + p(x)(y'_p + c_1y'_1 + c_2y'_2) \\ &\quad + q(x)(y_p + c_1y_1 + c_2y_2) \\ &= (y''_p + p(x)y'_p + q(x)y_p) + c_1(y''_1 + p(x)y'_1 + q(x)y_1) \\ &\quad + c_2(y''_2 + p(x)y'_2 + q(x)y_2) \\ &= f + c_1 \cdot 0 + c_2 \cdot 0 = f, \end{aligned}$$

since y_p satisfies (6.58) and y_1 and y_2 satisfy (6.59).

Now we'll show that every solution of (6.58) has the form (6.60) for some choice of the constants c_1 and c_2 . Suppose y is a solution of (6.58). We'll show that $y - y_p$ is a solution of (6.59), and therefore of the form $y - y_p = c_1 y_1 + c_2 y_2$, which implies (6.60). To see this, we compute

$$\begin{aligned}(y - y_p)'' + p(x)(y - y_p)' + q(x)(y - y_p) &= (y'' - y_p'') + p(x)(y' - y_p') \\ &\quad + q(x)(y - y_p) \\ &= (y'' + p(x)y' + q(x)y) \\ &\quad - (y_p'' + p(x)y_p' + q(x)y_p) \\ &= f(x) - f(x) = 0,\end{aligned}$$

since y and y_p both satisfy (6.58). ■

We say that (6.60) is the *general solution of (6.58) on (a, b)* .

If P_0 , P_1 , and F are continuous and P_0 has no zeros on (a, b) , then Theorem 6.3.2 implies that the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \quad (6.61)$$

on (a, b) is $y = y_p + c_1 y_1 + c_2 y_2$, where y_p is a particular solution of (6.61) on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

on (a, b) . To see this, we rewrite (6.61) as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = \frac{F(x)}{P_0(x)}$$

and apply Theorem 6.3.2 with $p = P_1/P_0$, $q = P_2/P_0$, and $f = F/P_0$.

To avoid awkward wording in examples and exercises, we won't specify the interval (a, b) when we ask for the general solution of a specific linear second order equation, or for a fundamental set of solutions of a homogeneous linear second order equation. Let's agree that this always means that we want the general solution (or a fundamental set of solutions, as the case may be) on every open interval on which p , q , and f are continuous if the equation is of the form (6.58), or on which P_0 , P_1 , P_2 , and F are continuous and P_0 has no zeros, if the equation is of the form (6.61). We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0 , P_1 , P_2 , and F are all continuous on an open interval (a, b) , but P_0 *does* have a zero in (a, b) , then (6.61) may fail to have a general solution on (a, b) in the sense just defined. Exercises 6.1.42–6.1.44 illustrate this point for a homogeneous equation.

In this section we limit ourselves to applications of Theorem 6.3.2 where we can guess at the form of the particular solution.

Example 6.3.1:

- (a) Find the general solution of

$$y'' + y = 1. \quad (6.62)$$

- (b) Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \quad (6.63)$$

Solution (a): We can apply Theorem 6.3.2 with $(a, b) = (-\infty, \infty)$, since the functions $p \equiv 0$, $q \equiv 1$, and $f \equiv 1$ in (6.62) are continuous on $(-\infty, \infty)$. By inspection we see that $y_p \equiv 1$ is a particular solution of (6.62). Since $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation $y'' + y = 0$, the general solution of (6.62) is

$$y = 1 + c_1 \cos x + c_2 \sin x. \quad (6.64)$$

Solution (b): Imposing the initial condition $y(0) = 2$ in (6.64) yields $2 = 1 + c_1$, so $c_1 = 1$. Differentiating (6.64) yields

$$y' = -c_1 \sin x + c_2 \cos x.$$

Imposing the initial condition $y'(0) = 7$ here yields $c_2 = 7$, so the solution of (6.63) is

$$y = 1 + \cos x + 7 \sin x.$$

Figure 6.4 is a graph of this function.

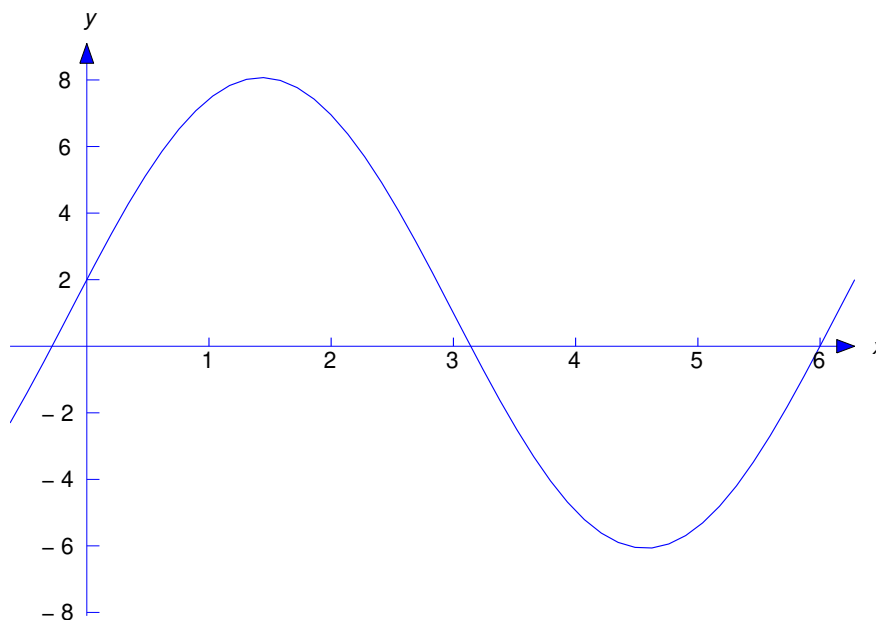


Figure 6.4: $y = 1 + \cos x + 7 \sin x$

Example 6.3.2:

- (a) Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \quad (6.65)$$

- (b) Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \quad (6.66)$$

Solution (a): The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $r^2 - 2r + 1 = (r - 1)^2$, so $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation. To guess a form for a particular solution of (6.65), we note that substituting a second degree polynomial $y_p = A + Bx + Cx^2$ into the left side of (6.65) will produce another second degree polynomial with coefficients that depend upon A , B , and C . The trick is to choose A , B , and C so the polynomials on the two sides of (6.65) have the same coefficients; thus, if

$$y_p = A + Bx + Cx^2 \quad \text{then} \quad y'_p = B + 2Cx \quad \text{and} \quad y''_p = 2C,$$

so

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2C - 2(B + 2Cx) + (A + Bx + Cx^2) \\ &= (2C - 2B + A) + (-4C + B)x + Cx^2 = -3 - x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} C &= 1 \\ B - 4C &= -1 \\ A - 2B + 2C &= -3, \end{aligned}$$

so $C = 1$, $B = -1 + 4C = 3$, and $A = -3 - 2C + 2B = 1$. Therefore $y_p = 1 + 3x + x^2$ is a particular solution of (6.65) and Theorem 6.3.2 implies that

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2x) \quad (6.67)$$

is the general solution of (6.65).

Solution (b): Imposing the initial condition $y(0) = -2$ in (6.67) yields $-2 = 1 + c_1$, so $c_1 = -3$. Differentiating (6.67) yields

$$y' = 3 + 2x + e^x(c_1 + c_2x) + c_2e^x,$$

and imposing the initial condition $y'(0) = 1$ here yields $1 = 3 + c_1 + c_2$, so $c_2 = 1$. Therefore the solution of (6.66) is

$$y = 1 + 3x + x^2 - e^x(3 - x).$$

Figure 6.5 is a graph of this solution.

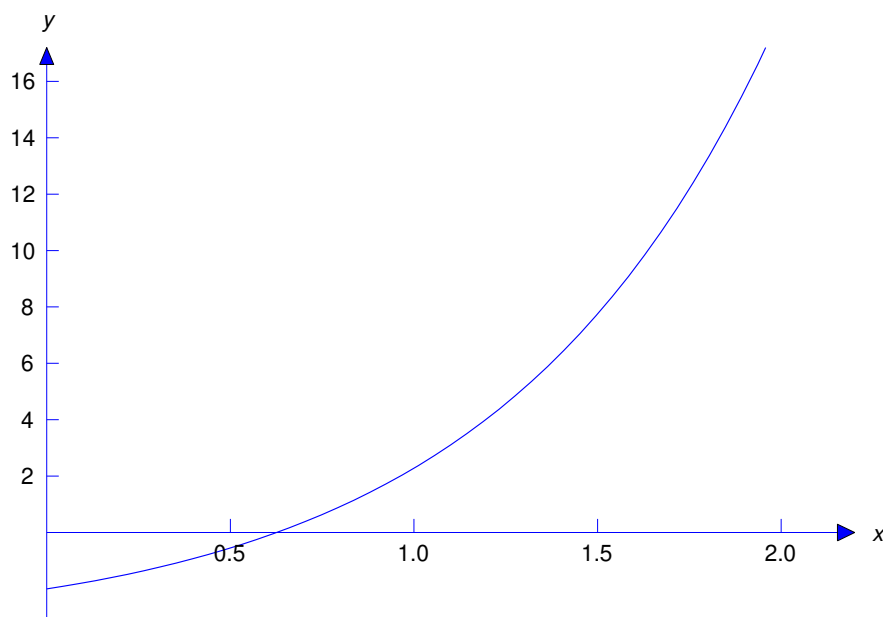


Figure 6.5: $y = 1 + 3x + x^2 - e^x(3 - x)$

Example 6.3.3: Find the general solution of

$$x^2y'' + xy' - 4y = 2x^4 \quad (6.68)$$

on $(-\infty, 0)$ and $(0, \infty)$.

Solution: In Example 6.1.3, we verified that $y_1 = x^2$ and $y_2 = 1/x^2$ form a fundamental set of solutions of the complementary equation

$$x^2 y'' + x y' - 4y = 0$$

on $(-\infty, 0)$ and $(0, \infty)$. To find a particular solution of (6.68), we note that if $y_p = Ax^4$, where A is a constant then both sides of (6.68) will be constant multiples of x^4 and we may be able to choose A so the two sides are equal. This is true in this example, since if $y_p = Ax^4$ then

$$x^2 y_p'' + x y_p' - 4y_p = x^2(12Ax^2) + x(4Ax^3) - 4Ax^4 = 12Ax^4 = 2x^4$$

if $A = 1/6$; therefore, $y_p = x^4/6$ is a particular solution of (6.68) on $(-\infty, \infty)$. Theorem 6.3.2 implies that the general solution of (6.68) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{x^4}{6} + c_1 x^2 + \frac{c_2}{x^2}.$$

6.3.1 The Principle of Superposition

The next theorem enables us to break a nonhomogeneous equation into simpler parts, find a particular solution for each part, and then combine their solutions to obtain a particular solution of the original problem.

Theorem 6.3.3. [*The Principle of Superposition*] Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b) . Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b) .

Proof. If $y_p = y_{p_1} + y_{p_2}$ then

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= (y_{p_1} + y_{p_2})'' + p(x)(y_{p_1} + y_{p_2})' + q(x)(y_{p_1} + y_{p_2}) \\ &= \left(y_{p_1}'' + p(x)y_{p_1}' + q(x)y_{p_1} \right) + \left(y_{p_2}'' + p(x)y_{p_2}' + q(x)y_{p_2} \right) \\ &= f_1(x) + f_2(x). \quad \blacksquare \end{aligned}$$

It's easy to generalize Theorem 6.3.3 to the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{6.69}$$

where

$$f = f_1 + f_2 + \cdots + f_k;$$

thus, if y_{p_i} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_i(x)$$

on (a, b) for $i = 1, 2, \dots, k$, then $y_{p_1} + y_{p_2} + \dots + y_{p_k}$ is a particular solution of (6.69) on (a, b) . Moreover, by a proof similar to the proof of Theorem 6.3.3 we can formulate the principle of superposition in terms of a linear equation written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

(Exercise 6.3.39); that is, if y_{p_1} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b) , then $y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b) .

Example 6.3.4: The function $y_{p_1} = x^4/15$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 \quad (6.70)$$

on $(-\infty, \infty)$ and $y_{p_2} = x^2/3$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 4x^2 \quad (6.71)$$

on $(-\infty, \infty)$. Use the principle of superposition to find a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2 \quad (6.72)$$

on $(-\infty, \infty)$.

Solution: The right side $F(x) = 2x^4 + 4x^2$ in (6.72) is the sum of the right sides

$$F_1(x) = 2x^4 \quad \text{and} \quad F_2(x) = 4x^2.$$

in (6.70) and (6.71). Therefore the principle of superposition implies that

$$y_p = y_{p_1} + y_{p_2} = \frac{x^4}{15} + \frac{x^2}{3}$$

is a particular solution of (6.72).

6.3.2 Exercises

In Exercises 6.3.1–6.3.6 find a particular solution by the method used in Example 6.3.2. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

1. $y'' + 5y' - 6y = 22 + 18x - 18x^2$
2. $y'' - 4y' + 5y = 1 + 5x$
3. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3$
4. $y'' - 4y' + 4y = 2 + 8x - 4x^2$
5. C/G $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3, \quad y(0) = 2, \quad y'(0) = 9$

6. C/G $y'' + 6y' + 10y = 22 + 20x$, $y(0) = 2$, $y'(0) = -2$
 7. Show that the method used in Example 6.3.2 won't yield a particular solution of

$$y'' + y' = 1 + 2x + x^2; \quad (\text{A})$$

that is, (A) doesn't have a particular solution of the form $y_p = A + Bx + Cx^2$, where A , B , and C are constants.

In Exercises 6.3.8–6.3.13 find a particular solution by the method used in Example 6.3.3.

8. $x^2y'' + 7xy' + 8y = \frac{6}{x}$ 9. $x^2y'' - 7xy' + 7y = 13x^{1/2}$
 10. $x^2y'' - xy' + y = 2x^3$ 11. $x^2y'' + 5xy' + 4y = \frac{1}{x^3}$
 12. $x^2y'' + xy' + y = 10x^{1/3}$ 13. $x^2y'' - 3xy' + 13y = 2x^4$
 14. Show that the method suggested for finding a particular solution in Exercises 6.3.8–6.3.13 won't yield a particular solution of

$$x^2y'' + 3xy' - 3y = \frac{1}{x^3}; \quad (\text{A})$$

that is, (A) doesn't have a particular solution of the form $y_p = A/x^3$.

15. Prove: If a , b , c , α , and M are constants and $M \neq 0$ then

$$ax^2y'' + bxy' + cy = Mx^\alpha$$

has a particular solution $y_p = Ax^\alpha$ ($A = \text{constant}$) if and only if $a\alpha(\alpha - 1) + b\alpha + c \neq 0$.

If a , b , c , and α are constants, then

$$a(e^{\alpha x})'' + b(e^{\alpha x})' + ce^{\alpha x} = (a\alpha^2 + b\alpha + c)e^{\alpha x}.$$

Use this in Exercises 6.3.16–6.3.21 to find a particular solution. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

16. $y'' + 5y' - 6y = 6e^{3x}$ 17. $y'' - 4y' + 5y = e^{2x}$
 18. C/G $y'' + 8y' + 7y = 10e^{-2x}$, $y(0) = -2$, $y'(0) = 10$
 19. C/G $y'' - 4y' + 4y = e^x$, $y(0) = 2$, $y'(0) = 0$
 20. $y'' + 2y' + 10y = e^{x/2}$ 21. $y'' + 6y' + 10y = e^{-3x}$
 22. Show that the method suggested for finding a particular solution in Exercises 6.3.16–6.3.21 won't yield a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}; \quad (\text{A})$$

that is, (A) doesn't have a particular solution of the form $y_p = Ae^{4x}$.

23. Prove: If α and M are constants and $M \neq 0$ then constant coefficient equation

$$ay'' + by' + cy = Me^{\alpha x}$$

has a particular solution $y_p = Ae^{\alpha x}$ ($A = \text{constant}$) if and only if $e^{\alpha x}$ isn't a solution of the complementary equation.

If ω is a constant, differentiating a linear combination of $\cos \omega x$ and $\sin \omega x$ with respect to x yields another linear combination of $\cos \omega x$ and $\sin \omega x$. In Exercises 6.3.24–6.3.29 use this to find a particular solution of the equation. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

24. $y'' - 8y' + 16y = 23 \cos x - 7 \sin x$
25. $y'' + y' = -8 \cos 2x + 6 \sin 2x$
26. $y'' - 2y' + 3y = -6 \cos 3x + 6 \sin 3x$
27. $y'' + 6y' + 13y = 18 \cos x + 6 \sin x$
28. C/G $y'' + 7y' + 12y = -2 \cos 2x + 36 \sin 2x, \quad y(0) = -3, \quad y'(0) = 3$
29. C/G $y'' - 6y' + 9y = 18 \cos 3x + 18 \sin 3x, \quad y(0) = 2, \quad y'(0) = 2$
30. Find the general solution of

$$y'' + \omega_0^2 y = M \cos \omega x + N \sin \omega x,$$

where M and N are constants and ω and ω_0 are distinct positive numbers.

31. Show that the method suggested for finding a particular solution in Exercises 6.3.24–6.3.29 won't yield a particular solution of

$$y'' + y = \cos x + \sin x; \tag{A}$$

that is, (A) does not have a particular solution of the form $y_p = A \cos x + B \sin x$.

32. Prove: If M, N are constants (not both zero) and $\omega > 0$, the constant coefficient equation

$$ay'' + by' + cy = M \cos \omega x + N \sin \omega x \tag{A}$$

has a particular solution that's a linear combination of $\cos \omega x$ and $\sin \omega x$ if and only if the left side of (A) is not of the form $a(y'' + \omega^2 y)$, so that $\cos \omega x$ and $\sin \omega x$ are solutions of the complementary equation.

In Exercises 6.3.33–6.3.38 refer to the cited exercises and use the principal of superposition to find a particular solution. Then find the general solution.

33. $y'' + 5y' - 6y = 22 + 18x - 18x^2 + 6e^{3x}$ (See Exercises 6.3.1 and 6.3.16.)
34. $y'' - 4y' + 5y = 1 + 5x + e^{2x}$ (See Exercises 6.3.2 and 6.3.17.)
35. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3 + 10e^{-2x}$ (See Exercises 6.3.3 and 6.3.18.)
36. $y'' - 4y' + 4y = 2 + 8x - 4x^2 + e^x$ (See Exercises 6.3.4 and 6.3.19.)
37. $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3 + e^{x/2}$ (See Exercises 6.3.5 and 6.3.20.)
38. $y'' + 6y' + 10y = 22 + 20x + e^{-3x}$ (See Exercises 6.3.6 and 6.3.21.)
39. Prove: If y_{p_1} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b) , then $y_p = y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b) .

40. Suppose p , q , and f are continuous on (a, b) . Let y_1 , y_2 , and y_p be twice differentiable on (a, b) , such that $y = c_1y_1 + c_2y_2 + y_p$ is a solution of

$$y'' + p(x)y' + q(x)y = f$$

on (a, b) for every choice of the constants c_1, c_2 . Show that y_1 and y_2 are solutions of the complementary equation on (a, b) .

6.4 The Method of Undetermined Coefficients I

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x), \quad (6.73)$$

where α is a constant and G is a polynomial.

From Theorem 6.3.2, the general solution of (6.73) is $y = y_p + c_1y_1 + c_2y_2$, where y_p is a particular solution of (6.73) and $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation

$$ay'' + by' + cy = 0.$$

In Section 5.2 we showed how to find $\{y_1, y_2\}$. In this section we'll show how to find y_p . The procedure that we'll use is called *the method of undetermined coefficients*.

Our first example is similar to Exercises 6.3.16–6.3.21.

Example 6.4.1: Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \quad (6.74)$$

Then find the general solution.

Solution: Substituting $y_p = Ae^{2x}$ for y in (6.74) will produce a constant multiple of Ae^{2x} on the left side of (6.74), so it may be possible to choose A so that y_p is a solution of (6.74). Let's try it; if $y_p = Ae^{2x}$ then

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

if $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution of (6.74). To find the general solution, we note that the characteristic polynomial of the complementary equation

$$y'' - 7y' + 12y = 0 \quad (6.75)$$

is $p(r) = r^2 - 7r + 12 = (r - 3)(r - 4)$, so $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions of (6.75). Therefore the general solution of (6.74) is

$$y = 2e^{2x} + c_1e^{3x} + c_2e^{4x}.$$

Example 6.4.2: Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \quad (6.76)$$

Then find the general solution.

Solution: Fresh from our success in finding a particular solution of (6.74) — where we chose $y_p = Ae^{2x}$ because the right side of (6.74) is a constant multiple of e^{2x} — it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of (6.76). However, this won't work, since we saw in Example 6.4.1 that e^{4x} is a solution of the complementary equation (6.75), so substituting $y_p = Ae^{4x}$ into the left side of (6.76) produces zero on the left, no matter how we choose A . To discover a suitable form for y_p , we use the same approach that we used in Section 5.2 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of (6.76) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \quad (6.77)$$

into (6.76) and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 7(u' + 4u) + 12u = 5,$$

or

$$u'' + u' = 5.$$

By inspection we see that $u_p = 5x$ is a particular solution of this equation, so $y_p = 5xe^{4x}$ is a particular solution of (6.76). Therefore

$$y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$$

is the general solution.

Example 6.4.3: Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \quad (6.78)$$

Solution: Since the characteristic polynomial of the complementary equation

$$y'' - 8y' + 16y = 0 \quad (6.79)$$

is $p(r) = r^2 - 8r + 16 = (r - 4)^2$, both $y_1 = e^{4x}$ and $y_2 = xe^{4x}$ are solutions of (6.79). Therefore (6.78) does not have a solution of the form $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$. As in Example 6.4.2, we look for solutions of (6.78) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting from (6.77) into (6.78) and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 8(u' + 4u) + 16u = 2,$$

or

$$u'' = 2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p = x^2$ is a particular solution of this equation, so $y_p = x^2e^{4x}$ is a particular solution of (6.76). Therefore

$$y = e^{4x}(x^2 + c_1 + c_2x)$$

is the general solution. ■

The preceding examples illustrate the following facts concerning the form of a particular solution y_p of a constant coefficient equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

(a) If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \quad (6.80)$$

then $y_p = Ae^{\alpha x}$, where A is a constant. (See Example 6.4.1).

(b) If $e^{\alpha x}$ is a solution of (6.80) but $xe^{\alpha x}$ is not, then $y_p = Axe^{\alpha x}$, where A is a constant. (See Example 6.4.2.)

(c) If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of (6.80), then $y_p = Ax^2e^{\alpha x}$, where A is a constant. (See Example 6.4.3.) See Exercise 6.4.30 for the proofs of these facts.

In all three cases you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = ke^{\alpha x},$$

and solve for the constant A , as we did in Example 6.4.1. (See Exercises 6.4.31–6.4.33.) However, if the equation is

$$ay'' + by' + cy = ke^{\alpha x}G(x),$$

where G is a polynomial of degree greater than zero, we recommend that you use the substitution $y = ue^{\alpha x}$ as we did in Examples 6.4.2 and 6.4.3. The equation for u will turn out to be

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x), \quad (6.81)$$

where $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation and $p'(r) = 2ar + b$ (Exercise 6.4.30); however, you shouldn't memorize this since it's easy to derive the equation for u in any particular case. Note, however, that if $e^{\alpha x}$ is a solution of the complementary equation then $p(\alpha) = 0$, so (6.81) reduces to

$$au'' + p'(\alpha)u' = G(x),$$

while if both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation then $p(r) = a(r - \alpha)^2$ and $p'(r) = 2a(r - \alpha)$, so $p(\alpha) = p'(\alpha) = 0$ and (6.81) reduces to

$$au'' = G(x).$$

Example 6.4.4: Find a particular solution of

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2). \quad (6.82)$$

Solution: Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and} \quad y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into (6.82) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 3(u' + 3u) + 2u = -1 + 2x + x^2,$$

or

$$u'' + 3u' + 2u = -1 + 2x + x^2. \quad (6.83)$$

As in Example 6.3.2, in order to guess a form for a particular solution of (6.83), we note that substituting a second degree polynomial $u_p = A + Bx + Cx^2$ for u in the left side of (6.83) produces another second degree polynomial with coefficients that depend upon A , B , and C ; thus,

$$\text{if } u_p = A + Bx + Cx^2 \quad \text{then} \quad u'_p = B + 2Cx \quad \text{and} \quad u''_p = 2C.$$

If u_p is to satisfy (6.83), we must have

$$\begin{aligned} u''_p + 3u'_p + 2u_p &= 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) \\ &= (2C + 3B + 2A) + (6C + 2B)x + 2Cx^2 = -1 + 2x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} 2C &= 1 \\ 2B + 6C &= 2 \\ 2A + 3B + 2C &= -1. \end{aligned}$$

Solving these equations for C , B , and A (in that order) yields $C = 1/2$, $B = -1/2$, $A = -1/4$. Therefore

$$u_p = -\frac{1}{4}(1 + 2x - 2x^2)$$

is a particular solution of (6.83), and

$$y_p = u_p e^{3x} = -\frac{e^{3x}}{4}(1 + 2x - 2x^2)$$

is a particular solution of (6.82).

Example 6.4.5: Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2). \quad (6.84)$$

Solution: Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and} \quad y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into (6.84) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

or

$$u'' + 2u' = 6 + 8x + 12x^2. \quad (6.85)$$

There's no u term in this equation, since e^{3x} is a solution of the complementary equation for (6.84). (See Exercise 6.4.30.) Therefore (6.85) does not have a particular solution of the form $u_p = A + Bx + Cx^2$ that we used successfully in Example 6.4.4, since with this choice of u_p ,

$$u_p'' + 2u_p' = 2C + (B + 2Cx)$$

can't contain the last term ($12x^2$) on the right side of (6.85). Instead, let's try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of x that appear on the right side of (6.85).

Substituting these expressions in place of u' and u'' in (6.85) yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2 = 6 + 8x + 12x^2.$$

Comparing coefficients of like powers of x on the two sides of the last equality shows that u_p satisfies (6.85) if

$$\begin{aligned} 6C &= 12 \\ 4B + 6C &= 8 \\ 2A + 2B &= 6. \end{aligned}$$

Solving these equations successively yields $C = 2$, $B = -1$, and $A = 4$. Therefore

$$u_p = x(4 - x + 2x^2)$$

is a particular solution of (6.85), and

$$y_p = u_p e^{3x} = x e^{3x} (4 - x + 2x^2)$$

is a particular solution of (6.84).

Example 6.4.6: Find a particular solution of

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2). \quad (6.86)$$

Solution: Substituting

$$y = ue^{-x/2}, \quad y' = u'e^{-x/2} - \frac{1}{2}ue^{-x/2}, \quad \text{and} \quad y'' = u''e^{-x/2} - u'e^{-x/2} + \frac{1}{4}ue^{-x/2}$$

into (6.86) and canceling $e^{-x/2}$ yields

$$4\left(u'' - u' + \frac{u}{4}\right) + 4\left(u' - \frac{u}{2}\right) + u = 4u'' = -8 + 48x + 144x^2,$$

or

$$u'' = -2 + 12x + 36x^2, \quad (6.87)$$

which does not contain u or u' because $e^{-x/2}$ and $xe^{-x/2}$ are both solutions of the complementary equation. (See Exercise 6.4.30.) To obtain a particular solution of (6.87) we integrate twice, taking the constants of integration to be zero; thus,

$$u'_p = -2x + 6x^2 + 12x^3 \quad \text{and} \quad u_p = -x^2 + 2x^3 + 3x^4 = x^2(-1 + 2x + 3x^2).$$

Therefore

$$y_p = u_p e^{-x/2} = x^2 e^{-x/2} (-1 + 2x + 3x^2)$$

is a particular solution of (6.86).

6.4.1 Summary

The preceding examples illustrate the following facts concerning particular solutions of a constant coefficient equation of the form

$$ay'' + by' + cy = e^{\alpha x} G(x),$$

where G is a polynomial (see Exercise 6.4.30):

(a) If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \quad (6.88)$$

then $y_p = e^{\alpha x} Q(x)$, where Q is a polynomial of the same degree as G . (See Example 6.4.4).

(b) If $e^{\alpha x}$ is a solution of (6.88) but $xe^{\alpha x}$ is not, then $y_p = xe^{\alpha x} Q(x)$, where Q is a polynomial of the same degree as G . (See Example 6.4.5.)

(c) If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of (6.88), then $y_p = x^2 e^{\alpha x} Q(x)$, where Q is a polynomial of the same degree as G . (See Example 6.4.6.)

In all three cases, you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = e^{\alpha x} G(x),$$

and solve for the coefficients of the polynomial Q . However, if you try this you will see that the computations are more tedious than those that you encounter by making the substitution $y = ue^{\alpha x}$ and finding a particular solution of the resulting equation for u . (See Exercises 6.4.34–6.4.36.) In Case (a) the equation for u will be of the form

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x),$$

with a particular solution of the form $u_p = Q(x)$, a polynomial of the same degree as G , whose coefficients can be found by the method used in Example 6.4.4. In Case (b) the equation for u will be of the form

$$au'' + p'(\alpha)u' = G(x)$$

(no u term on the left), with a particular solution of the form $u_p = xQ(x)$, where Q is a polynomial of the same degree as G whose coefficients can be found by the method used in Example 6.4.5. In Case (c) the equation for u will be of the form

$$au'' = G(x)$$

with a particular solution of the form $u_p = x^2 Q(x)$ that can be obtained by integrating $G(x)/a$ twice and taking the constants of integration to be zero, as in Example 6.4.6.

6.4.2 Using the Principle of Superposition

The next example shows how to combine the method of undetermined coefficients and Theorem 6.3.3, the principle of superposition.

Example 6.4.7: Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \quad (6.89)$$

Solution: In Example 6.4.1 we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example 6.4.2 we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that $y_p = 2e^{2x} + 5xe^{4x}$ is a particular solution of (6.89).

6.4.3 Exercises

In Exercises 6.4.1–6.4.14 find a particular solution.

1. $y'' - 3y' + 2y = e^{3x}(1 + x)$
2. $y'' - 6y' + 5y = e^{-3x}(35 - 8x)$
3. $y'' - 2y' - 3y = e^x(-8 + 3x)$
4. $y'' + 2y' + y = e^{2x}(-7 - 15x + 9x^2)$
5. $y'' + 4y = e^{-x}(7 - 4x + 5x^2)$
6. $y'' - y' - 2y = e^x(9 + 2x - 4x^2)$
7. $y'' - 4y' - 5y = -6xe^{-x}$
8. $y'' - 3y' + 2y = e^x(3 - 4x)$
9. $y'' + y' - 12y = e^{3x}(-6 + 7x)$
10. $2y'' - 3y' - 2y = e^{2x}(-6 + 10x)$
11. $y'' + 2y' + y = e^{-x}(2 + 3x)$
12. $y'' - 2y' + y = e^x(1 - 6x)$
13. $y'' - 4y' + 4y = e^{2x}(1 - 3x + 6x^2)$
14. $9y'' + 6y' + y = e^{-x/3}(2 - 4x + 4x^2)$

In Exercises 6.4.15–6.4.19 find the general solution.

15. $y'' - 3y' + 2y = e^{3x}(1 + x)$
16. $y'' - 6y' + 8y = e^x(11 - 6x)$
17. $y'' + 6y' + 9y = e^{2x}(3 - 5x)$
18. $y'' + 2y' - 3y = -16xe^x$
19. $y'' - 2y' + y = e^x(2 - 12x)$

In Exercises 6.4.20–6.4.23 solve the initial value problem and plot the solution.

20. C/G $y'' - 4y' - 5y = 9e^{2x}(1 + x), \quad y(0) = 0, \quad y'(0) = -10$

21. C/G $y'' + 3y' - 4y = e^{2x}(7 + 6x), \quad y(0) = 2, \quad y'(0) = 8$

22. C/G $y'' + 4y' + 3y = -e^{-x}(2 + 8x), \quad y(0) = 1, \quad y'(0) = 2$

23. C/G $y'' - 3y' - 10y = 7e^{-2x}, \quad y(0) = 1, \quad y'(0) = -17$

In Exercises 6.4.24–6.4.29 use the principle of superposition to find a particular solution.

24. $y'' + y' + y = xe^x + e^{-x}(1 + 2x)$

25. $y'' - 7y' + 12y = -e^x(17 - 42x) - e^{3x}$

26. $y'' - 8y' + 16y = 6xe^{4x} + 2 + 16x + 16x^2$

27. $y'' - 3y' + 2y = -e^{2x}(3 + 4x) - e^x$

28. $y'' - 2y' + 2y = e^x(1 + x) + e^{-x}(2 - 8x + 5x^2)$

29. $y'' + y = e^{-x}(2 - 4x + 2x^2) + e^{3x}(8 - 12x - 10x^2)$

30. (a) Prove that y is a solution of the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x) \quad (\text{A})$$

if and only if $y = ue^{\alpha x}$, where u satisfies

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x) \quad (\text{B})$$

and $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation

$$ay'' + by' + cy = 0.$$

For the rest of this exercise, let G be a polynomial. Give the requested proofs for the case where

$$G(x) = g_0 + g_1x + g_2x^2 + g_3x^3.$$

- (b) Prove that if $e^{\alpha x}$ isn't a solution of the complementary equation then (B) has a particular solution of the form $u_p = A(x)$, where A is a polynomial of the same degree as G , as in Example 6.4.4. Conclude that (A) has a particular solution of the form $y_p = e^{\alpha x}A(x)$.
- (c) Show that if $e^{\alpha x}$ is a solution of the complementary equation and $xe^{\alpha x}$ isn't, then (B) has a particular solution of the form $u_p = xA(x)$, where A is a polynomial of the same degree as G , as in Example 6.4.5. Conclude that (A) has a particular solution of the form $y_p = xe^{\alpha x}A(x)$.
- (d) Show that if $e^{\alpha x}$ and $xe^{\alpha x}$ are both solutions of the complementary equation then (B) has a particular solution of the form $u_p = x^2A(x)$, where A is a polynomial of the same degree as G , and $x^2A(x)$ can be obtained by integrating G/a twice, taking the constants of integration to be zero, as in Example 6.4.6. Conclude that (A) has a particular solution of the form $y_p = x^2e^{\alpha x}A(x)$.

Exercises 6.4.31–6.4.36 treat the equations considered in Examples 6.4.1–6.4.6. Substitute the suggested form of y_p into the equation and equate the resulting coefficients of like functions on the two sides of the resulting equation to derive a set of simultaneous equations for the coefficients in y_p . Then solve for the coefficients to obtain y_p . Compare the work you've done with the work required to obtain the same results in Examples 6.4.1–6.4.6.

31. Compare with Example 6.4.1:

$$y'' - 7y' + 12y = 4e^{2x}; \quad y_p = Ae^{2x}$$

32. Compare with Example 6.4.2:

$$y'' - 7y' + 12y = 5e^{4x}; \quad y_p = Axe^{4x}$$

33. Compare with Example 6.4.3.

$$y'' - 8y' + 16y = 2e^{4x}; \quad y_p = Ax^2e^{4x}$$

34. Compare with Example 6.4.4:

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2), \quad y_p = e^{3x}(A + Bx + Cx^2)$$

35. Compare with Example 6.4.5:

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2), \quad y_p = e^{3x}(Ax + Bx^2 + Cx^3)$$

36. Compare with Example 6.4.6:

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2), \quad y_p = e^{-x/2}(Ax^2 + Bx^3 + Cx^4)$$

37. Write $y = ue^{\alpha x}$ to find the general solution.

$$(a) \quad y'' + 2y' + y = \frac{e^{-x}}{\sqrt{x}}$$

$$(b) \quad y'' + 6y' + 9y = e^{-3x} \ln x$$

$$(c) \quad y'' - 4y' + 4y = \frac{e^{2x}}{1+x}$$

$$(d) \quad 4y'' + 4y' + y = 4e^{-x/2} \left(\frac{1}{x} + x \right)$$

38. Suppose $\alpha \neq 0$ and k is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x} dx$ are evaluated by integrating by parts k times. This exercise presents another method. Let

$$y = \int e^{\alpha x} P(x) dx$$

with

$$P(x) = p_0 + p_1x + \cdots + p_kx^k, \quad (\text{where } p_k \neq 0).$$

- (a) Show that $y = e^{\alpha x}u$, where

$$u' + \alpha u = P(x). \tag{A}$$

- (b) Show that (A) has a particular solution of the form

$$u_p = A_0 + A_1x + \cdots + A_kx^k,$$

where $A_k, A_{k-1}, \dots, A_0$ can be computed successively by equating coefficients of $x^k, x^{k-1}, \dots, 1$ on both sides of the equation

$$u'_p + \alpha u_p = P(x).$$

- (c) Conclude that

$$\int e^{\alpha x} P(x) dx = (A_0 + A_1x + \cdots + A_kx^k) e^{\alpha x} + c,$$

where c is a constant of integration.

39. Use the method of Exercise 6.4.38 to evaluate the integral.

$$(a) \quad \int e^x(4+x) dx$$

$$(b) \quad \int e^{-x}(-1+x^2) dx$$

$$(c) \quad \int x^3 e^{-2x} dx$$

$$(d) \quad \int e^x(1+x)^2 dx$$

$$(e) \quad \int e^{3x}(-14+30x+27x^2) dx$$

$$(f) \quad \int e^{-x}(1+6x^2-14x^3+3x^4) dx$$

40. Use the method suggested in Exercise 6.4.38 to evaluate $\int x^k e^{\alpha x} dx$, where k is an arbitrary positive integer and $\alpha \neq 0$.

6.5 The Method of Undetermined Coefficients II

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x} (P(x) \cos \omega x + Q(x) \sin \omega x) \quad (6.90)$$

where λ and ω are real numbers, $\omega \neq 0$, and P and Q are polynomials. We want to find a particular solution of (6.90). As in Section 5.4, the procedure that we will use is called *the method of undetermined coefficients*.

6.5.1 Forcing Functions Without Exponential Factors

We begin with the case where $\lambda = 0$ in (6.90); thus, we want to find a particular solution of

$$ay'' + by' + cy = P(x) \cos \omega x + Q(x) \sin \omega x, \quad (6.91)$$

where P and Q are polynomials.

Differentiating $x^r \cos \omega x$ and $x^r \sin \omega x$ yields

$$\frac{d}{dx} x^r \cos \omega x = -\omega x^r \sin \omega x + r x^{r-1} \cos \omega x$$

and

$$\frac{d}{dx} x^r \sin \omega x = \omega x^r \cos \omega x + r x^{r-1} \sin \omega x.$$

This implies that if

$$y_p = A(x) \cos \omega x + B(x) \sin \omega x$$

where A and B are polynomials, then

$$ay_p'' + by_p' + cy_p = F(x) \cos \omega x + G(x) \sin \omega x,$$

where F and G are polynomials with coefficients that can be expressed in terms of the coefficients of A and B . This suggests that we try to choose A and B so that $F = P$ and $G = Q$, respectively. Then y_p will be a particular solution of (6.91). The next theorem tells us how to choose the proper form for y_p . For the proof see Exercise 6.5.37.

Theorem 6.5.1. *Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q . Then the equation*

$$ay'' + by' + cy = P(x) \cos \omega x + Q(x) \sin \omega x$$

has a particular solution

$$y_p = A(x) \cos \omega x + B(x) \sin \omega x, \quad (6.92)$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k,$$

provided that $\cos \omega x$ and $\sin \omega x$ are not solutions of the complementary equation. The solutions of

$$a(y'' + \omega^2 y) = P(x) \cos \omega x + Q(x) \sin \omega x$$

(for which $\cos \omega x$ and $\sin \omega x$ are solutions of the complementary equation) are of the form (6.92), where

$$A(x) = A_0 x + A_1 x^2 + \cdots + A_k x^{k+1} \quad \text{and} \quad B(x) = B_0 x + B_1 x^2 + \cdots + B_k x^{k+1}.$$

For an analog of this theorem that's applicable to (6.90), see Exercise 6.5.38.

Example 6.5.1: Find a particular solution of

$$y'' - 2y' + y = 5 \cos 2x + 10 \sin 2x. \quad (6.93)$$

Solution: In (6.93) the coefficients of $\cos 2x$ and $\sin 2x$ are both zero degree polynomials (constants). Therefore Theorem 6.5.1 implies that (6.93) has a particular solution

$$y_p = A \cos 2x + B \sin 2x.$$

Since

$$y_p' = -2A \sin 2x + 2B \cos 2x \quad \text{and} \quad y_p'' = -4(A \cos 2x + B \sin 2x),$$

replacing y by y_p in (6.93) yields

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -4(A \cos 2x + B \sin 2x) - 4(-A \sin 2x + B \cos 2x) \\ &\quad + (A \cos 2x + B \sin 2x) \\ &= (-3A - 4B) \cos 2x + (4A - 3B) \sin 2x. \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ here with the corresponding coefficients on the right side of (6.93) shows that y_p is a solution of (6.93) if

$$\begin{aligned} -3A - 4B &= 5 \\ 4A - 3B &= 10. \end{aligned}$$

Solving these equations yields $A = 1$, $B = -2$. Therefore

$$y_p = \cos 2x - 2 \sin 2x$$

is a particular solution of (6.93).

Example 6.5.2: Find a particular solution of

$$y'' + 4y = 8 \cos 2x + 12 \sin 2x. \quad (6.94)$$

Solution: The procedure used in Example 6.5.1 doesn't work here; substituting $y_p = A \cos 2x + B \sin 2x$ for y in (6.94) yields

$$y_p'' + 4y_p = -4(A \cos 2x + B \sin 2x) + 4(A \cos 2x + B \sin 2x) = 0$$

for any choice of A and B , since $\cos 2x$ and $\sin 2x$ are both solutions of the complementary equation for (6.94). We're dealing with the second case mentioned in Theorem 6.5.1, and should therefore try a particular solution of the form

$$y_p = x(A \cos 2x + B \sin 2x). \quad (6.95)$$

Then

$$\begin{aligned} y_p' &= A \cos 2x + B \sin 2x + 2x(-A \sin 2x + B \cos 2x) \\ \text{and} \quad y_p'' &= -4A \sin 2x + 4B \cos 2x - 4x(A \cos 2x + B \sin 2x) \\ &= -4A \sin 2x + 4B \cos 2x - 4y_p \quad (\text{see (6.95)}), \end{aligned}$$

so

$$y_p'' + 4y_p = -4A \sin 2x + 4B \cos 2x.$$

Therefore y_p is a solution of (6.94) if

$$-4A \sin 2x + 4B \cos 2x = 8 \cos 2x + 12 \sin 2x,$$

which holds if $A = -3$ and $B = 2$. Therefore

$$y_p = -x(3 \cos 2x - 2 \sin 2x)$$

is a particular solution of (6.94).

Example 6.5.3: Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x) \cos x + 10 \sin x. \quad (6.96)$$

Solution: The coefficients of $\cos x$ and $\sin x$ in (6.96) are polynomials of degree one and zero, respectively. Therefore Theorem 6.5.1 tells us to look for a particular solution of (6.96) of the form

$$y_p = (A_0 + A_1x) \cos x + (B_0 + B_1x) \sin x. \quad (6.97)$$

Then

$$y'_p = (A_1 + B_0 + B_1x) \cos x + (B_1 - A_0 - A_1x) \sin x \quad (6.98)$$

and

$$y''_p = (2B_1 - A_0 - A_1x) \cos x - (2A_1 + B_0 + B_1x) \sin x, \quad (6.99)$$

so

$$\begin{aligned} y''_p + 3y'_p + 2y_p &= [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x] \cos x \\ &\quad + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x] \sin x. \end{aligned} \quad (6.100)$$

Comparing the coefficients of $x \cos x$, $x \sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in (6.96) shows that y_p is a solution of (6.96) if

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

Solving the first two equations yields $A_1 = 2$, $B_1 = 6$. Substituting these into the last two equations yields

$$\begin{aligned} A_0 + 3B_0 &= 16 - 3A_1 - 2B_1 = -2 \\ -3A_0 + B_0 &= 10 + 2A_1 - 3B_1 = -4. \end{aligned}$$

Solving these equations yields $A_0 = 1$, $B_0 = -1$. Substituting $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$ into (6.97) shows that

$$y_p = (1 + 2x) \cos x - (1 - 6x) \sin x$$

is a particular solution of (6.96).

6.5.2 A Useful Observation

In (6.98), (6.99), and (6.100) the polynomials multiplying $\sin x$ can be obtained by replacing A_0, A_1, B_0 , and B_1 by $B_0, B_1, -A_0$, and $-A_1$, respectively, in the polynomials multiplying $\cos x$. An analogous result applies in general, as follows (Exercise 6.5.36).

Theorem 6.5.2. *If*

$$y_p = A(x) \cos \omega x + B(x) \sin \omega x,$$

where $A(x)$ and $B(x)$ are polynomials with coefficients $A_0, \dots, A_k$ and $B_0, \dots, B_k$, then the polynomials multiplying $\sin \omega x$ in

$$y'_p, \quad y''_p, \quad ay''_p + by'_p + cy_p \quad \text{and} \quad y''_p + \omega^2 y_p$$

can be obtained by replacing $A_0, \dots, A_k$ by $B_0, \dots, B_k$ and $B_0, \dots, B_k$ by $-A_0, \dots, -A_k$ in the corresponding polynomials multiplying $\cos \omega x$.

We won't use this theorem in our examples, but we recommend that you use it to check your manipulations when you work the exercises.

Example 6.5.4: Find a particular solution of

$$y'' + y = (8 - 4x) \cos x - (8 + 8x) \sin x. \quad (6.101)$$

Solution: According to Theorem 6.5.1, we should look for a particular solution of the form

$$y_p = (A_0x + A_1x^2) \cos x + (B_0x + B_1x^2) \sin x, \quad (6.102)$$

since $\cos x$ and $\sin x$ are solutions of the complementary equation. However, let's try

$$y_p = (A_0 + A_1x) \cos x + (B_0 + B_1x) \sin x \quad (6.103)$$

first, so you can see why it doesn't work. From (6.99),

$$y_p'' = (2B_1 - A_0 - A_1x) \cos x - (2A_1 + B_0 + B_1x) \sin x,$$

which together with (6.103) implies that

$$y_p'' + y_p = 2B_1 \cos x - 2A_1 \sin x.$$

Since the right side of this equation does not contain $x \cos x$ or $x \sin x$, (6.103) can't satisfy (6.101) no matter how we choose A_0 , A_1 , B_0 , and B_1 .

Now let y_p be as in (6.102). Then

$$\begin{aligned} y_p' &= [A_0 + (2A_1 + B_0)x + B_1x^2] \cos x \\ &\quad + [B_0 + (2B_1 - A_0)x - A_1x^2] \sin x \\ \text{and } y_p'' &= [2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1x^2] \cos x \\ &\quad + [2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1x^2] \sin x, \end{aligned}$$

so

$$y_p'' + y_p = (2A_1 + 2B_0 + 4B_1x) \cos x + (2B_1 - 2A_0 - 4A_1x) \sin x.$$

Comparing the coefficients of $\cos x$ and $\sin x$ here with the corresponding coefficients in (6.101) shows that y_p is a solution of (6.101) if

$$\begin{aligned} 4B_1 &= -4 \\ -4A_1 &= -8 \\ 2B_0 + 2A_1 &= 8 \\ -2A_0 + 2B_1 &= -8. \end{aligned}$$

The solution of this system is $A_1 = 2$, $B_1 = -1$, $A_0 = 3$, $B_0 = 2$. Therefore

$$y_p = x [(3 + 2x) \cos x + (2 - x) \sin x]$$

is a particular solution of (6.101).

6.5.3 Forcing Functions with Exponential Factors

To find a particular solution of

$$ay'' + by' + cy = e^{\lambda x} (P(x) \cos \omega x + Q(x) \sin \omega x) \quad (6.104)$$

when $\lambda \neq 0$, we recall from Section 5.4 that substituting $y = ue^{\lambda x}$ into (6.104) will produce a constant coefficient equation for u with the forcing function $P(x) \cos \omega x + Q(x) \sin \omega x$. We can find a particular

solution u_p of this equation by the procedure that we used in Examples 6.5.1–6.5.4. Then $y_p = u_p e^{\lambda x}$ is a particular solution of (6.104).

Example 6.5.5: Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x} [2 \cos 3x - (34 - 150x) \sin 3x]. \quad (6.105)$$

Solution: Let $y = u e^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x} [(u'' - 4u' + 4u) - 3(u' - 2u) + 2u] \\ &= e^{-2x} (u'' - 7u' + 12u) \\ &= e^{-2x} [2 \cos 3x - (34 - 150x) \sin 3x] \end{aligned}$$

if

$$u'' - 7u' + 12u = 2 \cos 3x - (34 - 150x) \sin 3x. \quad (6.106)$$

Since $\cos 3x$ and $\sin 3x$ aren't solutions of the complementary equation

$$u'' - 7u' + 12u = 0,$$

Theorem 6.5.1 tells us to look for a particular solution of (6.106) of the form

$$u_p = (A_0 + A_1 x) \cos 3x + (B_0 + B_1 x) \sin 3x. \quad (6.107)$$

Then

$$u'_p = (A_1 + 3B_0 + 3B_1 x) \cos 3x + (B_1 - 3A_0 - 3A_1 x) \sin 3x$$

and

$$u''_p = (-9A_0 + 6B_1 - 9A_1 x) \cos 3x - (9B_0 + 6A_1 + 9B_1 x) \sin 3x,$$

so

$$\begin{aligned} u''_p - 7u'_p + 12u_p &= [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x] \cos 3x \\ &\quad + [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x] \sin 3x. \end{aligned}$$

Comparing the coefficients of $x \cos 3x$, $x \sin 3x$, $\cos 3x$, and $\sin 3x$ here with the corresponding coefficients on the right side of (6.106) shows that u_p is a solution of (6.106) if

$$\begin{aligned} 3A_1 - 21B_1 &= 0 \\ 21A_1 + 3B_1 &= 150 \\ 3A_0 - 21B_0 - 7A_1 + 6B_1 &= 2 \\ 21A_0 + 3B_0 - 6A_1 - 7B_1 &= -34. \end{aligned} \quad (6.108)$$

Solving the first two equations yields $A_1 = 7$, $B_1 = 1$. Substituting these values into the last two equations of (6.108) yields

$$\begin{aligned} 3A_0 - 21B_0 &= 2 + 7A_1 - 6B_1 = 45 \\ 21A_0 + 3B_0 &= -34 + 6A_1 + 7B_1 = 15. \end{aligned}$$

Solving this system yields $A_0 = 1$, $B_0 = -2$. Substituting $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$ into (6.107) shows that

$$u_p = (1 + 7x) \cos 3x - (2 - x) \sin 3x$$

is a particular solution of (6.106). Therefore

$$y_p = e^{-2x} [(1 + 7x) \cos 3x - (2 - x) \sin 3x]$$

is a particular solution of (6.105).

Example 6.5.6: Find a particular solution of

$$y'' + 2y' + 5y = e^{-x} [(6 - 16x) \cos 2x - (8 + 8x) \sin 2x]. \quad (6.109)$$

Solution: Let $y = ue^{-x}$. Then

$$\begin{aligned} y'' + 2y' + 5y &= e^{-x} [(u'' - 2u' + u) + 2(u' - u) + 5u] \\ &= e^{-x} (u'' + 4u) \\ &= e^{-x} [(6 - 16x) \cos 2x - (8 + 8x) \sin 2x] \end{aligned}$$

if

$$u'' + 4u = (6 - 16x) \cos 2x - (8 + 8x) \sin 2x. \quad (6.110)$$

Since $\cos 2x$ and $\sin 2x$ are solutions of the complementary equation

$$u'' + 4u = 0,$$

Theorem 6.5.1 tells us to look for a particular solution of (6.110) of the form

$$u_p = (A_0x + A_1x^2) \cos 2x + (B_0x + B_1x^2) \sin 2x.$$

Then

$$\begin{aligned} u'_p &= [A_0 + (2A_1 + 2B_0)x + 2B_1x^2] \cos 2x \\ &\quad + [B_0 + (2B_1 - 2A_0)x - 2A_1x^2] \sin 2x \end{aligned}$$

and

$$\begin{aligned} u''_p &= [2A_1 + 4B_0 - (4A_0 - 8B_1)x - 4A_1x^2] \cos 2x \\ &\quad + [2B_1 - 4A_0 - (4B_0 + 8A_1)x - 4B_1x^2] \sin 2x, \end{aligned}$$

so

$$u''_p + 4u_p = (2A_1 + 4B_0 + 8B_1x) \cos 2x + (2B_1 - 4A_0 - 8A_1x) \sin 2x.$$

Equating the coefficients of $x \cos 2x$, $x \sin 2x$, $\cos 2x$, and $\sin 2x$ here with the corresponding coefficients on the right side of (6.110) shows that u_p is a solution of (6.110) if

$$\begin{aligned} 8B_1 &= -16 \\ -8A_1 &= -8 \\ 4B_0 + 2A_1 &= 6 \\ -4A_0 + 2B_1 &= -8. \end{aligned} \quad (6.111)$$

The solution of this system is $A_1 = 1$, $B_1 = -2$, $B_0 = 1$, $A_0 = 1$. Therefore

$$u_p = x[(1 + x) \cos 2x + (1 - 2x) \sin 2x]$$

is a particular solution of (6.110), and

$$y_p = xe^{-x} [(1 + x) \cos 2x + (1 - 2x) \sin 2x]$$

is a particular solution of (6.109). ■

You can also find a particular solution of (6.109) by substituting

$$y_p = xe^{-x} [(A_0 + A_1x) \cos 2x + (B_0 + B_1x) \sin 2x]$$

for y in (6.109) and equating the coefficients of $xe^{-x} \cos 2x$, $xe^{-x} \sin 2x$, $e^{-x} \cos 2x$, and $e^{-x} \sin 2x$ in the resulting expression for

$$y''_p + 2y'_p + 5y_p$$

with the corresponding coefficients on the right side of (6.109). (See Exercise 6.5.38). This leads to the same system (6.111) of equations for A_0 , A_1 , B_0 , and B_1 that we obtained in Example 6.5.6. However, if you try this approach you'll see that deriving (6.111) this way is much more tedious than the way we did it in Example 6.5.6.

6.5.4 Exercises

In Exercises 6.5.1–6.5.17 find a particular solution.

1. $y'' + 3y' + 2y = 7 \cos x - \sin x$
2. $y'' + 3y' + y = (2 - 6x) \cos x - 9 \sin x$
3. $y'' + 2y' + y = e^x(6 \cos x + 17 \sin x)$
4. $y'' + 3y' - 2y = -e^{2x}(5 \cos 2x + 9 \sin 2x)$
5. $y'' - y' + y = e^x(2 + x) \sin x$
6. $y'' + 3y' - 2y = e^{-2x}[(4 + 20x) \cos 3x + (26 - 32x) \sin 3x]$
7. $y'' + 4y = -12 \cos 2x - 4 \sin 2x$
8. $y'' + y = (-4 + 8x) \cos x + (8 - 4x) \sin x$
9. $4y'' + y = -4 \cos x/2 - 8x \sin x/2$
10. $y'' + 2y' + 2y = e^{-x}(8 \cos x - 6 \sin x)$
11. $y'' - 2y' + 5y = e^x[(6 + 8x) \cos 2x + (6 - 8x) \sin 2x]$
12. $y'' + 2y' + y = 8x^2 \cos x - 4x \sin x$
13. $y'' + 3y' + 2y = (12 + 20x + 10x^2) \cos x + 8x \sin x$
14. $y'' + 3y' + 2y = (1 - x - 4x^2) \cos 2x - (1 + 7x + 2x^2) \sin 2x$
15. $y'' - 5y' + 6y = -e^x[(4 + 6x - x^2) \cos x - (2 - 4x + 3x^2) \sin x]$
16. $y'' - 2y' + y = -e^x[(3 + 4x - x^2) \cos x + (3 - 4x - x^2) \sin x]$
17. $y'' - 2y' + 2y = e^x[(2 - 2x - 6x^2) \cos x + (2 - 10x + 6x^2) \sin x]$

In Exercises 6.5.1–6.5.17 find a particular solution and graph it.

18. C/G $y'' + 2y' + y = e^{-x}[(5 - 2x) \cos x - (3 + 3x) \sin x]$
19. C/G $y'' + 9y = -6 \cos 3x - 12 \sin 3x$
20. C/G $y'' + 3y' + 2y = (1 - x - 4x^2) \cos 2x - (1 + 7x + 2x^2) \sin 2x$
21. C/G $y'' + 4y' + 3y = e^{-x}[(2 + x + x^2) \cos x + (5 + 4x + 2x^2) \sin x]$

In Exercises 6.5.22–6.5.26 solve the initial value problem.

22. $y'' - 7y' + 6y = -e^x(17 \cos x - 7 \sin x), \quad y(0) = 4, \quad y'(0) = 2$
23. $y'' - 2y' + 2y = -e^x(6 \cos x + 4 \sin x), \quad y(0) = 1, \quad y'(0) = 4$
24. $y'' + 6y' + 10y = -40e^x \sin x, \quad y(0) = 2, \quad y'(0) = -3$
25. $y'' - 6y' + 10y = -e^{3x}(6 \cos x + 4 \sin x), \quad y(0) = 2, \quad y'(0) = 7$
26. $y'' - 3y' + 2y = e^{3x}[21 \cos x - (11 + 10x) \sin x], \quad y(0) = 0, \quad y'(0) = 6$

In Exercises 6.5.27–6.5.32 use the principle of superposition to find a particular solution. Where indicated, solve the initial value problem.

27. $y'' - 2y' - 3y = 4e^{3x} + e^x(\cos x - 2 \sin x)$
28. $y'' + y = 4 \cos x - 2 \sin x + xe^x + e^{-x}$
29. $y'' - 3y' + 2y = xe^x + 2e^{2x} + \sin x$
30. $y'' - 2y' + 2y = 4xe^x \cos x + xe^{-x} + 1 + x^2$

31. $y'' - 4y' + 4y = e^{2x}(1 + x) + e^{2x}(\cos x - \sin x) + 3e^{3x} + 1 + x$

32. $y'' - 4y' + 4y = 6e^{2x} + 25 \sin x, \quad y(0) = 5, \quad y'(0) = 3$

In Exercises 6.5.33–6.5.35 solve the initial value problem and graph the solution.

33. C/G $y'' + 4y = -e^{-2x}[(4 - 7x)\cos x + (2 - 4x)\sin x], \quad y(0) = 3, \quad y'(0) = 1$

34. C/G $y'' + 4y' + 4y = 2\cos 2x + 3\sin 2x + e^{-x}, \quad y(0) = -1, \quad y'(0) = 2$

35. C/G $y'' + 4y = e^x(11 + 15x) + 8\cos 2x - 12\sin 2x, \quad y(0) = 3, \quad y'(0) = 5$

36. (a) Verify that if

$$y_p = A(x)\cos \omega x + B(x)\sin \omega x$$

where A and B are twice differentiable, then

$$\begin{aligned} y_p' &= (A' + \omega B)\cos \omega x + (B' - \omega A)\sin \omega x \text{ and} \\ y_p'' &= (A'' + 2\omega B' - \omega^2 A)\cos \omega x + (B'' - 2\omega A' - \omega^2 B)\sin \omega x. \end{aligned}$$

(b) Use the results of (a) to verify that

$$\begin{aligned} ay_p'' + by_p' + cy_p &= [(c - a\omega^2)A + b\omega B + 2a\omega B' + bA' + aA'']\cos \omega x + \\ &\quad [-b\omega A + (c - a\omega^2)B - 2a\omega A' + bB' + aB'']\sin \omega x. \end{aligned}$$

(c) Use the results of (a) to verify that

$$y_p'' + \omega^2 y_p = (A'' + 2\omega B')\cos \omega x + (B'' - 2\omega A')\sin \omega x.$$

(d) Prove Theorem 6.5.2.

37. Let a, b, c , and ω be constants, with $a \neq 0$ and $\omega > 0$, and let

$$P(x) = p_0 + p_1x + \cdots + p_kx^k \quad \text{and} \quad Q(x) = q_0 + q_1x + \cdots + q_kx^k,$$

where at least one of the coefficients p_k, q_k is nonzero, so k is the larger of the degrees of P and Q .

(a) Show that if $\cos \omega x$ and $\sin \omega x$ are not solutions of the complementary equation

$$ay'' + by' + cy = 0,$$

then there are polynomials

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k \quad (\text{A})$$

such that

$$\begin{aligned} (c - a\omega^2)A + b\omega B + 2a\omega B' + bA' + aA'' &= P \\ -b\omega A + (c - a\omega^2)B - 2a\omega A' + bB' + aB'' &= Q, \end{aligned}$$

where $(A_k, B_k), (A_{k-1}, B_{k-1}), \dots, (A_0, B_0)$ can be computed successively by solving the systems

$$\begin{aligned} (c - a\omega^2)A_k + b\omega B_k &= p_k \\ -b\omega A_k + (c - a\omega^2)B_k &= q_k, \end{aligned}$$

and, if $1 \leq r \leq k$,

$$\begin{aligned} (c - a\omega^2)A_{k-r} + b\omega B_{k-r} &= p_{k-r} + \cdots \\ -b\omega A_{k-r} + (c - a\omega^2)B_{k-r} &= q_{k-r} + \cdots, \end{aligned}$$

where the terms indicated by "... " depend upon the previously computed coefficients with subscripts greater than $k - r$. Conclude from this and Exercise 6.5.36(b) that

$$y_p = A(x) \cos \omega x + B(x) \sin \omega x \quad (\text{B})$$

is a particular solution of

$$ay'' + by' + cy = P(x) \cos \omega x + Q(x) \sin \omega x.$$

(b) Conclude from Exercise 6.5.36(c) that the equation

$$a(y'' + \omega^2 y) = P(x) \cos \omega x + Q(x) \sin \omega x \quad (\text{C})$$

does not have a solution of the form (B) with A and B as in (A). Then show that there are polynomials

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}$$

such that

$$a(A'' + 2\omega B') = P$$

$$a(B'' - 2\omega A') = Q,$$

where the pairs $(A_k, B_k), (A_{k-1}, B_{k-1}), \dots, (A_0, B_0)$ can be computed successively as follows:

$$A_k = -\frac{q_k}{2a\omega(k+1)}$$

$$B_k = \frac{p_k}{2a\omega(k+1)},$$

and, if $k \geq 1$,

$$A_{k-j} = -\frac{1}{2\omega} \left[\frac{q_{k-j}}{a(k-j+1)} - (k-j+2)B_{k-j+1} \right]$$

$$B_{k-j} = \frac{1}{2\omega} \left[\frac{p_{k-j}}{a(k-j+1)} - (k-j+2)A_{k-j+1} \right]$$

for $1 \leq j \leq k$. Conclude that (B) with this choice of the polynomials A and B is a particular solution of (C).

38. Show that Theorem 6.5.1 implies the next theorem: *Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q . Then the equation*

$$ay'' + by' + cy = e^{\lambda x} (P(x) \cos \omega x + Q(x) \sin \omega x)$$

has a particular solution

$$y_p = e^{\lambda x} (A(x) \cos \omega x + B(x) \sin \omega x), \quad (\text{A})$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $e^{\lambda x} \cos \omega x$ and $e^{\lambda x} \sin \omega x$ are not solutions of the complementary equation. The equation

$$a[y'' - 2\lambda y' + (\lambda^2 + \omega^2)y] = e^{\lambda x} (P(x) \cos \omega x + Q(x) \sin \omega x)$$

(for which $e^{\lambda x} \cos \omega x$ and $e^{\lambda x} \sin \omega x$ are solutions of the complementary equation) has a particular solution of the form (A), where

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

39. This exercise presents a method for evaluating the integral

$$y = \int e^{\lambda x} (P(x) \cos \omega x + Q(x) \sin \omega x) dx$$

where $\omega \neq 0$ and

$$P(x) = p_0 + p_1x + \cdots + p_kx^k, \quad Q(x) = q_0 + q_1x + \cdots + q_kx^k.$$

- (a) Show that $y = e^{\lambda x}u$, where

$$u' + \lambda u = P(x) \cos \omega x + Q(x) \sin \omega x. \quad (\text{A})$$

- (b) Show that (A) has a particular solution of the form

$$u_p = A(x) \cos \omega x + B(x) \sin \omega x,$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k, \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

and the pairs of coefficients $(A_k, B_k), (A_{k-1}, B_{k-1}), \dots, (A_0, B_0)$ can be computed successively as the solutions of pairs of equations obtained by equating the coefficients of $x^r \cos \omega x$ and $x^r \sin \omega x$ for $r = k, k-1, \dots, 0$.

- (c) Conclude that

$$\int e^{\lambda x} (P(x) \cos \omega x + Q(x) \sin \omega x) dx = e^{\lambda x} (A(x) \cos \omega x + B(x) \sin \omega x) + c,$$

where c is a constant of integration.

40. Use the method of Exercise 6.5.39 to evaluate the integral.

(a) $\int x^2 \cos x dx$

(b) $\int x^2 e^x \cos x dx$

(c) $\int x e^{-x} \sin 2x dx$

(d) $\int x^2 e^{-x} \sin x dx$

(e) $\int x^3 e^x \sin x dx$

(f) $\int e^x [x \cos x - (1 + 3x) \sin x] dx$

(g) $\int e^{-x} [(1 + x^2) \cos x + (1 - x^2) \sin x] dx$

6.6 Reduction of Order

In this section we give a method for finding the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \quad (6.112)$$

if we know a nontrivial solution y_1 of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \quad (6.113)$$

The method is called *reduction of order* because it reduces the task of solving (6.112) to solving a first order equation. Unlike the method of undetermined coefficients, it does not require P_0 , P_1 , and P_2 to be constants, or F to be of any special form.

By now you shouldn't be surprised that we look for solutions of (6.112) in the form

$$y = uy_1 \quad (6.114)$$

where u is to be determined so that y satisfies (6.112). Substituting (6.114) and

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

into (6.112) yields

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)uy_1 = F(x).$$

Collecting the coefficients of u , u' , and u'' yields

$$(P_0y_1)u'' + (2P_0y_1' + P_1y_1)u' + (P_0y_1'' + P_1y_1' + P_2y_1)u = F. \quad (6.115)$$

However, the coefficient of u is zero, since y_1 satisfies (6.113). Therefore (6.115) reduces to

$$Q_0(x)u'' + Q_1(x)u' = F, \quad (6.116)$$

with

$$Q_0 = P_0y_1 \quad \text{and} \quad Q_1 = 2P_0y_1' + P_1y_1.$$

(It isn't worthwhile to memorize the formulas for Q_0 and Q_1 !) Since (6.116) is a linear first order equation in u' , we can solve it for u' by variation of parameters as in Section 1.2, integrate the solution to obtain u , and then obtain y from (6.114).

Example 6.6.1:

(a) Find the general solution of

$$xy'' - (2x + 1)y' + (x + 1)y = x^2, \quad (6.117)$$

given that $y_1 = e^x$ is a solution of the complementary equation

$$xy'' - (2x + 1)y' + (x + 1)y = 0. \quad (6.118)$$

(b) As a byproduct of (a), find a fundamental set of solutions of (6.118).

Solution (a): If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy'' - (2x + 1)y' + (x + 1)y &= x(u''e^x + 2u'e^x + ue^x) \\ &\quad - (2x + 1)(u'e^x + ue^x) + (x + 1)ue^x \\ &= (xu'' - u')e^x. \end{aligned}$$

Therefore $y = ue^x$ is a solution of (6.117) if and only if

$$(xu'' - u')e^x = x^2,$$

which is a first order equation in u' . We rewrite it as

$$u'' - \frac{u'}{x} = xe^{-x}. \quad (6.119)$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that (6.119) becomes

$$z' - \frac{z}{x} = xe^{-x}. \quad (6.120)$$

We leave it to you to show (by separation of variables) that $z_1 = x$ is a solution of the complementary equation

$$z' - \frac{z}{x} = 0$$

for (6.120). By applying variation of parameters as in Section 1.2, we can now see that every solution of (6.120) is of the form

$$z = vx \quad \text{where} \quad v'x = xe^{-x}, \quad \text{so} \quad v' = e^{-x} \quad \text{and} \quad v = -e^{-x} + C_1.$$

Since $u' = z = vx$, u is a solution of (6.119) if and only if

$$u' = vx = -xe^{-x} + C_1x.$$

Integrating this yields

$$u = (x + 1)e^{-x} + \frac{C_1}{2}x^2 + C_2.$$

Therefore the general solution of (6.117) is

$$y = ue^x = x + 1 + \frac{C_1}{2}x^2e^x + C_2e^x. \quad (6.121)$$

Solution (b): By letting $C_1 = C_2 = 0$ in (6.121), we see that $y_{p_1} = x + 1$ is a solution of (6.117). By letting $C_1 = 2$ and $C_2 = 0$, we see that $y_{p_2} = x + 1 + x^2e^x$ is also a solution of (6.117). Since the difference of two solutions of (6.117) is a solution of (6.118), $y_2 = y_{p_1} - y_{p_2} = x^2e^x$ is a solution of (6.118). Since y_2/y_1 is nonconstant and we already know that $y_1 = e^x$ is a solution of (6.117), Theorem 6.1.6 implies that $\{e^x, x^2e^x\}$ is a fundamental set of solutions of (6.118). ■

Although (6.121) is a correct form for the general solution of (6.117), it's silly to leave the arbitrary coefficient of x^2e^x as $C_1/2$ where C_1 is an arbitrary constant. Moreover, it's sensible to make the subscripts of the coefficients of $y_1 = e^x$ and $y_2 = x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite (6.121) as

$$y = x + 1 + c_1e^x + c_2x^2e^x$$

by simply renaming the arbitrary constants. We'll also do this in the next two examples, and in the answers to the exercises.

Example 6.6.2:

(a) Find the general solution of

$$x^2y'' + xy' - y = x^2 + 1,$$

given that $y_1 = x$ is a solution of the complementary equation

$$x^2y'' + xy' - y = 0. \quad (6.122)$$

As a byproduct of this result, find a fundamental set of solutions of (6.122).

(b) Solve the initial value problem

$$x^2 y'' + x y' - y = x^2 + 1, \quad y(1) = 2, \quad y'(1) = -3. \quad (6.123)$$

Solution (a): If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2 y'' + x y' - y &= x^2(u''x + 2u') + x(u'x + u) - ux \\ &= x^3 u'' + 3x^2 u'. \end{aligned}$$

Therefore $y = ux$ is a solution of (6.123) if and only if

$$x^3 u'' + 3x^2 u' = x^2 + 1,$$

which is a first order equation in u' . We rewrite it as

$$u'' + \frac{3}{x}u' = \frac{1}{x} + \frac{1}{x^3}. \quad (6.124)$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that (6.124) becomes

$$z' + \frac{3}{x}z = \frac{1}{x} + \frac{1}{x^3}. \quad (6.125)$$

We leave it to you to show by separation of variables that $z_1 = 1/x^3$ is a solution of the complementary equation

$$z' + \frac{3}{x}z = 0$$

for (6.125). By variation of parameters, every solution of (6.125) is of the form

$$z = \frac{v}{x^3} \quad \text{where} \quad \frac{v'}{x^3} = \frac{1}{x} + \frac{1}{x^3}, \quad \text{so} \quad v' = x^2 + 1 \quad \text{and} \quad v = \frac{x^3}{3} + x + C_1.$$

Since $u' = z = v/x^3$, u is a solution of (6.125) if and only if

$$u' = \frac{v}{x^3} = \frac{1}{3} + \frac{1}{x^2} + \frac{C_1}{x^3}.$$

Integrating this yields

$$u = \frac{x}{3} - \frac{1}{x} - \frac{C_1}{2x^2} + C_2.$$

Therefore the general solution of (6.123) is

$$y = ux = \frac{x^2}{3} - 1 - \frac{C_1}{2x} + C_2x. \quad (6.126)$$

Reasoning as in the solution of Example 6.6.1(a), we conclude that $y_1 = x$ and $y_2 = 1/x$ form a fundamental set of solutions for (6.122).

As we explained above, we rename the constants in (6.126) and rewrite it as

$$y = \frac{x^2}{3} - 1 + c_1x + \frac{c_2}{x}. \quad (6.127)$$

Solution (b): Differentiating (6.127) yields

$$y' = \frac{2x}{3} + c_1 - \frac{c_2}{x^2}. \quad (6.128)$$

Setting $x = 1$ in (6.127) and (6.128) and imposing the initial conditions $y(1) = 2$ and $y'(1) = -3$ yields

$$\begin{aligned} c_1 + c_2 &= \frac{8}{3} \\ c_1 - c_2 &= -\frac{11}{3}. \end{aligned}$$

Solving these equations yields $c_1 = -1/2$, $c_2 = 19/6$. Therefore the solution of (6.123) is

$$y = \frac{x^2}{3} - 1 - \frac{x}{2} + \frac{19}{6x}.$$

Using reduction of order to find the general solution of a homogeneous linear second order equation leads to a homogeneous linear first order equation in u' that can be solved by separation of variables. The next example illustrates this.

Example 6.6.3: Find the general solution and a fundamental set of solutions of

$$x^2 y'' - 3xy' + 3y = 0, \quad (6.129)$$

given that $y_1 = x$ is a solution.

Solution: If $y = ux$ then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2 y'' - 3xy' + 3y &= x^2(u''x + 2u') - 3x(u'x + u) + 3ux \\ &= x^3 u'' - x^2 u'. \end{aligned}$$

Therefore $y = ux$ is a solution of (6.129) if and only if

$$x^3 u'' - x^2 u' = 0.$$

Separating the variables u' and x yields

$$\frac{u''}{u'} = \frac{1}{x},$$

so

$$\ln|u'| = \ln|x| + k, \quad \text{or, equivalently,} \quad u' = C_1 x.$$

Therefore

$$u = \frac{C_1}{2} x^2 + C_2,$$

so the general solution of (6.129) is

$$y = ux = \frac{C_1}{2} x^3 + C_2 x,$$

which we rewrite as

$$y = c_1 x + c_2 x^3.$$

Therefore $\{x, x^3\}$ is a fundamental set of solutions of (6.129).

6.6.1 Exercises

In Exercises 6.6.1–6.6.17 find the general solution, given that y_1 satisfies the complementary equation. As a byproduct, find a fundamental set of solutions of the complementary equation.

1. $(2x + 1)y'' - 2y' - (2x + 3)y = (2x + 1)^2$; $y_1 = e^{-x}$
2. $x^2 y'' + xy' - y = \frac{4}{x^2}$; $y_1 = x$

3. $x^2 y'' - xy' + y = x; \quad y_1 = x$
4. $y'' - 3y' + 2y = \frac{1}{1 + e^{-x}}; \quad y_1 = e^{2x}$
5. $y'' - 2y' + y = 7x^{3/2}e^x; \quad y_1 = e^x$
6. $4x^2 y'' + (4x - 8x^2)y' + (4x^2 - 4x - 1)y = 4x^{1/2}e^x(1 + 4x); \quad y_1 = x^{1/2}e^x$
7. $y'' - 2y' + 2y = e^x \sec x; \quad y_1 = e^x \cos x$
8. $y'' + 4xy' + (4x^2 + 2)y = 8e^{-x(x+2)}; \quad y_1 = e^{-x^2}$
9. $x^2 y'' + xy' - 4y = -6x - 4; \quad y_1 = x^2$
10. $x^2 y'' + 2x(x - 1)y' + (x^2 - 2x + 2)y = x^3 e^{2x}; \quad y_1 = xe^{-x}$
11. $x^2 y'' - x(2x - 1)y' + (x^2 - x - 1)y = x^2 e^x; \quad y_1 = xe^x$
12. $(1 - 2x)y'' + 2y' + (2x - 3)y = (1 - 4x + 4x^2)e^x; \quad y_1 = e^x$
13. $x^2 y'' - 3xy' + 4y = 4x^4; \quad y_1 = x^2$
14. $2xy'' + (4x + 1)y' + (2x + 1)y = 3x^{1/2}e^{-x}; \quad y_1 = e^{-x}$
15. $xy'' - (2x + 1)y' + (x + 1)y = -e^x; \quad y_1 = e^x$
16. $4x^2 y'' - 4x(x + 1)y' + (2x + 3)y = 4x^{5/2}e^{2x}; \quad y_1 = x^{1/2}$
17. $x^2 y'' - 5xy' + 8y = 4x^2; \quad y_1 = x^2$

In Exercises 6.6.18–6.6.30 find a fundamental set of solutions, given that y_1 is a solution.

18. $xy'' + (2 - 2x)y' + (x - 2)y = 0; \quad y_1 = e^x$
19. $x^2 y'' - 4xy' + 6y = 0; \quad y_1 = x^2$
20. $x^2(\ln|x|)^2 y'' - (2x \ln|x|)y' + (2 + \ln|x|)y = 0; \quad y_1 = \ln|x|$
21. $4xy'' + 2y' + y = 0; \quad y_1 = \sin \sqrt{x}$
22. $xy'' - (2x + 2)y' + (x + 2)y = 0; \quad y_1 = e^x$
23. $x^2 y'' - (2a - 1)xy' + a^2 y = 0; \quad y_1 = x^a$
24. $x^2 y'' - 2xy' + (x^2 + 2)y = 0; \quad y_1 = x \sin x$
25. $xy'' - (4x + 1)y' + (4x + 2)y = 0; \quad y_1 = e^{2x}$
26. $4x^2(\sin x)y'' - 4x(x \cos x + \sin x)y' + (2x \cos x + 3 \sin x)y = 0; \quad y_1 = x^{1/2}$
27. $4x^2 y'' - 4xy' + (3 - 16x^2)y = 0; \quad y_1 = x^{1/2}e^{2x}$
28. $(2x + 1)xy'' - 2(2x^2 - 1)y' - 4(x + 1)y = 0; \quad y_1 = 1/x$
29. $(x^2 - 2x)y'' + (2 - x^2)y' + (2x - 2)y = 0; \quad y_1 = e^x$
30. $xy'' - (4x + 1)y' + (4x + 2)y = 0; \quad y_1 = e^{2x}$

In Exercises 6.6.31–6.6.33 solve the initial value problem, given that y_1 satisfies the complementary equation.

31. $x^2 y'' - 3xy' + 4y = 4x^4, \quad y(-1) = 7, \quad y'(-1) = -8; \quad y_1 = x^2$
32. $(3x - 1)y'' - (3x + 2)y' - (6x - 8)y = 0, \quad y(0) = 2, \quad y'(0) = 3; \quad y_1 = e^{2x}$
33. $(x + 1)^2 y'' - 2(x + 1)y' - (x^2 + 2x - 1)y = (x + 1)^3 e^x, \quad y(0) = 1, \quad y'(0) = -1;$
 $y_1 = (x + 1)e^x$

In Exercises 6.6.34 and 6.6.35 solve the initial value problem and graph the solution, given that y_1 satisfies the complementary equation.

34. C/G $x^2 y'' + 2xy' - 2y = x^2, \quad y(1) = \frac{5}{4}, \quad y'(1) = \frac{3}{2}; \quad y_1 = x$

35. C/G $(x^2 - 4)y'' + 4xy' + 2y = x + 2, \quad y(0) = -\frac{1}{3}, \quad y'(0) = -1; \quad y_1 = \frac{1}{x-2}$

36. Suppose p_1 and p_2 are continuous on (a, b) . Let y_1 be a solution of

$$y'' + p_1(x)y' + p_2(x)y = 0 \quad (\text{A})$$

that has no zeros on (a, b) , and let x_0 be in (a, b) . Use reduction of order to show that y_1 and

$$y_2(x) = y_1(x) \int_{x_0}^x \frac{1}{y_1^2(t)} \exp \left(- \int_{x_0}^t p_1(s) ds \right) dt$$

form a fundamental set of solutions of (A) on (a, b) . (NOTE: This exercise is related to Exercise 6.1.9.)

37. The nonlinear first order equation

$$y' + y^2 + p(x)y + q(x) = 0 \quad (\text{A})$$

is a *Riccati equation*. (See Exercise 2.4.55.) Assume that p and q are continuous.

(a) Show that y is a solution of (A) if and only if $y = z'/z$, where

$$z'' + p(x)z' + q(x)z = 0. \quad (\text{B})$$

(b) Show that the general solution of (A) is

$$y = \frac{c_1 z'_1 + c_2 z'_2}{c_1 z_1 + c_2 z_2}, \quad (\text{C})$$

where $\{z_1, z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

(c) Does the formula (C) imply that the first order equation (A) has a two-parameter family of solutions? Explain your answer.

38. Use a method suggested by Exercise 6.6.37 to find all solutions. of the equation.

(a) $y' + y^2 + k^2 = 0$

(b) $y' + y^2 - 3y + 2 = 0$

(c) $y' + y^2 + 5y - 6 = 0$

(d) $y' + y^2 + 8y + 7 = 0$

(e) $y' + y^2 + 14y + 50 = 0$

(f) $6y' + 6y^2 - y - 1 = 0$

(g) $36y' + 36y^2 - 12y + 1 = 0$

39. Use a method suggested by Exercise 6.6.37 and reduction of order to find all solutions of the equation, given that y_1 is a solution.

(a) $x^2(y' + y^2) - x(x+2)y + x + 2 = 0; \quad y_1 = 1/x$

(b) $y' + y^2 + 4xy + 4x^2 + 2 = 0; \quad y_1 = -2x$

(c) $(2x+1)(y' + y^2) - 2y - (2x+3) = 0; \quad y_1 = -1$

(d) $(3x-1)(y' + y^2) - (3x+2)y - 6x + 8 = 0; \quad y_1 = 2$

(e) $x^2(y' + y^2) + xy + x^2 - \frac{1}{4} = 0; \quad y_1 = -\tan x - \frac{1}{2x}$

(f) $x^2(y' + y^2) - 7xy + 7 = 0; \quad y_1 = 1/x$

40. The nonlinear first order equation

$$y' + r(x)y^2 + p(x)y + q(x) = 0 \quad (\text{A})$$

is the *generalized Riccati equation*. (See Exercise 2.4. 2.4.55.) Assume that p and q are continuous and r is differentiable.

- (a) Show that y is a solution of (A) if and only if $y = z'/rz$, where

$$z'' + \left[p(x) - \frac{r'(x)}{r(x)} \right] z' + r(x)q(x)z = 0. \quad (\text{B})$$

- (b) Show that the general solution of (A) is

$$y = \frac{c_1 z'_1 + c_2 z'_2}{r(c_1 z_1 + c_2 z_2)},$$

where $\{z_1, z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

6.7 Variation of Parameters

In this section we give a method called *variation of parameters* for finding a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \quad (6.130)$$

if we know a fundamental set $\{y_1, y_2\}$ of solutions of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \quad (6.131)$$

Having found a particular solution y_p by this method, we can write the general solution of (6.130) as

$$y = y_p + c_1y_1 + c_2y_2.$$

Since we need only one nontrivial solution of (6.131) to find the general solution of (6.130) by reduction of order, it's natural to ask why we're interested in variation of parameters, which requires two linearly independent solutions of (6.131) to achieve the same goal. Here's the answer:

- If we already know two linearly independent solutions of (6.131) then variation of parameters will probably be simpler than reduction of order.
- Variation of parameters generalizes naturally to a method for finding particular solutions of higher order linear equations (Section 9.4) and linear systems of equations (Section 10.7), while reduction of order doesn't.
- Variation of parameters is a powerful theoretical tool used by researchers in differential equations. Although a detailed discussion of this is beyond the scope of this book, you can get an idea of what it means from Exercises 6.7.37–6.7.39.

We'll now derive the method. As usual, we consider solutions of (6.130) and (6.131) on an interval (a, b) where P_0, P_1, P_2 , and F are continuous and P_0 has no zeros. Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation (6.131). We look for a particular solution of (6.130) in the form

$$y_p = u_1y_1 + u_2y_2 \quad (6.132)$$

where u_1 and u_2 are functions to be determined so that y_p satisfies (6.130). You may not think this is a good idea, since there are now two unknown functions to be determined, rather than one. However, since u_1 and u_2 have to satisfy only one condition (that y_p is a solution of (6.130)), we can impose a second condition that produces a convenient simplification, as follows.

Differentiating (6.132) yields

$$y'_p = u_1y'_1 + u_2y'_2 + u'_1y_1 + u'_2y_2. \quad (6.133)$$

As our second condition on u_1 and u_2 we require that

$$u'_1y_1 + u'_2y_2 = 0. \quad (6.134)$$

Then (6.133) becomes

$$y'_p = u_1y'_1 + u_2y'_2; \quad (6.135)$$

that is, (6.134) permits us to differentiate y_p (once!) as if u_1 and u_2 are constants. Differentiating (6.133) yields

$$y''_p = u_1y''_1 + u_2y''_2 + u'_1y'_1 + u'_2y'_2. \quad (6.136)$$

(There are no terms involving u''_1 and u''_2 here, as there would be if we hadn't required (6.134).) Substituting (6.132), (6.135), and (6.136) into (6.130) and collecting the coefficients of u_1 and u_2 yields

$$u_1(P_0y''_1 + P_1y'_1 + P_2y_1) + u_2(P_0y''_2 + P_1y'_2 + P_2y_2) + P_0(u'_1y'_1 + u'_2y'_2) = F.$$

As in the derivation of the method of reduction of order, the coefficients of u_1 and u_2 here are both zero because y_1 and y_2 satisfy the complementary equation. Hence, we can rewrite the last equation as

$$P_0(u_1'y_1' + u_2'y_2') = F. \quad (6.137)$$

Therefore y_p in (6.132) satisfies (6.130) if

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \frac{F}{P_0}, \end{aligned} \quad (6.138)$$

where the first equation is the same as (6.134) and the second is from (6.137).

We'll now show that you can always solve (6.138) for u_1' and u_2' . (The method that we use here will always work, but simpler methods usually work when you're dealing with specific equations.) To obtain u_1' , multiply the first equation in (6.138) by y_2' and the second equation by y_2 . This yields

$$\begin{aligned} u_1'y_1y_2' + u_2'y_2y_2' &= 0 \\ u_1'y_1'y_2 + u_2'y_2'y_2 &= \frac{Fy_2}{P_0}. \end{aligned}$$

Subtracting the second equation from the first yields

$$u_1'(y_1y_2' - y_1'y_2) = -\frac{Fy_2}{P_0}. \quad (6.139)$$

Since $\{y_1, y_2\}$ is a fundamental set of solutions of (6.131) on (a, b) , Theorem 6.1.6 implies that the Wronskian $y_1y_2' - y_1'y_2$ has no zeros on (a, b) . Therefore we can solve (6.139) for u_1' , to obtain

$$u_1' = -\frac{Fy_2}{P_0(y_1y_2' - y_1'y_2)}. \quad (6.140)$$

We leave it to you to start from (6.138) and show by a similar argument that

$$u_2' = \frac{Fy_1}{P_0(y_1y_2' - y_1'y_2)}. \quad (6.141)$$

We can now obtain u_1 and u_2 by integrating u_1' and u_2' . The constants of integration can be taken to be zero, since any choice of u_1 and u_2 in (6.132) will suffice.

You should not memorize (6.140) and (6.141). On the other hand, you don't want to rederive the whole procedure for every specific problem. We recommend the a compromise:

(a) Write

$$y_p = u_1y_1 + u_2y_2 \quad (6.142)$$

to remind yourself of what you're doing.

(b) Write the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \frac{F}{P_0} \end{aligned} \quad (6.143)$$

for the specific problem you're trying to solve.

(c) Solve (6.143) for u_1' and u_2' by any convenient method.

(d) Obtain u_1 and u_2 by integrating u_1' and u_2' , taking the constants of integration to be zero.

(e) Substitute u_1 and u_2 into (6.142) to obtain y_p .

Example 6.7.1: Find a particular solution y_p of

$$x^2 y'' - 2xy' + 2y = x^{9/2}, \quad (6.144)$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation

$$x^2 y'' - 2xy' + 2y = 0.$$

Then find the general solution of (6.144).

Solution: We set

$$y_p = u_1 x + u_2 x^2,$$

where

$$\begin{aligned} u_1' x + u_2' x^2 &= 0 \\ u_1' + 2u_2' x &= \frac{x^{9/2}}{x^2} = x^{5/2}. \end{aligned}$$

From the first equation, $u_1' = -u_2' x$. Substituting this into the second equation yields $u_2' x = x^{5/2}$, so $u_2' = x^{3/2}$ and therefore $u_1' = -u_2' x = -x^{5/2}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{2}{7}x^{7/2} \quad \text{and} \quad u_2 = \frac{2}{5}x^{5/2}.$$

Therefore

$$y_p = u_1 x + u_2 x^2 = -\frac{2}{7}x^{7/2}x + \frac{2}{5}x^{5/2}x^2 = \frac{4}{35}x^{9/2},$$

and the general solution of (6.144) is

$$y = \frac{4}{35}x^{9/2} + c_1 x + c_2 x^2.$$

Example 6.7.2: Find a particular solution y_p of

$$(x-1)y'' - xy' + y = (x-1)^2, \quad (6.145)$$

given that $y_1 = x$ and $y_2 = e^x$ are solutions of the complementary equation

$$(x-1)y'' - xy' + y = 0.$$

Then find the general solution of (6.145).

Solution: We set

$$y_p = u_1 x + u_2 e^x,$$

where

$$\begin{aligned} u_1' x + u_2' e^x &= 0 \\ u_1' + u_2' e^x &= \frac{(x-1)^2}{x-1} = x-1. \end{aligned}$$

Subtracting the first equation from the second yields $-u_1'(x-1) = x-1$, so $u_1' = -1$. From this and the first equation, $u_2' = -xe^{-x}u_1' = xe^{-x}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -x \quad \text{and} \quad u_2 = -(x+1)e^{-x}.$$

Therefore

$$y_p = u_1 x + u_2 e^x = (-x)x + (-(x+1)e^{-x})e^x = -x^2 - x - 1,$$

so the general solution of (6.145) is

$$y = y_p + c_1x + c_2e^x = -x^2 - x - 1 + c_1x + c_2e^x = -x^2 - 1 + (c_1 - 1)x + c_2e^x. \quad (6.146)$$

However, since c_1 is an arbitrary constant, so is $c_1 - 1$; therefore, we improve the appearance of this result by renaming the constant and writing the general solution as

$$y = -x^2 - 1 + c_1x + c_2e^x. \quad (6.147)$$

There's nothing *wrong* with leaving the general solution of (6.145) in the form (6.146); however, we think you'll agree that (6.147) is preferable. We can also view the transition from (6.146) to (6.147) differently. In this example the particular solution $y_p = -x^2 - x - 1$ contained the term $-x$, which satisfies the complementary equation. We can drop this term and redefine $y_p = -x^2 - 1$, since $-x^2 - x - 1$ is a solution of (6.145) and x is a solution of the complementary equation; hence, $-x^2 - 1 = (-x^2 - x - 1) + x$ is also a solution of (6.145). In general, it's always legitimate to drop linear combinations of $\{y_1, y_2\}$ from particular solutions obtained by variation of parameters. (See Exercise 6.7.36 for a general discussion of this question.) We'll do this in the following examples and in the answers to exercises that ask for a particular solution. Therefore, don't be concerned if your answer to such an exercise differs from ours only by a solution of the complementary equation.

Example 6.7.3: Find a particular solution of

$$y'' + 3y' + 2y = \frac{1}{1 + e^x}. \quad (6.148)$$

Then find the general solution.

Solution:

The characteristic polynomial of the complementary equation

$$y'' + 3y' + 2y = 0 \quad (6.149)$$

is $p(r) = r^2 + 3r + 2 = (r + 1)(r + 2)$, so $y_1 = e^{-x}$ and $y_2 = e^{-2x}$ form a fundamental set of solutions of (6.149). We look for a particular solution of (6.148) in the form

$$y_p = u_1e^{-x} + u_2e^{-2x},$$

where

$$\begin{aligned} u_1'e^{-x} + u_2'e^{-2x} &= 0 \\ -u_1'e^{-x} - 2u_2'e^{-2x} &= \frac{1}{1 + e^x}. \end{aligned}$$

Adding these two equations yields

$$-u_2'e^{-2x} = \frac{1}{1 + e^x}, \quad \text{so} \quad u_2' = -\frac{e^{2x}}{1 + e^x}.$$

From the first equation,

$$u_1' = -u_2'e^{-x} = \frac{e^x}{1 + e^x}.$$

Integrating by means of the substitution $v = e^x$ and taking the constants of integration to be zero yields

$$u_1 = \int \frac{e^x}{1 + e^x} dx = \int \frac{dv}{1 + v} = \ln(1 + v) = \ln(1 + e^x)$$

and

$$\begin{aligned} u_2 &= - \int \frac{e^{2x}}{1+e^x} dx = - \int \frac{v}{1+v} dv = \int \left[\frac{1}{1+v} - 1 \right] dv \\ &= \ln(1+v) - v = \ln(1+e^x) - e^x. \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1 e^{-x} + u_2 e^{-2x} \\ &= [\ln(1+e^x)]e^{-x} + [\ln(1+e^x) - e^x]e^{-2x}, \end{aligned}$$

so

$$y_p = (e^{-x} + e^{-2x}) \ln(1+e^x) - e^{-x}.$$

Since the last term on the right satisfies the complementary equation, we drop it and redefine

$$y_p = (e^{-x} + e^{-2x}) \ln(1+e^x).$$

The general solution of (6.148) is

$$y = y_p + c_1 e^{-x} + c_2 e^{-2x} = (e^{-x} + e^{-2x}) \ln(1+e^x) + c_1 e^{-x} + c_2 e^{-2x}.$$

Example 6.7.4: Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = \frac{2}{x+1}, \quad y(0) = -1, \quad y'(0) = -5, \quad (6.150)$$

given that

$$y_1 = \frac{1}{x-1} \quad \text{and} \quad y_2 = \frac{1}{x+1}$$

are solutions of the complementary equation

$$(x^2 - 1)y'' + 4xy' + 2y = 0.$$

Solution: We first use variation of parameters to find a particular solution of

$$(x^2 - 1)y'' + 4xy' + 2y = \frac{2}{x+1}$$

on $(-1, 1)$ in the form

$$y_p = \frac{u_1}{x-1} + \frac{u_2}{x+1},$$

where

$$\begin{aligned} \frac{u_1'}{x-1} + \frac{u_2'}{x+1} &= 0 \\ -\frac{u_1'}{(x-1)^2} - \frac{u_2'}{(x+1)^2} &= \frac{2}{(x+1)(x^2-1)}. \end{aligned} \quad (6.151)$$

Multiplying the first equation by $1/(x-1)$ and adding the result to the second equation yields

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2} \right] u_2' = \frac{2}{(x+1)(x^2-1)}. \quad (6.152)$$

Since

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2} \right] = \frac{(x+1) - (x-1)}{(x+1)(x^2-1)} = \frac{2}{(x+1)(x^2-1)},$$

(6.152) implies that $u_2' = 1$. From (6.151),

$$u_1' = -\frac{x-1}{x+1}u_2' = -\frac{x-1}{x+1}.$$

Integrating and taking the constants of integration to be zero yields

$$\begin{aligned} u_1 &= -\int \frac{x-1}{x+1} dx = -\int \frac{x+1-2}{x+1} dx \\ &= \int \left[\frac{2}{x+1} - 1 \right] dx = 2\ln(x+1) - x \end{aligned}$$

and

$$u_2 = \int dx = x.$$

Therefore

$$\begin{aligned} y_p &= \frac{u_1}{x-1} + \frac{u_2}{x+1} = [2\ln(x+1) - x] \frac{1}{x-1} + x \frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + x \left[\frac{1}{x+1} - \frac{1}{x-1} \right] = \frac{2\ln(x+1)}{x-1} - \frac{2x}{(x+1)(x-1)}. \end{aligned}$$

However, since

$$\frac{2x}{(x+1)(x-1)} = \left[\frac{1}{x+1} + \frac{1}{x-1} \right]$$

is a solution of the complementary equation, we redefine

$$y_p = \frac{2\ln(x+1)}{x-1}.$$

Therefore the general solution of (6.153) is

$$y = \frac{2\ln(x+1)}{x-1} + \frac{c_1}{x-1} + \frac{c_2}{x+1}. \quad (6.153)$$

Differentiating this yields

$$y' = \frac{2}{x^2-1} - \frac{2\ln(x+1)}{(x-1)^2} - \frac{c_1}{(x-1)^2} - \frac{c_2}{(x+1)^2}.$$

Setting $x = 0$ in the last two equations and imposing the initial conditions $y(0) = -1$ and $y'(0) = -5$ yields the system

$$\begin{aligned} -c_1 + c_2 &= -1 \\ -2 - c_1 - c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = 2$, $c_2 = 1$. Substituting these into (6.153) yields

$$\begin{aligned} y &= \frac{2\ln(x+1)}{x-1} + \frac{2}{x-1} + \frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1} \end{aligned}$$

as the solution of (6.150). Figure 6.6 is a graph of the solution.

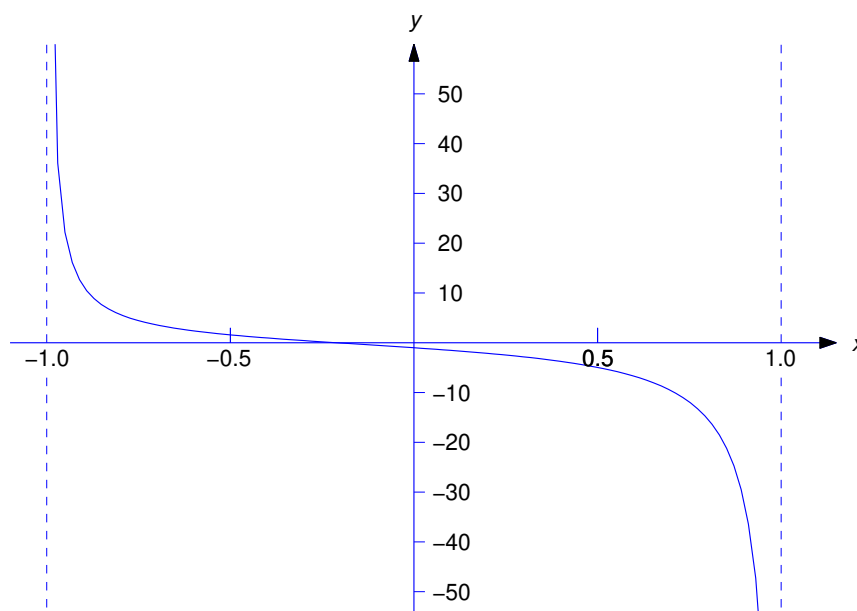


Figure 6.6: $y = \frac{2 \ln(x+1)}{x-1} + \frac{3x+1}{x^2-1}$

6.7.1 Comparison of Methods

We've now considered three methods for solving nonhomogeneous linear equations: undetermined coefficients, reduction of order, and variation of parameters. It's natural to ask which method is best for a given problem. The method of undetermined coefficients should be used for constant coefficient equations with forcing functions that are linear combinations of polynomials multiplied by functions of the form $e^{\alpha x}$, $e^{\lambda x} \cos \omega x$, or $e^{\lambda x} \sin \omega x$. Although the other two methods can be used to solve such problems, they will be more difficult except in the most trivial cases, because of the integrations involved.

If the equation isn't a constant coefficient equation or the forcing function isn't of the form just specified, the method of undetermined coefficients does not apply and the choice is necessarily between the other two methods. The case could be made that reduction of order is better because it requires only one solution of the complementary equation while variation of parameters requires two. However, variation of parameters will probably be easier if you already know a fundamental set of solutions of the complementary equation.

6.7.2 Exercises

In Exercises 6.7.1–6.7.6 use variation of parameters to find a particular solution.

1. $y'' + 9y = \tan 3x$
2. $y'' + 4y = \sin 2x \sec^2 2x$
3. $y'' - 3y' + 2y = \frac{4}{1 + e^{-x}}$
4. $y'' - 2y' + 2y = 3e^x \sec x$
5. $y'' - 2y' + y = 14x^{3/2}e^x$
6. $y'' - y = \frac{4e^{-x}}{1 - e^{-2x}}$

In Exercises 6.7.7–6.7.29 use variation of parameters to find a particular solution, given the solutions y_1, y_2 of the complementary equation.

7. $x^2y'' + xy' - y = 2x^2 + 2; \quad y_1 = x, \quad y_2 = \frac{1}{x}$
8. $xy'' + (2 - 2x)y' + (x - 2)y = e^{2x}; \quad y_1 = e^x, \quad y_2 = \frac{e^x}{x}$
9. $4x^2y'' + (4x - 8x^2)y' + (4x^2 - 4x - 1)y = 4x^{1/2}e^x, \quad x > 0;$
 $y_1 = x^{1/2}e^x, \quad y_2 = x^{-1/2}e^x$
10. $y'' + 4xy' + (4x^2 + 2)y = 4e^{-x(x+2)}; \quad y_1 = e^{-x^2}, \quad y_2 = xe^{-x^2}$
11. $x^2y'' - 4xy' + 6y = x^{5/2}, \quad x > 0; \quad y_1 = x^2, \quad y_2 = x^3$
12. $x^2y'' - 3xy' + 3y = 2x^4 \sin x; \quad y_1 = x, \quad y_2 = x^3$
13. $(2x + 1)y'' - 2y' - (2x + 3)y = (2x + 1)^2e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = xe^{-x}$
14. $4xy'' + 2y' + y = \sin \sqrt{x}; \quad y_1 = \cos \sqrt{x}, \quad y_2 = \sin \sqrt{x}$
15. $xy'' - (2x + 2)y' + (x + 2)y = 6x^3e^x; \quad y_1 = e^x, \quad y_2 = x^3e^x$
16. $x^2y'' - (2a - 1)xy' + a^2y = x^{a+1}; \quad y_1 = x^a, \quad y_2 = x^a \ln x$
17. $x^2y'' - 2xy' + (x^2 + 2)y = x^3 \cos x; \quad y_1 = x \cos x, \quad y_2 = x \sin x$
18. $xy'' - y' - 4x^3y = 8x^5; \quad y_1 = e^{x^2}, \quad y_2 = e^{-x^2}$
19. $(\sin x)y'' + (2 \sin x - \cos x)y' + (\sin x - \cos x)y = e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = e^{-x} \cos x$
20. $4x^2y'' - 4xy' + (3 - 16x^2)y = 8x^{5/2}; \quad y_1 = \sqrt{x}e^{2x}, \quad y_2 = \sqrt{x}e^{-2x}$
21. $4x^2y'' - 4xy' + (4x^2 + 3)y = x^{7/2}; \quad y_1 = \sqrt{x} \sin x, \quad y_2 = \sqrt{x} \cos x$
22. $x^2y'' - 2xy' - (x^2 - 2)y = 3x^4; \quad y_1 = xe^x, \quad y_2 = xe^{-x}$
23. $x^2y'' - 2x(x + 1)y' + (x^2 + 2x + 2)y = x^3e^x; \quad y_1 = xe^x, \quad y_2 = x^2e^x$
24. $x^2y'' - xy' - 3y = x^{3/2}; \quad y_1 = 1/x, \quad y_2 = x^3$
25. $x^2y'' - x(x + 4)y' + 2(x + 3)y = x^4e^x; \quad y_1 = x^2, \quad y_2 = x^2e^x$
26. $x^2y'' - 2x(x + 2)y' + (x^2 + 4x + 6)y = 2xe^x; \quad y_1 = x^2e^x, \quad y_2 = x^3e^x$
27. $x^2y'' - 4xy' + (x^2 + 6)y = x^4; \quad y_1 = x^2 \cos x, \quad y_2 = x^2 \sin x$
28. $(x - 1)y'' - xy' + y = 2(x - 1)^2e^x; \quad y_1 = x, \quad y_2 = e^x$
29. $4x^2y'' - 4x(x + 1)y' + (2x + 3)y = x^{5/2}e^x; \quad y_1 = \sqrt{x}, \quad y_2 = \sqrt{x}e^x$

In Exercises 6.7.30–6.7.32 use variation of parameters to solve the initial value problem, given y_1, y_2 are solutions of the complementary equation.

30. $(3x - 1)y'' - (3x + 2)y' - (6x - 8)y = (3x - 1)^2e^{2x}, \quad y(0) = 1, \quad y'(0) = 2;$
 $y_1 = e^{2x}, \quad y_2 = xe^{-x}$
31. $(x - 1)^2y'' - 2(x - 1)y' + 2y = (x - 1)^2, \quad y(0) = 3, \quad y'(0) = -6;$
 $y_1 = x - 1, \quad y_2 = x^2 - 1$
32. $(x - 1)^2y'' - (x^2 - 1)y' + (x + 1)y = (x - 1)^3e^x, \quad y(0) = 4, \quad y'(0) = -6;$
 $y_1 = (x - 1)e^x, \quad y_2 = x - 1$

In Exercises 6.7.33–6.7.35 use variation of parameters to solve the initial value problem and graph the solution, given that y_1, y_2 are solutions of the complementary equation.

33. C/G $(x^2 - 1)y'' + 4xy' + 2y = 2x, \quad y(0) = 0, \quad y'(0) = -2; \quad y_1 = \frac{1}{x-1}, \quad y_2 = \frac{1}{x+1}$

34. C/G $x^2y'' + 2xy' - 2y = -2x^2, \quad y(1) = 1, \quad y'(1) = -1; \quad y_1 = x, \quad y_2 = \frac{1}{x^2}$

35. C/G $(x+1)(2x+3)y'' + 2(x+2)y' - 2y = (2x+3)^2, \quad y(0) = 0, \quad y'(0) = 0;$
 $y_1 = x+2, \quad y_2 = \frac{1}{x+1}$

36. Suppose

$$y_p = \bar{y} + a_1y_1 + a_2y_2$$

is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x), \tag{A}$$

where y_1 and y_2 are solutions of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0.$$

Show that $\bar{y}$ is also a solution of (A).

37. Suppose p, q , and f are continuous on (a, b) and let x_0 be in (a, b) . Let y_1 and y_2 be the solutions of

$$y'' + p(x)y' + q(x)y = 0$$

such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0, \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Use variation of parameters to show that the solution of the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \quad y'(x_0) = k_1,$$

is

$$y(x) = k_0y_1(x) + k_1y_2(x) + \int_{x_0}^x (y_1(t)y_2(x) - y_1(x)y_2(t)) f(t) \exp\left(\int_{x_0}^t p(s) ds\right) dt.$$

HINT: Use Abel's formula for the Wronskian of $\{y_1, y_2\}$, and integrate u_1' and u_2' from x_0 to x . Show also that

$$y'(x) = k_0y_1'(x) + k_1y_2'(x) + \int_{x_0}^x (y_1(t)y_2'(x) - y_1'(x)y_2(t)) f(t) \exp\left(\int_{x_0}^t p(s) ds\right) dt.$$

38. Suppose f is continuous on an open interval that contains $x_0 = 0$. Use variation of parameters to find a formula for the solution of the initial value problem

$$y'' - y = f(x), \quad y(0) = k_0, \quad y'(0) = k_1.$$

39. Suppose f is continuous on (a, ∞) , where $a < 0$, so $x_0 = 0$ is in (a, ∞) .

(a) Use variation of parameters to find a formula for the solution of the initial value problem

$$y'' + y = f(x), \quad y(0) = k_0, \quad y'(0) = k_1.$$

HINT: You will need the addition formulas for the sine and cosine:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

For the rest of this exercise assume that the improper integral $\int_0^\infty f(t) dt$ is absolutely convergent.

(b) Show that if y is a solution of

$$y'' + y = f(x) \tag{A}$$

on (a, ∞) , then

$$\lim_{x \rightarrow \infty} (y(x) - A_0 \cos x - A_1 \sin x) = 0 \tag{B}$$

and

$$\lim_{x \rightarrow \infty} (y'(x) + A_0 \sin x - A_1 \cos x) = 0, \tag{C}$$

where

$$A_0 = k_0 - \int_0^\infty f(t) \sin t dt \quad \text{and} \quad A_1 = k_1 + \int_0^\infty f(t) \cos t dt.$$

HINT: Recall from calculus that if $\int_0^\infty f(t) dt$ converges absolutely, then $\lim_{x \rightarrow \infty} \int_x^\infty |f(t)| dt = 0$.

(c) Show that if A_0 and A_1 are arbitrary constants, then there's a unique solution of $y'' + y = f(x)$ on (a, ∞) that satisfies (B) and (C).

Chapter 7

Higher Order Linear ODEs

7.1 Second Order Linear ODEs

Consider the general *second order linear differential equation*

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We usually divide through by $A(x)$ to get

$$y'' + p(x)y' + q(x)y = f(x), \quad (7.1)$$

where $p(x) = B(x)/A(x)$, $q(x) = C(x)/A(x)$, and $f(x) = F(x)/A(x)$. The word *linear* means that the equation contains no powers nor functions of y , y' , and y'' .

In the special case when $f(x) = 0$, we have a so-called *homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0. \quad (7.2)$$

We have already seen some second order linear homogeneous equations.

$y'' + k^2y = 0$	Two solutions are: $y_1 = \cos(kx)$, $y_2 = \sin(kx)$.
$y'' - k^2y = 0$	Two solutions are: $y_1 = e^{kx}$, $y_2 = e^{-kx}$.

If we know two solutions of a linear homogeneous equation, we know many more of them.

Theorem 7.1.1 (Superposition). Suppose y_1 and y_2 are two solutions of the homogeneous equation (7.2). Then

$$y(x) = C_1y_1(x) + C_2y_2(x),$$

also solves (7.2) for arbitrary constants C_1 and C_2 .

That is, we can add solutions together and multiply them by constants to obtain new and different solutions. We call the expression $C_1y_1 + C_2y_2$ a *linear combination* of y_1 and y_2 . Let us prove this theorem; the proof is very enlightening and illustrates how linear equations work.

Proof: Let $y = C_1y_1 + C_2y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1y_1 + C_2y_2)'' + p(C_1y_1 + C_2y_2)' + q(C_1y_1 + C_2y_2) \\ &= C_1y_1'' + C_2y_2'' + C_1py_1' + C_2py_2' + C_1qy_1 + C_2qy_2 \\ &= C_1(y_1'' + py_1' + qy_1) + C_2(y_2'' + py_2' + qy_2) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0. \quad \square \end{aligned}$$

The proof becomes even simpler to state if we use the operator notation. An *operator* is an object that eats functions and spits out functions (kind of like what a function is, but a function eats numbers and spits out numbers). Define the operator L by

$$Ly = y'' + py' + qy.$$

The differential equation now becomes $Ly = 0$. The operator (and the equation) L being *linear* means that $L(C_1y_1 + C_2y_2) = C_1Ly_1 + C_2Ly_2$. It is almost as if we were “multiplying” by L . The proof above becomes

$$Ly = L(C_1y_1 + C_2y_2) = C_1Ly_1 + C_2Ly_2 = C_1 \cdot 0 + C_2 \cdot 0 = 0.$$

Two different solutions to the second equation $y'' - k^2y = 0$ are $y_1 = \cosh(kx)$ and $y_2 = \sinh(kx)$. Recalling the definition of $\sinh$ and $\cosh$, we note that these are solutions by superposition as they are linear combinations of the two exponential solutions: $\cosh(kx) = \frac{e^{kx} + e^{-kx}}{2} = (1/2)e^{kx} + (1/2)e^{-kx}$ and $\sinh(kx) = \frac{e^{kx} - e^{-kx}}{2} = (1/2)e^{kx} - (1/2)e^{-kx}$.

The functions $\sinh$ and $\cosh$ are sometimes more convenient to use than the exponential. Let us review some of their properties:

$$\begin{aligned} \cosh 0 &= 1, & \sinh 0 &= 0, \\ \frac{d}{dx} [\cosh x] &= \sinh x, & \frac{d}{dx} [\sinh x] &= \cosh x, \\ \cosh^2 x - \sinh^2 x &= 1. \end{aligned}$$

Exercise 7.1.1: Derive these properties using the definitions of $\sinh$ and $\cosh$ in terms of exponentials.

Linear equations have nice and simple answers to the existence and uniqueness question.

Theorem 7.1.2 (Existence and uniqueness). Suppose p, q, f are continuous functions on some interval I , a is a number in I , and b_0, b_1 are constants. Then the equation

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1.$$

For example, the equation $y'' + k^2y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

The equation $y'' - k^2y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k} \sinh(kx).$$

Using $\cosh$ and $\sinh$ in this solution allows us to solve for the initial conditions in a cleaner way than if we have used the exponentials.

The initial conditions for a second order ODE consist of two equations. Common sense tells us that if we have two arbitrary constants and two equations, then we should be able to solve for the constants and find a solution to the differential equation satisfying the initial conditions.

Question: Suppose we find two different solutions y_1 and y_2 to the homogeneous equation (7.2). Can every solution be written (using superposition) in the form $y = C_1y_1 + C_2y_2$?

Answer is affirmative! Provided that y_1 and y_2 are different enough in the following sense. We say y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem 7.1.3. Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation (7.2). Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2.$$

That is, $y = C_1 y_1 + C_2 y_2$ is the general solution.

For example, we found the solutions $y_1 = \sin x$ and $y_2 = \cos x$ for the equation $y'' + y = 0$. It is not hard to see that sine and cosine are not constant multiples of each other. Indeed, if $\sin x = A \cos x$ for some constant A , plugging in $x = 0$ would imply $A = 0$. But then $\sin x = 0$ for all x , which is preposterous. So y_1 and y_2 are linearly independent. Hence,

$$y = C_1 \cos x + C_2 \sin x$$

is the general solution to $y'' + y = 0$.

For two functions, checking linear independence is rather simple. Let us see another example. Consider $y'' - 2x^{-2}y = 0$. Then $y_1 = x^2$ and $y_2 = 1/x$ are solutions. To see that they are linearly independent, suppose one is a multiple of the other: $y_1 = Ay_2$, we just have to find out that A cannot be a constant. In this case we have $A = y_1/y_2 = x^3$, this most decidedly not a constant. So $y = C_1 x^2 + C_2 1/x$ is the general solution.

If you have one solution to a second order linear homogeneous equation, then you can find another one. This is the *reduction of order method*. The idea is that if we somehow found y_1 as a solution of $y'' + p(x)y' + q(x)y = 0$, then we try a second solution of the form $y_2(x) = y_1(x)v(x)$. We just need to find v . We plug y_2 into the equation:

$$\begin{aligned} 0 &= y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'y_1v' + y_1v''}_{y_2''} + \underbrace{p(x)(y_1'v + y_1v')}_{y_2'} + \underbrace{q(x)y_1v}_{y_2} \\ &= y_1v'' + (2y_1' + p(x)y_1)v' + \underbrace{(y_1'' + p(x)y_1' + q(x)y_1)}_{=0}v. \end{aligned}$$

In other words, $y_1v'' + (2y_1' + p(x)y_1)v' = 0$. Using $w = v'$, we have the first order linear equation $y_1w' + (2y_1' + p(x)y_1)w = 0$. After solving this equation for w (integrating factor), we find v by antidifferentiating w . We then form y_2 by computing y_1v . For example, suppose we somehow know $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$. The equation for w is then $xw' + 3w = 0$. We find a solution, $w = Cx^{-3}$, and we find an antiderivative $v = \frac{-C}{2x^2}$. Hence $y_2 = y_1v = \frac{-C}{2x}$. Any C works and so $C = -2$ makes $y_2 = 1/x$. Thus, the general solution is $y = C_1x + C_21/x$.

Since we have a formula for the solution to the first order linear equation, we can write a formula for y_2 :

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2} dx$$

However, it is much easier to remember that we just need to try $y_2(x) = y_1(x)v(x)$ and find $v(x)$ as we did above. Also, the technique works for higher order equations too: You get to reduce the order for each solution you find. So it is better to remember how to do it rather than a specific formula.

We will study the solution of nonhomogeneous equations in § 7.5. We will first focus on finding general solutions to homogeneous equations.

7.1.1 Exercises

Exercise 7.1.2: Show that $y = e^x$ and $y = e^{2x}$ are linearly independent.

Exercise 7.1.3: Take $y'' + 5y = 10x + 5$. Find (guess!) a solution.

Exercise 7.1.4: Prove the superposition principle for nonhomogeneous equations. Suppose that y_1 is a solution to $Ly_1 = f(x)$ and y_2 is a solution to $Ly_2 = g(x)$ (same linear operator L). Show that $y = y_1 + y_2$ solves $Ly = f(x) + g(x)$.

Exercise 7.1.5: For the equation $x^2y'' - xy' = 0$, find two solutions, show that they are linearly independent and find the general solution. Hint: Try $y = x^r$.

Equations of the form $ax^2y'' + bxy' + cy = 0$ are called *Euler's equations* or *Cauchy–Euler equations*. They are solved by trying $y = x^r$ and solving for r (assume $x \geq 0$ for simplicity).

Exercise 7.1.6: Suppose that $(b - a)^2 - 4ac > 0$.

- Find a formula for the general solution of Euler's equation (see above) $ax^2y'' + bxy' + cy = 0$. Hint: Try $y = x^r$ and find a formula for r .
- What happens when $(b - a)^2 - 4ac = 0$ or $(b - a)^2 - 4ac < 0$?

We will revisit the case when $(b - a)^2 - 4ac < 0$ later.

Exercise 7.1.7: Same equation as in [Exercise 7.1.6](#). Suppose $(b - a)^2 - 4ac = 0$. Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$. Hint: Try $y = x^r \ln x$ for the second solution.

Exercise 7.1.8 (reduction of order): Suppose y_1 is a solution to $y'' + p(x)y' + q(x)y = 0$. By directly plugging in to the equation, show that

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x) dx}}{(y_1(x))^2} dx$$

is also a solution.

Exercise 7.1.9 (Chebyshev's equation of order 1): Take $(1 - x^2)y'' - xy' + y = 0$.

- Show that $y = x$ is a solution.
- Use reduction of order to find a second linearly independent solution.
- Write down the general solution.

Exercise 7.1.10 (Hermite's equation of order 2): Take $y'' - 2xy' + 4y = 0$.

- Show that $y = 1 - 2x^2$ is a solution.
- Use reduction of order to find a second linearly independent solution. (It's OK to leave a definite integral in the formula.)
- Write down the general solution.

Exercise 7.1.101: Are $\sin(x)$ and e^x linearly independent? Justify.

Exercise 7.1.102: Are e^x and e^{x+2} linearly independent? Justify.

Exercise 7.1.103: Guess a solution to $y'' + y' + y = 5$.

Exercise 7.1.104: Find the general solution to $xy'' + y' = 0$. Hint: It is a first order ODE in y' .

Exercise 7.1.105: Write down an equation (guess) for which we have the solutions e^x and e^{2x} . Hint: Try an equation of the form $y'' + Ay' + By = 0$ for constants A and B , plug in both e^x and e^{2x} and solve for A and B .

7.2 Constant Coefficient Second Order Linear ODEs

7.2.1 Solving Constant Coefficient Equations

Consider the problem

$$y'' - 6y' + 8y = 0, \quad y(0) = -2, \quad y'(0) = 6.$$

This is a second order linear homogeneous equation with constant coefficients. *Constant coefficients* means that the functions in front of y'' , y' , and y are constants, they do not depend on x .

To guess a solution, think of a function that stays essentially the same when we differentiate it, so that we can take the function and its derivatives, add some multiples of these together, and end up with zero. Yes, we are talking about the exponential.

Let us try* a solution of the form $y = e^{rx}$. Then $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Plug in to get

$$\begin{aligned} y'' - 6y' + 8y &= 0, \\ \underbrace{r^2e^{rx}}_{y''} - 6 \underbrace{re^{rx}}_{y'} + 8 \underbrace{e^{rx}}_y &= 0, \\ r^2 - 6r + 8 &= 0 \quad (\text{divide through by } e^{rx}), \\ (r - 2)(r - 4) &= 0. \end{aligned}$$

Hence, if $r = 2$ or $r = 4$, then e^{rx} is a solution. So let $y_1 = e^{2x}$ and $y_2 = e^{4x}$.

Exercise 7.2.1: Check that y_1 and y_2 are solutions.

The functions e^{2x} and e^{4x} are linearly independent. If they were not linearly independent, we could write $e^{4x} = Ce^{2x}$ for some constant C , implying that $e^{2x} = C$ for all x , which is clearly not possible. Hence, we can write the general solution as

$$y = C_1e^{2x} + C_2e^{4x}.$$

We need to solve for C_1 and C_2 . To apply the initial conditions, we first find $y' = 2C_1e^{2x} + 4C_2e^{4x}$. We plug $x = 0$ into y and y' and solve.

$$\begin{aligned} -2 &= y(0) = C_1 + C_2, \\ 6 &= y'(0) = 2C_1 + 4C_2. \end{aligned}$$

Either apply some matrix algebra, or just solve these by high school math. For example, divide the second equation by 2 to obtain $3 = C_1 + 2C_2$, and subtract the two equations to get $5 = C_2$. Then $C_1 = -7$ as $-2 = C_1 + 5$. Hence, the solution we are looking for is

$$y = -7e^{2x} + 5e^{4x}.$$

We generalize this example into a method. Suppose that we have an equation

$$ay'' + by' + cy = 0, \tag{7.3}$$

where a, b, c are constants. Try the solution $y = e^{rx}$ to obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Divide by e^{rx} to obtain the so-called *characteristic equation* of the ODE:

$$ar^2 + br + c = 0.$$

*Making an educated guess with some parameters to solve for is such a central technique in differential equations, that people sometimes use a fancy name for such a guess: *ansatz*, German for “initial placement of a tool at a work piece.” Yes, the Germans have a word for that.

Solve for the r by using the quadratic formula:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So $e^{r_1 x}$ and $e^{r_2 x}$ are solutions. There is still a difficulty if $r_1 = r_2$, but it is not hard to overcome.

Theorem 7.2.1. Suppose that r_1 and r_2 are the roots of the characteristic equation.

(i) If r_1 and r_2 are distinct and real (when $b^2 - 4ac > 0$), then (7.3) has the general solution

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

(ii) If $r_1 = r_2$ (happens when $b^2 - 4ac = 0$), then (7.3) has the general solution

$$y = (C_1 + C_2 x) e^{r_1 x}.$$

Example 7.2.1: Solve

$$y'' - k^2 y = 0.$$

The characteristic equation is $r^2 - k^2 = 0$ or $(r - k)(r + k) = 0$. Consequently, e^{-kx} and e^{kx} are the two linearly independent solutions, and the general solution is

$$y = C_1 e^{kx} + C_2 e^{-kx}.$$

Since $\cosh s = \frac{e^s + e^{-s}}{2}$ and $\sinh s = \frac{e^s - e^{-s}}{2}$, we can also write the general solution as

$$y = D_1 \cosh(kx) + D_2 \sinh(kx).$$

Example 7.2.2: Find the general solution of

$$y'' - 8y' + 16y = 0.$$

The characteristic equation is $r^2 - 8r + 16 = (r - 4)^2 = 0$. The equation has a double root $r_1 = r_2 = 4$. The general solution is, therefore,

$$y = (C_1 + C_2 x) e^{4x} = C_1 e^{4x} + C_2 x e^{4x}.$$

Exercise 7.2.2: Check that e^{4x} and $x e^{4x}$ are linearly independent.

It is good to check your work. That e^{4x} solves the equation is clear. Let us check that $x e^{4x}$ solves the equation. Compute $y' = e^{4x} + 4x e^{4x}$ and $y'' = 8e^{4x} + 16x e^{4x}$. Plug in,

$$y'' - 8y' + 16y = 8e^{4x} + 16x e^{4x} - 8(e^{4x} + 4x e^{4x}) + 16x e^{4x} = 0.$$

In some sense, a doubled root rarely happens. If coefficients are picked randomly, a doubled root is unlikely. There are, however, some real-world problems where a doubled root does happen naturally (e.g., critically damped mass-spring system as we will see).

Let us give a short argument for why the solution $x e^{r x}$ works for a doubled root. This case is a limiting case of two distinct but very close roots. Note that $\frac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1}$ is a solution when the roots are distinct. When we take the limit as r_1 goes to r_2 , we are really taking the derivative of $e^{r x}$ using r as the variable. Therefore, the limit is $x e^{r x}$, and hence this is a solution in the doubled root case. We remark that in some numerical computations, two very close roots may lead to numerical instability while a doubled root will not.

7.2.2 Complex Numbers and Euler's Formula

A polynomial may have complex roots. The equation $r^2 + 1 = 0$ has no real roots, but it does have two complex roots. Here we review some properties of complex numbers.

Complex numbers may seem a strange concept, especially because of the terminology. There is nothing imaginary or really complicated about complex numbers. A complex number is simply a pair of real numbers, (a, b) . Think of a complex number as a point in the plane. We add complex numbers in the straightforward way: $(a, b) + (c, d) = (a + c, b + d)$. We define multiplication by

$$(a, b) \times (c, d) \stackrel{\text{def}}{=} (ac - bd, ad + bc).$$

It turns out that with this multiplication rule, all the standard properties of arithmetic hold. Further, and most importantly $(0, 1) \times (0, 1) = (-1, 0)$.

Generally we write (a, b) as $a + ib$, and we treat i as if it were an unknown. When b is zero, then $(a, 0)$ is just the number a . We do arithmetic with complex numbers just as we would with polynomials. The property we just mentioned becomes $i^2 = -1$. So whenever we see i^2 , we replace it by -1 . For example,

$$(2 + 3i)(4i) - 5i = (2 \times 4)i + (3 \times 4)i^2 - 5i = 8i + 12(-1) - 5i = -12 + 3i.$$

The numbers i and $-i$ are the two roots of $r^2 + 1 = 0$. Some engineers use the letter j instead of i for the square root of -1 . We use the mathematicians' convention and use i .

Exercise 7.2.3: Make sure you understand (that you can justify) the following identities:

$$a) \quad i^2 = -1, i^3 = -i, i^4 = 1,$$

$$b) \quad \frac{1}{i} = -i,$$

$$c) \quad (3 - 7i)(-2 - 9i) = \cdots = -69 - 13i,$$

$$d) \quad (3 - 2i)(3 + 2i) = 3^2 - (2i)^2 = 3^2 + 2^2 = 13,$$

$$e) \quad \frac{1}{3-2i} = \frac{1}{3-2i} \frac{3+2i}{3+2i} = \frac{3+2i}{13} = \frac{3}{13} + \frac{2}{13}i.$$

We also define the exponential e^{a+ib} of a complex number. We do this by writing down the Taylor series and plugging in the complex number. Because most properties of the exponential can be proved by looking at the Taylor series, these properties still hold for the complex exponential. For example the very important property: $e^{x+y} = e^x e^y$. This means that $e^{a+ib} = e^a e^{ib}$. Hence if we can compute e^{ib} , we can compute e^{a+ib} . For e^{ib} we use the so-called *Euler's formula*.

Theorem 7.2.2 (Euler's formula).

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta.$$

In other words, $e^{a+ib} = e^a (\cos(b) + i \sin(b)) = e^a \cos(b) + i e^a \sin(b)$.

Exercise 7.2.4: Using Euler's formula, check the identities:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Exercise 7.2.5: Double angle identities: Start with $e^{i(2\theta)} = (e^{i\theta})^2$. Use Euler on each side and deduce:

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta \quad \text{and} \quad \sin(2\theta) = 2 \sin \theta \cos \theta.$$

For a complex number $a + ib$ we call a the *real part* and b the *imaginary part* of the number. Often the following notation is used,

$$\operatorname{Re}(a + ib) = a \quad \text{and} \quad \operatorname{Im}(a + ib) = b.$$

7.2.3 Complex Roots

Suppose the differential equation $ay'' + by' + cy = 0$ has the characteristic equation $ar^2 + br + c = 0$ that has complex roots. By the quadratic formula, the roots are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. These roots are complex if $b^2 - 4ac < 0$. In this case, we can write the roots as

$$r_1, r_2 = \frac{-b}{2a} \pm i \frac{\sqrt{4ac - b^2}}{2a}.$$

As you can see, we always get a pair of roots of the form $\alpha \pm i\beta$. We can still write the solution as

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}.$$

However, the exponential is now complex-valued. We need to allow C_1 and C_2 to be complex numbers to obtain a real-valued solution (which is what we are after). While there is nothing particularly wrong with this approach, it can make calculations harder and it is generally preferred to find two real-valued solutions.

[Euler's formula](#) comes to the rescue. Let

$$y_1 = e^{(\alpha + i\beta)x} \quad \text{and} \quad y_2 = e^{(\alpha - i\beta)x}.$$

Then

$$\begin{aligned} y_1 &= e^{\alpha x} \cos(\beta x) + i e^{\alpha x} \sin(\beta x), \\ y_2 &= e^{\alpha x} \cos(\beta x) - i e^{\alpha x} \sin(\beta x). \end{aligned}$$

Linear combinations of solutions are also solutions. Hence,

$$\begin{aligned} y_3 &= \frac{y_1 + y_2}{2} = e^{\alpha x} \cos(\beta x), \\ y_4 &= \frac{y_1 - y_2}{2i} = e^{\alpha x} \sin(\beta x), \end{aligned}$$

are also solutions. Furthermore, they are real-valued. It is not hard to see that they are linearly independent (not multiples of each other). We summarize what we found as a theorem.

Theorem 7.2.3. *Take the equation*

$$ay'' + by' + cy = 0.$$

If the characteristic equation has the roots $\alpha \pm i\beta$ (when $b^2 - 4ac < 0$), then the general solution is

$$y = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x).$$

Example 7.2.3: Find the general solution of $y'' + k^2 y = 0$, for a constant $k > 0$.

The characteristic equation is $r^2 + k^2 = 0$. Therefore, the roots are $r = \pm ik$, and by the theorem, we have the general solution

$$y = C_1 \cos(kx) + C_2 \sin(kx).$$

Example 7.2.4: Find the solution of $y'' - 6y' + 13y = 0$, $y(0) = 0$, $y'(0) = 10$.

The characteristic equation is $r^2 - 6r + 13 = 0$. By completing the square we get $(r - 3)^2 + 2^2 = 0$ and hence the roots are $r = 3 \pm 2i$. By the theorem we have the general solution

$$y = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x).$$

To find the solution satisfying the initial conditions, we first plug in zero to get

$$0 = y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1.$$

Hence, $C_1 = 0$ and $y = C_2 e^{3x} \sin(2x)$. We differentiate,

$$y' = 3C_2 e^{3x} \sin(2x) + 2C_2 e^{3x} \cos(2x).$$

We again plug in the initial condition and obtain $10 = y'(0) = 2C_2$, or $C_2 = 5$. The solution we are seeking is

$$y = 5e^{3x} \sin(2x).$$

7.2.4 Exercises

Exercise 7.2.6: Find the general solution of $2y'' + 2y' - 4y = 0$.

Exercise 7.2.7: Find the general solution of $y'' + 9y' - 10y = 0$.

Exercise 7.2.8: Solve $y'' - 8y' + 16y = 0$ for $y(0) = 2$, $y'(0) = 0$.

Exercise 7.2.9: Solve $y'' + 9y' = 0$ for $y(0) = 1$, $y'(0) = 1$.

Exercise 7.2.10: Find the general solution of $2y'' + 50y = 0$.

Exercise 7.2.11: Find the general solution of $y'' + 6y' + 13y = 0$.

Exercise 7.2.12: Find the general solution of $y'' = 0$ using the methods of this section.

Exercise 7.2.13: The method of this section applies to equations of other orders than two. We will see higher orders later. Solve the first order equation $2y' + 3y = 0$ using the methods of this section.

Exercise 7.2.14: Let us revisit the Cauchy–Euler equations of [Exercise 7.1.6](#) on page 304. Suppose now that $(b - a)^2 - 4ac < 0$. Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$. Hint: Note that $x^r = e^{r \ln x}$.

Exercise 7.2.15: Find the solution to $y'' - (2\alpha)y' + \alpha^2y = 0$, $y(0) = a$, $y'(0) = b$, where α , a , and b are real numbers.

Exercise 7.2.16: Construct an equation such that $y = C_1e^{-2x} \cos(3x) + C_2e^{-2x} \sin(3x)$ is the general solution.

Exercise 7.2.101: Find the general solution to $y'' + 4y' + 2y = 0$.

Exercise 7.2.102: Find the general solution to $y'' - 6y' + 9y = 0$.

Exercise 7.2.103: Find the solution to $2y'' + y' + y = 0$, $y(0) = 1$, $y'(0) = -2$.

Exercise 7.2.104: Find the solution to $2y'' + y' - 3y = 0$, $y(0) = a$, $y'(0) = b$.

Exercise 7.2.105: Find the solution to $z''(t) = -2z'(t) - 2z(t)$, $z(0) = 2$, $z'(0) = -2$.

Exercise 7.2.106: Find the solution to $y'' - (\alpha + \beta)y' + \alpha\beta y = 0$, $y(0) = a$, $y'(0) = b$, where α , β , a , and b are real numbers, and $\alpha \neq \beta$.

Exercise 7.2.107: Construct an equation such that $y = C_1e^{3x} + C_2e^{-2x}$ is the general solution.

7.3 Higher Order Linear ODEs

We briefly study higher order equations. Equations appearing in applications tend to be second order. Higher order equations do appear from time to time, but generally the world around us is “second order.”

The basic results about linear ODEs of higher order are essentially the same as for second order equations, with 2 replaced by n . The important concept of linear independence is somewhat more complicated when more than two functions are involved. For higher order constant coefficient ODEs, the methods developed are also somewhat harder to apply, but we will not dwell on these complications. It is also possible to use the methods for systems of linear equations from [chapter 14](#) to solve higher order constant coefficient equations.

Let us start with a general homogeneous linear equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \quad (7.4)$$

Theorem 7.3.1 (Superposition). *If $y_1, y_2, \dots, y_n$ are solutions of the homogeneous equation (7.4), then*

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x)$$

also solves (7.4) for arbitrary constants $C_1, C_2, \dots, C_n$.

That is, a *linear combination* of solutions to (7.4) is a solution to (7.4). There is also the existence and uniqueness theorem for linear equations, including nonhomogeneous ones.

Theorem 7.3.2 (Existence and uniqueness). *Suppose $p_0, p_1, \dots, p_{n-1}$, and f are continuous functions on some interval I , a is a number in I , and $b_0, b_1, \dots, b_{n-1}$ are constants. Then the equation*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \dots, \quad y^{(n-1)}(a) = b_{n-1}.$$

7.3.1 Linear Independence

When we had two functions y_1 and y_2 , we said they were linearly independent if one was not a multiple of the other. Same idea holds for n functions, although in this case it is easier to state as follows. The functions $y_1, y_2, \dots, y_n$ are *linearly independent* if the equation

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$, where the equation must hold for all x . If we can solve the equation with some constants $c_1, c_2, \dots, c_n$, where for example $c_1 \neq 0$, then we can solve for y_1 as a linear combination of the others. If the functions are not linearly independent, they are *linearly dependent*.

Example 7.3.1: Verify that e^x, e^{2x}, e^{3x} are linearly independent.

Let us give several ways to show this fact. Many textbooks (including [EP] and [F]) introduce Wronskians, but it is difficult to see why they work and they are not really necessary here.

Consider

$$c_1e^x + c_2e^{2x} + c_3e^{3x} = 0.$$

We use rules of exponentials and write $z = e^x$. Hence $z^2 = e^{2x}$ and $z^3 = e^{3x}$. Then we have

$$c_1z + c_2z^2 + c_3z^3 = 0.$$

The left-hand side is a third degree polynomial in z . It is either identically zero, or it has at most 3 zeros. Therefore, it is identically zero, $c_1 = c_2 = c_3 = 0$, and the functions are linearly independent.

Let us try another way. As before we write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

This equation has to hold for all x . We divide through by e^{3x} to get

$$c_1 e^{-2x} + c_2 e^{-x} + c_3 = 0.$$

As the equation is true for all x , let $x \rightarrow \infty$. After taking the limit we see that $c_3 = 0$. Hence our equation becomes

$$c_1 e^x + c_2 e^{2x} = 0.$$

Rinse, repeat!

How about yet another way. We again write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

We can evaluate the equation and its derivatives at different values of x to obtain equations for c_1 , c_2 , and c_3 . Let us first divide by e^x for simplicity.

$$c_1 + c_2 e^x + c_3 e^{2x} = 0.$$

We set $x = 0$ to get the equation $c_1 + c_2 + c_3 = 0$. Now differentiate both sides

$$c_2 e^x + 2c_3 e^{2x} = 0.$$

We set $x = 0$ to get $c_2 + 2c_3 = 0$. We divide by e^x again and differentiate to get $2c_3 e^x = 0$. It is clear that c_3 is zero. Then c_2 must be zero as $c_2 = -2c_3$, and c_1 must be zero because $c_1 + c_2 + c_3 = 0$.

There is no one best way to do it. All of these methods are perfectly valid. The important thing is to understand why the functions are linearly independent.

Example 7.3.2: On the other hand, the functions e^x , e^{-x} , and $\cosh x$ are linearly dependent. Simply apply definition of the hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{or} \quad 2 \cosh x - e^x - e^{-x} = 0.$$

Once we have enough linearly independent solutions, we have the general solution to the homogeneous equation, just as we did for second order equations.

Theorem 7.3.3. If $y_1, y_2, \dots, y_n$ are linearly independent solutions of the homogeneous equation (7.4), then the general solution to (7.4) can be written as

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x).$$

7.3.2 Constant Coefficient Higher Order ODEs

When we have a higher order constant coefficient homogeneous linear equation, the song and dance is exactly the same as it was for second order. We just need to find more solutions. If the equation is n^{th} order, we need to find n linearly independent solutions. It is best seen by example.

Example 7.3.3: Find the general solution to

$$y''' - 3y'' - y' + 3y = 0. \tag{7.5}$$

Try: $y = e^{rx}$. We plug in and get

$$\underbrace{r^3 e^{rx}}_{y'''} - 3 \underbrace{r^2 e^{rx}}_{y''} - \underbrace{r e^{rx}}_{y'} + 3 \underbrace{e^{rx}}_y = 0.$$

We divide through by e^{rx} . Then

$$r^3 - 3r^2 - r + 3 = 0.$$

The trick now is to find the roots. There is a formula for the roots of degree 3 and 4 polynomials but it is very complicated. There is no formula for higher degree polynomials. That does not mean that the roots do not exist. There are always n roots for an n^{th} degree polynomial. They may be repeated and they may be complex. Computers are pretty good at finding roots approximately for reasonable size polynomials.

A good place to start is to plot the polynomial and check where it is zero. We can also simply try plugging in. We just start plugging in numbers $r = -2, -1, 0, 1, 2, \dots$ and see if we get a hit (we can also try complex numbers). Even if we do not get a hit, we may get an indication of where the root is. For example, we plug $r = -2$ into our polynomial and get -15 ; we plug in $r = 0$ and get 3 . That means there is a root between $r = -2$ and $r = 0$, because the sign changed. If we find one root, say r_1 , then we know $(r - r_1)$ is a factor of our polynomial. Polynomial long division can then be used.

A good strategy is to begin with $r = 0, 1$, or -1 . These are easy to compute. Our polynomial has two such roots, $r_1 = -1$ and $r_2 = 1$. There should be 3 roots and the last root is reasonably easy to find. The constant term in a monic* polynomial such as this is the multiple of the negations of all the roots because $r^3 - 3r^2 - r + 3 = (r - r_1)(r - r_2)(r - r_3)$. So

$$3 = (-r_1)(-r_2)(-r_3) = (1)(-1)(-r_3) = r_3.$$

You should check that $r_3 = 3$ really is a root. Hence e^{-x} , e^x and e^{3x} are solutions to (7.5). They are linearly independent as can easily be checked, and there are 3 of them, which happens to be exactly the number we need. So the general solution is

$$y = C_1 e^{-x} + C_2 e^x + C_3 e^{3x}.$$

Suppose we were given some initial conditions $y(0) = 1$, $y'(0) = 2$, and $y''(0) = 3$. Then

$$1 = y(0) = C_1 + C_2 + C_3,$$

$$2 = y'(0) = -C_1 + C_2 + 3C_3,$$

$$3 = y''(0) = C_1 + C_2 + 9C_3.$$

It is possible to find the solution by high school algebra, but it would be a pain. The sensible way to solve a system of equations such as this is to use matrix algebra, see § 14.2 or appendix A. For now we note that the solution is $C_1 = -1/4$, $C_2 = 1$, and $C_3 = 1/4$. The specific solution to the ODE is

$$y = \frac{-1}{4} e^{-x} + e^x + \frac{1}{4} e^{3x}.$$

Next, suppose that we have real roots, but they are repeated. Let us say we have a root r repeated k times. In the spirit of the second order solution, and for the same reasons, we have the solutions

$$e^{rx}, \quad x e^{rx}, \quad x^2 e^{rx}, \quad \dots, \quad x^{k-1} e^{rx}.$$

We take a linear combination of these solutions to find the general solution.

Example 7.3.4: Solve

$$y^{(4)} - 3y''' + 3y'' - y' = 0.$$

We note that the characteristic equation is

$$r^4 - 3r^3 + 3r^2 - r = 0.$$

By inspection, we note that $r^4 - 3r^3 + 3r^2 - r = r(r - 1)^3$. Hence the roots given with multiplicity are $r = 0, 1, 1, 1$. Thus the general solution is

$$y = \underbrace{(C_1 + C_2 x + C_3 x^2) e^x}_{\text{terms coming from } r=1} + \underbrace{C_4}_{\text{from } r=0}.$$

*The word monic means that the coefficient of the top degree r^d , in our case r^3 , is 1.

The case of complex roots is similar to second order equations. Complex roots always come in pairs $r = \alpha \pm i\beta$. Suppose we have two such complex roots, each repeated k times. The corresponding solution is

$$(C_0 + C_1x + \cdots + C_{k-1}x^{k-1})e^{\alpha x} \cos(\beta x) + (D_0 + D_1x + \cdots + D_{k-1}x^{k-1})e^{\alpha x} \sin(\beta x),$$

where $C_0, \dots, C_{k-1}, D_0, \dots, D_{k-1}$ are arbitrary constants.

Example 7.3.5: Solve

$$y^{(4)} - 4y''' + 8y'' - 8y' + 4y = 0.$$

The characteristic equation is

$$r^4 - 4r^3 + 8r^2 - 8r + 4 = 0,$$

$$(r^2 - 2r + 2)^2 = 0,$$

$$((r - 1)^2 + 1)^2 = 0.$$

Hence the roots are $1 \pm i$, both with multiplicity 2. Hence the general solution to the ODE is

$$y = (C_1 + C_2x)e^x \cos x + (C_3 + C_4x)e^x \sin x.$$

The way we solved the characteristic equation above is really by guessing or by inspection. It is not so easy in general. We could also have asked a computer or an advanced calculator for the roots.

7.3.3 Exercises

Exercise 7.3.1: Find the general solution for $y''' - y'' + y' - y = 0$.

Exercise 7.3.2: Find the general solution for $y^{(4)} - 5y''' + 6y'' = 0$.

Exercise 7.3.3: Find the general solution for $y''' + 2y'' + 2y' = 0$.

Exercise 7.3.4: Suppose the characteristic equation for an ODE is $(r - 1)^2(r - 2)^2 = 0$.

a) Find such a differential equation.

b) Find its general solution.

Exercise 7.3.5: Suppose that a fourth order equation has a solution $y = 2e^{4x}x \cos x$.

a) Find such an equation.

b) Find the initial conditions that the given solution satisfies.

Exercise 7.3.6: Find the general solution for the equation of [Exercise 7.3.5](#).

Exercise 7.3.7: Let $f(x) = e^x - \cos x$, $g(x) = e^x + \cos x$, and $h(x) = \cos x$. Are $f(x)$, $g(x)$, and $h(x)$ linearly independent? If so, show it, if not, find a linear combination that works.

Exercise 7.3.8: Let $f(x) = 0$, $g(x) = \cos x$, and $h(x) = \sin x$. Are $f(x)$, $g(x)$, and $h(x)$ linearly independent? If so, show it, if not, find a linear combination that works.

Exercise 7.3.9: Are x , x^2 , and x^4 linearly independent? If so, show it, if not, find a linear combination that works.

Exercise 7.3.10: Are e^x , xe^x , and x^2e^x linearly independent? If so, show it, if not, find a linear combination that works.

Exercise 7.3.11: Find an equation such that $y = xe^{-2x} \sin(3x)$ is a solution.

Exercise 7.3.101: Find the general solution of $y^{(5)} - y^{(4)} = 0$.

Exercise 7.3.102: Suppose that the characteristic equation of a third order differential equation has roots $\pm 2i$ and 3.

- a) What is the characteristic equation?
- b) Find the corresponding differential equation.
- c) Find the general solution.

Exercise 7.3.103: Solve $1001y''' + 3.2y'' + \pi y' - \sqrt{4}y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$.

Exercise 7.3.104: Are e^x , e^{x+1} , e^{2x} , $\sin(x)$ linearly independent? If so, show it, if not find a linear combination that works.

Exercise 7.3.105: Are $\sin(x)$, x , $x \sin(x)$ linearly independent? If so, show it, if not find a linear combination that works.

Exercise 7.3.106: Find an equation such that $y = \cos(x)$, $y = \sin(x)$, $y = e^x$ are solutions.

7.4 Mechanical Vibrations

Let us look at some applications of linear second order constant coefficient equations.

7.4.1 Some Examples

Our first example is a mass on a spring. Suppose we have a mass $m > 0$ (in kilograms) connected by a spring with spring constant $k > 0$ (in newtons per meter) to a fixed wall. There may be some external force $F(t)$ (in newtons) acting on the mass. Here, t is time in seconds. Finally, there is some friction measured by $c \geq 0$ (in newton-seconds per meter) as the mass slides along the floor (or perhaps a damper is connected).

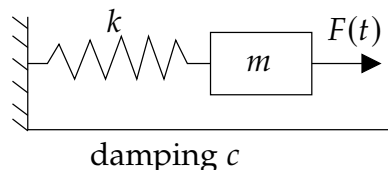
Let x be the displacement of the mass in meters ($x = 0$ is the rest position), with x growing to the right (away from the wall). The force exerted by the spring is proportional to the compression of the spring by Hooke's law. Therefore, it is kx in the negative direction. Similarly the force exerted by friction is proportional to the velocity of the mass. By Newton's second law we know that force equals mass times acceleration and hence $mx'' = F(t) - cx' - kx$ or

$$mx'' + cx' + kx = F(t).$$

This is a linear second order constant coefficient ODE. We say the motion is

- (i) *forced* if $F \not\equiv 0$ (if F is not identically zero),
- (ii) *unforced* or *free* if $F \equiv 0$ (if F is identically zero),
- (iii) *damped* if $c > 0$, and
- (iv) *undamped* if $c = 0$.

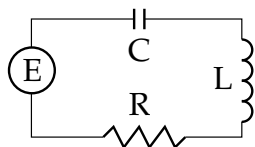
This system appears in lots of applications even if it does not at first seem like it. Many real-world scenarios can be simplified to a mass on a spring. For example, a bungee jump setup is essentially a mass and spring system (you are the mass). It would be good if someone did the math before you jump off the bridge, right? Let us give two other examples.



Here is an example for electrical engineers. Consider the pictured RLC circuit. There is a resistor with a resistance of R ohms, an inductor with an inductance of L henries, and a capacitor with a capacitance of C farads. There is also an electric source (such as a battery) giving a voltage of $E(t)$ volts at time t (measured in seconds). Let $Q(t)$ be the charge (in coulombs) on the capacitor and $I(t)$ be the current (in amperes) in the circuit. The relation between the two is $Q' = I$. By elementary principles, we find $LI' + RI + Q/C = E$. We differentiate to get

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t).$$

This is a nonhomogeneous second order constant coefficient linear equation. As L , R , and C are all positive, this system behaves just like the mass and spring system. Position of the mass is replaced by current. Mass is replaced by inductance, damping is replaced by resistance, and the spring constant is replaced by one over the capacitance. The change in voltage becomes the forcing function—for constant voltage this is an unforced motion.

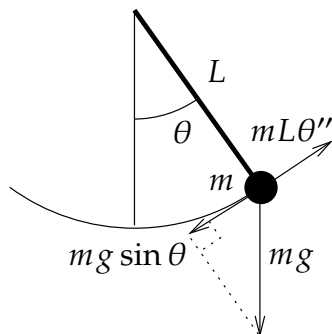


Our next example behaves like a mass and spring system only approximately. Suppose a mass m hangs on a pendulum of length L . We seek an equation for the angle $\theta(t)$ (in radians). Let g be the force of gravity. Elementary physics mandates that the equation is

$$\theta'' + \frac{g}{L} \sin \theta = 0.$$

Let us derive this equation using Newton's second law: force equals mass times acceleration. The acceleration is $L\theta''$ and mass is m . So $mL\theta''$ has to be equal to the tangential component of the force given by the gravity, which is $mg \sin \theta$ in the opposite direction. So $mL\theta'' = -mg \sin \theta$. The m curiously cancels from the equation.

Now we make our approximation. For small θ we have that approximately $\sin \theta \approx \theta$. This can be seen by looking at the graph. In [Figure 7.1](#) on page 318 we can see that for approximately $-0.5 < \theta < 0.5$ (in radians) the graphs of $\sin \theta$ and θ are almost the same.



Therefore, when the swings are small, θ is small and we can model the behavior by the simpler linear equation

$$\theta'' + \frac{g}{L} \theta = 0.$$

The errors from this approximation build up. So after a long time, the state of the real-world system might be substantially different from our solution. Also we will see that in a mass-spring system, the amplitude is independent of the period. This is not true for a pendulum. Nevertheless, for reasonably short periods of time and small swings (that is, only small angles θ), the approximation is reasonably good.

In real-world problems it is often necessary to make these types of simplifications. We must understand both the mathematics and the physics of the situation to see if the simplification is valid in the context of the questions we are trying to answer.

7.4.2 Free Undamped Motion

In this section we only consider free or unforced motion, as we do not know yet how to solve nonhomogeneous equations. Let us start with undamped motion where $c = 0$. The equation is

$$mx'' + kx = 0.$$

We divide by m and let $\omega_0 = \sqrt{k/m}$ to rewrite the equation as

$$x'' + \omega_0^2 x = 0.$$

The general solution to this equation is

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t).$$

By a trigonometric identity

$$A \cos(\omega_0 t) + B \sin(\omega_0 t) = C \cos(\omega_0 t - \gamma),$$

for two different constants C and γ . It is not hard to compute that $C = \sqrt{A^2 + B^2}$ and $\tan \gamma = B/A$. Therefore, we let C and γ be our arbitrary constants and write $x(t) = C \cos(\omega_0 t - \gamma)$.

Exercise 7.4.1: Justify the identity $A \cos(\omega_0 t) + B \sin(\omega_0 t) = C \cos(\omega_0 t - \gamma)$ and verify the equations for C and γ . Hint: Start with $\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta)$ and multiply by C . Then what should α and β be?

While it is generally easier to use the first form with A and B to solve for the initial conditions, the second form is much more natural. The constants C and γ have nice physical interpretation. Write the solution as

$$x(t) = C \cos(\omega_0 t - \gamma).$$

This is a pure-frequency oscillation (a sine wave). The *amplitude* is C , ω_0 is the (angular) *frequency*, and γ is the so-called *phase shift*. The phase shift just shifts the graph left or right. We call ω_0 the *natural (angular) frequency*. This entire setup is called *simple harmonic motion*.

Let us pause to explain the word *angular* before the word *frequency*. The units of ω_0 are radians per unit time, not cycles per unit time as is the usual measure of frequency. Because one cycle is 2π radians, the usual frequency is given by $\frac{\omega_0}{2\pi}$. It is simply a matter of where we put the constant 2π , and that is a matter of taste.

The *period* of the motion is one over the frequency (in cycles per unit time) and hence it is $\frac{2\pi}{\omega_0}$. That is the amount of time it takes to complete one full cycle.

Example 7.4.1: Suppose that $m = 2$ kg and $k = 8$ N/m. The whole mass and spring setup is sitting on a truck that was traveling at 1 m/s. The truck crashes and hence stops. The mass was held in place 0.5 meters forward from the rest position. During the crash the mass gets loose. That is, the mass is now moving forward at 1 m/s, while the other end of the spring is held in place. The mass therefore starts oscillating. What is the frequency of the resulting oscillation? What is the amplitude? The units are the mks units (meters-kilograms-seconds).

The setup means that the mass was at half a meter in the positive direction during the crash and relative to the wall the spring is mounted to, the mass was moving forward (in the positive direction) at 1 m/s. This gives us the initial conditions.

So the equation with initial conditions is

$$2x'' + 8x = 0, \quad x(0) = 0.5, \quad x'(0) = 1.$$

We directly compute $\omega_0 = \sqrt{k/m} = \sqrt{4} = 2$. Hence the angular frequency is 2. The usual frequency in Hertz (cycles per second) is $2/2\pi = 1/\pi \approx 0.318$.

The general solution is

$$x(t) = A \cos(2t) + B \sin(2t).$$

Letting $x(0) = 0.5$ means $A = 0.5$. Then $x'(t) = -2(0.5) \sin(2t) + 2B \cos(2t)$. Letting $x'(0) = 1$ we get $B = 0.5$. Therefore, the amplitude is $C = \sqrt{A^2 + B^2} = \sqrt{0.25 + 0.25} = \sqrt{0.5} \approx 0.707$. The solution is

$$x(t) = 0.5 \cos(2t) + 0.5 \sin(2t).$$

A plot of $x(t)$ is shown in [Figure 7.2](#) on page 319.

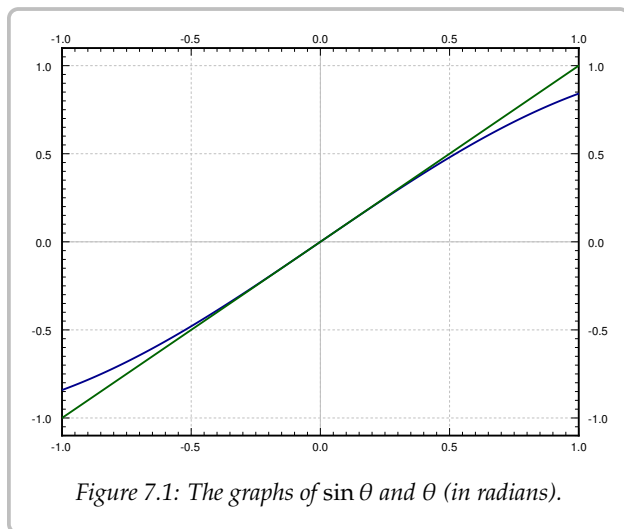


Figure 7.1: The graphs of $\sin \theta$ and θ (in radians).

In general, for free undamped motion, a solution of the form

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t),$$

corresponds to the initial conditions $x(0) = A$ and $x'(0) = \omega_0 B$. So it is easy to figure out A and B from the initial conditions. The amplitude and the phase shift are computed from A and B . In the example, we found the amplitude C . Let us compute the phase shift. We know $\tan \gamma = B/A = 1$. We take the arctangent of 1 and get $\pi/4$ or approximately 0.785. We need to check if $\pi/4$ gives the correct quadrant (same quadrant as the point (A, B)) and add π to get γ if it is not. Since both A and B are positive, then γ should be in the first quadrant, $\pi/4$ radians is in the first quadrant, so $\gamma = \pi/4$.

Note: Many calculators and computer software have not only the atan function for arctangent, but also what is sometimes called atan2 . This function takes two arguments, B and A , and returns a γ in the correct quadrant for you.

7.4.3 Free Damped Motion

Let us now focus on damped motion. We rewrite the equation

$$mx'' + cx' + kx = 0,$$

as

$$x'' + 2px' + \omega_0^2 x = 0,$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad p = \frac{c}{2m}.$$

The characteristic equation is

$$r^2 + 2pr + \omega_0^2 = 0.$$

Using the quadratic formula, we get that the roots are

$$r = -p \pm \sqrt{p^2 - \omega_0^2}.$$

The form of the solution depends on whether we get complex or real roots. We get real roots if and only if the following number is nonnegative:

$$p^2 - \omega_0^2 = \left(\frac{c}{2m}\right)^2 - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}.$$

The sign of $p^2 - \omega_0^2$ is the same as the sign of $c^2 - 4km$. Thus we get real roots if and only if $c^2 - 4km$ is nonnegative, or in other words if $c^2 \geq 4km$.

Overdamping

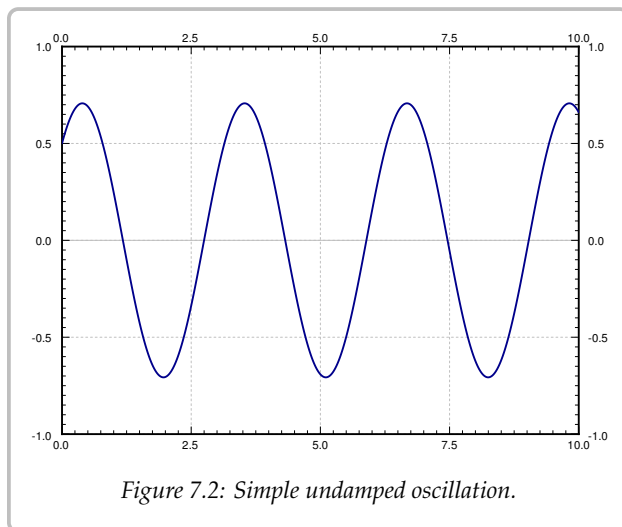


Figure 7.2: Simple undamped oscillation.

When $c^2 - 4km > 0$, the system is *overdamped*. In this case, there are two distinct real roots r_1 and r_2 . Both roots are negative: As $\sqrt{p^2 - \omega_0^2}$ is always less than p , then $-p \pm \sqrt{p^2 - \omega_0^2}$ is negative in either case.

The solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Since r_1, r_2 are negative, $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Thus the mass will tend towards the rest position as time goes to infinity. For a few sample plots for different initial conditions, see [Figure 7.3](#) on the next page.

No oscillation happens. In fact, the graph crosses the t -axis at most once. To see why, we try to solve $0 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. Therefore, $C_1 e^{r_1 t} = -C_2 e^{r_2 t}$ and using laws of exponents we obtain

$$\frac{-C_1}{C_2} = e^{(r_2 - r_1)t}.$$

This equation has at most one solution $t \geq 0$. For some initial conditions the graph never crosses the t -axis, as is evident from the sample graphs.

Example 7.4.2: Suppose the mass is released from rest. That is, $x(0) = x_0$ and $x'(0) = 0$. Then

$$x(t) = \frac{x_0}{r_1 - r_2} (r_1 e^{r_2 t} - r_2 e^{r_1 t}).$$

It is not hard to see that this satisfies the initial conditions.

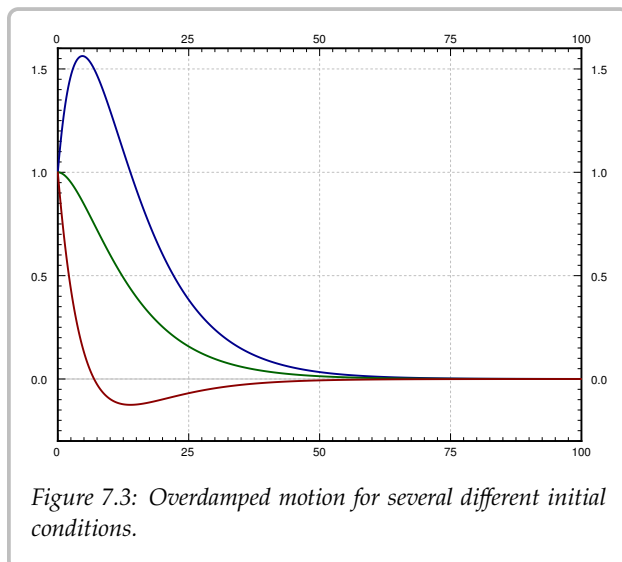
Critical damping

When $c^2 - 4km = 0$, the system is *critically damped*. In this case, there is one root of multiplicity 2 and this root is $-p$. Our solution is

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt}.$$

The behavior of a critically damped system is very similar to an overdamped system. After all a critically damped system is in some sense a limit of overdamped systems. Since these equations are really only an approximation to the real world, in reality we are never critically damped, it is a place we can only reach in theory. We are always a little bit underdamped or a little bit overdamped. It is better not to dwell on critical damping.

Underdamping



When $c^2 - 4km < 0$, the system is *underdamped*. In this case, the roots are complex.

$$\begin{aligned} r &= -p \pm \sqrt{p^2 - \omega_0^2} \\ &= -p \pm \sqrt{-1} \sqrt{\omega_0^2 - p^2} \\ &= -p \pm i\omega_1, \end{aligned}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. Our solution is

$$x(t) = e^{-pt} (A \cos(\omega_1 t) + B \sin(\omega_1 t)),$$

or

$$x(t) = Ce^{-pt} \cos(\omega_1 t - \gamma).$$

An example plot is given in [Figure 7.4](#) on page 322. Note that we still have that $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

The figure also shows the *envelope curves* Ce^{-pt} and $-Ce^{-pt}$. The solution is the oscillating line between the two envelope curves. The envelope curves give the maximum amplitude of the oscillation at any given point in time. For example, if you are bungee jumping, you are really interested in computing the envelope curve as not to hit the concrete with your head.

The phase shift γ shifts the oscillation left or right, but within the envelope curves (the envelope curves do not change if γ changes).

Notice that the angular *pseudo-frequency** ω_1 becomes smaller when the damping c (and hence p) becomes larger. This makes sense. When we change the damping just a little bit, we do not expect the behavior of the solution to change dramatically. If we keep making c larger, then at some point the solution should start looking like the solution for critical damping or overdamping, where no oscillation happens. So if c^2 approaches $4km$, we find that ω_1 approaches 0.

On the other hand, when c gets smaller, ω_1 approaches ω_0 (ω_1 is always smaller than ω_0), and the solution looks more and more like the steady periodic motion of the undamped case. The envelope curves become flatter and flatter as c (and hence p) goes to 0.

*We do not call ω_1 a frequency since the solution is not really a periodic function.

7.4.4 Exercises

Exercise 7.4.2: Consider a mass and spring system with a mass $m = 2$, spring constant $k = 3$, and damping constant $c = 1$.

- Set up and find the general solution of the system.
- Is the system underdamped, overdamped or critically damped?
- If the system is not critically damped, find a c that makes the system critically damped.

Exercise 7.4.3: Do [Exercise 7.4.2](#) for $m = 3$, $k = 12$, and $c = 12$.

Exercise 7.4.4: Using the mks units (meters-kilograms-seconds), suppose you have a spring with spring constant 4 N/m . You want to use it to weigh items. Assume no friction. You place the mass on the spring and put it in motion.

- You count and find that the frequency is 0.8 Hz (cycles per second). What is the mass?
- Find a formula for the mass m given the frequency ω in Hz .

Exercise 7.4.5: Suppose we add possible friction to [Exercise 7.4.4](#). Further, suppose you do not know the spring constant, but you have two reference weights 1 kg and 2 kg to calibrate your setup. You put each in motion on your spring and measure the frequency. For the 1 kg weight you measured 1.1 Hz , for the 2 kg weight you measured 0.8 Hz .

- Find k (spring constant) and c (damping constant).
- Find a formula for the mass in terms of the frequency in Hz . Note that there may be more than one possible mass for a given frequency.
- For an unknown object you measured 0.2 Hz , what is the mass of the object? Suppose that you know that the mass of the unknown object is more than a kilogram.

Exercise 7.4.6: Suppose you wish to measure the friction a mass of 0.1 kg experiences as it slides along a floor (you wish to find c). You have a spring with spring constant $k = 5 \text{ N/m}$. You take the spring, you attach it to the mass and fix it to a wall. Then you pull on the spring and let the mass go. You find that the mass oscillates with frequency 1 Hz . What is the friction?

Exercise 7.4.101: A mass of 2 kilograms is on a spring with spring constant k newtons per meter with no damping. Suppose the system is at rest and at time $t = 0$ the mass is kicked and starts traveling at $2 \text{ meters per second}$. How large does k have to be to so that the mass does not go further than 3 meters from the rest position?

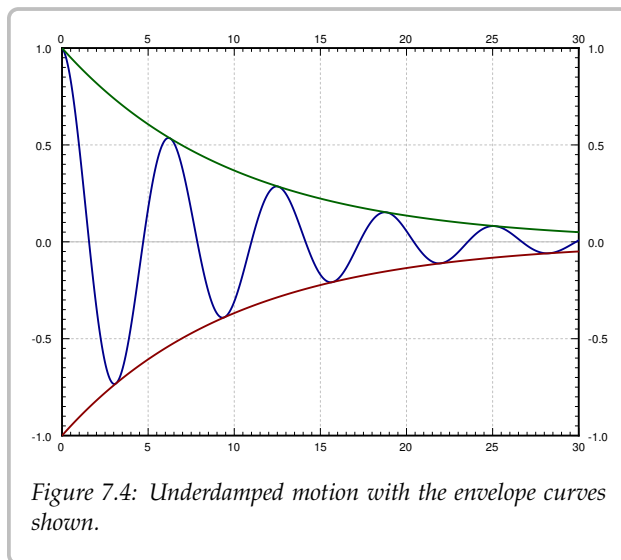
Exercise 7.4.102: Suppose we have an RLC circuit with a resistor of 100 milliohms (0.1 ohms), inductor of inductance of 50 millihenries (0.05 henries), and a capacitor of 5 farads , with constant voltage.

- Set up the ODE equation for the current I .
- Find the general solution.
- Solve for $I(0) = 10$ and $I'(0) = 0$.

Exercise 7.4.103: A 5000 kg railcar hits a bumper (a spring) at 1 m/s , and the spring compresses by 0.1 m . Assume no damping.

- Find k .
- How far does the spring compress when a 10000 kg railcar hits the spring at the same speed?
- If the spring would break if it compresses further than 0.3 m , what is the maximum mass of a railcar that can hit it at 1 m/s ?
- What is the maximum mass of a railcar that can hit the spring without breaking at 2 m/s ?

Exercise 7.4.104: A mass of $m \text{ kg}$ is on a spring with $k = 3 \text{ N/m}$ and $c = 2 \text{ Ns/m}$. Find the mass m_0 for which there is critical damping. If $m < m_0$, does the system oscillate or not, that is, is it underdamped or overdamped?



7.5 Nonhomogeneous Equations

7.5.1 Solving Nonhomogeneous Equations

We have solved linear constant coefficient homogeneous equations. What about nonhomogeneous linear ODEs? For example, the equations for forced mechanical vibrations. That is, suppose we have an equation such as

$$y'' + 5y' + 6y = 2x + 1. \quad (7.6)$$

We will write $Ly = 2x + 1$ when the exact form of the operator is not important. We solve (7.6) in the following manner. First, we find the general solution y_c to the *associated homogeneous equation*

$$y'' + 5y' + 6y = 0. \quad (7.7)$$

We call y_c the *complementary solution*. Next, we find a single *particular solution* y_p to (7.6) in some way. Then

$$y = y_c + y_p$$

is the general solution to (7.6). We have $Ly_c = 0$ and $Ly_p = 2x + 1$. As L is a *linear operator* we verify that y is a solution, $Ly = L(y_c + y_p) = Ly_c + Ly_p = 0 + (2x + 1)$. Let us see why we obtain the *general* solution.

Let y_p and $\tilde{y}_p$ be two different particular solutions to (7.6). Write the difference as $w = y_p - \tilde{y}_p$. Then plug w into the left-hand side of the equation to get

$$w'' + 5w' + 6w = (y_p'' + 5y_p' + 6y_p) - (\tilde{y}_p'' + 5\tilde{y}_p' + 6\tilde{y}_p) = (2x + 1) - (2x + 1) = 0.$$

Using the operator notation the calculation becomes simpler. As L is a linear operator we write

$$Lw = L(y_p - \tilde{y}_p) = Ly_p - L\tilde{y}_p = (2x + 1) - (2x + 1) = 0.$$

So $w = y_p - \tilde{y}_p$ is a solution to (7.7), that is $Lw = 0$. Any two solutions of (7.6) differ by a solution to the homogeneous equation (7.7). The solution $y = y_c + y_p$ includes *all* solutions to (7.6), since y_c is the general solution to the associated homogeneous equation.

Theorem 7.5.1. Let $Ly = f(x)$ be a linear ODE (not necessarily constant coefficient). Let y_c be the complementary solution (the general solution to the associated homogeneous equation $Ly = 0$) and let y_p be any particular solution to $Ly = f(x)$. Then the general solution to $Ly = f(x)$ is

$$y = y_c + y_p.$$

The moral of the story is that we can find the particular solution in any old way. If we find a different particular solution (by a different method, or simply by guessing), then we still get the same general solution. The formula may look different, and the constants we have to choose to satisfy the initial conditions may be different, but it is the same solution.

7.5.2 Undetermined Coefficients

The trick is to somehow, in a smart way, guess one particular solution to (7.6). Note that $2x + 1$ is a polynomial, and the left-hand side of the equation will be a polynomial if we let y be a polynomial of the same degree. Let us try

$$y_p = Ax + B.$$

We plug y_p into the left hand side to obtain

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= (Ax + B)'' + 5(Ax + B)' + 6(Ax + B) \\ &= 0 + 5A + 6Ax + 6B = 6Ax + (5A + 6B). \end{aligned}$$

So $6Ax + (5A + 6B) = 2x + 1$. Therefore, $A = 1/3$ and $B = -1/9$. That means $y_p = \frac{1}{3}x - \frac{1}{9} = \frac{3x-1}{9}$. Solving the complementary problem (exercise!) we get

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}.$$

Hence the general solution to (7.6) is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{3x-1}{9}.$$

Now suppose we are further given some initial conditions. For example, $y(0) = 0$ and $y'(0) = 1/3$. First find $y' = -2C_1 e^{-2x} - 3C_2 e^{-3x} + 1/3$. Then

$$0 = y(0) = C_1 + C_2 - \frac{1}{9}, \quad \frac{1}{3} = y'(0) = -2C_1 - 3C_2 + \frac{1}{3}.$$

We solve to get $C_1 = 1/3$ and $C_2 = -2/9$. The particular solution we want is

$$y = \frac{1}{3}e^{-2x} - \frac{2}{9}e^{-3x} + \frac{3x-1}{9} = \frac{3e^{-2x} - 2e^{-3x} + 3x - 1}{9}.$$

Exercise 7.5.1: Check that y really solves the equation (7.6) and the given initial conditions.

Note: A common mistake is to solve for constants using the initial conditions with y_c and only add the particular solution y_p after that. That will *not* work. You need to first compute $y = y_c + y_p$ and *only then* solve for the constants using the initial conditions.

Another important remark is that you should not forget the lower degree terms, even if they do not appear on the right hand side. If the equation is $Ly = x^3 + 1$, you must try $y_p = Ax^3 + Bx^2 + Cx + D$, even though there is no x^2 nor x on the right hand side of the equation. It is a general polynomial of degree 3 that must be tried.

A right-hand side consisting of exponentials, sines, and cosines can be handled similarly. For example,

$$y'' + 2y' + 2y = \cos(2x).$$

Let us find some y_p . We start by guessing the solution includes some multiple of $\cos(2x)$. We may have to also add a multiple of $\sin(2x)$ to our guess since derivatives of cosine are sines. We try

$$y_p = A \cos(2x) + B \sin(2x).$$

We plug y_p into the equation and we get

$$\underbrace{-4A \cos(2x) - 4B \sin(2x)}_{y_p''} + 2 \underbrace{(-2A \sin(2x) + 2B \cos(2x))}_{y_p'} + 2 \underbrace{(A \cos(2x) + B \sin(2x))}_{y_p} = \cos(2x),$$

or

$$(-4A + 4B + 2A) \cos(2x) + (-4B - 4A + 2B) \sin(2x) = \cos(2x).$$

The left-hand side must equal to right-hand side. Namely, $-4A + 4B + 2A = 1$ and $-4B - 4A + 2B = 0$. So $-2A + 4B = 1$ and $2A + B = 0$ and hence $A = -1/10$ and $B = 1/5$. So

$$y_p = A \cos(2x) + B \sin(2x) = \frac{-\cos(2x) + 2 \sin(2x)}{10}.$$

Similarly, if the right-hand side contains exponentials we try exponentials. If

$$Ly = e^{3x},$$

we try $y_p = Ae^{3x}$ as our guess and try to solve for A .

When the right-hand side is a multiple of sines, cosines, exponentials, and polynomials, we can use the product rule for differentiation to come up with a guess. We need to guess a form for y_p such that Ly_p is of the same form, and has all the terms needed to for the right-hand side. For example,

$$Ly = (1 + 3x^2)e^{-x} \cos(\pi x).$$

For this equation, we guess

$$y_p = (A + Bx + Cx^2)e^{-x} \cos(\pi x) + (D + Ex + Fx^2)e^{-x} \sin(\pi x).$$

We plug in and then hopefully get equations that we can solve for A, B, C, D, E , and F . As you can see this can make for a very long and tedious calculation very quickly. *C'est la vie!*

There is one hiccup in all this. It could be that our guess actually solves the associated homogeneous equation. That is, consider

$$y'' - 9y = e^{3x}.$$

We would love to guess $y_p = Ae^{3x}$, but if we plug this into the left-hand side of the equation we get

$$y_p'' - 9y_p = 9Ae^{3x} - 9Ae^{3x} = 0 \neq e^{3x}.$$

There is no way to choose A to make the left-hand side be e^{3x} . The trick in this case is to multiply our guess by x to get rid of duplication with the complementary solution. That is, first we find y_c (solution to $Ly = 0$)

$$y_c = C_1e^{-3x} + C_2e^{3x},$$

and we note that the e^{3x} term is a duplicate with our desired guess. We modify our guess to $y_p = Axe^{3x}$ so that there is no duplication anymore. Let us try: $y_p' = Ae^{3x} + 3Axe^{3x}$ and $y_p'' = 6Ae^{3x} + 9Axe^{3x}$, so

$$y_p'' - 9y_p = 6Ae^{3x} + 9Axe^{3x} - 9Axe^{3x} = 6Ae^{3x}.$$

Thus $6Ae^{3x}$ is supposed to equal e^{3x} . Hence, $6A = 1$ and so $A = 1/6$. We can now write the general solution as

$$y = y_c + y_p = C_1e^{-3x} + C_2e^{3x} + \frac{1}{6}xe^{3x}.$$

It is possible that multiplying by x does not get rid of all duplication. For example,

$$y'' - 6y' + 9y = e^{3x}.$$

The complementary solution is $y_c = C_1e^{3x} + C_2xe^{3x}$. Guessing $y_p = Axe^{3x}$ does not get us anywhere. We want to guess $y_p = Ax^2e^{3x}$. Basically, we multiply our guess by x until all duplication is gone. *But no more!* Multiplying too many times will not work.

Finally, what if the right-hand side has several terms, such as

$$Ly = e^{2x} + \cos x.$$

In this case we find u that solves $Lu = e^{2x}$ and v that solves $Lv = \cos x$ (that is, do each term separately). If we set $y = u + v$, then we find our desired solution $Ly = e^{2x} + \cos x$. This is because L is linear; we have $Ly = L(u + v) = Lu + Lv = e^{2x} + \cos x$.

7.5.3 Variation of Parameters

The method of undetermined coefficients works for many basic problems that crop up. But it does not work all the time. It only works when the right-hand side of the equation $Ly = f(x)$ has finitely many linearly independent derivatives, so that we can write a guess that consists of them all. Some equations are a bit tougher. Consider

$$y'' + y = \tan x.$$

Each new derivative of $\tan x$ looks completely different and cannot be written as a linear combination of the previous derivatives. If we start differentiating $\tan x$, we get:

$$\begin{aligned} \sec^2 x, \quad 2\sec^2 x \tan x, \quad 4\sec^2 x \tan^2 x + 2\sec^4 x, \\ 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x, \quad 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x, \quad \dots \end{aligned}$$

This equation calls for a different method. We present the method of *variation of parameters*, which handles any equation of the form $Ly = f(x)$, provided we can solve certain integrals. For simplicity, we restrict ourselves to second order constant coefficient equations, but the method works for higher order equations just as well (the computations become more tedious). The method also works for equations with nonconstant coefficients, provided we can solve the associated homogeneous equation.

Perhaps it is best to explain this method by example. Let us try to solve the equation

$$Ly = y'' + y = \tan x.$$

First we find the complementary solution (solution to $Ly_c = 0$). We get $y_c = C_1 y_1 + C_2 y_2$, where $y_1 = \cos x$ and $y_2 = \sin x$. To find a particular solution to the nonhomogeneous equation we try

$$y_p = y = u_1 y_1 + u_2 y_2,$$

where u_1 and u_2 are *functions* and not constants. We are trying to satisfy $Ly = \tan x$. That gives us one condition on the functions u_1 and u_2 . Compute (note the product rule!)

$$y' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2').$$

We can still impose one more condition at our discretion to simplify computations (we have two unknown functions, so we should be allowed two conditions). We require that $(u_1' y_1 + u_2' y_2) = 0$. This makes computing the second derivative easier.

$$\begin{aligned} y' &= u_1 y_1' + u_2 y_2', \\ y'' &= (u_1' y_1' + u_2' y_2') + (u_1 y_1'' + u_2 y_2''). \end{aligned}$$

Since y_1 and y_2 are solutions to $y'' + y = 0$, we find $y_1'' = -y_1$ and $y_2'' = -y_2$. (If the equation was a more general $y'' + p(x)y' + q(x)y = 0$, we would have $y_i'' = -p(x)y_i' - q(x)y_i$.) So

$$y'' = (u_1' y_1' + u_2' y_2') - (u_1 y_1 + u_2 y_2).$$

We have $(u_1 y_1 + u_2 y_2) = y$ and so

$$y'' = (u_1' y_1' + u_2' y_2') - y,$$

and hence

$$y'' + y = Ly = u_1' y_1' + u_2' y_2'.$$

For y to satisfy $Ly = f(x)$, we must have $f(x) = u_1' y_1' + u_2' y_2'$.

What we need to solve are the two equations (conditions) we imposed on u_1 and u_2 :

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f(x). \end{aligned}$$

We solve for u_1' and u_2' in terms of $f(x)$, y_1 and y_2 . We always get these formulas for any $Ly = f(x)$, where $Ly = y'' + p(x)y' + q(x)y$. There is a general formula for the solution we could just plug into, but instead of memorizing that, it is better, and easier, to just repeat what we do below. In our case, the two equations are

$$\begin{aligned} u_1' \cos x + u_2' \sin x &= 0, \\ -u_1' \sin x + u_2' \cos x &= \tan x. \end{aligned}$$

Hence

$$\begin{aligned} u_1' \cos x \sin x + u_2' \sin^2 x &= 0, \\ -u_1' \sin x \cos x + u_2' \cos^2 x &= \tan x \cos x = \sin x. \end{aligned}$$

And thus

$$\begin{aligned} u_2'(\sin^2 x + \cos^2 x) &= \sin x, \\ u_2' &= \sin x, \\ u_1' &= \frac{-\sin^2 x}{\cos x} = \cos x - \sec x. \end{aligned}$$

We integrate u_1' and u_2' to get u_1 and u_2 .

$$\begin{aligned} u_1 &= \int u_1' dx = \int (\cos x - \sec x) dx = \sin x - \ln|\sec x + \tan x|, \\ u_2 &= \int u_2' dx = \int \sin x dx = -\cos x. \end{aligned}$$

We are looking for a particular solution, so we forget about the constants of integration. So our particular solution is

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 = \cos x \sin x - \cos x \ln|\sec x + \tan x| - \cos x \sin x \\ &= -\cos x \ln|\sec x + \tan x|. \end{aligned}$$

The general solution to $y'' + y = \tan x$ is, therefore,

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|.$$

7.5.4 Exercises

Exercise 7.5.2: Find a particular solution of $y'' - y' - 6y = e^{2x}$.

Exercise 7.5.3: Find a particular solution of $y'' - 4y' + 4y = e^{2x}$.

Exercise 7.5.4: Solve the initial value problem $y'' + 9y = \cos(3x) + \sin(3x)$ for $y(0) = 2$, $y'(0) = 1$.

Exercise 7.5.5: Set up the form of the particular solution but do not solve for the coefficients for $y^{(4)} - 2y''' + y'' = e^x$.

Exercise 7.5.6: Set up the form of the particular solution but do not solve for the coefficients for $y^{(4)} - 2y''' + y'' = e^x + x + \sin x$.

Exercise 7.5.7:

- Using variation of parameters find a particular solution of $y'' - 2y' + y = e^x$.
- Find a particular solution using undetermined coefficients.
- Are the two solutions you found the same? See also [Exercise 7.5.10](#).

Exercise 7.5.8: Find a particular solution of $y'' - 2y' + y = \sin(x^2)$. It is OK to leave the answer as a definite integral.

Exercise 7.5.9: For an arbitrary constant c find a particular solution to $y'' - y = e^{cx}$. Hint: Make sure to handle every possible real c .

Exercise 7.5.10:

- a) Using variation of parameters find a particular solution of $y'' - y = e^x$.
- b) Find a particular solution using undetermined coefficients.
- c) Are the two solutions you found the same? What is going on?

Exercise 7.5.11: Find a polynomial $P(x)$, so that $y = 2x^2 + 3x + 4$ solves $y'' + 5y' + y = P(x)$.

Exercise 7.5.101: Find a particular solution to $y'' - y' + y = 2 \sin(3x)$.

Exercise 7.5.102:

- a) Find a particular solution to $y'' + 2y = e^x + x^3$.
- b) Find the general solution.

Exercise 7.5.103: Solve $y'' + 2y' + y = x^2$, $y(0) = 1$, $y'(0) = 2$.

Exercise 7.5.104: Use variation of parameters to find a particular solution of $y'' - y = \frac{1}{e^x + e^{-x}}$.

Exercise 7.5.105: For an arbitrary constant c find the general solution to $y'' - 2y = \sin(x + c)$.

Exercise 7.5.106: Undetermined coefficients can sometimes be used to guess a particular solution to other equations than constant coefficients. Find a polynomial $y(x)$ that solves $y' + xy = x^3 + 2x^2 + 5x + 2$.

Note: Not every right hand side will allow a polynomial solution, for example, $y' + xy = 1$ does not, but a technique based on undetermined coefficients does work, see [chapter 10](#).

7.6 Forced Oscillations and Resonance

Let us return back to the example of a mass on a spring. We examine the case of forced oscillations, which we did not yet handle. That is, we consider the equation

$$mx'' + cx' + kx = F(t),$$

for some nonzero $F(t)$. The setup is again: m is mass, c is friction, k is the spring constant, and $F(t)$ is an external force acting on the mass.

We are interested in periodic forcing, such as noncentered rotating parts, or perhaps loud sounds, or other sources of periodic force. Once we learn about Fourier series in chapter ??, we will see that we cover all periodic functions by simply considering $F(t) = F_0 \cos(\omega t)$ (or sine instead of cosine, the calculations are essentially the same).

7.6.1 Undamped Forced Motion and Resonance

First, let us consider undamped ($c = 0$) motion. We have the equation

$$mx'' + kx = F_0 \cos(\omega t).$$

This equation has the complementary solution (solution to the associated homogeneous equation)

$$x_c = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t),$$

where $\omega_0 = \sqrt{k/m}$ is the *natural frequency* (angular). It is the frequency at which the system “wants to oscillate” without external interference.

Suppose that $\omega_0 \neq \omega$. We solve using the method of undetermined coefficients. We try the solution $x_p = A \cos(\omega t)$ and solve for A . We do not need a sine in our trial solution as after plugging in we only have cosines. If you include a sine, it is fine; you will find that its coefficient is zero (I could not find a second rhyme). We plug into the equation and solving for A to find

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

We leave it as an exercise to do the algebra required.

The general solution is

$$x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Written another way

$$x = C \cos(\omega_0 t - \gamma) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

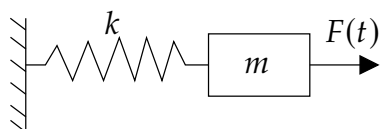
The solution is a superposition of two cosine waves at different frequencies.

Example 7.6.1: Take

$$0.5x'' + 8x = 10 \cos(\pi t), \quad x(0) = 0, \quad x'(0) = 0.$$

Let us compute. First we read off the parameters: $\omega = \pi$, $\omega_0 = \sqrt{8/0.5} = 4$, $F_0 = 10$, $m = 0.5$. The general solution is

$$x = C_1 \cos(4t) + C_2 \sin(4t) + \frac{20}{16 - \pi^2} \cos(\pi t).$$



damping c

Solve for C_1 and C_2 using the initial conditions: $C_1 = \frac{-20}{16-\pi^2}$ and $C_2 = 0$. Hence

$$x = \frac{20}{16-\pi^2} (\cos(\pi t) - \cos(4t)).$$

Do notice the “beating” behavior in Figure 7.5. To see it, use the trigonometric identity

$$2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right) = \cos B - \cos A$$

to get

$$x = \frac{20}{16-\pi^2} \left(2 \sin\left(\frac{4-\pi}{2}t\right) \sin\left(\frac{4+\pi}{2}t\right) \right).$$

The function x is a high frequency wave modulated by a low frequency wave.

Now suppose $\omega_0 = \omega$. We notice that $\cos(\omega t)$ solves the associated homogeneous equation. Hence, we cannot try the solution $A \cos(\omega t)$ with the method of undetermined coefficients. Therefore, we try $x_p = At \cos(\omega t) + Bt \sin(\omega t)$. This time we do need the sine term, since the second derivative of $t \cos(\omega t)$ contains sines. We write the equation

$$x'' + \omega^2 x = \frac{F_0}{m} \cos(\omega t).$$

Plugging x_p into the left-hand side we get

$$2B\omega \cos(\omega t) - 2A\omega \sin(\omega t) = \frac{F_0}{m} \cos(\omega t).$$

Hence $A = 0$ and $B = \frac{F_0}{2m\omega}$. Our particular solution is $\frac{F_0}{2m\omega} t \sin(\omega t)$ and our general solution is

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{F_0}{2m\omega} t \sin(\omega t).$$

The important term is the last one (the particular solution we found). This term grows without bound as $t \rightarrow \infty$. In fact it oscillates between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$. The first two terms only oscillate between $\pm \sqrt{C_1^2 + C_2^2}$, which becomes smaller and smaller in proportion to the oscillations of the last term as t gets larger. In Figure 7.6 on page 332, we see the graph with $C_1 = C_2 = 0$, $F_0 = 2$, $m = 1$, $\omega = \pi$.

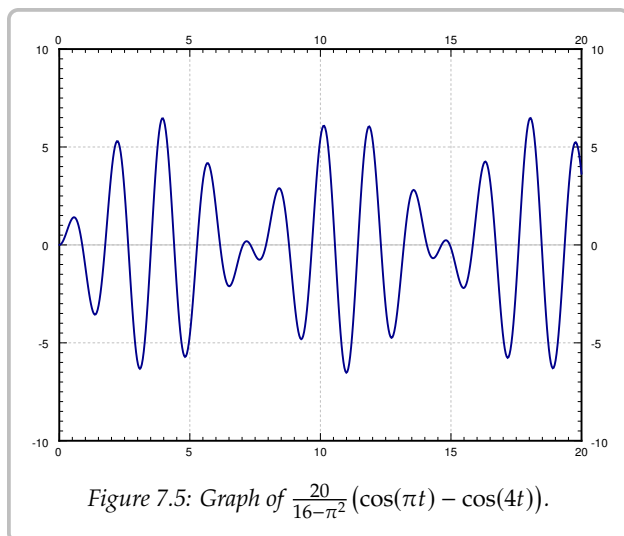


Figure 7.5: Graph of $\frac{20}{16-\pi^2} (\cos(\pi t) - \cos(4t))$.

By forcing the system in just the right frequency we produce very wild oscillations. This kind of behavior is called *resonance* or perhaps *pure resonance*. Sometimes resonance is desired. For example, remember when as a kid you could start swinging by just moving back and forth on the swing seat in the “correct frequency”? You were trying to achieve resonance. The force of each one of your moves was small, but after a while it produced large swings.

On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is due to different buildings having different resonance frequencies. So figuring out the resonance frequency can be very important.

A common (but wrong) example of destructive force of resonance is the Tacoma Narrows bridge failure. It turns out there was a different phenomenon at play*.

7.6.2 Damped Forced Motion and Practical Resonance

In real life things are not as simple as they were above. There is, of course, some damping. Our equation becomes

$$mx'' + cx' + kx = F_0 \cos(\omega t), \quad (7.8)$$

for some $c > 0$. We solved the homogeneous problem before. We let

$$p = \frac{c}{2m}, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

We replace equation (7.8) with

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t).$$

The roots of the characteristic equation of the associated homogeneous problem are $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$. The form of the general solution of the associated homogeneous equation depends on the sign of $p^2 - \omega_0^2$, or equivalently on the sign of $c^2 - 4km$, as before:

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & \text{if } c^2 > 4km, \\ C_1 e^{-pt} + C_2 t e^{-pt} & \text{if } c^2 = 4km, \\ e^{-pt} (C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)) & \text{if } c^2 < 4km, \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. In any case, we see that $x_c(t) \rightarrow 0$ as $t \rightarrow \infty$.

Let us find a particular solution. There can be no conflicts when trying to solve for the undetermined coefficients by trying $x_p = A \cos(\omega t) + B \sin(\omega t)$. Let us plug in and solve for A and B . We get (the tedious details are left to reader)

$$((\omega_0^2 - \omega^2)B - 2\omega p A) \sin(\omega t) + ((\omega_0^2 - \omega^2)A + 2\omega p B) \cos(\omega t) = \frac{F_0}{m} \cos(\omega t).$$

We solve for A and B :

$$A = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2},$$

$$B = \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}.$$

We also compute $C = \sqrt{A^2 + B^2}$ to be

$$C = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}}.$$

*K. Billah and R. Scanlan, *Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks*, American Journal of Physics, 59(2), 1991, 118–124, <http://www.ketchum.org/billah/Billah-Scanlan.pdf>

Thus our particular solution is

$$x_p = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \sin(\omega t).$$

Or in the alternative notation we have amplitude C and phase shift γ where (if $\omega \neq \omega_0$)

$$\tan \gamma = \frac{B}{A} = \frac{2\omega p}{\omega_0^2 - \omega^2}.$$

Hence,

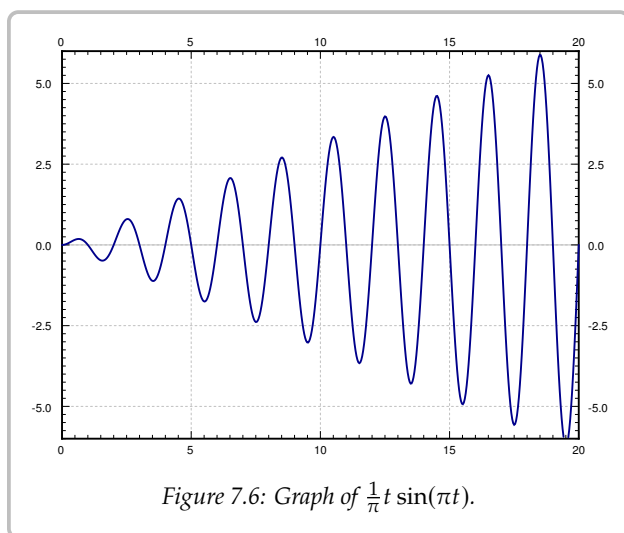
$$x_p = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}} \cos(\omega t - \gamma).$$

If $\omega = \omega_0$, then $A = 0$, $B = C = \frac{F_0}{2m\omega p}$, and $\gamma = \pi/2$.

For reasons we will explain in a moment, we call x_c the *transient solution* and denote it by x_{tr} . We call the x_p from above the *steady periodic solution* and denote it by x_{sp} . The general solution is

$$x = x_c + x_p = x_{tr} + x_{sp}.$$

The transient solution $x_{tr} = x_c$ goes to zero as $t \rightarrow \infty$, as all the terms involve an exponential with a negative exponent. So for large t , the effect of x_{tr} is negligible and we see essentially only x_{sp} . Hence the name *transient*. Notice that x_{sp} involves no arbitrary constants, and the initial conditions only affect x_{tr} . Thus, the effect of the initial conditions is negligible after some period of time. We might as well focus on the steady periodic solution and ignore the transient solution. See Figure 7.7 on page 335 for a graph given several different initial conditions.



The speed at which x_{tr} goes to zero depends on p (and hence c). The bigger p is (the bigger c is), the “faster” x_{tr} becomes negligible. So the smaller the damping, the longer the “transient region.” This is consistent with the observation that when $c = 0$, the initial conditions affect the behavior for all time (i.e. an infinite “transient region”).

Let us describe what we mean by resonance when damping is present. Since there were no conflicts when solving with undetermined coefficient, there is no term that goes to infinity. We look instead at the maximum value of the amplitude of the steady periodic solution. Let C be the amplitude of x_{sp} . If we plot C as a function of ω (with all other parameters fixed), we can find its maximum. We call the ω that achieves

this maximum the *practical resonance frequency*. We call the maximal amplitude $C(\omega)$ the *practical resonance amplitude*. Thus when damping is present we talk of *practical resonance* rather than pure resonance. A sample plot for three different values of c is given in Figure 7.8. As you can see the practical resonance amplitude grows as damping gets smaller, and practical resonance can disappear altogether when damping is large.

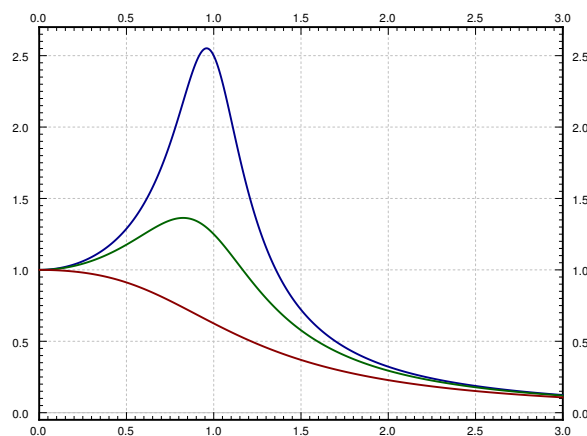


Figure 7.8: Graph of $C(\omega)$ showing practical resonance with parameters $k = 1$, $m = 1$, $F_0 = 1$. The top line is with $c = 0.4$, the middle line with $c = 0.8$, and the bottom line with $c = 1.6$.

To find the maximum we need to find the derivative $C'(\omega)$. Computation shows

$$C'(\omega) = \frac{-2\omega(2p^2 + \omega^2 - \omega_0^2)F_0}{m((2\omega p)^2 + (\omega_0^2 - \omega^2)^2)^{3/2}}.$$

This is zero either when $\omega = 0$ or when $2p^2 + \omega^2 - \omega_0^2 = 0$. In other words, $C'(\omega) = 0$ when

$$\omega = \sqrt{\omega_0^2 - 2p^2} \quad \text{or} \quad \omega = 0.$$

If $\omega_0^2 - 2p^2$ is positive, then $\sqrt{\omega_0^2 - 2p^2}$ is the practical resonance frequency (that is the point where $C(\omega)$ is maximal). This follows by the first derivative test for example as then $C'(\omega) > 0$ for small ω in this case. If on the other hand $\omega_0^2 - 2p^2$ is not positive, then $C(\omega)$ achieves its maximum at $\omega = 0$, and there is no practical resonance since we assume $\omega > 0$ in our system. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance occurs, the frequency is smaller than ω_0 . As the damping c (and hence p) becomes smaller, the practical resonance frequency goes to ω_0 . So when damping is very small, ω_0 is a good estimate of the practical resonance frequency. This behavior agrees with the observation that when $c = 0$, then ω_0 is the resonance frequency.

Another interesting observation to make is that when $\omega \rightarrow \infty$, then $C \rightarrow 0$. This means that if the forcing frequency gets too high it does not manage to get the mass moving in the mass-spring system. This is quite reasonable intuitively. If we wiggle back and forth really fast while sitting on a swing, we will not get it moving at all, no matter how forceful. Fast vibrations just cancel each other out before the mass has any chance of responding by moving one way or the other.

The behavior is more complicated if the forcing function is not an exact cosine wave, but for example a square wave. A general periodic function will be the sum (superposition) of many cosine waves of different frequencies. The reader is encouraged to come back to this section once we have learned about the Fourier series.

7.6.3 Exercises

Exercise 7.6.1: Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \sin(\omega t)$. Assume $c > 0$.

Exercise 7.6.2: Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \cos(\omega t) + F_1 \cos(3\omega t)$. Assume $c > 0$.

Exercise 7.6.3: Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $m > 0$, $k > 0$, and $F_0 > 0$. Consider the function $C(\omega)$. For what values of c (solve in terms of m , k , and F_0) will there be no practical resonance (that is, for what values of c is there no maximum of $C(\omega)$ for $\omega > 0$)?

Exercise 7.6.4: Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $c > 0$, $k > 0$, and $F_0 > 0$. Consider the function $C(\omega)$. For what values of m (solve in terms of c , k , and F_0) will there be no practical resonance (that is, for what values of m is there no maximum of $C(\omega)$ for $\omega > 0$)?

Exercise 7.6.5: A water tower in an earthquake acts as a mass-spring system. Assume that the container on top is full and the water does not move around. The container then acts as the mass and the support acts as the spring, where the induced vibrations are horizontal. The container with water has a mass of $m = 10,000$ kg. It takes a force of 1000 newtons to displace the container 1 meter. For simplicity assume no friction. When the earthquake hits the water tower is at rest (it is not moving). The earthquake induces an external force $F(t) = mA\omega^2 \cos(\omega t)$.

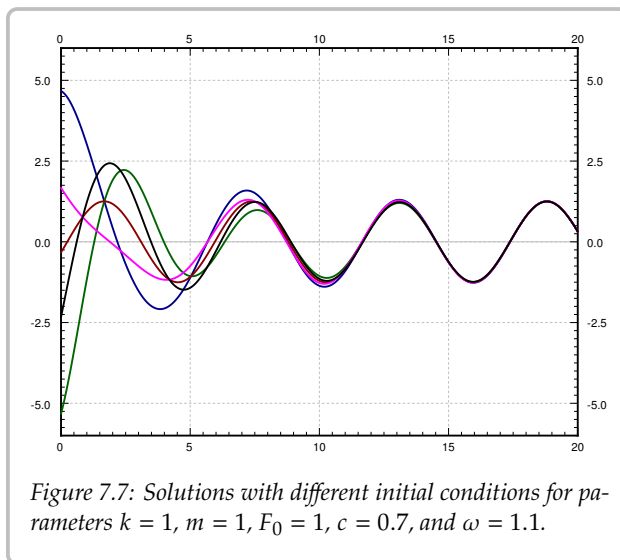
- What is the natural frequency of the water tower?
- If ω is not the natural frequency, find a formula for the maximal amplitude of the resulting oscillations of the water container (the maximal deviation from the rest position). The motion will be a high frequency wave modulated by a low frequency wave, so simply find the constant in front of the sines.
- Suppose $A = 1$ and an earthquake with frequency 0.5 cycles per second comes. What is the amplitude of the oscillations? Suppose that if the water tower moves more than 1.5 meter from the rest position, the tower collapses. Will the tower collapse?

Exercise 7.6.101: A mass of 4 kg on a spring with $k = 4$ N/m and a damping constant $c = 1$ Ns/m. Suppose that $F_0 = 2$ N. Using forcing function $F_0 \cos(\omega t)$, find the ω that causes practical resonance and find the amplitude.

Exercise 7.6.102: Derive a formula for x_{sp} for $mx'' + cx' + kx = F_0 \cos(\omega t) + A$, where A is some constant. Assume $c > 0$.

Exercise 7.6.103: Suppose there is no damping in a mass and spring system with $m = 5$, $k = 20$, and $F_0 = 5$. Suppose ω is chosen to be precisely the resonance frequency.

- Find ω .
- Find the amplitude of the oscillations at time $t = 100$, given the system is at rest at $t = 0$.



Chapter 8

Applications of Linear Second Order Equations

8.1 Spring Problems I

We consider the motion of an object of mass m , suspended from a spring of negligible mass. We say that the spring–mass system is in *equilibrium* when the object is at rest and the forces acting on it sum to zero. The position of the object in this case is the *equilibrium position*. We define y to be the displacement of the object from its equilibrium position (Figure 8.1), measured positive upward.

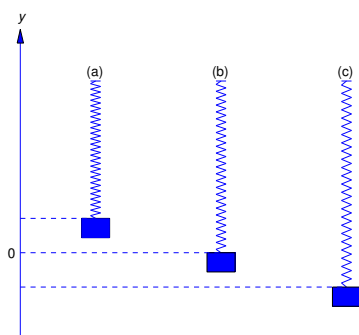


Figure 8.1: (a) $y > 0$ (b) $y = 0$, (c) $y < 0$



Figure 8.2: A spring – mass system with damping

Our model accounts for the following kinds of forces acting on the object:

- The force $-mg$, due to gravity.
- A force F_s exerted by the spring resisting change in its length. The *natural length* of the spring is its length with no mass attached. We assume that the spring obeys *Hooke's law*: If the length of the spring is changed by an amount ΔL from its natural length, then the spring exerts a force $F_s = k\Delta L$, where k is a positive number called the *spring constant*. If the spring is stretched then $\Delta L > 0$ and $F_s > 0$, so the

spring force is upward, while if the spring is compressed then $\Delta L < 0$ and $F_s < 0$, so the spring force is downward.

- A **damping force** $F_d = -cy'$ that resists the motion with a force proportional to the velocity of the object. It may be due to air resistance or friction in the spring. However, a convenient way to visualize a damping force is to assume that the object is rigidly attached to a piston with negligible mass immersed in a cylinder (called a **dashpot**) filled with a viscous liquid (Figure 8.2). As the piston moves, the liquid exerts a damping force. We say that the motion is **undamped** if $c = 0$, or **damped** if $c > 0$.
- An external force F , other than the force due to gravity, that may vary with t , but is independent of displacement and velocity. We say that the motion is **free** if $F \equiv 0$, or **forced** if $F \neq 0$.

From Newton's second law of motion,

$$my'' = -mg + F_d + F_s + F = -mg - cy' + F_s + F. \quad (8.1)$$

We must now relate F_s to y . In the absence of external forces the object stretches the spring by an amount Δl to assume its equilibrium position (Figure 8.3). Since the sum of the forces acting on the object is then zero, Hooke's Law implies that $mg = k\Delta l$. If the object is displaced y units from its equilibrium position, the total change in the length of the spring is $\Delta L = \Delta l - y$, so Hooke's law implies that

$$F_s = k\Delta L = k\Delta l - ky.$$

Substituting this into (8.1) yields

$$my'' = -mg - cy' + k\Delta l - ky + F.$$

Since $mg = k\Delta l$ this can be written as

$$my'' + cy' + ky = F. \quad (8.2)$$

We call this **the equation of motion**.

8.1.1 Simple Harmonic Motion

Throughout the rest of this section we'll consider spring-mass systems without damping; that is, $c = 0$. We'll consider systems with damping in the next section.

We first consider the case where the motion is also free; that is, $F=0$. We begin with an example.

Example 8.1.1: An object stretches a spring 6 inches in equilibrium.

- Set up the equation of motion and find its general solution.
- Find the displacement of the object for $t > 0$ if it's initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.

Solution (a): Setting $c = 0$ and $F = 0$ in (8.2) yields the equation of motion

$$my'' + ky = 0,$$

which we rewrite as

$$y'' + \frac{k}{m}y = 0. \quad (8.3)$$

Although we would need the weight of the object to obtain k from the equation $mg = k\Delta l$ we can obtain k/m from Δl alone; thus, $k/m = g/\Delta l$. Consistent with the units used in the problem statement, we take $g = 32 \text{ ft/s}^2$. Although Δl is stated in inches, we must convert it to feet to be consistent with this choice of g ; that is, $\Delta l = 1/2 \text{ ft}$. Therefore

$$\frac{k}{m} = \frac{32}{1/2} = 64$$

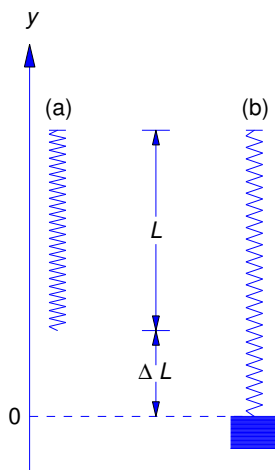


Figure 8.3: (a) Natural length of spring (b) Spring stretched by mass

and (8.3) becomes

$$y'' + 64y = 0. \quad (8.4)$$

The characteristic equation of (8.4) is

$$r^2 + 64 = 0,$$

which has the zeros $r = \pm 8i$. Therefore the general solution of (8.4) is

$$y = c_1 \cos 8t + c_2 \sin 8t. \quad (8.5)$$

Solution (b): The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus,

$$y(0) = \frac{3}{2} \quad \text{and} \quad y'(0) = -3.$$

Differentiating (8.5) yields

$$y' = -8c_1 \sin 8t + 8c_2 \cos 8t. \quad (8.6)$$

Setting $t = 0$ in (8.5) and (8.6) and imposing the initial conditions shows that $c_1 = 3/2$ and $c_2 = -3/8$. Therefore

$$y = \frac{3}{2} \cos 8t - \frac{3}{8} \sin 8t,$$

where y is in feet (Figure 8.4).

We'll now consider the equation

$$my'' + ky = 0$$

where m and k are arbitrary positive numbers. Dividing through by m and defining $\omega_0 = \sqrt{k/m}$ yields

$$y'' + \omega_0^2 y = 0.$$

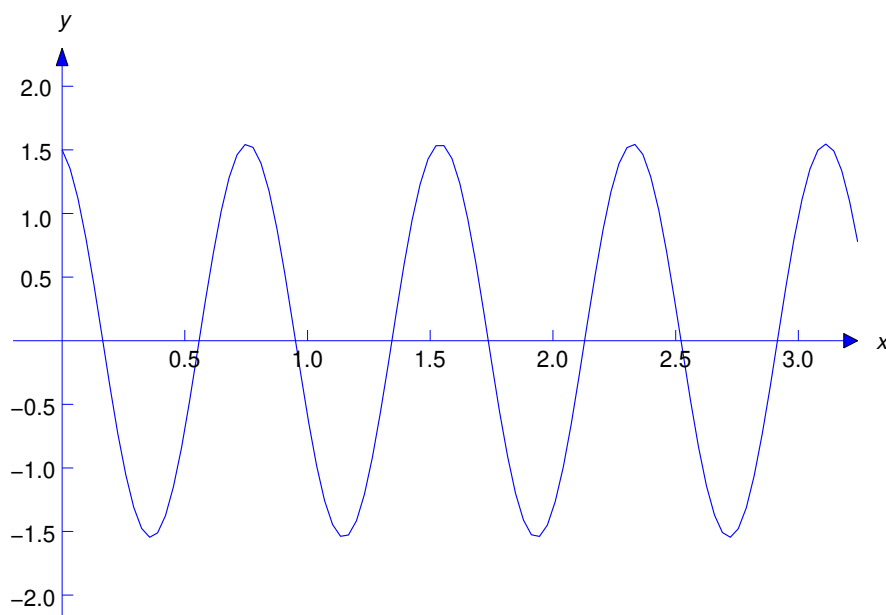


Figure 8.4: $y = \frac{3}{2} \cos 8t - \frac{3}{8} \sin 8t$

The general solution of this equation is

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t. \quad (8.7)$$

We can rewrite this in a more useful form by defining

$$R = \sqrt{c_1^2 + c_2^2}, \quad (8.8)$$

and

$$c_1 = R \cos \phi \quad \text{and} \quad c_2 = R \sin \phi. \quad (8.9)$$

Substituting from (8.9) into (8.7) and applying the identity

$$\cos \omega_0 t \cos \phi + \sin \omega_0 t \sin \phi = \cos(\omega_0 t - \phi)$$

yields

$$y = R \cos(\omega_0 t - \phi). \quad (8.10)$$

From (8.8) and (8.9) we see that the R and ϕ can be interpreted as polar coordinates of the point with rectangular coordinates (c_1, c_2) (Figure 8.5). Given c_1 and c_2 , we can compute R from (8.8). From (8.8) and (8.9), we see that ϕ is related to c_1 and c_2 by

$$\cos \phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \sin \phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.$$

There are infinitely many angles ϕ , differing by integer multiples of 2π , that satisfy these equations. We will always choose ϕ so that $-\pi \leq \phi < \pi$.

The motion described by (8.7) or (8.10) is *simple harmonic motion*. We see from either of these equations that the motion is periodic, with period

$$T = 2\pi/\omega_0.$$

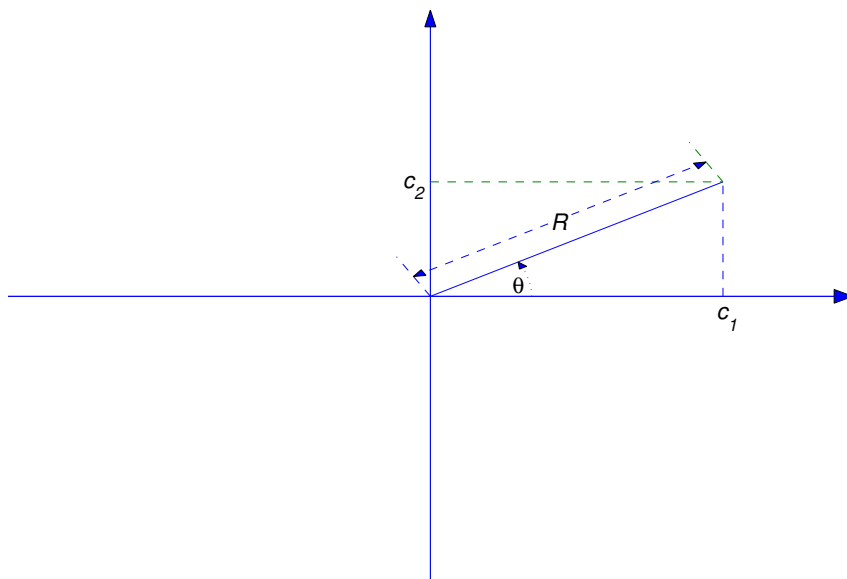


Figure 8.5: $R = \sqrt{c_1^2 + c_2^2}$; $c_1 = R \cos \phi$; $c_2 = R \sin \phi$

This is the time required for the object to complete one full cycle of oscillation (for example, to move from its highest position to its lowest position and back to its highest position). Since the highest and lowest positions of the object are $y = R$ and $y = -R$, we say that R is the *amplitude* of the oscillation. The angle ϕ in (8.10) is the *phase angle*. It's measured in radians. Equation (8.10) is the *amplitude–phase form* of the displacement. If t is in seconds then ω_0 is in radians per second (rad/s); it's the *frequency* of the motion. It is also called the *natural frequency* of the spring–mass system without damping.

Example 8.1.2: We found the displacement of the object in Example 8.1.1 to be

$$y = \frac{3}{2} \cos 8t - \frac{3}{8} \sin 8t.$$

Find the frequency, period, amplitude, and phase angle of the motion.

Solution: The frequency is $\omega_0 = 8$ rad/s, and the period is $T = 2\pi/\omega_0 = \pi/4$ s. Since $c_1 = 3/2$ and $c_2 = -3/8$, the amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{8}\right)^2} = \frac{3}{8} \sqrt{17}.$$

The phase angle is determined by

$$\cos \phi = \frac{\frac{3}{2}}{\frac{3}{8} \sqrt{17}} = \frac{4}{\sqrt{17}} \quad (8.11)$$

and

$$\sin \phi = \frac{-\frac{3}{8}}{\frac{3}{8} \sqrt{17}} = -\frac{1}{\sqrt{17}}. \quad (8.12)$$

Using a calculator, we see from (8.11) that

$$\phi \approx \pm .245 \text{ rad.}$$

Since $\sin \phi < 0$ (see (8.12)), the minus sign applies here; that is,

$$\phi \approx -.245 \text{ rad.}$$

Example 8.1.3: The natural length of a spring is 1 m. An object is attached to it and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find the displacement for $t > 0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion. Express the answers in cgs units.

Solution: In cgs units $g = 980 \text{ cm/s}^2$. Since $\Delta l = 2 \text{ cm}$, $\omega_0^2 = g/\Delta l = 490$. Therefore

$$y'' + 490y = 0, \quad y(0) = -1, \quad y'(0) = 14.$$

The general solution of the differential equation is

$$y = c_1 \cos 7\sqrt{10}t + c_2 \sin 7\sqrt{10}t,$$

so

$$y' = 7\sqrt{10} \left(-c_1 \sin 7\sqrt{10}t + c_2 \cos 7\sqrt{10}t \right).$$

Substituting the initial conditions into the last two equations yields $c_1 = -1$ and $c_2 = 2/\sqrt{10}$. Hence,

$$y = -\cos 7\sqrt{10}t + \frac{2}{\sqrt{10}} \sin 7\sqrt{10}t.$$

The frequency is $7\sqrt{10} \text{ rad/s}$, and the period is $T = 2\pi/(7\sqrt{10}) \text{ s}$. The amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + \left(\frac{2}{\sqrt{10}}\right)^2} = \sqrt{\frac{7}{5}} \text{ cm.}$$

The phase angle is determined by

$$\cos \phi = \frac{c_1}{R} = -\sqrt{\frac{5}{7}} \quad \text{and} \quad \sin \phi = \frac{c_2}{R} = \sqrt{\frac{2}{7}}.$$

Therefore ϕ is in the second quadrant and

$$\phi = \cos^{-1} \left(-\sqrt{\frac{5}{7}} \right) \approx 2.58 \text{ rad.}$$

8.1.2 Undamped Forced Oscillation

In many mechanical problems a device is subjected to periodic external forces. For example, soldiers marching in cadence on a bridge cause periodic disturbances in the bridge, and the engines of a propeller driven aircraft cause periodic disturbances in its wings. In the absence of sufficient damping forces, such disturbances – even if small in magnitude – can cause structural breakdown if they are at certain critical frequencies. To illustrate, this we'll consider the motion of an object in a spring-mass system without damping, subject to an external force

$$F(t) = F_0 \cos \omega t$$

where F_0 is a constant. In this case the equation of motion (8.2) is

$$my'' + ky = F_0 \cos \omega t,$$

which we rewrite as

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t \quad (8.13)$$

with $\omega_0 = \sqrt{k/m}$. We'll see from the next two examples that the solutions of (8.13) with $\omega \neq \omega_0$ behave very differently from the solutions with $\omega = \omega_0$.

Example 8.1.4: Solve the initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t, \quad y(0) = 0, \quad y'(0) = 0, \quad (8.14)$$

given that $\omega \neq \omega_0$.

Solution: We first obtain a particular solution of (8.13) by the method of undetermined coefficients. Since $\omega \neq \omega_0$, $\cos \omega t$ isn't a solution of the complementary equation

$$y'' + \omega_0^2 y = 0.$$

Therefore (8.13) has a particular solution of the form

$$y_p = A \cos \omega t + B \sin \omega t.$$

Since

$$y_p'' = -\omega^2(A \cos \omega t + B \sin \omega t),$$

$$y_p'' + \omega_0^2 y_p = \frac{F_0}{m} \cos \omega t$$

if and only if

$$(\omega_0^2 - \omega^2)(A \cos \omega t + B \sin \omega t) = \frac{F_0}{m} \cos \omega t.$$

This holds if and only if

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad B = 0,$$

so

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

The general solution of (8.13) is

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \quad (8.15)$$

so

$$y' = \frac{-\omega F_0}{m(\omega_0^2 - \omega^2)} \sin \omega t + \omega_0(-c_1 \sin \omega_0 t + c_2 \cos \omega_0 t).$$

The initial conditions $y(0) = 0$ and $y'(0) = 0$ in (8.14) imply that

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad c_2 = 0.$$

Substituting these into (8.15) yields

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t). \quad (8.16)$$

It is revealing to write this in a different form. We start with the trigonometric identities

$$\begin{aligned}\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta.\end{aligned}$$

Subtracting the second identity from the first yields

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta \quad (8.17)$$

Now let

$$\alpha - \beta = \omega t \quad \text{and} \quad \alpha + \beta = \omega_0 t, \quad (8.18)$$

so that

$$\alpha = \frac{(\omega_0 + \omega)t}{2} \quad \text{and} \quad \beta = \frac{(\omega_0 - \omega)t}{2}. \quad (8.19)$$

Substituting (8.18) and (8.19) into (8.17) yields

$$\cos \omega t - \cos \omega_0 t = 2 \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2},$$

and substituting this into (8.16) yields

$$y = R(t) \sin \frac{(\omega_0 + \omega)t}{2}, \quad (8.20)$$

where

$$R(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}. \quad (8.21)$$

From (8.20) we can regard y as a sinusoidal variation with frequency $(\omega_0 + \omega)/2$ and variable amplitude $|R(t)|$. In Figure 8.6 the dashed curve above the t axis is $y = |R(t)|$, the dashed curve below the t axis is $y = -|R(t)|$, and the displacement y appears as an oscillation bounded by them. The oscillation of y for t on an interval between successive zeros of $R(t)$ is called a *beat*.

You can see from (8.20) and (8.21) that

$$|y(t)| \leq \frac{2|F_0|}{m|\omega_0^2 - \omega^2|};$$

moreover, if $\omega + \omega_0$ is sufficiently large compared with $\omega - \omega_0$, then $|y|$ assumes values close to (perhaps equal to) this upper bound during each beat. However, the oscillation remains bounded for all t . (This assumes that the spring can withstand deflections of this size and continue to obey Hooke's law.) The next example shows that this isn't so if $\omega = \omega_0$.

Example 8.1.5: Find the general solution of

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t. \quad (8.22)$$

Solution: We first obtain a particular solution y_p of (8.22). Since $\cos \omega_0 t$ is a solution of the complementary equation, the form for y_p is

$$y_p = t(A \cos \omega_0 t + B \sin \omega_0 t). \quad (8.23)$$

Then

$$y'_p = A \cos \omega_0 t + B \sin \omega_0 t + \omega_0 t(-A \sin \omega_0 t + B \cos \omega_0 t)$$

and

$$y''_p = 2\omega_0(-A \sin \omega_0 t + B \cos \omega_0 t) - \omega_0^2 t(A \cos \omega_0 t + B \sin \omega_0 t). \quad (8.24)$$

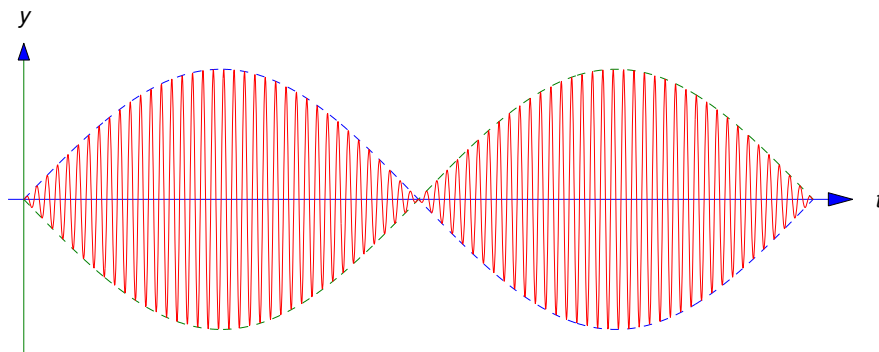


Figure 8.6: Undamped oscillation with beats

From (8.23) and (8.24), we see that y_p satisfies (8.22) if

$$-2A\omega_0 \sin \omega_0 t + 2B\omega_0 \cos \omega_0 t = \frac{F_0}{m} \cos \omega_0 t;$$

that is, if

$$A = 0 \quad \text{and} \quad B = \frac{F_0}{2m\omega_0}.$$

Therefore

$$y_p = \frac{F_0 t}{2m\omega_0} \sin \omega_0 t$$

is a particular solution of (8.22). The general solution of (8.22) is

$$y = \frac{F_0 t}{2m\omega_0} \sin \omega_0 t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

The graph of y_p is shown in Figure 8.7, where it can be seen that y_p oscillates between the dashed lines

$$y = \frac{F_0 t}{2m\omega_0} \quad \text{and} \quad y = -\frac{F_0 t}{2m\omega_0}$$

with increasing amplitude that approaches ∞ as $t \rightarrow \infty$. Of course, this means that the spring must eventually fail to obey Hooke's law or break. ■

This phenomenon of unbounded displacements of a spring-mass system in response to a periodic forcing function at its natural frequency is called *resonance*. More complicated mechanical structures can also exhibit resonance-like phenomena. For example, rhythmic oscillations of a suspension bridge by wind forces or of an airplane wing by periodic vibrations of reciprocating engines can cause damage or even failure if the frequencies of the disturbances are close to critical frequencies determined by the parameters of the mechanical system in question.

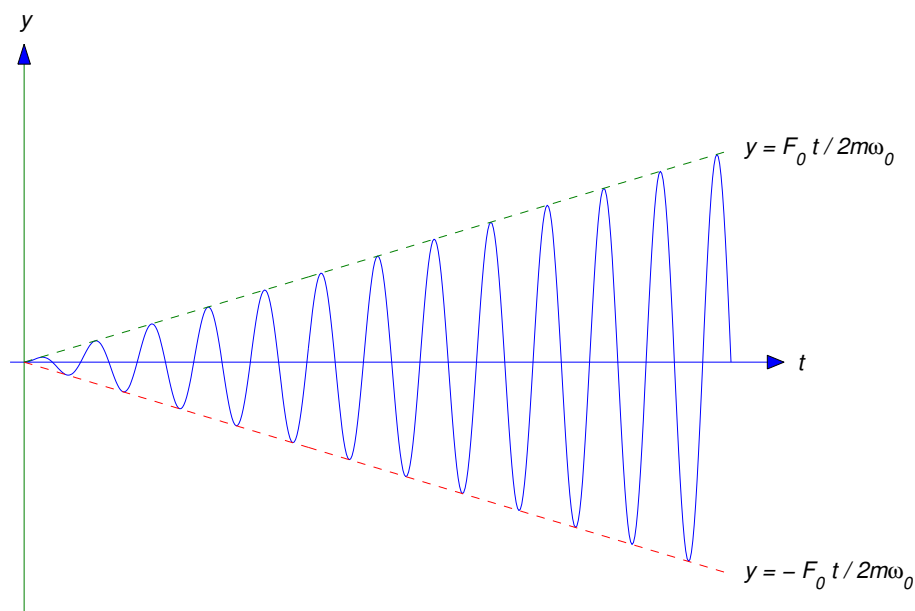


Figure 8.7: Unbounded displacement due to resonance

8.1.3 Exercises

In the following exercises assume that there's no damping.

1. C/G An object stretches a spring 4 inches in equilibrium. Find and graph its displacement for $t > 0$ if it's initially displaced 36 inches above equilibrium and given a downward velocity of 2 ft/s.
2. An object stretches a string 1.2 inches in equilibrium. Find its displacement for $t > 0$ if it's initially displaced 3 inches below equilibrium and given a downward velocity of 2 ft/s.
3. A spring with natural length .5 m has length 50.5 cm with a mass of 2 gm suspended from it. The mass is initially displaced 1.5 cm below equilibrium and released with zero velocity. Find its displacement for $t > 0$.
4. An object stretches a spring 6 inches in equilibrium. Find its displacement for $t > 0$ if it's initially displaced 3 inches above equilibrium and given a downward velocity of 6 inches/s. Find the frequency, period, amplitude and phase angle of the motion.
5. C/G An object stretches a spring 5 cm in equilibrium. It is initially displaced 10 cm above equilibrium and given an upward velocity of .25 m/s. Find and graph its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.
6. A 10 kg mass stretches a spring 70 cm in equilibrium. Suppose a 2 kg mass is attached to the spring, initially displaced 25 cm below equilibrium, and given an upward velocity of 2 m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.
7. A weight stretches a spring 1.5 inches in equilibrium. The weight is initially displaced 8 inches above equilibrium and given a downward velocity of 4 ft/s. Find its displacement for $t > 0$.
8. A weight stretches a spring 6 inches in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/s. Find its displacement for $t > 0$.
9. A spring-mass system has natural frequency $7\sqrt{10}$ rad/s. The natural length of the spring is .7 m. What is the length of the spring when the mass is in equilibrium?

10. A 64 lb weight is attached to a spring with constant $k = 8$ lb/ft and subjected to an external force $F(t) = 2 \sin t$. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 2 ft/s. Find its displacement for $t > 0$.
11. A unit mass hangs in equilibrium from a spring with constant $k = 1/16$. Starting at $t = 0$, a force $F(t) = 3 \sin t$ is applied to the mass. Find its displacement for $t > 0$.
12. C/G A 4 lb weight stretches a spring 1 ft in equilibrium. An external force $F(t) = .25 \sin 8t$ lb is applied to the weight, which is initially displaced 4 inches above equilibrium and given a downward velocity of 1 ft/s. Find and graph its displacement for $t > 0$.
13. A 2 lb weight stretches a spring 6 inches in equilibrium. An external force $F(t) = \sin 8t$ lb is applied to the weight, which is released from rest 2 inches below equilibrium. Find its displacement for $t > 0$.
14. A 10 gm mass suspended on a spring moves in simple harmonic motion with period 4 s. Find the period of the simple harmonic motion of a 20 gm mass suspended from the same spring.
15. A 6 lb weight stretches a spring 6 inches in equilibrium. Suppose an external force $F(t) = \frac{3}{16} \sin \omega t + \frac{3}{8} \cos \omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.
16. C/G A 6 lb weight stretches a spring 4 inches in equilibrium. Suppose an external force $F(t) = 4 \sin \omega t - 6 \cos \omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find and graph the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.
17. A mass of one kg is attached to a spring with constant $k = 4$ N/m. An external force $F(t) = -\cos \omega t - 2 \sin \omega t$ n is applied to the mass. Find the displacement y for $t > 0$ if ω equals the natural frequency of the spring-mass system. Assume that the mass is initially displaced 3 m above equilibrium and given an upward velocity of 450 cm/s.
18. An object is in simple harmonic motion with frequency ω_0 , with $y(0) = y_0$ and $y'(0) = v_0$. Find its displacement for $t > 0$. Also, find the amplitude of the oscillation and give formulas for the sine and cosine of the initial phase angle.
19. Two objects suspended from identical springs are set into motion. The period of one object is twice the period of the other. How are the weights of the two objects related?
20. Two objects suspended from identical springs are set into motion. The weight of one object is twice the weight of the other. How are the periods of the resulting motions related?
21. Two identical objects suspended from different springs are set into motion. The period of one motion is 3 times the period of the other. How are the two spring constants related?

8.2 Spring Problems II

8.2.1 Free Vibrations With Damping

In this section we consider the motion of an object in a spring–mass system with damping. We start with unforced motion, so the equation of motion is

$$my'' + cy' + ky = 0. \quad (8.25)$$

Now suppose the object is displaced from equilibrium and given an initial velocity. Intuition suggests that if the damping force is sufficiently weak the resulting motion will be oscillatory, as in the undamped case considered in the previous section, while if it's sufficiently strong the object may just move slowly toward the equilibrium position without ever reaching it. We'll now confirm these intuitive ideas mathematically. The characteristic equation of (8.25) is

$$mr^2 + cr + k = 0.$$

The roots of this equation are

$$r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}. \quad (8.26)$$

In Section 5.2 we saw that the form of the solution of (8.25) depends upon whether $c^2 - 4mk$ is positive, negative, or zero. We'll now consider these three cases.

8.2.2 Underdamped Motion

We say the motion is *underdamped* if $c < \sqrt{4mk}$. In this case r_1 and r_2 in (8.26) are complex conjugates, which we write as

$$r_1 = -\frac{c}{2m} - i\omega_1 \quad \text{and} \quad r_2 = -\frac{c}{2m} + i\omega_1,$$

where

$$\omega_1 = \frac{\sqrt{4mk - c^2}}{2m}.$$

The general solution of (8.25) in this case is

$$y = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

By the method used in Section 6.1 to derive the amplitude–phase form of the displacement of an object in simple harmonic motion, we can rewrite this equation as

$$y = Re^{-ct/2m} \cos(\omega_1 t - \phi), \quad (8.27)$$

where

$$R = \sqrt{c_1^2 + c_2^2}, \quad R \cos \phi = c_1, \quad \text{and} \quad R \sin \phi = c_2.$$

The factor $Re^{-ct/2m}$ in (8.27) is called the *time-varying amplitude* of the motion, the quantity ω_1 is called the *frequency*, and $T = 2\pi/\omega_1$ (which is the period of the cosine function in (8.27) is called the *quasi-period*. A typical graph of (8.27) is shown in Figure 8.8. As illustrated in that figure, the graph of y oscillates between the dashed exponential curves $y = \pm Re^{-ct/2m}$.

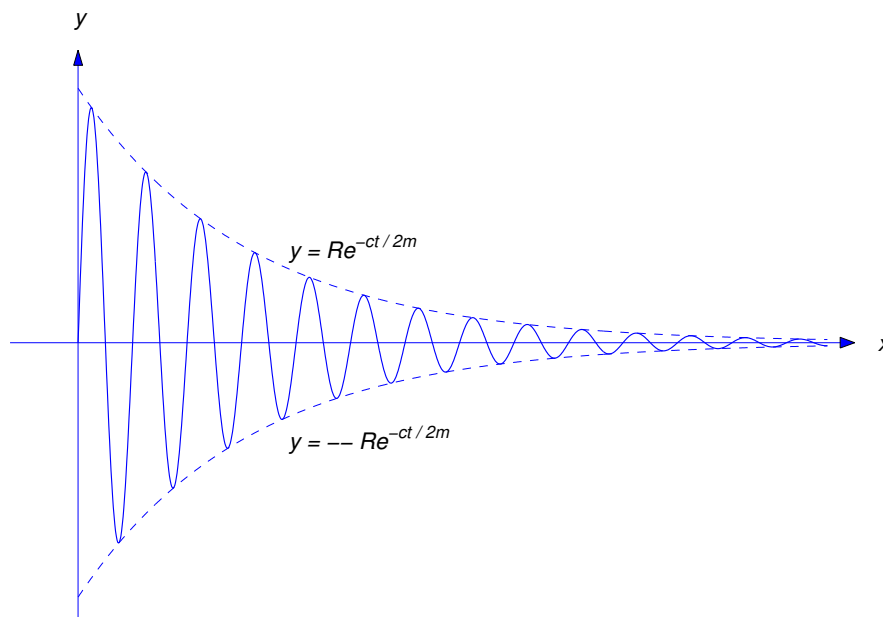


Figure 8.8: Underdamped motion

8.2.3 Overdamped Motion

We say the motion is *overdamped* if $c > \sqrt{4mk}$. In this case the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1 < r_2 < 0$ (see (8.26)), and the general solution of (8.25) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Again $\lim_{t \rightarrow \infty} y(t) = 0$ as in the underdamped case, but the motion isn't oscillatory, since y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$. (Exercise 8.2.23.)

8.2.4 Critically Damped Motion

We say the motion is *critically damped* if $c = \sqrt{4mk}$. In this case $r_1 = r_2 = -c/2m$ and the general solution of (8.25) is

$$y = e^{-ct/2m}(c_1 + c_2 t).$$

Again $\lim_{t \rightarrow \infty} y(t) = 0$ and the motion is nonoscillatory, since y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$. (Exercise 8.2.22).

Example 8.2.1: Suppose a 64 lb weight stretches a spring 6 inches in equilibrium and a dashpot provides a damping force of c lb for each ft/sec of velocity.

- Write the equation of motion of the object and determine the value of c for which the motion is critically damped.
- Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = 20$.
- Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = -20$.

Solution (a): Here $m = 2$ slugs and $k = 64/.5 = 128$ lb/ft. Therefore the equation of motion (8.25) is

$$2y'' + cy' + 128y = 0. \quad (8.28)$$

The characteristic equation is

$$2r^2 + cr + 128 = 0,$$

which has roots

$$r = \frac{-c \pm \sqrt{c^2 - 8 \cdot 128}}{4}.$$

Therefore the damping is critical if

$$c = \sqrt{8 \cdot 128} = 32 \text{ lb-sec/ft}.$$

Solution (b): Setting $c = 32$ in (8.28) and cancelling the common factor 2 yields

$$y'' + 16y + 64y = 0.$$

The characteristic equation is

$$r^2 + 16r + 64 = (r + 8)^2 = 0.$$

Hence, the general solution is

$$y = e^{-8t}(c_1 + c_2 t). \quad (8.29)$$

Differentiating this yields

$$y' = -8y + c_2 e^{-8t}. \quad (8.30)$$

Imposing the initial conditions $y(0) = 1$ and $y'(0) = 20$ in the last two equations shows that $1 = c_1$ and $20 = -8 + c_2$. Hence, the solution of the initial value problem is

$$y = e^{-8t}(1 + 28t).$$

Therefore the object approaches equilibrium from above as $t \rightarrow \infty$. There's no oscillation.

Solution (c): Imposing the initial conditions $y(0) = 1$ and $y'(0) = -20$ in (8.29) and (8.30) yields $1 = c_1$ and $-20 = -8 + c_2$. Hence, the solution of this initial value problem is

$$y = e^{-8t}(1 - 12t).$$

Therefore the object moves downward through equilibrium just once, and then approaches equilibrium from below as $t \rightarrow \infty$. Again, there's no oscillation. The solutions of these two initial value problems are graphed in Figure 8.9.

Example 8.2.2: Find the displacement of the object in Example 8.2.1 if the damping constant is $c = 4$ lb-sec/ft and the initial conditions are $y(0) = 1.5$ ft and $y'(0) = -3$ ft/sec.

Solution: With $c = 4$, the equation of motion (8.28) becomes

$$y'' + 2y' + 64y = 0 \quad (8.31)$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 2r + 64 = 0$$

has complex conjugate roots

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 64}}{2} = -1 \pm 3\sqrt{7}i.$$

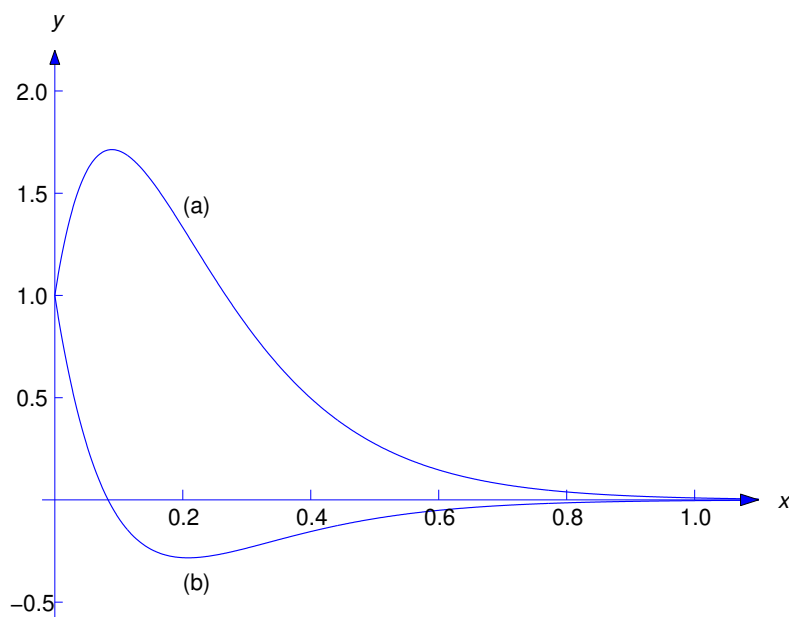


Figure 8.9: (a) $y = e^{-8t}(1 + 28t)$ (b) $y = e^{-8t}(1 - 12t)$

Therefore the motion is underdamped and the general solution of (8.31) is

$$y = e^{-t}(c_1 \cos 3\sqrt{7}t + c_2 \sin 3\sqrt{7}t).$$

Differentiating this yields

$$y' = -y + 3\sqrt{7}e^{-t}(-c_1 \sin 3\sqrt{7}t + c_2 \cos 3\sqrt{7}t).$$

Imposing the initial conditions $y(0) = 1.5$ and $y'(0) = -3$ in the last two equations yields $1.5 = c_1$ and $-3 = -1.5 + 3\sqrt{7}c_2$. Hence, the solution of the initial value problem is

$$y = e^{-t} \left(\frac{3}{2} \cos 3\sqrt{7}t - \frac{1}{2\sqrt{7}} \sin 3\sqrt{7}t \right). \quad (8.32)$$

The amplitude of the function in parentheses is

$$R = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2\sqrt{7}}\right)^2} = \sqrt{\frac{9}{4} + \frac{1}{4 \cdot 7}} = \sqrt{\frac{64}{4 \cdot 7}} = \frac{4}{\sqrt{7}}.$$

Therefore we can rewrite (8.32) as

$$y = \frac{4}{\sqrt{7}} e^{-t} \cos(3\sqrt{7}t - \phi),$$

where

$$\cos \phi = \frac{3}{2R} = \frac{3\sqrt{7}}{8} \quad \text{and} \quad \sin \phi = -\frac{1}{2\sqrt{7}R} = -\frac{1}{8}.$$

Therefore $\phi \cong -.125$ radians.

Example 8.2.3: Let the damping constant in Example 1 be $c = 40$ lb-sec/ft. Find the displacement y for $t > 0$ if $y(0) = 1$ and $y'(0) = 1$.

Solution: With $c = 40$, the equation of motion (8.28) reduces to

$$y'' + 20y' + 64y = 0 \quad (8.33)$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 20r + 64 = (r + 16)(r + 4) = 0$$

has the roots $r_1 = -4$ and $r_2 = -16$. Therefore the general solution of (8.33) is

$$y = c_1 e^{-4t} + c_2 e^{-16t}. \quad (8.34)$$

Differentiating this yields

$$y' = -4c_1 e^{-4t} - 16c_2 e^{-16t}.$$

The last two equations and the initial conditions $y(0) = 1$ and $y'(0) = 1$ imply that

$$\begin{aligned} c_1 + c_2 &= 1 \\ -4c_1 - 16c_2 &= 1. \end{aligned}$$

The solution of this system is $c_1 = 17/12$, $c_2 = -5/12$. Substituting these into (8.34) yields

$$y = \frac{17}{12} e^{-4t} - \frac{5}{12} e^{-16t}$$

as the solution of the given initial value problem (Figure 8.10).

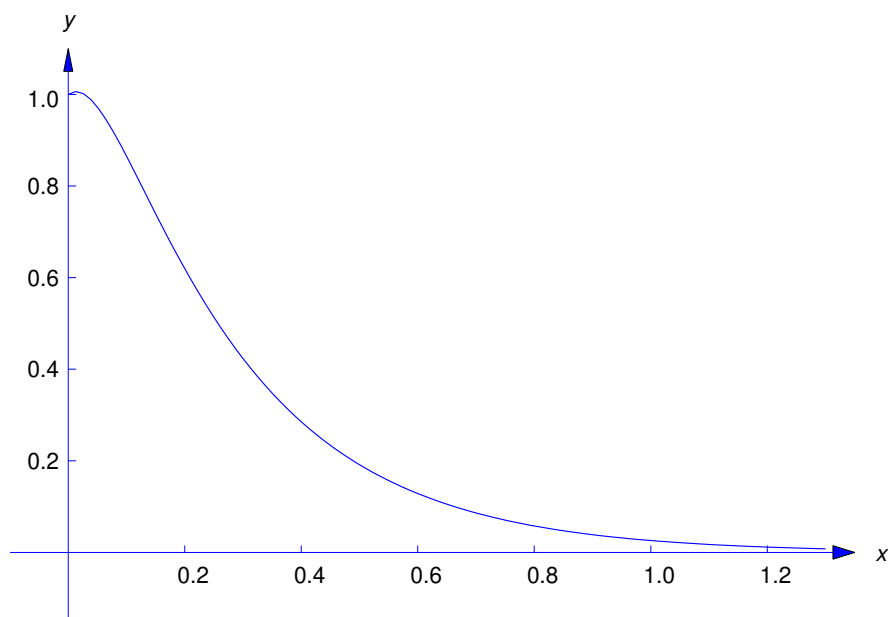


Figure 8.10: $y = \frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$

8.2.5 Forced Vibrations With Damping

Now we consider the motion of an object in a spring-mass system with damping, under the influence of a periodic forcing function $F(t) = F_0 \cos \omega t$, so that the equation of motion is

$$my'' + cy' + ky = F_0 \cos \omega t. \quad (8.35)$$

In Section 6.1 we considered this equation with $c = 0$ and found that the resulting displacement y assumed arbitrarily large values in the case of resonance (that is, when $\omega = \omega_0 = \sqrt{k/m}$). Here we'll see that in the presence of damping the displacement remains bounded for all t , and the initial conditions have little effect on the motion as $t \rightarrow \infty$. In fact, we'll see that for large t the displacement is closely approximated by a function of the form

$$y = R \cos(\omega t - \phi), \quad (8.36)$$

where the amplitude R depends upon m, c, k, F_0 , and ω . We're interested in the following question:

QUESTION: Assuming that m, c, k , and F_0 are held constant, what value of ω produces the largest amplitude R in (8.36), and what is this largest amplitude? To answer this question, we must solve (8.35) and determine R in terms of F_0, ω_0, ω , and c . We can obtain a particular solution of (8.35) by the method of undetermined coefficients. Since $\cos \omega t$ does not satisfy the complementary equation

$$my'' + cy' + ky = 0,$$

we can obtain a particular solution of (8.35) in the form

$$y_p = A \cos \omega t + B \sin \omega t. \quad (8.37)$$

Differentiating this yields

$$y'_p = \omega(-A \sin \omega t + B \cos \omega t)$$

and

$$y''_p = -\omega^2(A \cos \omega t + B \sin \omega t).$$

From the last three equations,

$$my''_p + cy'_p + ky_p = (-m\omega^2 A + c\omega B + kA) \cos \omega t + (-m\omega^2 B - c\omega A + kB) \sin \omega t,$$

so y_p satisfies (8.35) if

$$\begin{aligned} (k - m\omega^2)A + c\omega B &= F_0 \\ -c\omega A + (k - m\omega^2)B &= 0. \end{aligned}$$

Solving for A and B and substituting the results into (8.37) yields

$$y_p = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2} [(k - m\omega^2) \cos \omega t + c\omega \sin \omega t],$$

which can be written in amplitude-phase form as

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \phi), \quad (8.38)$$

where

$$\cos \phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}. \quad (8.39)$$

To compare this with the undamped forced vibration that we considered in Section 6.1 it's useful to write

$$k - m\omega^2 = m\left(\frac{k}{m} - \omega^2\right) = m(\omega_0^2 - \omega^2), \quad (8.40)$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular frequency of the undamped simple harmonic motion of an object with mass m on a spring with constant k . Substituting (8.40) into (8.38) yields

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \cos(\omega t - \phi). \quad (8.41)$$

The solution of an initial value problem

$$my'' + cy' + ky = F_0 \cos \omega t, \quad y(0) = y_0, \quad y'(0) = v_0,$$

is of the form $y = y_c + y_p$, where y_c has one of the three forms

$$\begin{aligned} y_c &= e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t), \\ y_c &= e^{-ct/2m}(c_1 + c_2 t), \\ y_c &= c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad (r_1, r_2 < 0). \end{aligned}$$

In all three cases $\lim_{t \rightarrow \infty} y_c(t) = 0$ for any choice of c_1 and c_2 . For this reason we say that y_c is the *transient component* of the solution y . The behavior of y for large t is determined by y_p , which we call the *steady state component* of y . Thus, for large t the motion is like simple harmonic motion at the frequency of the external force.

The amplitude of y_p in (8.41) is

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}, \quad (8.42)$$

which is finite for all ω ; that is, the presence of damping precludes the phenomenon of resonance that we encountered in studying undamped vibrations under a periodic forcing function. We'll now find the value $\omega_{\max}$ of ω for which R is maximized. This is the value of ω for which the function

$$\rho(\omega) = m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2$$

in the denominator of (8.42) attains its minimum value. By rewriting this as

$$\rho(\omega) = m^2(\omega_0^4 + \omega^4) + (c^2 - 2m^2\omega_0^2)\omega^2, \quad (8.43)$$

you can see that ρ is a strictly increasing function of ω^2 if

$$c \geq \sqrt{2m^2\omega_0^2} = \sqrt{2mk}.$$

(Recall that $\omega_0^2 = k/m$). Therefore $\omega_{\max} = 0$ if this inequality holds. From (8.39), you can see that $\phi = 0$ if $\omega = 0$. In this case, (8.38) reduces to

$$y_p = \frac{F_0}{\sqrt{m^2\omega_0^4}} = \frac{F_0}{k},$$

which is consistent with Hooke's law: if the mass is subjected to a constant force F_0 , its displacement should approach a constant y_p such that $ky_p = F_0$. Now suppose $c < \sqrt{2mk}$. Then, from (8.43),

$$\rho'(\omega) = 2\omega(2m^2\omega^2 + c^2 - 2m^2\omega_0^2),$$

and $\omega_{\max}$ is the value of ω for which the expression in parentheses equals zero; that is,

$$\omega_{\max} = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}} = \sqrt{\frac{k}{m} \left(1 - \frac{c^2}{2km}\right)}.$$

(To see that $\rho(\omega_{\max})$ is the minimum value of $\rho(\omega)$, note that $\rho'(\omega) < 0$ if $\omega < \omega_{\max}$ and $\rho'(\omega) > 0$ if $\omega > \omega_{\max}$.) Substituting $\omega = \omega_{\max}$ in (8.42) and simplifying shows that the maximum amplitude $R_{\max}$ is

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}} \quad \text{if} \quad c < \sqrt{2mk}.$$

We summarize our results as follows.

Theorem 8.2.1. *Suppose we consider the amplitude R of the steady state component of the solution of*

$$my'' + cy' + ky = F_0 \cos \omega t$$

as a function of ω .

- (a) *If $c \geq \sqrt{2mk}$, the maximum amplitude is $R_{\max} = F_0/k$ and it's attained when $\omega = \omega_{\max} = 0$.*
 (b) *If $c < \sqrt{2mk}$, the maximum amplitude is*

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}}, \quad (8.44)$$

and it's attained when

$$\omega = \omega_{\max} = \sqrt{\frac{k}{m} \left(1 - \frac{c^2}{2km} \right)}. \quad (8.45)$$

Note that $R_{\max}$ and $\omega_{\max}$ are continuous functions of c , for $c \geq 0$, since (8.44) and (8.45) reduce to $R_{\max} = F_0/k$ and $\omega_{\max} = 0$ if $c = \sqrt{2km}$.

8.2.6 Exercises

1. A 64 lb object stretches a spring 4 ft in equilibrium. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The object is initially displaced 18 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement and time-varying amplitude for $t > 0$.
2. C/G A 16 lb weight is attached to a spring with natural length 5 ft. With the weight attached, the spring measures 8.2 ft. The weight is initially displaced 3 ft below equilibrium and given an upward velocity of 2 ft/sec. Find and graph its displacement for $t > 0$ if the medium resists the motion with a force of one lb for each ft/sec of velocity. Also, find its time-varying amplitude.
3. C/G An 8 lb weight stretches a spring 1.5 inches. It is attached to a dashpot with damping constant $c=8$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given an upward velocity of 6 ft/sec. Find and graph its displacement for $t > 0$.
4. A 96 lb weight stretches a spring 3.2 ft in equilibrium. It is attached to a dashpot with damping constant $c=18$ lb-sec/ft. The weight is initially displaced 15 inches below equilibrium and given a downward velocity of 12 ft/sec. Find its displacement for $t > 0$.
5. A 16 lb weight stretches a spring 6 inches in equilibrium. It is attached to a damping mechanism with constant c . Find all values of c such that the free vibration of the weight has infinitely many oscillations.
6. An 8 lb weight stretches a spring .32 ft. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 4 ft/sec. Find its displacement for $t > 0$ if the medium exerts a damping force of 1.5 lb for each ft/sec of velocity.
7. A 32 lb weight stretches a spring 2 ft in equilibrium. It is attached to a dashpot with constant $c = 8$ lb-sec/ft. The weight is initially displaced 8 inches below equilibrium and released from rest. Find its displacement for $t > 0$.
8. A mass of 20 gm stretches a spring 5 cm. The spring is attached to a dashpot with damping constant 400 dyne sec/cm. Determine the displacement for $t > 0$ if the mass is initially displaced 9 cm above equilibrium and released from rest.
9. A 64 lb weight is suspended from a spring with constant $k = 25$ lb/ft. It is initially displaced 18 inches above equilibrium and released from rest. Find its displacement for $t > 0$ if the medium resists the motion with 6 lb of force for each ft/sec of velocity.
10. A 32 lb weight stretches a spring 1 ft in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/sec. Find its displacement for $t > 0$ if the medium resists the motion with a force equal to 3 times the speed in ft/sec.
11. An 8 lb weight stretches a spring 2 inches. It is attached to a dashpot with damping constant $c=4$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement for $t > 0$.
12. C/G A 2 lb weight stretches a spring .32 ft. The weight is initially displaced 4 inches below equilibrium and given an upward velocity of 5 ft/sec. The medium provides damping with constant $c = 1/8$ lb-sec/ft. Find and graph the displacement for $t > 0$.
13. An 8 lb weight stretches a spring 8 inches in equilibrium. It is attached to a dashpot with damping constant $c = .5$ lb-sec/ft and subjected to an external force $F(t) = 4 \cos 2t$ lb. Determine the steady state component of the displacement for $t > 0$.
14. A 32 lb weight stretches a spring 1 ft in equilibrium. It is attached to a dashpot with constant $c = 12$ lb-sec/ft. The weight is initially displaced 8 inches above equilibrium and released from rest. Find its displacement for $t > 0$.
15. A mass of one kg stretches a spring 49 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 4 N for each m/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

16. A mass of 100 grams stretches a spring 98 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 600 dynes for each cm/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.
17. A 192 lb weight is suspended from a spring with constant $k = 6$ lb/ft and subjected to an external force $F(t) = 8 \cos 3t$ lb. Find the steady state component of the displacement for $t > 0$ if the medium resists the motion with a force equal to 8 times the speed in ft/sec.
18. A 2 gm mass is attached to a spring with constant 20 dyne/cm. Find the steady state component of the displacement if the mass is subjected to an external force $F(t) = 3 \cos 4t - 5 \sin 4t$ dynes and a dashpot supplies 4 dynes of damping for each cm/sec of velocity.
19. C/G A 96 lb weight is attached to a spring with constant 12 lb/ft. Find and graph the steady state component of the displacement if the mass is subjected to an external force $F(t) = 18 \cos t - 9 \sin t$ lb and a dashpot supplies 24 lb of damping for each ft/sec of velocity.
20. A mass of one kg stretches a spring 49 cm in equilibrium. It is attached to a dashpot that supplies a damping force of 4 N for each m/sec of speed. Find the steady state component of its displacement if it's subjected to an external force $F(t) = 8 \sin 2t - 6 \cos 2t$ N.
21. A mass m is suspended from a spring with constant k and subjected to an external force $F(t) = \alpha \cos \omega_0 t + \beta \sin \omega_0 t$, where ω_0 is the natural frequency of the spring-mass system without damping. Find the steady state component of the displacement if a dashpot with constant c supplies damping.
22. Show that if c_1 and c_2 are not both zero then

$$y = e^{r_1 t}(c_1 + c_2 t)$$

can't equal zero for more than one value of t .

23. Show that if c_1 and c_2 are not both zero then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

can't equal zero for more than one value of t .

24. Find the solution of the initial value problem

$$m y'' + c y' + k y = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

given that the motion is underdamped, so the general solution of the equation is

$$y = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

25. Find the solution of the initial value problem

$$m y'' + c y' + k y = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

given that the motion is overdamped, so the general solution of the equation is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad (r_1, r_2 < 0).$$

26. Find the solution of the initial value problem

$$m y'' + c y' + k y = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

given that the motion is critically damped, so that the general solution of the equation is of the form

$$y = e^{r_1 t}(c_1 + c_2 t) \quad (r_1 < 0).$$

8.3 The *RLC* Circuit

In this section we consider the *RLC circuit*, shown schematically in Figure 8.11. As we'll see, the *RLC* circuit is an electrical analog of a spring-mass system with damping.

Nothing happens while the switch is open (dashed line). When the switch is closed (solid line) we say that the *circuit is closed*. Differences in electrical potential in a closed circuit cause current to flow in the circuit. The battery or generator in Figure 8.11 creates a difference in electrical potential $E = E(t)$ between its two terminals, which we've marked arbitrarily as positive and negative. (We could just as well interchange the markings.) We'll say that $E(t) > 0$ if the potential at the positive terminal is greater than the potential at the negative terminal, $E(t) < 0$ if the potential at the positive terminal is less than the potential at the negative terminal, and $E(t) = 0$ if the potential is the same at the two terminals. We call E the *impressed voltage*.

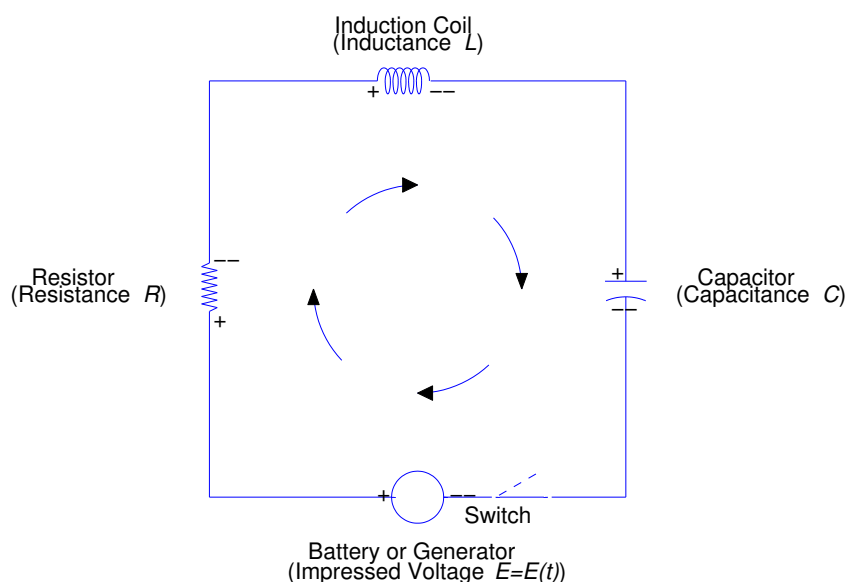


Figure 8.11: An *RLC* circuit

At any time t , the same current flows in all points of the circuit. We denote current by $I = I(t)$. We say that $I(t) > 0$ if the direction of flow is around the circuit from the positive terminal of the battery or generator back to the negative terminal, as indicated by the arrows in Figure 8.11 $I(t) < 0$ if the flow is in the opposite direction, and $I(t) = 0$ if no current flows at time t .

Differences in potential occur at the resistor, induction coil, and capacitor in Figure 8.11. Note that the two sides of each of these components are also identified as positive and negative. The *voltage drop across* each component is defined to be the potential on the positive side of the component minus the potential on the negative side. This terminology is somewhat misleading, since “drop” suggests a decrease even though changes in potential are signed quantities and therefore may be increases. Nevertheless, we'll go along with tradition and call them voltage drops. The voltage drop across the resistor in Figure 8.11 is given by

$$V_R = IR, \quad (8.46)$$

where I is current and R is a positive constant, the *resistance* of the resistor. The voltage drop across the

induction coil is given by

$$V_I = L \frac{dI}{dt} = LI', \quad (8.47)$$

where L is a positive constant, the *inductance* of the coil.

A capacitor stores electrical charge $Q = Q(t)$, which is related to the current in the circuit by the equation

$$Q(t) = Q_0 + \int_0^t I(\tau) d\tau, \quad (8.48)$$

where Q_0 is the charge on the capacitor at $t = 0$. The voltage drop across a capacitor is given by

$$V_C = \frac{Q}{C}, \quad (8.49)$$

where C is a positive constant, the *capacitance* of the capacitor.

Table 8.3 names the units for the quantities that we've discussed. The units are defined so that

$$\begin{aligned} 1 \text{ volt} &= 1 \text{ ampere} \cdot 1 \text{ ohm} \\ &= 1 \text{ henry} \cdot 1 \text{ ampere/second} \\ &= 1 \text{ coulomb/farad} \end{aligned}$$

and

$$1 \text{ ampere} = 1 \text{ coulomb/second}.$$

Electrical Units

Symbol	Name	Unit
E	Impressed Voltage	volt
I	Current	ampere
Q	Charge	coulomb
R	Resistance	ohm
L	Inductance	henry
C	Capacitance	farad

According to *Kirchoff's law*, the sum of the voltage drops in a closed RLC circuit equals the impressed voltage. Therefore, from (8.46), (8.47), and (8.49),

$$LI' + RI + \frac{1}{C}Q = E(t). \quad (8.50)$$

This equation contains two unknowns, the current I in the circuit and the charge Q on the capacitor. However, (8.48) implies that $Q' = I$, so (8.50) can be converted into the second order equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \quad (8.51)$$

in Q . To find the current flowing in an RLC circuit, we solve (8.51) for Q and then differentiate the solution to obtain I .

In Sections 6.1 and 6.2 we encountered the equation

$$my'' + cy' + ky = F(t) \quad (8.52)$$

in connection with spring-mass systems. Except for notation this equation is the same as (8.51). The correspondence between electrical and mechanical quantities connected with (8.51) and (8.52) is shown in Table 8.3.

Electrical and Mechanical Units

Electrical		Mechanical	
charge	Q	displacement	y
curent	I	velocity	y'
impressed voltage	$E(t)$	external force	$F(t)$
inductance	L	Mass	m
resistance	R	damping	c
1/capacitance	$1/C$	cpring constant	k

The equivalence between (8.51) and (8.52) is an example of how mathematics unifies fundamental similarities in diverse physical phenomena. Since we've already studied the properties of solutions of (8.52) in Sections 6.1 and 6.2, we can obtain results concerning solutions of (8.51) by simply changing notation, according to Table 8.3.

8.3.1 Free Oscillations

We say that an RLC circuit is in *free oscillation* if $E(t) = 0$ for $t > 0$, so that (8.51) becomes

$$LQ'' + RQ' + \frac{1}{C}Q = 0. \quad (8.53)$$

The characteristic equation of (8.53) is

$$Lr^2 + Rr + \frac{1}{C} = 0,$$

with roots

$$r_1 = \frac{-R - \sqrt{R^2 - 4L/C}}{2L} \quad \text{and} \quad r_2 = \frac{-R + \sqrt{R^2 - 4L/C}}{2L}. \quad (8.54)$$

There are three cases to consider, all analogous to the cases considered in Section 6.2 for free vibrations of a damped spring-mass system.

CASE 1. The oscillation is *underdamped* if $R < \sqrt{4L/C}$. In this case, r_1 and r_2 in (8.54) are complex conjugates, which we write as

$$r_1 = -\frac{R}{2L} + i\omega_1 \quad \text{and} \quad r_2 = -\frac{R}{2L} - i\omega_1,$$

where

$$\omega_1 = \frac{\sqrt{4L/C - R^2}}{2L}.$$

The general solution of (8.53) is

$$Q = e^{-Rt/2L}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t),$$

which we can write as

$$Q = Ae^{-Rt/2L} \cos(\omega_1 t - \phi), \quad (8.55)$$

where

$$A = \sqrt{c_1^2 + c_2^2}, \quad A \cos \phi = c_1, \quad \text{and} \quad A \sin \phi = c_2.$$

In the idealized case where $R = 0$, the solution (8.55) reduces to

$$Q = A \cos\left(\frac{t}{\sqrt{LC}} - \phi\right),$$

which is analogous to the simple harmonic motion of an undamped spring-mass system in free vibration.

Actual RLC circuits are usually underdamped, so the case we've just considered is the most important. However, for completeness we'll consider the other two possibilities.

CASE 2. The oscillation is *overdamped* if $R > \sqrt{4L/C}$. In this case, the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1 < r_2 < 0$ (see (8.54)), and the general solution of (8.53) is

$$Q = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \quad (8.56)$$

CASE 3. The oscillation is *critically damped* if $R = \sqrt{4L/C}$. In this case, $r_1 = r_2 = -R/2L$ and the general solution of (8.53) is

$$Q = e^{-Rt/2L}(c_1 + c_2 t). \quad (8.57)$$

If $R \neq 0$, the exponentials in (8.55), (8.56), and (8.57) are negative, so the solution of any homogeneous initial value problem

$$LQ'' + RQ' + \frac{1}{C}Q = 0, \quad Q(0) = Q_0, \quad Q'(0) = I_0,$$

approaches zero exponentially as $t \rightarrow \infty$. Thus, all such solutions are *transient*, in the sense defined Section 6.2 in the discussion of forced vibrations of a spring-mass system with damping.

Example 8.3.1: At $t = 0$ a current of 2 amperes flows in an RLC circuit with resistance $R = 40$ ohms, inductance $L = .2$ henrys, and capacitance $C = 10^{-5}$ farads. Find the current flowing in the circuit at $t > 0$ if the initial charge on the capacitor is 1 coulomb. Assume that $E(t) = 0$ for $t > 0$.

Solution: The equation for the charge Q is

$$\frac{1}{5}Q'' + 40Q' + 10000Q = 0,$$

or

$$Q'' + 200Q' + 50000Q = 0. \quad (8.58)$$

Therefore we must solve the initial value problem

$$Q'' + 200Q' + 50000Q = 0, \quad Q(0) = 1, \quad Q'(0) = 2. \quad (8.59)$$

The desired current is the derivative of the solution of this initial value problem.

The characteristic equation of (8.58) is

$$r^2 + 200r + 50000 = 0,$$

which has complex zeros $r = -100 \pm 200i$. Therefore the general solution of (8.58) is

$$Q = e^{-100t}(c_1 \cos 200t + c_2 \sin 200t). \quad (8.60)$$

Differentiating this and collecting like terms yields

$$Q' = -e^{-100t}[(100c_1 - 200c_2) \cos 200t + (100c_2 + 200c_1) \sin 200t]. \quad (8.61)$$

To find the solution of the initial value problem (8.59), we set $t = 0$ in (8.60) and (8.61) to obtain

$$c_1 = Q(0) = 1 \quad \text{and} \quad -100c_1 + 200c_2 = Q'(0) = 2;$$

therefore, $c_1 = 1$ and $c_2 = 51/100$, so

$$Q = e^{-100t} \left(\cos 200t + \frac{51}{100} \sin 200t \right)$$

is the solution of (8.59). Differentiating this yields

$$I = e^{-100t}(2 \cos 200t - 251 \sin 200t).$$

8.3.2 Forced Oscillations With Damping

An initial value problem for (8.51) has the form

$$LQ'' + RQ' + \frac{1}{C}Q = E(t), \quad Q(0) = Q_0, \quad Q'(0) = I_0, \quad (8.62)$$

where Q_0 is the initial charge on the capacitor and I_0 is the initial current in the circuit. We've already seen that if $E \equiv 0$ then all solutions of (8.62) are transient. If $E \neq 0$, we know that the solution of (8.62) has the form $Q = Q_c + Q_p$, where Q_c satisfies the complementary equation, and approaches zero exponentially as $t \rightarrow \infty$ for any initial conditions, while Q_p depends only on E and is independent of the initial conditions. As in the case of forced oscillations of a spring-mass system with damping, we call Q_p the *steady state* charge on the capacitor of the RLC circuit. Since $I = Q' = Q'_c + Q'_p$ and Q'_c also tends to zero exponentially as $t \rightarrow \infty$, we say that $I_c = Q'_c$ is the *transient* current and $I_p = Q'_p$ is the *steady state* current. In most applications we're interested only in the steady state charge and current.

Example 8.3.2: Find the amplitude-phase form of the steady state current in the RLC circuit in Figure 8.11 if the impressed voltage, provided by an alternating current generator, is $E(t) = E_0 \cos \omega t$.

Solution: We'll first find the steady state charge on the capacitor as a particular solution of

$$LQ'' + RQ' + \frac{1}{C}Q = E_0 \cos \omega t.$$

To do, this we'll simply reinterpret a result obtained in Section 6.2, where we found that the steady state solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

is

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \phi),$$

where

$$\cos \phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}.$$

(See Equations (8.38) and (8.39).) By making the appropriate changes in the symbols (according to Table 2) yields the steady state charge

$$Q_p = \frac{E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \cos(\omega t - \phi),$$

where

$$\cos \phi = \frac{1/C - L\omega^2}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{R\omega}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}}.$$

Therefore the steady state current in the circuit is

$$I_p = Q'_p = -\frac{\omega E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \sin(\omega t - \phi).$$

8.3.3 Exercises

In Exercises 8.3.1-8.3.5 find the current in the RLC circuit, assuming that $E(t) = 0$ for $t > 0$.

1. $R = 3$ ohms; $L = .1$ henrys; $C = .01$ farads; $Q_0 = 0$ coulombs; $I_0 = 2$ amperes.
2. $R = 2$ ohms; $L = .05$ henrys; $C = .01$ farads; $Q_0 = 2$ coulombs; $I_0 = -2$ amperes.

3. $R = 2$ ohms; $L = .1$ henrys; $C = .01$ farads; $Q_0 = 2$ coulombs; $I_0 = 0$ amperes.
4. $R = 6$ ohms; $L = .1$ henrys; $C = .004$ farads; $Q_0 = 3$ coulombs; $I_0 = -10$ amperes.
5. $R = 4$ ohms; $L = .05$ henrys; $C = .008$ farads; $Q_0 = -1$ coulombs; $I_0 = 2$ amperes.

In Exercises 8.3.6-8.3.10 find the steady state current in the circuit described by the equation.

6. $\frac{1}{10}Q'' + 3Q' + 100Q = 5 \cos 10t - 5 \sin 10t$
7. $\frac{1}{20}Q'' + 2Q' + 100Q = 10 \cos 25t - 5 \sin 25t$
8. $\frac{1}{10}Q'' + 2Q' + 100Q = 3 \cos 50t - 6 \sin 50t$
9. $\frac{1}{10}Q'' + 6Q' + 250Q = 10 \cos 100t + 30 \sin 100t$
10. $\frac{1}{20}Q'' + 4Q' + 125Q = 15 \cos 30t - 30 \sin 30t$
11. Show that if $E(t) = U \cos \omega t + V \sin \omega t$ where U and V are constants then the steady state current in the RLC circuit shown in Figure 8.11 is

$$I_p = \frac{\omega^2 R E(t) + (1/C - L\omega^2)E'(t)}{\Delta},$$

where

$$\Delta = (1/C - L\omega^2)^2 + R^2 \omega^2.$$

12. Find the amplitude of the steady state current I_p in the RLC circuit shown in Figure 8.11 if $E(t) = U \cos \omega t + V \sin \omega t$, where U and V are constants. Then find the value ω_0 of ω maximizes the amplitude, and find the maximum amplitude.

In Exercises 8.3.13-8.3.17 plot the amplitude of the steady state current against ω . Estimate the value of ω that maximizes the amplitude of the steady state current, and estimate this maximum amplitude. HINT: You can confirm your results by doing Exercise 8.3.12.

13. $\boxed{\text{L}} \quad \frac{1}{10}Q'' + 3Q' + 100Q = U \cos \omega t + V \sin \omega t$
14. $\boxed{\text{L}} \quad \frac{1}{20}Q'' + 2Q' + 100Q = U \cos \omega t + V \sin \omega t$
15. $\boxed{\text{L}} \quad \frac{1}{10}Q'' + 2Q' + 100Q = U \cos \omega t + V \sin \omega t$
16. $\boxed{\text{L}} \quad \frac{1}{10}Q'' + 6Q' + 250Q = U \cos \omega t + V \sin \omega t$
17. $\boxed{\text{L}} \quad \frac{1}{20}Q'' + 4Q' + 125Q = U \cos \omega t + V \sin \omega t$

8.4 Motion Under a Central Force

We'll now study the motion of a object moving under the influence of a *central force*; that is, a force whose magnitude at any point P other than the origin depends only on the distance from P to the origin, and whose direction at P is parallel to the line connecting P and the origin, as indicated in Figure 8.12 for the case where the direction of the force at every point is toward the origin. Gravitational forces are central forces; for example, as mentioned in Section 4.3, if we assume that Earth is a perfect sphere with constant mass density then Newton's law of gravitation asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth, which we take to be the origin.

If the initial position and velocity vectors of an object moving under a central force are parallel, then the subsequent motion is along the line from the origin to the initial position. Here we'll assume that the initial position and velocity vectors are not parallel; in this case the subsequent motion is in the plane determined by them. For convenience we take this to be the xy -plane. We'll consider the problem of determining the curve traversed by the object. We call this curve the *orbit*.

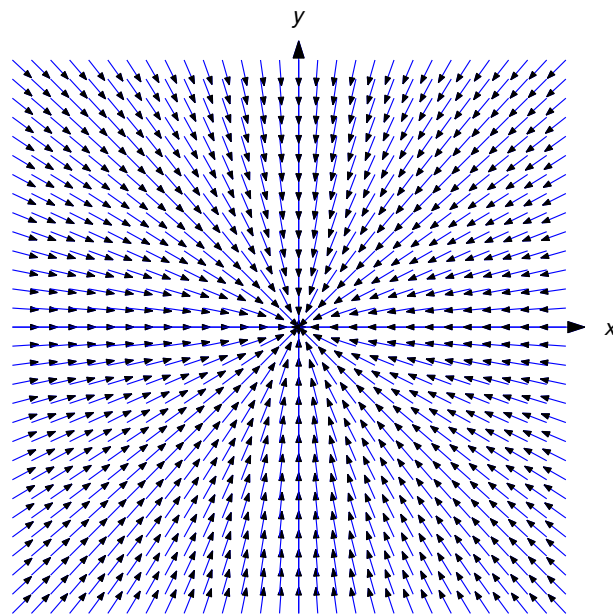


Figure 8.12

We can represent a central force in terms of polar coordinates

$$x = r \cos \theta, \quad y = r \sin \theta$$

as

$$\mathbf{F}(r, \theta) = f(r)(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}).$$

We assume that f is continuous for all $r > 0$. The magnitude of $\mathbf{F}$ at $(x, y) = (r \cos \theta, r \sin \theta)$ is $|f(r)|$, so it depends only on the distance r from the point to the origin the direction of $\mathbf{F}$ is from the point to the origin if $f(r) < 0$, or from the origin to the point if $f(r) > 0$. We'll show that the orbit of an object with mass m moving under this force is given by

$$r(\theta) = \frac{1}{u(\theta)},$$

where u is solution of the differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2}f(1/u), \quad (8.63)$$

and h is a constant defined below.

Newton's second law of motion ($\mathbf{F} = m\mathbf{a}$) says that the polar coordinates $r = r(t)$ and $\theta = \theta(t)$ of the particle satisfy the vector differential equation

$$m(r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j})'' = f(r)(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}). \quad (8.64)$$

To deal with this equation we introduce the unit vectors

$$\mathbf{e}_1 = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \quad \text{and} \quad \mathbf{e}_2 = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}.$$

Note that $\mathbf{e}_1$ points in the direction of increasing r and $\mathbf{e}_2$ points in the direction of increasing θ (Figure 8.13); moreover,

$$\frac{d\mathbf{e}_1}{d\theta} = \mathbf{e}_2, \quad \frac{d\mathbf{e}_2}{d\theta} = -\mathbf{e}_1, \quad (8.65)$$

and

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \cos \theta (-\sin \theta) + \sin \theta \cos \theta = 0,$$

so $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular. Recalling that the single prime (') stands for differentiation with respect to t , we see from (8.65) and the chain rule that

$$\mathbf{e}_1' = \theta' \mathbf{e}_2 \quad \text{and} \quad \mathbf{e}_2' = -\theta' \mathbf{e}_1. \quad (8.66)$$

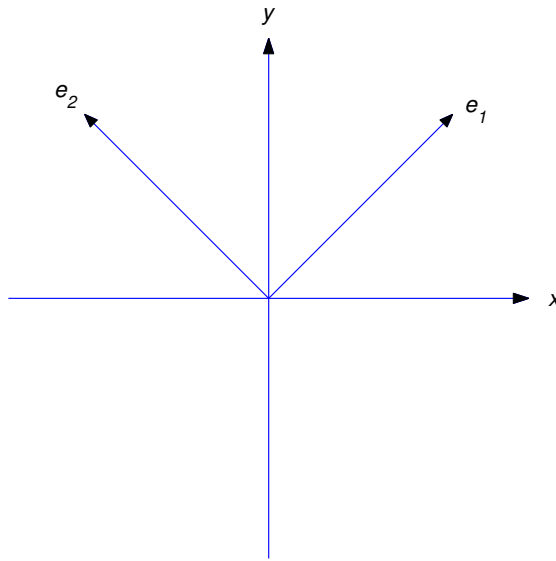


Figure 8.13

Now we can write (8.64) as

$$m(r\mathbf{e}_1)'' = f(r)\mathbf{e}_1. \quad (8.67)$$

But

$$(r\mathbf{e}_1)' = r'\mathbf{e}_1 + r\mathbf{e}_1' = r'\mathbf{e}_1 + r\theta'\mathbf{e}_2$$

(from (8.66)), and

$$\begin{aligned} (r\mathbf{e}_1)'' &= (r'\mathbf{e}_1 + r\theta'\mathbf{e}_2)' \\ &= r''\mathbf{e}_1 + r'\mathbf{e}_1' + (r\theta'' + r'\theta')\mathbf{e}_2 + r\theta'\mathbf{e}_2' \\ &= r''\mathbf{e}_1 + r'\theta'\mathbf{e}_2 + (r\theta'' + r'\theta')\mathbf{e}_2 - r(\theta')^2\mathbf{e}_1 \quad (\text{from (8.66)}) \\ &= (r'' - r(\theta')^2)\mathbf{e}_1 + (r\theta'' + 2r'\theta')\mathbf{e}_2. \end{aligned}$$

Substituting this into (8.67) yields

$$m(r'' - r(\theta')^2)\mathbf{e}_1 + m(r\theta'' + 2r'\theta')\mathbf{e}_2 = f(r)\mathbf{e}_1.$$

By equating the coefficients of $\mathbf{e}_1$ and $\mathbf{e}_2$ on the two sides of this equation we see that

$$m(r'' - r(\theta')^2) = f(r) \tag{8.68}$$

and

$$r\theta'' + 2r'\theta' = 0.$$

Multiplying the last equation by r yields

$$r^2\theta'' + 2rr'\theta' = (r^2\theta')' = 0,$$

so

$$r^2\theta' = h, \tag{8.69}$$

where h is a constant that we can write in terms of the initial conditions as

$$h = r^2(0)\theta'(0).$$

Since the initial position and velocity vectors are

$$r(0)\mathbf{e}_1(0) \quad \text{and} \quad r'(0)\mathbf{e}_1(0) + r(0)\theta'(0)\mathbf{e}_2(0),$$

our assumption that these two vectors are not parallel implies that $\theta'(0) \neq 0$, so $h \neq 0$.

Now let $u = 1/r$. Then $u^2 = \theta'/h$ (from (8.69)) and

$$r' = -\frac{u'}{u^2} = -h\left(\frac{u'}{\theta'}\right),$$

which implies that

$$r' = -h\frac{du}{d\theta}, \tag{8.70}$$

since

$$\frac{u'}{\theta'} = \frac{du}{dt} \bigg/ \frac{d\theta}{dt} = \frac{du}{d\theta}.$$

Differentiating (8.70) with respect to t yields

$$r'' = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\theta',$$

which implies that

$$r'' = -h^2u^2\frac{d^2u}{d\theta^2} \quad \text{since} \quad \theta' = hu^2.$$

Substituting from these equalities into (8.68) and recalling that $r = 1/u$ yields

$$-m \left(h^2 u^2 \frac{d^2 u}{d\theta^2} + \frac{1}{u} h^2 u^4 \right) = f(1/u),$$

and dividing through by $-mh^2 u^2$ yields (8.63).

Eqn. (8.69) has the following geometrical interpretation, which is known as *Kepler's Second Law*.

Theorem 8.4.1. *The position vector of an object moving under a central force sweeps out equal areas in equal times; more precisely, if $\theta(t_1) \leq \theta(t_2)$ then the (signed) area of the sector*

$$\{(x, y) = (r \cos \theta, r \sin \theta) : 0 \leq r \leq r(\theta), \theta(t_1) \leq \theta(t_2)\}$$

(Figure 8.14) is given by

$$A = \frac{h(t_2 - t_1)}{2},$$

where $h = r^2 \theta'$, which we have shown to be constant.

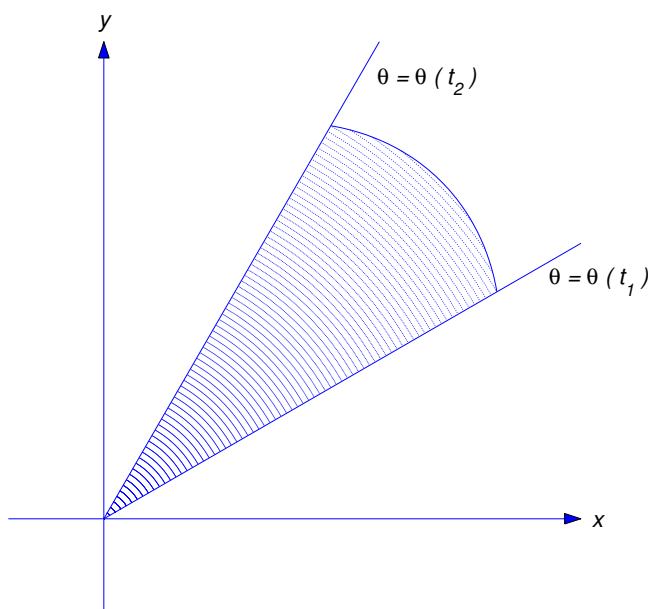


Figure 8.14

Proof. Recall from calculus that the area of the shaded sector in Figure 8.14 is

$$A = \frac{1}{2} \int_{\theta(t_1)}^{\theta(t_2)} r^2(\theta) d\theta,$$

where $r = r(\theta)$ is the polar representation of the orbit. Making the change of variable $\theta = \theta(t)$ yields

$$A = \frac{1}{2} \int_{t_1}^{t_2} r^2(\theta(t)) \theta'(t) dt. \quad (8.71)$$

But (8.69) and (8.71) imply that

$$A = \frac{1}{2} \int_{t_1}^{t_2} h dt = \frac{h(t_2 - t_1)}{2},$$

which completes the proof.

8.4.1 Motion Under an Inverse Square Law Force

In the special case where $f(r) = -mk/r^2 = -mku^2$, so $\mathbf{F}$ can be interpreted as a gravitational force, (8.63) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2}. \quad (8.72)$$

The general solution of the complementary equation

$$\frac{d^2u}{d\theta^2} + u = 0$$

can be written in amplitude–phase form as

$$u = A \cos(\theta - \phi),$$

where $A \geq 0$ and ϕ is a phase angle. Since $u_p = k/h^2$ is a particular solution of (8.72), the general solution of (8.72) is

$$u = A \cos(\theta - \phi) + \frac{k}{h^2};$$

hence, the orbit is given by

$$r = \left(A \cos(\theta - \phi) + \frac{k}{h^2} \right)^{-1},$$

which we rewrite as

$$r = \frac{\rho}{1 + e \cos(\theta - \phi)}, \quad (8.73)$$

where

$$\rho = \frac{h^2}{k} \quad \text{and} \quad e = A\rho.$$

A curve satisfying (8.73) is a conic section with a focus at the origin (Exercise 8.4.1). The nonnegative constant e is the *eccentricity* of the orbit, which is an ellipse if $e < 1$ ellipse (a circle if $e = 0$), a parabola if $e = 1$, or a hyperbola if $e > 1$.

If the orbit is an ellipse, then the minimum and maximum values of r are

$$\begin{aligned} r_{\min} &= \frac{\rho}{1 + e} \quad (\text{the } \textit{perihelion distance}, \text{ attained when } \theta = \phi) \\ r_{\max} &= \frac{\rho}{1 - e} \quad (\text{the } \textit{aphelion distance}, \text{ attained when } \theta = \phi + \pi). \end{aligned}$$

Figure 8.15 shows a typical elliptic orbit. The point P on the orbit where $r = r_{\min}$ is the *perigee* and the point A where $r = r_{\max}$ is the *apogee*.

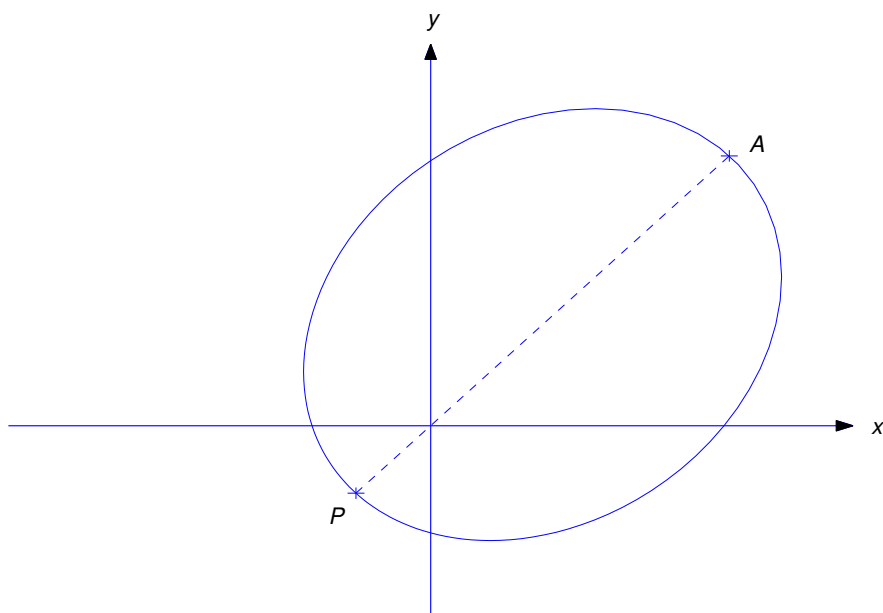


Figure 8.15

For example, Earth's orbit around the Sun is approximately an ellipse with $e \approx .017$, $r_{\min} \approx 91 \times 10^6$ miles, and $r_{\max} \approx 95 \times 10^6$ miles. Halley's comet has a very elongated approximately elliptical orbit around the sun, with $e \approx .967$, $r_{\min} \approx 55 \times 10^6$ miles, and $r_{\max} \approx 33 \times 10^8$ miles. Some comets (the nonrecurring type) have parabolic or hyperbolic orbits.

8.4.2 Exercises

1. Find the equation of the curve

$$r = \frac{\rho}{1 + e \cos(\theta - \phi)} \quad (\text{A})$$

in terms of $(X, Y) = (r \cos(\theta - \phi), r \sin(\theta - \phi))$, which are rectangular coordinates with respect to the axes shown in Figure 8.16. Use your results to verify that (A) is the equation of an ellipse if $0 < e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$. If $e < 1$, leave your answer in the form

$$\frac{(X - X_0)^2}{a^2} + \frac{(Y - Y_0)^2}{b^2} = 1,$$

and show that the area of the ellipse is

$$A = \frac{\pi \rho^2}{(1 - e^2)^{3/2}}.$$

Then use Theorem 8.4.1 to show that the time required for the object to traverse the entire orbit is

$$T = \frac{2\pi \rho^2}{h(1 - e^2)^{3/2}}.$$

(This is *Kepler's third law*; T is called the *period* of the orbit.)

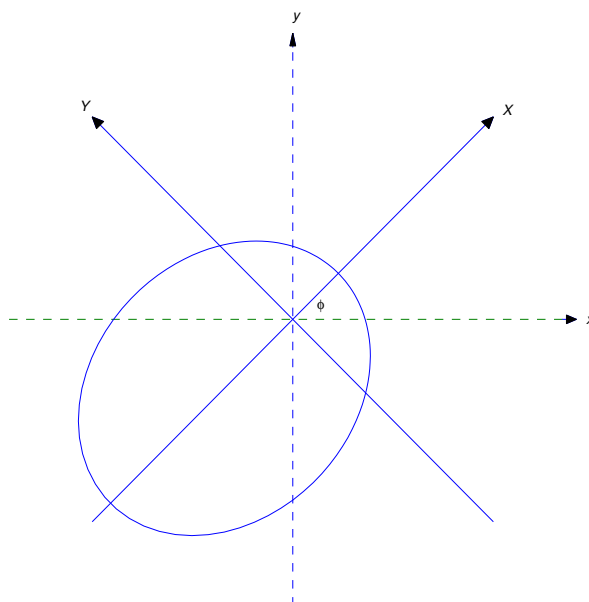


Figure 8.16

2. Suppose an object with mass m moves in the xy -plane under the central force

$$\mathbf{F}(r, \theta) = -\frac{mk}{r^2}(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}),$$

where k is a positive constant. As we shown, the orbit of the object is given by

$$r = \frac{\rho}{1 + e \cos(\theta - \phi)}.$$

Determine ρ , e , and ϕ in terms of the initial conditions

$$r(0) = r_0, \quad r'(0) = r'_0, \quad \text{and} \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta'_0.$$

Assume that the initial position and velocity vectors are not collinear.

3. Suppose we wish to put a satellite with mass m into an elliptical orbit around Earth. Assume that the only force acting on the object is Earth's gravity, given by

$$\mathbf{F}(r, \theta) = -mg \left(\frac{R^2}{r^2} \right) (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}),$$

where R is Earth's radius, g is the acceleration due to gravity at Earth's surface, and r and θ are polar coordinates in the plane of the orbit, with the origin at Earth's center.

- (a) Find the eccentricity required to make the aphelion and perihelion distances equal to $R\gamma_1$ and $R\gamma_2$, respectively, where $1 < \gamma_1 < \gamma_2$.
 (b) Find the initial conditions

$$r(0) = r_0, \quad r'(0) = r'_0, \quad \text{and} \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta'_0$$

required to make the initial point the perigee, and the motion along the orbit in the direction of increasing θ . HINT: Use the results of Exercise 8.4.2.

4. An object with mass m moves in a spiral orbit $r = c\theta^2$ under a central force

$$\mathbf{F}(r, \theta) = f(r)(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}).$$

Find f .

5. An object with mass m moves in the orbit $r = r_0 e^{\gamma\theta}$ under a central force

$$\mathbf{F}(r, \theta) = f(r)(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}).$$

Find f .

6. Suppose an object with mass m moves under the central force

$$\mathbf{F}(r, \theta) = -\frac{mk}{r^3}(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}),$$

with

$$r(0) = r_0, \quad r'(0) = r'_0, \quad \text{and} \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta'_0,$$

where $h = r_0^2 \theta'_0 \neq 0$.

- (a) Set up a second order initial value problem for $u = 1/r$ as a function of θ .
 (b) Determine $r = r(\theta)$ if (i) $h^2 < k$; (ii) $h^2 = k$; (iii) $h^2 > k$.

Chapter 9

Power Series Solutions of Linear Second Order Equations

9.1 Review of Power Series

Many applications give rise to differential equations with solutions that can't be expressed in terms of elementary functions such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric functions. The solutions of some of the most important of these equations can be expressed in terms of power series. We'll study such equations in this chapter. In this section we review relevant properties of power series. We'll omit proofs, which can be found in any standard calculus text.

Definition 9.1.1 An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n, \quad (9.1)$$

where x_0 and $a_0, a_1, \dots, a_n, \dots$ are constants, is called a *power series in $x - x_0$* . We say that the power series (9.1) *converges* for a given x if the limit

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n(x - x_0)^n$$

exists; otherwise, we say that the power series *diverges* for the given x .

A power series in $x - x_0$ must converge if $x = x_0$, since the positive powers of $x - x_0$ are all zero in this case. This may be the only value of x for which the power series converges. However, the next theorem shows that if the power series converges for some $x \neq x_0$ then the set of all values of x for which it converges forms an interval.

Theorem 9.1.2. For any power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

exactly one of the these statements is true:

- (i) The power series converges only for $x = x_0$.
- (ii) The power series converges for all values of x .
- (iii) There's a positive number R such that the power series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.

In case (iii) we say that R is the *radius of convergence* of the power series. For convenience, we include the other two cases in this definition by defining $R = 0$ in case (i) and $R = \infty$ in case (ii). We define the *open interval of convergence* of $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ to be

$$(x_0 - R, x_0 + R) \quad \text{if } 0 < R < \infty, \quad \text{or} \quad (-\infty, \infty) \quad \text{if } R = \infty.$$

If R is finite, no general statement can be made concerning convergence at the endpoints $x = x_0 \pm R$ of the open interval of convergence; the series may converge at one or both points, or diverge at both.

Recall from calculus that a series of constants $\sum_{n=0}^{\infty} \alpha_n$ is said to *converge absolutely* if the series of absolute values $\sum_{n=0}^{\infty} |\alpha_n|$ converges. It can be shown that a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ with a positive radius of convergence R converges absolutely in its open interval of convergence; that is, the series

$$\sum_{n=0}^{\infty} |a_n| |x - x_0|^n$$

of absolute values converges if $|x - x_0| < R$. However, if $R < \infty$, the series may fail to converge absolutely at an endpoint $x_0 \pm R$, even if it converges there.

The next theorem provides a useful method for determining the radius of convergence of a power series. It's derived in calculus by applying the ratio test to the corresponding series of absolute values. For related theorems see Exercises 9.1.2 and 9.1.4.

Theorem 9.1.3. *Suppose there's an integer N such that $a_n \neq 0$ if $n \geq N$ and*

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

where $0 \leq L \leq \infty$. Then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is $R = 1/L$, which should be interpreted to mean that $R = 0$ if $L = \infty$, or $R = \infty$ if $L = 0$.

Example 9.1.1: Find the radius of convergence of the series:

$$\text{(a)} \quad \sum_{n=0}^{\infty} n! x^n \quad \text{(b)} \quad \sum_{n=10}^{\infty} (-1)^n \frac{x^n}{n!} \quad \text{(c)} \quad \sum_{n=0}^{\infty} 2^n n^2 (x - 1)^n.$$

Solution (a): Here $a_n = n!$, so

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty.$$

Hence, $R = 0$.

Solution (b): Here $a_n = (1)^n / n!$ for $n \geq N = 10$, so

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Hence, $R = \infty$.

Solution (c): Here $a_n = 2^n n^2$, so

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2^{n+1} (n+1)^2}{2^n n^2} = 2 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^2 = 2.$$

Hence, $R = 1/2$.

9.1.1 Taylor Series

If a function f has derivatives of all orders at a point $x = x_0$, then the *Taylor series of f about x_0* is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

In the special case where $x_0 = 0$, this series is also called the *Maclaurin series of f* .

Taylor series for most of the common elementary functions converge to the functions on their open intervals of convergence. For example, you are probably familiar with the following Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty, \quad (9.2)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty, \quad (9.3)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty, \quad (9.4)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1. \quad (9.5)$$

9.1.2 Differentiation of Power Series

A power series with a positive radius of convergence defines a function

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

on its open interval of convergence. We say that the series *represents* f on the open interval of convergence. A function f represented by a power series may be a familiar elementary function as in (9.2)–(9.5); however, it often happens that f isn't a familiar function, so the series actually *defines* f .

The next theorem shows that a function represented by a power series has derivatives of all orders on the open interval of convergence of the power series, and provides power series representations of the derivatives.

Theorem 9.1.4. *A power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with positive radius of convergence R has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \quad (9.6)$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \quad (9.7)$$

$\vdots$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (x - x_0)^{n-k}. \quad (9.8)$$

Moreover, all of these series have the same radius of convergence R .

Example 9.1.2: Let $f(x) = \sin x$. From (9.3),

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

From (9.6),

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx} \left[\frac{x^{2n+1}}{(2n+1)!} \right] = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

which is the series (9.4) for $\cos x$.

9.1.3 Uniqueness of Power Series

The next theorem shows that if f is *defined* by a power series in $x - x_0$ with a positive radius of convergence, then the power series is the Taylor series of f about x_0 .

Theorem 9.1.5. *If the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

has a positive radius of convergence, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}; \quad (9.9)$$

that is, $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is the Taylor series of f about x_0 .

This result can be obtained by setting $x = x_0$ in (9.8), which yields

$$f^{(k)}(x_0) = k(k-1) \cdots 1 \cdot a_k = k! a_k.$$

This implies that

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

Except for notation, this is the same as (9.9).

The next theorem lists two important properties of power series that follow from Theorem 9.1.5.

Theorem 9.1.6.

(a) *If*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

for all x in an open interval that contains x_0 , then $a_n = b_n$ for $n = 0, 1, 2, \dots$

(b) *If*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0$$

for all x in an open interval that contains x_0 , then $a_n = 0$ for $n = 0, 1, 2, \dots$

To obtain (a) we observe that the two series represent the same function f on the open interval; hence, Theorem 9.1.5 implies that

$$a_n = b_n = \frac{f^{(n)}(x_0)}{n!}, \quad n = 0, 1, 2, \dots$$

(b) can be obtained from (a) by taking $b_n = 0$ for $n = 0, 1, 2, \dots$

9.1.4 Taylor Polynomials

If f has N derivatives at a point x_0 , we say that

$$T_N(x) = \sum_{n=0}^N \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

is the N -th Taylor polynomial of f about x_0 . This definition and Theorem 9.1.5 imply that if

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where the power series has a positive radius of convergence, then the Taylor polynomials of f about x_0 are given by

$$T_N(x) = \sum_{n=0}^N a_n (x - x_0)^n.$$

In numerical applications, we use the Taylor polynomials to approximate f on subintervals of the open interval of convergence of the power series. For example, (9.2) implies that the Taylor polynomial T_N of $f(x) = e^x$ is

$$T_N(x) = \sum_{n=0}^N \frac{x^n}{n!}.$$

The solid curve in Figure 9.1 is the graph of $y = e^x$ on the interval $[0, 5]$. The dotted curves in Figure 9.1 are the graphs of the Taylor polynomials $T_1, \dots, T_6$ of $y = e^x$ about $x_0 = 0$. From this figure, we conclude that the accuracy of the approximation of $y = e^x$ by its Taylor polynomial T_N improves as N increases.

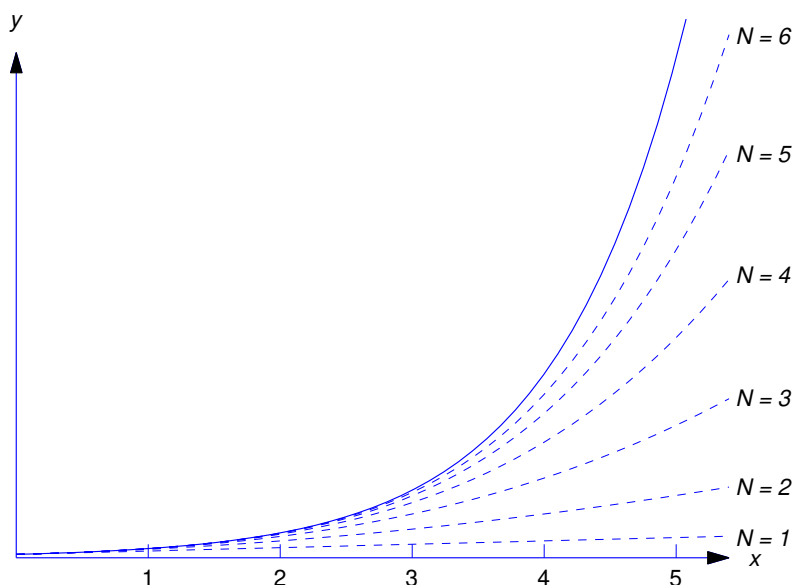


Figure 9.1: Approximation of $y = e^x$ by Taylor polynomials about $x = 0$

9.1.5 Shifting the Summation Index

In Definition 9.1.1 of a power series in $x - x_0$, the n -th term is a constant multiple of $(x - x_0)^n$. This isn't true in (9.6), (9.7), and (9.8), where the general terms are constant multiples of $(x - x_0)^{n-1}$, $(x - x_0)^{n-2}$, and $(x - x_0)^{n-k}$, respectively. However, these series can all be rewritten so that their n -th terms are constant multiples of $(x - x_0)^n$. For example, letting $n = k + 1$ in the series in (9.6) yields

$$f'(x) = \sum_{k=0}^{\infty} (k+1)a_{k+1}(x - x_0)^k, \quad (9.10)$$

where we start the new summation index k from zero so that the first term in (9.10) (obtained by setting $k = 0$) is the same as the first term in (9.6) (obtained by setting $n = 1$). However, the sum of a series is independent of the symbol used to denote the summation index, just as the value of a definite integral is independent of the symbol used to denote the variable of integration. Therefore we can replace k by n in (9.10) to obtain

$$f'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x - x_0)^n, \quad (9.11)$$

where the general term is a constant multiple of $(x - x_0)^n$.

It isn't really necessary to introduce the intermediate summation index k . We can obtain (9.11) directly from (9.6) by replacing n by $n + 1$ in the general term of (9.6) and subtracting 1 from the lower limit of (9.6). More generally, we use the following procedure for shifting indices.

Shifting the Summation Index in a Power Series

For any integer k , the power series

$$\sum_{n=n_0}^{\infty} b_n(x - x_0)^{n-k}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} b_{n+k}(x - x_0)^n;$$

that is, replacing n by $n + k$ in the general term and subtracting k from the lower limit of summation leaves the series unchanged.

Example 9.1.3: Rewrite the following power series from (9.7) and (9.8) so that the general term in each is a constant multiple of $(x - x_0)^n$:

$$\text{(a)} \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2} \quad \text{(b)} \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x - x_0)^{n-k}.$$

Solution (a): Replacing n by $n + 2$ in the general term and subtracting 2 from the lower limit of summation yields

$$\sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x - x_0)^n.$$

Solution (b): Replacing n by $n + k$ in the general term and subtracting k from the lower limit of summation yields

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x - x_0)^{n-k} = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1)a_{n+k}(x - x_0)^n.$$

Example 9.1.4: Given that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

write the function xf'' as a power series in which the general term is a constant multiple of x^n .

Solution: From Theorem 9.1.4 with $x_0 = 0$,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$xf''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}.$$

Replacing n by $n+1$ in the general term and subtracting 1 from the lower limit of summation yields

$$xf''(x) = \sum_{n=1}^{\infty} (n+1)na_{n+1}x^n.$$

We can also write this as

$$xf''(x) = \sum_{n=0}^{\infty} (n+1)na_{n+1}x^n,$$

since the first term in this last series is zero. (We'll see later that sometimes it's useful to include zero terms at the beginning of a series.)

9.1.6 Linear Combinations of Power Series

If a power series is multiplied by a constant, then the constant can be placed inside the summation; that is,

$$c \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} ca_n (x - x_0)^n.$$

Two power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

with positive radii of convergence can be added term by term at points common to their open intervals of convergence; thus, if the first series converges for $|x - x_0| < R_1$ and the second converges for $|x - x_0| < R_2$, then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$

for $|x - x_0| < R$, where R is the smaller of R_1 and R_2 . More generally, linear combinations of power series can be formed term by term; for example,

$$c_1 f(x) + c_2 g(x) = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n)(x - x_0)^n.$$

Example 9.1.5: Find the Maclaurin series for $\cosh x$ as a linear combination of the Maclaurin series for e^x and e^{-x} .

Solution: By definition,

$$\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$

Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!},$$

it follows that

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{2} [1 + (-1)^n] \frac{x^n}{n!}. \quad (9.12)$$

Since

$$\frac{1}{2} [1 + (-1)^n] = \begin{cases} 1 & \text{if } n = 2m, \text{ an even integer,} \\ 0 & \text{if } n = 2m + 1, \text{ an odd integer,} \end{cases}$$

we can rewrite (9.12) more simply as

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}.$$

This result is valid on $(-\infty, \infty)$, since this is the open interval of convergence of the Maclaurin series for e^x and e^{-x} .

Example 9.1.6: Suppose

$$y = \sum_{n=0}^{\infty} a_n x^n$$

on an open interval I that contains the origin.

(a) Express

$$(2 - x)y'' + 2y$$

as a power series in x on I .

(b) Use the result of (a) to find necessary and sufficient conditions on the coefficients $\{a_n\}$ for y to be a solution of the homogeneous equation

$$(2 - x)y'' + 2y = 0 \quad (9.13)$$

on I .

Solution (a): From (9.7) with $x_0 = 0$,

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$\begin{aligned} (2 - x)y'' + 2y &= 2y'' - xy'' + 2y \\ &= \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n. \end{aligned} \quad (9.14)$$

To combine the three series we shift indices in the first two to make their general terms constant multiples of x^n ; thus,

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n \quad (9.15)$$

and

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1)na_{n+1} x^n = \sum_{n=0}^{\infty} (n+1)na_{n+1} x^n, \quad (9.16)$$

where we added a zero term in the last series so that when we substitute from (9.15) and (9.16) into (9.14) all three series will start with $n = 0$; thus,

$$(2-x)y'' + 2y = \sum_{n=0}^{\infty} [2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n]x^n. \quad (9.17)$$

Solution (b): From (9.17) we see that y satisfies (9.13) on I if

$$2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n = 0, \quad n = 0, 1, 2, \dots \quad (9.18)$$

Conversely, Theorem 9.1.6 (b) implies that if $y = \sum_{n=0}^{\infty} a_n x^n$ satisfies (9.13) on I , then (9.18) holds.

Example 9.1.7: Suppose

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

on an open interval I that contains $x_0 = 1$. Express the function

$$(1+x)y'' + 2(x-1)^2 y' + 3y \quad (9.19)$$

as a power series in $x-1$ on I .

Solution: Since we want a power series in $x-1$, we rewrite the coefficient of y'' in (9.19) as $1+x = 2+(x-1)$, so (9.19) becomes

$$2y'' + (x-1)y'' + 2(x-1)^2 y' + 3y.$$

From (9.6) and (9.7) with $x_0 = 1$,

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}.$$

Therefore

$$\begin{aligned} 2y'' &= \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2}, \\ (x-1)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1}, \\ 2(x-1)^2 y' &= \sum_{n=1}^{\infty} 2n a_n (x-1)^{n+1}, \\ 3y &= \sum_{n=0}^{\infty} 3a_n (x-1)^n. \end{aligned}$$

Before adding these four series we shift indices in the first three so that their general terms become constant multiples of $(x-1)^n$. This yields

$$2y'' = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n, \quad (9.20)$$

$$(x-1)y'' = \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n, \quad (9.21)$$

$$2(x-1)^2 y' = \sum_{n=1}^{\infty} 2(n-1)a_{n-1}(x-1)^n, \quad (9.22)$$

$$3y = \sum_{n=0}^{\infty} 3a_n (x-1)^n, \quad (9.23)$$

where we added initial zero terms to the series in (9.21) and (9.22). Adding (9.20)–(9.23) yields

$$\begin{aligned}(1+x)y'' + 2(x-1)^2y' + 3y &= 2y'' + (x-1)y'' + 2(x-1)^2y' + 3y \\ &= \sum_{n=0}^{\infty} b_n(x-1)^n,\end{aligned}$$

where

$$b_0 = 4a_2 + 3a_0, \quad (9.24)$$

$$b_n = 2(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2(n-1)a_{n-1} + 3a_n, \quad n \geq 1. \quad (9.25)$$

The formula (9.24) for b_0 can't be obtained by setting $n = 0$ in (9.25), since the summation in (9.22) begins with $n = 1$, while those in (9.20), (9.21), and (9.23) begin with $n = 0$.

9.1.7 Exercises

- For each power series use Theorem 9.1.3 to find the radius of convergence R . If $R > 0$, find the open interval of convergence.

$$(a) \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n} (x-1)^n$$

$$(b) \sum_{n=0}^{\infty} 2^n n (x-2)^n$$

$$(c) \sum_{n=0}^{\infty} \frac{n!}{9^n} x^n$$

$$(d) \sum_{n=0}^{\infty} \frac{n(n+1)}{16^n} (x-2)^n$$

$$(e) \sum_{n=0}^{\infty} (-1)^n \frac{7^n}{n!} x^n$$

$$(f) \sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}(n+1)^2} (x+7)^n$$

- Suppose there's an integer M such that $b_m \neq 0$ for $m \geq M$, and

$$\lim_{m \rightarrow \infty} \left| \frac{b_{m+1}}{b_m} \right| = L,$$

where $0 \leq L \leq \infty$. Show that the radius of convergence of

$$\sum_{m=0}^{\infty} b_m (x - x_0)^{2m}$$

is $R = 1/\sqrt{L}$, which is interpreted to mean that $R = 0$ if $L = \infty$ or $R = \infty$ if $L = 0$. *HINT: Apply Theorem 9.1.3 to the series $\sum_{m=0}^{\infty} b_m z^m$ and then let $z = (x - x_0)^2$.*

- For each power series, use the result of Exercise 9.1.2 to find the radius of convergence R . If $R > 0$, find the open interval of convergence.

$$(a) \sum_{m=0}^{\infty} (-1)^m (3m+1) (x-1)^{2m+1}$$

$$(b) \sum_{m=0}^{\infty} (-1)^m \frac{m(2m+1)}{2^m} (x+2)^{2m}$$

$$(c) \sum_{m=0}^{\infty} \frac{m!}{(2m)!} (x-1)^{2m}$$

$$(d) \sum_{m=0}^{\infty} (-1)^m \frac{m!}{9^m} (x+8)^{2m}$$

$$(e) \sum_{m=0}^{\infty} (-1)^m \frac{(2m-1)}{3^m} x^{2m+1}$$

$$(f) \sum_{m=0}^{\infty} (x-1)^{2m}$$

- Suppose there's an integer M such that $b_m \neq 0$ for $m \geq M$, and

$$\lim_{m \rightarrow \infty} \left| \frac{b_{m+1}}{b_m} \right| = L,$$

where $0 \leq L \leq \infty$. Let k be a positive integer. Show that the radius of convergence of

$$\sum_{m=0}^{\infty} b_m (x - x_0)^{km}$$

is $R = 1/\sqrt[k]{L}$, which is interpreted to mean that $R = 0$ if $L = \infty$ or $R = \infty$ if $L = 0$. **HINT:** Apply Theorem 9.1.3 to the series $\sum_{m=0}^{\infty} b_m z^m$ and then let $z = (x - x_0)^k$.

5. For each power series use the result of Exercise 9.1.4 to find the radius of convergence R . If $R > 0$, find the open interval of convergence.

(a) $\sum_{m=0}^{\infty} \frac{(-1)^m}{(27)^m} (x - 3)^{3m+2}$

(b) $\sum_{m=0}^{\infty} \frac{x^{7m+6}}{m}$

(c) $\sum_{m=0}^{\infty} \frac{9^m (m+1)}{(m+2)} (x-3)^{4m+2}$

(d) $\sum_{m=0}^{\infty} (-1)^m \frac{2^m}{m!} x^{4m+3}$

(e) $\sum_{m=0}^{\infty} \frac{m!}{(26)^m} (x+1)^{4m+3}$

(f) $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m m(m+1)} (x-1)^{3m+1}$

6. L Graph $y = \sin x$ and the Taylor polynomial

$$T_{2M+1}(x) = \sum_{n=0}^M \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1, 2, 3, \dots$, until you find a value of M for which there's no perceptible difference between the two graphs.

7. L Graph $y = \cos x$ and the Taylor polynomial

$$T_{2M}(x) = \sum_{n=0}^M \frac{(-1)^n x^{2n}}{(2n)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1, 2, 3, \dots$, until you find a value of M for which there's no perceptible difference between the two graphs.

8. L Graph $y = 1/(1-x)$ and the Taylor polynomial

$$T_N(x) = \sum_{n=0}^N x^n$$

on the interval $[0, .95]$ for $N = 1, 2, 3, \dots$, until you find a value of N for which there's no perceptible difference between the two graphs. Choose the scale on the y -axis so that $0 \leq y \leq 20$.

9. L Graph $y = \cosh x$ and the Taylor polynomial

$$T_{2M}(x) = \sum_{n=0}^M \frac{x^{2n}}{(2n)!}$$

on the interval $(-5, 5)$ for $M = 1, 2, 3, \dots$, until you find a value of M for which there's no perceptible difference between the two graphs. Choose the scale on the y -axis so that $0 \leq y \leq 75$.

10. L Graph $y = \sinh x$ and the Taylor polynomial

$$T_{2M+1}(x) = \sum_{n=0}^M \frac{x^{2n+1}}{(2n+1)!}$$

on the interval $(-5, 5)$ for $M = 0, 1, 2, \dots$, until you find a value of M for which there's no perceptible difference between the two graphs. Choose the scale on the y -axis so that $-75 \leq y \leq 75$.

In Exercises 9.1.11–9.1.15 find a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

11. $(2+x)y'' + xy' + 3y$

12. $(1+3x^2)y'' + 3x^2y' - 2y$

13. $(1+2x^2)y'' + (2-3x)y' + 4y$

14. $(1+x^2)y'' + (2-x)y' + 3y$

15. $(1+3x^2)y'' - 2xy' + 4y$

16. Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x+1)^n$ on an open interval that contains $x_0 = -1$. Find a power series in $x+1$ for

$$xy'' + (4+2x)y' + (2+x)y.$$

17. Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x-2)^n$ on an open interval that contains $x_0 = 2$. Find a power series in $x-2$ for

$$x^2y'' + 2xy' - 3xy.$$

18. **L** Do the following experiment for various choices of real numbers a_0 and a_1 .

- (a) Use differential equations software to solve the initial value problem

$$(2-x)y'' + 2y = 0, \quad y(0) = a_0, \quad y'(0) = a_1,$$

numerically on $(-1.95, 1.95)$. Choose the most accurate method your software package provides. (See Section 10.1 for a brief discussion of one such method.)

- (b) For $N = 2, 3, 4, \dots$, compute $a_2, \dots, a_N$ from Eqn.(9.18) and graph

$$T_N(x) = \sum_{n=0}^N a_n x^n$$

and the solution obtained in (a) on the same axes. Continue increasing N until it's obvious that there's no point in continuing. (This sounds vague, but you'll know when to stop.)

19. **L** Follow the directions of Exercise 9.1.18 for the initial value problem

$$(1+x)y'' + 2(x-1)^2y' + 3y = 0, \quad y(1) = a_0, \quad y'(1) = a_1,$$

on the interval $(0, 2)$. Use Eqns. (9.24) and (9.25) to compute $\{a_n\}$.

20. Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ converges on an open interval $(-R, R)$, let r be an arbitrary real number, and define

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

on $(0, R)$. Use Theorem 9.1.4 and the rule for differentiating the product of two functions to show that

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1},$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2},$$

$\vdots$

$$y^{(k)}(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) \cdots (n+r-k) a_n x^{n+r-k}$$

on $(0, R)$

In Exercises 9.1.21–9.1.26 let y be as defined in Exercise 9.1.20, and write the given expression in the form $x^r \sum_{n=0}^{\infty} b_n x^n$.

- 21. $x^2(1-x)y'' + x(4+x)y' + (2-x)y$
- 22. $x^2(1+x)y'' + x(1+2x)y' - (4+6x)y$
- 23. $x^2(1+x)y'' - x(1-6x-x^2)y' + (1+6x+x^2)y$
- 24. $x^2(1+3x)y'' + x(2+12x+x^2)y' + 2x(3+x)y$
- 25. $x^2(1+2x^2)y'' + x(4+2x^2)y' + 2(1-x^2)y$
- 26. $x^2(2+x^2)y'' + 2x(5+x^2)y' + 2(3-x^2)y$

9.2 Series Solution Near an Ordinary Point I

Many physical applications give rise to second order homogeneous linear differential equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad (9.26)$$

where P_0 , P_1 , and P_2 are polynomials. Usually the solutions of these equations can't be expressed in terms of familiar elementary functions. Therefore we'll consider the problem of representing solutions of (9.26) with series.

We assume throughout that P_0 , P_1 and P_2 have no common factors. Then we say that x_0 is an *ordinary point* of (9.26) if $P_0(x_0) \neq 0$, or a *singular point* if $P_0(x_0) = 0$. For Legendre's equation,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad (9.27)$$

$x_0 = 1$ and $x_0 = -1$ are singular points and all other points are ordinary points. For Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

$x_0 = 0$ is a singular point and all other points are ordinary points. If P_0 is a nonzero constant as in Airy's equation,

$$y'' - xy = 0, \quad (9.28)$$

then every point is an ordinary point.

Since polynomials are continuous everywhere, P_1/P_0 and P_2/P_0 are continuous at any point x_0 that isn't a zero of P_0 . Therefore, if x_0 is an ordinary point of (9.26) and a_0 and a_1 are arbitrary real numbers, then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1 \quad (9.29)$$

has a unique solution on the largest open interval that contains x_0 and does not contain any zeros of P_0 . To see this, we rewrite the differential equation in (9.29) as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 6.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$. In this section and the next we consider the problem of representing solutions of (9.26) by power series that converge for values of x near an ordinary point x_0 .

We state the next theorem without proof.

Theorem 9.2.1. *Suppose P_0 , P_1 , and P_2 are polynomials with no common factor and P_0 isn't identically zero. Let x_0 be a point such that $P_0(x_0) \neq 0$, and let ρ be the distance from x_0 to the nearest zero of P_0 in the complex plane. (If P_0 is constant, then $\rho = \infty$.) Then every solution of*

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \quad (9.30)$$

can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad (9.31)$$

that converges at least on the open interval $(x_0 - \rho, x_0 + \rho)$. (If P_0 is nonconstant, so that ρ is necessarily finite, then the open interval of convergence of (9.31) may be larger than $(x_0 - \rho, x_0 + \rho)$. If P_0 is constant then $\rho = \infty$ and $(x_0 - \rho, x_0 + \rho) = (-\infty, \infty)$.)

We call (9.31) a *power series solution in $x - x_0$* of (9.30). We'll now develop a method for finding power series solutions of (9.30). For this purpose we write (9.30) as $Ly = 0$, where

$$Ly = P_0 y'' + P_1 y' + P_2 y. \quad (9.32)$$

Theorem 9.2.1 implies that every solution of $Ly = 0$ on $(x_0 - \rho, x_0 + \rho)$ can be written as

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Setting $x = x_0$ in this series and in the series

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

shows that $y(x_0) = a_0$ and $y'(x_0) = a_1$. Since every initial value problem (9.29) has a unique solution, this means that a_0 and a_1 can be chosen arbitrarily, and $a_2, a_3, \dots$ are uniquely determined by them.

To find $a_2, a_3, \dots$, we write P_0, P_1 , and P_2 in powers of $x - x_0$, substitute

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

into (9.32), and collect the coefficients of like powers of $x - x_0$. This yields

$$Ly = \sum_{n=0}^{\infty} b_n (x - x_0)^n, \quad (9.33)$$

where $\{b_0, b_1, \dots, b_n, \dots\}$ are expressed in terms of $\{a_0, a_1, \dots, a_n, \dots\}$ and the coefficients of P_0, P_1 , and P_2 , written in powers of $x - x_0$. Since (9.33) and (a) of Theorem 9.1.6 imply that $Ly = 0$ if and only if $b_n = 0$ for $n \geq 0$, all power series solutions in $x - x_0$ of $Ly = 0$ can be obtained by choosing a_0 and a_1 arbitrarily and computing $a_2, a_3, \dots$, successively so that $b_n = 0$ for $n \geq 0$. For simplicity, we call the power series obtained this way *the power series in $x - x_0$ for the general solution* of $Ly = 0$, without explicitly identifying the open interval of convergence of the series.

Example 9.2.1: Let x_0 be an arbitrary real number. Find the power series in $x - x_0$ for the general solution of

$$y'' + y = 0. \quad (9.34)$$

Solution: Here

$$Ly = y'' + y.$$

If

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

so

$$Ly = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + \sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

To collect coefficients of like powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-x_0)^n + \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n,$$

with

$$b_n = (n+2)(n+1)a_{n+2} + a_n.$$

Therefore $Ly = 0$ if and only if

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0, \quad (9.35)$$

where a_0 and a_1 are arbitrary. Since the indices on the left and right sides of (9.35) differ by two, we write (9.35) separately for n even ($n = 2m$) and n odd ($n = 2m + 1$). This yields

$$a_{2m+2} = \frac{-a_{2m}}{(2m+2)(2m+1)}, \quad m \geq 0, \quad (9.36)$$

and

$$a_{2m+3} = \frac{-a_{2m+1}}{(2m+3)(2m+2)}, \quad m \geq 0. \quad (9.37)$$

Computing the coefficients of the even powers of $x - x_0$ from (9.36) yields

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 1} \\ a_4 &= -\frac{a_2}{4 \cdot 3} = -\frac{1}{4 \cdot 3} \left(-\frac{a_0}{2 \cdot 1} \right) = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \\ a_6 &= -\frac{a_4}{6 \cdot 5} = -\frac{1}{6 \cdot 5} \left(\frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} \right) = -\frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, \end{aligned}$$

and, in general,

$$a_{2m} = (-1)^m \frac{a_0}{(2m)!}, \quad m \geq 0. \quad (9.38)$$

Computing the coefficients of the odd powers of $x - x_0$ from (9.37) yields

$$\begin{aligned} a_3 &= -\frac{a_1}{3 \cdot 2} \\ a_5 &= -\frac{a_3}{5 \cdot 4} = -\frac{1}{5 \cdot 4} \left(-\frac{a_1}{3 \cdot 2} \right) = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \\ a_7 &= -\frac{a_5}{7 \cdot 6} = -\frac{1}{7 \cdot 6} \left(\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2} \right) = -\frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}, \end{aligned}$$

and, in general,

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!}, \quad m \geq 0. \quad (9.39)$$

Thus, the general solution of (9.34) can be written as

$$y = \sum_{m=0}^{\infty} a_{2m}(x-x_0)^{2m} + \sum_{m=0}^{\infty} a_{2m+1}(x-x_0)^{2m+1},$$

or, from (9.38) and (9.39), as

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m+1}}{(2m+1)!}. \quad (9.40)$$

If we recall from calculus that

$$\sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m}}{(2m)!} = \cos(x - x_0) \quad \text{and} \quad \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m+1}}{(2m+1)!} = \sin(x - x_0),$$

then (9.40) becomes

$$y = a_0 \cos(x - x_0) + a_1 \sin(x - x_0),$$

which should look familiar. ■

Equations like (9.35), (9.36), and (9.37), which define a given coefficient in the sequence $\{a_n\}$ in terms of one or more coefficients with lesser indices are called *recurrence relations*. When we use a recurrence relation to compute terms of a sequence we're computing *recursively*.

In the remainder of this section we consider the problem of finding power series solutions in $x - x_0$ for equations of the form

$$(1 + \alpha(x - x_0)^2) y'' + \beta(x - x_0) y' + \gamma y = 0. \quad (9.41)$$

Many important equations that arise in applications are of this form with $x_0 = 0$, including Legendre's equation (9.27), Airy's equation (9.28), *Chebyshev's equation*,

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

and *Hermite's equation*,

$$y'' - 2xy' + 2\alpha y = 0.$$

Since

$$P_0(x) = 1 + \alpha(x - x_0)^2$$

in (9.41), the point x_0 is an ordinary point of (9.41), and Theorem 9.2.1 implies that the solutions of (9.41) can be written as power series in $x - x_0$ that converge on the interval $(x_0 - 1/\sqrt{|\alpha|}, x_0 + 1/\sqrt{|\alpha|})$ if $\alpha \neq 0$, or on $(-\infty, \infty)$ if $\alpha = 0$. We'll see that the coefficients in these power series can be obtained by methods similar to the one used in Example 9.2.1.

To simplify finding the coefficients, we introduce some notation for products:

$$\prod_{j=r}^s b_j = b_r b_{r+1} \cdots b_s \quad \text{if } s \geq r.$$

Thus,

$$\prod_{j=2}^7 b_j = b_2 b_3 b_4 b_5 b_6 b_7,$$

$$\prod_{j=0}^4 (2j+1) = (1)(3)(5)(7)(9) = 945,$$

and

$$\prod_{j=2}^2 j^2 = 2^2 = 4.$$

We define

$$\prod_{j=r}^s b_j = 1 \quad \text{if } s < r,$$

no matter what the form of b_j .

Example 9.2.2: Find the power series in x for the general solution of

$$(1 + 2x^2)y'' + 6xy' + 2y = 0. \quad (9.42)$$

Solution: Here

$$Ly = (1 + 2x^2)y'' + 6xy' + 2y.$$

If

$$y = \sum_{n=0}^{\infty} a_n x^n$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

so

$$\begin{aligned} Ly &= (1 + 2x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} [2n(n-1) + 6n + 2] a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} (n+1)^2 a_n x^n. \end{aligned}$$

To collect coefficients of x^n , we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2 \sum_{n=0}^{\infty} (n+1)^2 a_n x^n = \sum_{n=0}^{\infty} b_n x^n,$$

with

$$b_n = (n+2)(n+1) a_{n+2} + 2(n+1)^2 a_n, \quad n \geq 0.$$

To obtain solutions of (9.42), we set $b_n = 0$ for $n \geq 0$. This is equivalent to the recurrence relation

$$a_{n+2} = -2 \frac{n+1}{n+2} a_n, \quad n \geq 0. \quad (9.43)$$

Since the indices on the left and right differ by two, we write (9.43) separately for $n = 2m$ and $n = 2m + 1$, as in Example 9.2.1. This yields

$$a_{2m+2} = -2 \frac{2m+1}{2m+2} a_{2m} = -\frac{2m+1}{m+1} a_{2m}, \quad m \geq 0, \quad (9.44)$$

and

$$a_{2m+3} = -2 \frac{2m+2}{2m+3} a_{2m+1} = -4 \frac{m+1}{2m+3} a_{2m+1}, \quad m \geq 0. \quad (9.45)$$

Computing the coefficients of even powers of x from (9.44) yields

$$\begin{aligned} a_2 &= -\frac{1}{1}a_0, \\ a_4 &= -\frac{3}{2}a_2 = \left(-\frac{3}{2}\right)\left(-\frac{1}{1}\right)a_0 = \frac{1 \cdot 3}{1 \cdot 2}a_0, \\ a_6 &= -\frac{5}{3}a_4 = -\frac{5}{3}\left(\frac{1 \cdot 3}{1 \cdot 2}\right)a_0 = -\frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3}a_0, \\ a_8 &= -\frac{7}{4}a_6 = -\frac{7}{4}\left(-\frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3}\right)a_0 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4}a_0. \end{aligned}$$

In general,

$$a_{2m} = (-1)^m \frac{\prod_{j=1}^m (2j-1)}{m!} a_0, \quad m \geq 0. \quad (9.46)$$

(Note that (9.46) is correct for $m = 0$ because we defined $\prod_{j=1}^0 b_j = 1$ for any b_j .)

Computing the coefficients of odd powers of x from (9.45) yields

$$\begin{aligned} a_3 &= -4\frac{1}{3}a_1, \\ a_5 &= -4\frac{2}{5}a_3 = -4\frac{2}{5}\left(-4\frac{1}{3}\right)a_1 = 4^2\frac{1 \cdot 2}{3 \cdot 5}a_1, \\ a_7 &= -4\frac{3}{7}a_5 = -4\frac{3}{7}\left(4^2\frac{1 \cdot 2}{3 \cdot 5}\right)a_1 = -4^3\frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}a_1, \\ a_9 &= -4\frac{4}{9}a_7 = -4\frac{4}{9}\left(-4^3\frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}\right)a_1 = 4^4\frac{1 \cdot 2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7 \cdot 9}a_1. \end{aligned}$$

In general,

$$a_{2m+1} = \frac{(-1)^m 4^m m!}{\prod_{j=1}^m (2j+1)} a_1, \quad m \geq 0. \quad (9.47)$$

From (9.46) and (9.47),

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^m (2j-1)}{m!} x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{4^m m!}{\prod_{j=1}^m (2j+1)} x^{2m+1}.$$

is the power series in x for the general solution of (9.42). Since $P_0(x) = 1+2x^2$ has no real zeros, Theorem 6.1.1 implies that every solution of (9.42) is defined on $(-\infty, \infty)$. However, since $P_0(\pm i/\sqrt{2}) = 0$, Theorem 9.2.1 implies only that the power series converges in $(-1/\sqrt{2}, 1/\sqrt{2})$ for any choice of a_0 and a_1 .

The results in Examples 9.2.1 and 9.2.2 are consequences of the following general theorem.

Theorem 9.2.2. *The coefficients $\{a_n\}$ in any solution $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ of*

$$(1 + \alpha(x-x_0)^2) y'' + \beta(x-x_0)y' + \gamma y = 0 \quad (9.48)$$

satisfy the recurrence relation

$$a_{n+2} = -\frac{p(n)}{(n+2)(n+1)} a_n, \quad n \geq 0, \quad (9.49)$$

where

$$p(n) = \alpha n(n-1) + \beta n + \gamma. \quad (9.50)$$

Moreover, the coefficients of the even and odd powers of $x - x_0$ can be computed separately as

$$a_{2m+2} = -\frac{p(2m)}{(2m+2)(2m+1)}a_{2m}, \quad m \geq 0 \quad (9.51)$$

and

$$a_{2m+3} = -\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1}, \quad m \geq 0, \quad (9.52)$$

where a_0 and a_1 are arbitrary.

Proof. Here

$$Ly = (1 + \alpha(x - x_0)^2)y'' + \beta(x - x_0)y' + \gamma y.$$

If

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

then

$$y' = \sum_{n=1}^{\infty} n a_n(x - x_0)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2}.$$

Hence,

$$\begin{aligned} Ly &= \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2} + \sum_{n=0}^{\infty} [\alpha n(n-1) + \beta n + \gamma] a_n(x - x_0)^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2} + \sum_{n=0}^{\infty} p(n)a_n(x - x_0)^n, \end{aligned}$$

from (9.50). To collect coefficients of powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + p(n)a_n] (x - x_0)^n.$$

Thus, $Ly = 0$ if and only if

$$(n+2)(n+1)a_{n+2} + p(n)a_n = 0, \quad n \geq 0,$$

which is equivalent to (9.49). Writing (9.49) separately for the cases where $n = 2m$ and $n = 2m + 1$ yields (9.51) and (9.52).

Example 9.2.3: Find the power series in $x - 1$ for the general solution of

$$(2 + 4x - 2x^2)y'' - 12(x - 1)y' - 12y = 0. \quad (9.53)$$

Solution: We must first write the coefficient $P_0(x) = 2 + 4x - x^2$ in powers of $x - 1$. To do this, we write $x = (x - 1) + 1$ in $P_0(x)$ and then expand the terms, collecting powers of $x - 1$; thus,

$$\begin{aligned} 2 + 4x - 2x^2 &= 2 + 4[(x - 1) + 1] - 2[(x - 1) + 1]^2 \\ &= 4 - 2(x - 1)^2. \end{aligned}$$

Therefore we can rewrite (9.53) as

$$(4 - 2(x - 1)^2)y'' - 12(x - 1)y' - 12y = 0,$$

or, equivalently,

$$\left(1 - \frac{1}{2}(x - 1)^2\right)y'' - 3(x - 1)y' - 3y = 0.$$

This is of the form (9.48) with $\alpha = -1/2$, $\beta = -3$, and $\gamma = -3$. Therefore, from (9.50)

$$p(n) = -\frac{n(n-1)}{2} - 3n - 3 = -\frac{(n+2)(n+3)}{2}.$$

Hence, Theorem 9.2.2 implies that

$$\begin{aligned} a_{2m+2} &= -\frac{p(2m)}{(2m+2)(2m+1)} a_{2m} \\ &= \frac{(2m+2)(2m+3)}{2(2m+2)(2m+1)} a_{2m} = \frac{2m+3}{2(2m+1)} a_{2m}, \quad m \geq 0 \end{aligned}$$

and

$$\begin{aligned} a_{2m+3} &= -\frac{p(2m+1)}{(2m+3)(2m+2)} a_{2m+1} \\ &= \frac{(2m+3)(2m+4)}{2(2m+3)(2m+2)} a_{2m+1} = \frac{m+2}{2(m+1)} a_{2m+1}, \quad m \geq 0. \end{aligned}$$

We leave it to you to show that

$$a_{2m} = \frac{2m+1}{2^m} a_0 \quad \text{and} \quad a_{2m+1} = \frac{m+1}{2^m} a_1, \quad m \geq 0,$$

which implies that the power series in $x-1$ for the general solution of (9.53) is

$$y = a_0 \sum_{m=0}^{\infty} \frac{2m+1}{2^m} (x-1)^{2m} + a_1 \sum_{m=0}^{\infty} \frac{m+1}{2^m} (x-1)^{2m+1}. \blacksquare$$

In the examples considered so far we were able to obtain closed formulas for coefficients in the power series solutions. In some cases this is impossible, and we must settle for computing a finite number of terms in the series. The next example illustrates this with an initial value problem.

Example 9.2.4: Compute $a_0, a_1, \dots, a_7$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1+2x^2)y'' + 10xy' + 8y = 0, \quad y(0) = 2, \quad y'(0) = -3. \quad (9.54)$$

Solution: Since $\alpha = 2$, $\beta = 10$, and $\gamma = 8$ in (9.54),

$$p(n) = 2n(n-1) + 10n + 8 = 2(n+2)^2.$$

Therefore

$$a_{n+2} = -2 \frac{(n+2)^2}{(n+2)(n+1)} a_n = -2 \frac{n+2}{n+1} a_n, \quad n \geq 0.$$

Writing this equation separately for $n = 2m$ and $n = 2m+1$ yields

$$a_{2m+2} = -2 \frac{(2m+2)}{2m+1} a_{2m} = -4 \frac{m+1}{2m+1} a_{2m}, \quad m \geq 0 \quad (9.55)$$

and

$$a_{2m+3} = -2 \frac{2m+3}{2m+2} a_{2m+1} = -\frac{2m+3}{m+1} a_{2m+1}, \quad m \geq 0. \quad (9.56)$$

Starting with $a_0 = y(0) = 2$, we compute a_2, a_4 , and a_6 from (9.55):

$$\begin{aligned} a_2 &= -4 \frac{1}{1} 2 = -8, \\ a_4 &= -4 \frac{2}{3} (-8) = \frac{64}{3}, \\ a_6 &= -4 \frac{3}{5} \left(\frac{64}{3} \right) = -\frac{256}{5}. \end{aligned}$$

Starting with $a_1 = y'(0) = -3$, we compute a_3, a_5 and a_7 from (9.56):

$$\begin{aligned} a_3 &= -\frac{3}{1} (-3) = 9, \\ a_5 &= -\frac{5}{2} 9 = -\frac{45}{2}, \\ a_7 &= -\frac{7}{3} \left(-\frac{45}{2} \right) = \frac{105}{2}. \end{aligned}$$

Therefore the solution of (9.54) is

$$y = 2 - 3x - 8x^2 + 9x^3 + \frac{64}{3}x^4 - \frac{45}{2}x^5 - \frac{256}{5}x^6 + \frac{105}{2}x^7 + \dots$$

Computing coefficients recursively as in Example 9.2.4 is tedious. We recommend that you do this kind of computation by writing a short program to implement the appropriate recurrence relation on a calculator or computer. You may wish to do this in verifying examples and doing exercises (identified by the symbol ) in this chapter that call for numerical computation of the coefficients in series solutions. We obtained the answers to these exercises by using software that can produce answers in the form of rational numbers. However, it's perfectly acceptable - and more practical - to get your answers in decimal form. You can always check them by converting our fractions to decimals.

If you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficients is essentially irrelevant. For computational purposes it's usually more efficient to start with the given coefficients $a_0 = y(x_0)$ and $a_1 = y'(x_0)$, compute $a_2, \dots, a_N$ recursively, and then compute approximate values of the solution from the Taylor polynomial

$$T_N(x) = \sum_{n=0}^N a_n (x - x_0)^n.$$

The trick is to decide how to choose N so the approximation $y(x) \approx T_N(x)$ is sufficiently accurate on the subinterval of the interval of convergence that you're interested in. In the computational exercises in this and the next two sections, you will often be asked to obtain the solution of a given problem by numerical integration with software of your choice (see Section 10.1 for a brief discussion of one such method), and to compare the solution obtained in this way with the approximations obtained with T_N for various values of N . This is a typical textbook kind of exercise, designed to give you insight into how the accuracy of the approximation $y(x) \approx T_N(x)$ behaves as a function of N and the interval that you're working on. In real life, you would choose one or the other of the two methods (numerical integration or series solution). If you choose the method of series solution, then a practical procedure for determining a suitable value of N is to continue increasing N until the maximum of $|T_N - T_{N-1}|$ on the interval of interest is within the margin of error that you're willing to accept.

In doing computational problems that call for numerical solution of differential equations you should choose the most accurate numerical integration procedure your software supports, and experiment with the step size until you're confident that the numerical results are sufficiently accurate for the problem at hand.

9.2.1 Exercises

In Exercises 9.2.1–9.2.8 find the power series in x for the general solution.

1. $(1 + x^2)y'' + 6xy' + 6y = 0$

2. $(1 + x^2)y'' + 2xy' - 2y = 0$

3. $(1 + x^2)y'' - 8xy' + 20y = 0$

4. $(1 - x^2)y'' - 8xy' - 12y = 0$

5. $(1 + 2x^2)y'' + 7xy' + 2y = 0$

6. $(1 + x^2)y'' + 2xy' + \frac{1}{4}y = 0$

7. $(1 - x^2)y'' - 5xy' - 4y = 0$

8. $(1 + x^2)y'' - 10xy' + 28y = 0$

9. L

- (a) Find the power series in x for the general solution of $y'' + xy' + 2y = 0$.
 (b) For several choices of a_0 and a_1 , use differential equations software to solve the initial value problem

$$y'' + xy' + 2y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \quad (\text{A})$$

numerically on $(-5, 5)$.

- (c) For fixed r in $\{1, 2, 3, 4, 5\}$ graph

$$T_N(x) = \sum_{n=0}^N a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

10. L Follow the directions of Exercise 9.2.9 for the differential equation

$$y'' + 2xy' + 3y = 0.$$

In Exercises 9.2.11–9.2.13 find $a_0, \dots, a_N$ for N at least 7 in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

11. C $(1 + x^2)y'' + xy' + y = 0, \quad y(0) = 2, \quad y'(0) = -1$

12. C $(1 + 2x^2)y'' - 9xy' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1$

13. C $(1 + 8x^2)y'' + 2y = 0, \quad y(0) = 2, \quad y'(0) = -1$

14. L Do the next experiment for various choices of real numbers a_0, a_1 , and r , with $0 < r < 1/\sqrt{2}$.

- (a) Use differential equations software to solve the initial value problem

$$(1 - 2x^2)y'' - xy' + 3y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \quad (\text{A})$$

numerically on $(-r, r)$.

- (b) For $N = 2, 3, 4, \dots$, compute $a_2, \dots, a_N$ in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^N a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

15. L Do (a) and (b) for several values of r in $(0, 1)$:

- (a) Use differential equations software to solve the initial value problem

$$(1 + x^2)y'' + 10xy' + 14y = 0, \quad y(0) = 5, \quad y'(0) = 1, \quad (\text{A})$$

numerically on $(-r, r)$.

- (b) For $N = 2, 3, 4, \dots$, compute $a_2, \dots, a_N$ in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^N a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs. What happens to the required N as $r \rightarrow 1$?

- (c) Try (a) and (b) with $r = 1.2$. Explain your results.

In Exercises 9.2.16–9.2.20 find the power series in $x - x_0$ for the general solution.

16. $y'' - y = 0; \quad x_0 = 3$ 17. $y'' - (x - 3)y' - y = 0; \quad x_0 = 3$

18. $(1 - 4x + 2x^2)y'' + 10(x - 1)y' + 6y = 0; \quad x_0 = 1$

19. $(11 - 8x + 2x^2)y'' - 16(x - 2)y' + 36y = 0; \quad x_0 = 2$

20. $(5 + 6x + 3x^2)y'' + 9(x + 1)y' + 3y = 0; \quad x_0 = -1$

In Exercises 9.2.21–9.2.26 find $a_0, \dots, a_N$ for N at least 7 in the power series $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ for the solution of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

21. $\boxed{\text{C}}$ $(x^2 - 4)y'' - xy' - 3y = 0, \quad y(0) = -1, \quad y'(0) = 2$

22. $\boxed{\text{C}}$ $y'' + (x - 3)y' + 3y = 0, \quad y(3) = -2, \quad y'(3) = 3$

23. $\boxed{\text{C}}$ $(5 - 6x + 3x^2)y'' + (x - 1)y' + 12y = 0, \quad y(1) = -1, \quad y'(1) = 1$

24. $\boxed{\text{C}}$ $(4x^2 - 24x + 37)y'' + y = 0, \quad y(3) = 4, \quad y'(3) = -6$

25. $\boxed{\text{C}}$ $(x^2 - 8x + 14)y'' - 8(x - 4)y' + 20y = 0, \quad y(4) = 3, \quad y'(4) = -4$

26. $\boxed{\text{C}}$ $(2x^2 + 4x + 5)y'' - 20(x + 1)y' + 60y = 0, \quad y(-1) = 3, \quad y'(-1) = -3$

27. (a) Find a power series in x for the general solution of

$$(1 + x^2)y'' + 4xy' + 2y = 0. \quad (\text{A})$$

(b) Use (a) and the formula

$$\frac{1}{1 - r} = 1 + r + r^2 + \cdots + r^n + \cdots \quad (-1 < r < 1)$$

for the sum of a geometric series to find a closed form expression for the general solution of (A) on $(-1, 1)$.

(c) Show that the expression obtained in (b) is actually the general solution of (A) on $(-\infty, \infty)$.

28. Use Theorem 9.2.2 to show that the power series in x for the general solution of

$$(1 + \alpha x^2)y'' + \beta xy' + \gamma y = 0$$

is

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} p(2j) \right] \frac{x^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} p(2j+1) \right] \frac{x^{2m+1}}{(2m+1)!}.$$

29. Use Exercise 9.2.28 to show that all solutions of

$$(1 + \alpha x^2)y'' + \beta xy' + \gamma y = 0$$

are polynomials if and only if

$$\alpha n(n-1) + \beta n + \gamma = \alpha(n-2r)(n-2s-1),$$

where r and s are nonnegative integers.

30. (a) Use Exercise 9.2.28 to show that the power series in x for the general solution of

$$(1 - x^2)y'' - 2bxy' + \alpha(\alpha + 2b - 1)y = 0$$

is $y = a_0y_1 + a_1y_2$, where

$$y_1 = \sum_{m=0}^{\infty} \left[\prod_{j=0}^{m-1} (2j - \alpha)(2j + \alpha + 2b - 1) \right] \frac{x^{2m}}{(2m)!}$$

and

$$y_2 = \sum_{m=0}^{\infty} \left[\prod_{j=0}^{m-1} (2j + 1 - \alpha)(2j + \alpha + 2b) \right] \frac{x^{2m+1}}{(2m + 1)!}.$$

- (b) Suppose $2b$ isn't a negative odd integer and k is a nonnegative integer. Show that y_1 is a polynomial of degree $2k$ such that $y_1(-x) = y_1(x)$ if $\alpha = 2k$, while y_2 is a polynomial of degree $2k + 1$ such that $y_2(-x) = -y_2(x)$ if $\alpha = 2k + 1$. Conclude that if n is a nonnegative integer, then there's a polynomial P_n of degree n such that $P_n(-x) = (-1)^n P_n(x)$ and

$$(1 - x^2)P_n'' - 2bP_n' + n(n + 2b - 1)P_n = 0. \quad (\text{A})$$

- (c) Show that (A) implies that

$$[(1 - x^2)^b P_n']' = -n(n + 2b - 1)(1 - x^2)^{b-1} P_n,$$

and use this to show that if m and n are nonnegative integers, then

$$\begin{aligned} [(1 - x^2)^b P_n']' P_m - [(1 - x^2)^b P_m']' P_n = \\ [m(m + 2b - 1) - n(n + 2b - 1)] (1 - x^2)^{b-1} P_m P_n. \end{aligned} \quad (\text{B})$$

- (d) Now suppose $b > 0$. Use (B) and integration by parts to show that if $m \neq n$, then

$$\int_{-1}^1 (1 - x^2)^{b-1} P_m(x) P_n(x) dx = 0.$$

(We say that P_m and P_n are *orthogonal on* $(-1, 1)$ *with respect to the weighting function* $(1 - x^2)^{b-1}$.)

31. (a) Use Exercise 9.2.28 to show that the power series in x for the general solution of Hermite's equation

$$y'' - 2xy' + 2\alpha y = 0$$

is $y = a_0y_1 + a_1y_2$, where

$$y_1 = \sum_{m=0}^{\infty} \left[\prod_{j=0}^{m-1} (2j - \alpha) \right] \frac{2^m x^{2m}}{(2m)!}$$

and

$$y_2 = \sum_{m=0}^{\infty} \left[\prod_{j=0}^{m-1} (2j + 1 - \alpha) \right] \frac{2^m x^{2m+1}}{(2m + 1)!}.$$

- (b) Suppose k is a nonnegative integer. Show that y_1 is a polynomial of degree $2k$ such that $y_1(-x) = y_1(x)$ if $\alpha = 2k$, while y_2 is a polynomial of degree $2k+1$ such that $y_2(-x) = -y_2(x)$ if $\alpha = 2k+1$. Conclude that if n is a nonnegative integer then there's a polynomial P_n of degree n such that $P_n(-x) = (-1)^n P_n(x)$ and

$$P_n'' - 2xP_n' + 2nP_n = 0. \quad (\text{A})$$

- (c) Show that (A) implies that

$$[e^{-x^2} P_n']' = -2ne^{-x^2} P_n,$$

and use this to show that if m and n are nonnegative integers, then

$$[e^{-x^2} P_n']' P_m - [e^{-x^2} P_m']' P_n = 2(m-n)e^{-x^2} P_m P_n. \quad (\text{B})$$

- (d) Use (B) and integration by parts to show that if $m \neq n$, then

$$\int_{-\infty}^{\infty} e^{-x^2} P_m(x) P_n(x) dx = 0.$$

(We say that P_m and P_n are *orthogonal on* $(-\infty, \infty)$ *with respect to the weighting function* e^{-x^2} .)

32. Consider the equation

$$(1 + \alpha x^3) y'' + \beta x^2 y' + \gamma xy = 0, \quad (\text{A})$$

and let $p(n) = \alpha n(n-1) + \beta n + \gamma$. (The special case $y'' - xy = 0$ of (A) is Airy's equation.)

- (a) Modify the argument used to prove Theorem 9.2.2 to show that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

is a solution of (A) if and only if $a_2 = 0$ and

$$a_{n+3} = -\frac{p(n)}{(n+3)(n+2)} a_n, \quad n \geq 0.$$

- (b) Show from (a) that $a_n = 0$ unless $n = 3m$ or $n = 3m+1$ for some nonnegative integer m , and that

$$a_{3m+3} = -\frac{p(3m)}{(3m+3)(3m+2)} a_{3m}, \quad m \geq 0,$$

and

$$a_{3m+4} = -\frac{p(3m+1)}{(3m+4)(3m+3)} a_{3m+1}, \quad m \geq 0,$$

where a_0 and a_1 may be specified arbitrarily.

- (c) Conclude from (b) that the power series in x for the general solution of (A) is

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p(3j)}{3j+2} \right] \frac{x^{3m}}{3^m m!} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p(3j+1)}{3j+4} \right] \frac{x^{3m+1}}{3^m m!}.$$

In Exercises 9.2.33–9.2.37 use the method of Exercise 9.2.32 to find the power series in x for the general solution.

33. $y'' - xy = 0$ 34. $(1 - 2x^3)y'' - 10x^2y' - 8xy = 0$

35. $(1 + x^3)y'' + 7x^2y' + 9xy = 0$ 36. $(1 - 2x^3)y'' + 6x^2y' + 24xy = 0$

37. $(1 - x^3)y'' + 15x^2y' - 63xy = 0$

38. Consider the equation

$$(1 + \alpha x^{k+2})y'' + \beta x^{k+1}y' + \gamma x^k y = 0, \quad (\text{A})$$

where k is a positive integer, and let $p(n) = \alpha n(n-1) + \beta n + \gamma$.

(a) Modify the argument used to prove Theorem 9.2.2 to show that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

is a solution of (A) if and only if $a_n = 0$ for $2 \leq n \leq k+1$ and

$$a_{n+k+2} = -\frac{p(n)}{(n+k+2)(n+k+1)} a_n, \quad n \geq 0.$$

(b) Show from (a) that $a_n = 0$ unless $n = (k+2)m$ or $n = (k+2)m+1$ for some nonnegative integer m , and that

$$a_{(k+2)(m+1)} = -\frac{p((k+2)m)}{(k+2)(m+1)[(k+2)(m+1)-1]} a_{(k+2)m}, \quad m \geq 0,$$

and

$$a_{(k+2)(m+1)+1} = -\frac{p((k+2)m+1)}{[(k+2)(m+1)+1](k+2)(m+1)} a_{(k+2)m+1}, \quad m \geq 0,$$

where a_0 and a_1 may be specified arbitrarily.

(c) Conclude from (b) that the power series in x for the general solution of (A) is

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p((k+2)j)}{(k+2)(j+1)-1} \right] \frac{x^{(k+2)m}}{(k+2)^m m!} \\ + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p((k+2)j+1)}{(k+2)(j+1)+1} \right] \frac{x^{(k+2)m+1}}{(k+2)^m m!}.$$

In Exercises 9.2.39–9.2.44 use the method of Exercise 9.2.38 to find the power series in x for the general solution.

39. $(1 + 2x^5)y'' + 14x^4y' + 10x^3y = 0$

40. $y'' + x^2y = 0$ 41. $y'' + x^6y' + 7x^5y = 0$

42. $(1 + x^8)y'' - 16x^7y' + 72x^6y = 0$

43. $(1 - x^6)y'' - 12x^5y' - 30x^4y = 0$

44. $y'' + x^5y' + 6x^4y = 0$

9.3 Series Solution Near an Ordinary Point II

In this section we continue to find series solutions

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

of initial value problems

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1, \quad (9.57)$$

where P_0, P_1 , and P_2 are polynomials and $P_0(x_0) \neq 0$, so x_0 is an ordinary point of (9.57). However, here we consider cases where the differential equation in (9.57) is not of the form

$$(1 + \alpha(x - x_0)^2)y'' + \beta(x - x_0)y' + \gamma y = 0,$$

so Theorem 9.2.2 does not apply, and the computation of the coefficients $\{a_n\}$ is more complicated. For the equations considered here it's difficult or impossible to obtain an explicit formula for a_n in terms of n . Nevertheless, we can calculate as many coefficients as we wish. The next three examples illustrate this.

Example 9.3.1: Find the coefficients $a_0, \dots, a_7$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0, \quad y(0) = -1, \quad y'(0) = -2. \quad (9.58)$$

Solution: Here

$$Ly = (1 + x + 2x^2)y'' + (1 + 7x)y' + 2y.$$

The zeros $(-1 \pm i\sqrt{7})/4$ of $P_0(x) = 1 + x + 2x^2$ have absolute value $1/\sqrt{2}$, so Theorem 9.2.2 implies that the series solution converges to the solution of (9.58) on $(-1/\sqrt{2}, 1/\sqrt{2})$. Since

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}, \\ Ly &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + 2 \sum_{n=2}^{\infty} n(n-1) a_n x^n \\ &\quad + \sum_{n=1}^{\infty} n a_n x^{n-1} + 7 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Shifting indices so the general term in each series is a constant multiple of x^n yields

$$\begin{aligned} Ly &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) n a_{n+1} x^n + 2 \sum_{n=0}^{\infty} n(n-1) a_n x^n \\ &\quad + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 7 \sum_{n=0}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n, \end{aligned}$$

where

$$b_n = (n+2)(n+1) a_{n+2} + (n+1)^2 a_{n+1} + (n+2)(2n+1) a_n.$$

Therefore $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ if and only if

$$a_{n+2} = -\frac{n+1}{n+2} a_{n+1} - \frac{2n+1}{n+1} a_n, \quad n \geq 0. \quad (9.59)$$

From the initial conditions in (9.58), $a_0 = y(0) = -1$ and $a_1 = y'(0) = -2$. Setting $n = 0$ in (9.59) yields

$$a_2 = -\frac{1}{2} a_1 - a_0 = -\frac{1}{2}(-2) - (-1) = 2.$$

Setting $n = 1$ in (9.59) yields

$$a_3 = -\frac{2}{3} a_2 - \frac{3}{2} a_1 = -\frac{2}{3}(2) - \frac{3}{2}(-2) = \frac{5}{3}.$$

We leave it to you to compute a_4, a_5, a_6, a_7 from (9.59) and show that

$$y = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 - \frac{443}{56}x^7 + \cdots.$$

We also leave it to you (Exercise 9.3.13) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^N a_n x^n$ converge to the solution of (9.58) on $(-1/\sqrt{2}, 1/\sqrt{2})$.

Example 9.3.2: Find the coefficients $a_0, \dots, a_5$ in the series solution

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n$$

of the initial value problem

$$(3+x)y'' + (1+2x)y' - (2-x)y = 0, \quad y(-1) = 2, \quad y'(-1) = -3. \quad (9.60)$$

Solution: Since the desired series is in powers of $x+1$ we rewrite the differential equation in (9.60) as $Ly = 0$, with

$$Ly = (2 + (x+1))y'' - (1 - 2(x+1))y' - (3 - (x+1))y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n, \quad y' = \sum_{n=1}^{\infty} n a_n (x+1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-2},$$

$$\begin{aligned} Ly &= 2 \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-1} \\ &\quad - \sum_{n=1}^{\infty} n a_n (x+1)^{n-1} + 2 \sum_{n=1}^{\infty} n a_n (x+1)^n \\ &\quad - 3 \sum_{n=0}^{\infty} a_n (x+1)^n + \sum_{n=0}^{\infty} a_n (x+1)^{n+1}. \end{aligned}$$

Shifting indices so that the general term in each series is a constant multiple of $(x+1)^n$ yields

$$\begin{aligned} Ly &= 2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x+1)^n + \sum_{n=0}^{\infty} (n+1) n a_{n+1} (x+1)^n \\ &\quad - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x+1)^n + \sum_{n=0}^{\infty} (2n-3) a_n (x+1)^n + \sum_{n=1}^{\infty} a_{n-1} (x+1)^n \\ &= \sum_{n=0}^{\infty} b_n (x+1)^n, \end{aligned}$$

where

$$b_0 = 4a_2 - a_1 - 3a_0$$

and

$$b_n = 2(n+2)(n+1)a_{n+2} + (n^2 - 1)a_{n+1} + (2n - 3)a_n + a_{n-1}, \quad n \geq 1.$$

Therefore $y = \sum_{n=0}^{\infty} a_n(x+1)^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = \frac{1}{4}(a_1 + 3a_0) \quad (9.61)$$

and

$$a_{n+2} = -\frac{1}{2(n+2)(n+1)} [(n^2 - 1)a_{n+1} + (2n - 3)a_n + a_{n-1}], \quad n \geq 1. \quad (9.62)$$

From the initial conditions in (9.60), $a_0 = y(-1) = 2$ and $a_1 = y'(-1) = -3$. We leave it to you to compute $a_2, \dots, a_5$ with (9.61) and (9.62) and show that the solution of (9.60) is

$$y = -2 - 3(x+1) + \frac{3}{4}(x+1)^2 - \frac{5}{12}(x+1)^3 + \frac{7}{48}(x+1)^4 - \frac{1}{60}(x+1)^5 + \dots$$

We also leave it to you (Exercise 9.3.14) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^N a_n x^n$ converge to the solution of (9.60) on the interval of convergence of the power series solution.

Example 9.3.3: Find the coefficients $a_0, \dots, a_5$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$y'' + 3xy' + (4 + 2x^2)y = 0, \quad y(0) = 2, \quad y'(0) = -3. \quad (9.63)$$

Solution: Here

$$Ly = y'' + 3xy' + (4 + 2x^2)y.$$

Since

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}, \\ Ly &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3 \sum_{n=1}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^{n+2}. \end{aligned}$$

Shifting indices so that the general term in each series is a constant multiple of x^n yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (3n+4) a_n x^n + 2 \sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=0}^{\infty} b_n x^n$$

where

$$b_0 = 2a_2 + 4a_0, \quad b_1 = 6a_3 + 7a_1,$$

and

$$b_n = (n+2)(n+1) a_{n+2} + (3n+4) a_n + 2a_{n-2}, \quad n \geq 2.$$

Therefore $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = -2a_0, \quad a_3 = -\frac{7}{6}a_1, \quad (9.64)$$

and

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} [(3n+4)a_n + 2a_{n-2}], \quad n \geq 2. \quad (9.65)$$

From the initial conditions in (9.63), $a_0 = y(0) = 2$ and $a_1 = y'(0) = -3$. We leave it to you to compute $a_2, \dots, a_5$ with (9.64) and (9.65) and show that the solution of (9.63) is

$$y = 2 - 3x - 4x^2 + \frac{7}{2}x^3 + 3x^4 - \frac{79}{40}x^5 + \dots$$

We also leave it to you (Exercise 9.3.15) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^N a_n x^n$ converge to the solution of (9.63) on the interval of convergence of the power series solution.

9.3.1 Exercises

In Exercises 9.3.1–9.3.12 find the coefficients $a_0, \dots, a_N$ for N at least 7 in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

1. C $(1 + 3x)y'' + xy' + 2y = 0, \quad y(0) = 2, \quad y'(0) = -3$
2. C $(1 + x + 2x^2)y'' + (2 + 8x)y' + 4y = 0, \quad y(0) = -1, \quad y'(0) = 2$
3. C $(1 - 2x^2)y'' + (2 - 6x)y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$
4. C $(1 + x + 3x^2)y'' + (2 + 15x)y' + 12y = 0, \quad y(0) = 0, \quad y'(0) = 1$
5. C $(2 + x)y'' + (1 + x)y' + 3y = 0, \quad y(0) = 4, \quad y'(0) = 3$
6. C $(3 + 3x + x^2)y'' + (6 + 4x)y' + 2y = 0, \quad y(0) = 7, \quad y'(0) = 3$
7. C $(4 + x)y'' + (2 + x)y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 5$
8. C $(2 - 3x + 2x^2)y'' - (4 - 6x)y' + 2y = 0, \quad y(1) = 1, \quad y'(1) = -1$
9. C $(3x + 2x^2)y'' + 10(1 + x)y' + 8y = 0, \quad y(-1) = 1, \quad y'(-1) = -1$
10. C $(1 - x + x^2)y'' - (1 - 4x)y' + 2y = 0, \quad y(1) = 2, \quad y'(1) = -1$
11. C $(2 + x)y'' + (2 + x)y' + y = 0, \quad y(-1) = -2, \quad y'(-1) = 3$
12. C $x^2 y'' - (6 - 7x)y' + 8y = 0, \quad y(1) = 1, \quad y'(1) = -2$
13. L Do the following experiment for various choices of real numbers a_0, a_1 , and r , with $0 < r < 1/\sqrt{2}$.

(a) Use differential equations software to solve the initial value problem

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \quad (\text{A})$$

numerically on $(-r, r)$. (See Example 9.3.1.)

(b) For $N = 2, 3, 4, \dots$, compute $a_2, \dots, a_N$ in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^N a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

14. L Do the following experiment for various choices of real numbers a_0, a_1 , and r , with $0 < r < 2$.

(a) Use differential equations software to solve the initial value problem

$$(3 + x)y'' + (1 + 2x)y' - (2 - x)y = 0, \quad y(-1) = a_0, \quad y'(-1) = a_1, \quad (\text{A})$$

numerically on $(-1 - r, -1 + r)$. (See Example 9.3.2. Why this interval?)

(b) For $N = 2, 3, 4, \dots$, compute $a_2, \dots, a_N$ in the power series solution

$$y = \sum_{n=0}^{\infty} a_n (x + 1)^n$$

of (A), and graph

$$T_N(x) = \sum_{n=0}^N a_n (x + 1)^n$$

and the solution obtained in (a) on $(-1 - r, -1 + r)$. Continue increasing N until there's no perceptible difference between the two graphs.

15. L Do the following experiment for several choices of a_0 , a_1 , and r , with $r > 0$.

(a) Use differential equations software to solve the initial value problem

$$y'' + 3xy' + (4 + 2x^2)y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \quad (\text{A})$$

numerically on $(-r, r)$. (See Example 9.3.3.)

- (b) Find the coefficients $a_0, a_1, \dots, a_N$ in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^N a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

16. L Do the following experiment for several choices of a_0 and a_1 .

(a) Use differential equations software to solve the initial value problem

$$(1-x)y'' - (2-x)y' + y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \quad (\text{A})$$

numerically on $(-r, r)$.

- (b) Find the coefficients $a_0, a_1, \dots, a_N$ in the power series solution $y = \sum_{n=0}^N a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^N a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs. What happens as you let $r \rightarrow 1$?

17. L Follow the directions of Exercise 9.3.16 for the initial value problem

$$(1+x)y'' + 3y' + 32y = 0, \quad y(0) = a_0, \quad y'(0) = a_1.$$

18. L Follow the directions of Exercise 9.3.16 for the initial value problem

$$(1+x^2)y'' + y' + 2y = 0, \quad y(0) = a_0, \quad y'(0) = a_1.$$

In Exercises 9.3.19–9.3.28 find the coefficients $a_0, \dots, a_N$ for N at least 7 in the series solution

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

19. C $(2+4x)y'' - 4y' - (6+4x)y = 0, \quad y(0) = 2, \quad y'(0) = -7$
 20. C $(1+2x)y'' - (1-2x)y' - (3-2x)y = 0, \quad y(1) = 1, \quad y'(1) = -2$
 21. C $(5+2x)y'' - y' + (5+x)y = 0, \quad y(-2) = 2, \quad y'(-2) = -1$
 22. C $(4+x)y'' - (4+2x)y' + (6+x)y = 0, \quad y(-3) = 2, \quad y'(-3) = -2$
 23. C $(2+3x)y'' - xy' + 2xy = 0, \quad y(0) = -1, \quad y'(0) = 2$
 24. C $(3+2x)y'' + 3y' - xy = 0, \quad y(-1) = 2, \quad y'(-1) = -3$
 25. C $(3+2x)y'' - 3y' - (2+x)y = 0, \quad y(-2) = -2, \quad y'(-2) = 3$
 26. C $(10-2x)y'' + (1+x)y = 0, \quad y(2) = 2, \quad y'(2) = -4$
 27. C $(7+x)y'' + (8+2x)y' + (5+x)y = 0, \quad y(-4) = 1, \quad y'(-4) = 2$
 28. C $(6+4x)y'' + (1+2x)y = 0, \quad y(-1) = -1, \quad y'(-1) = 2$
 29. Show that the coefficients in the power series in x for the general solution of

$$(1 + \alpha x + \beta x^2)y'' + (\gamma + \delta x)y' + \epsilon y = 0$$

satisfy the recurrence relation

$$a_{n+2} = -\frac{\gamma + \alpha n}{n+2} a_{n+1} - \frac{\beta n(n-1) + \delta n + \epsilon}{(n+2)(n+1)} a_n.$$

30. (a) Let α and β be constants, with $\beta \neq 0$. Show that $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of

$$(1 + \alpha x + \beta x^2)y'' + (2\alpha + 4\beta x)y' + 2\beta y = 0 \quad (\text{A})$$

if and only if

$$a_{n+2} + \alpha a_{n+1} + \beta a_n = 0, \quad n \geq 0. \quad (\text{B})$$

An equation of this form is called a *second order homogeneous linear difference equation*. The polynomial $p(r) = r^2 + \alpha r + \beta$ is called the *characteristic polynomial* of (B). If r_1 and r_2 are the zeros of p , then $1/r_1$ and $1/r_2$ are the zeros of

$$P_0(x) = 1 + \alpha x + \beta x^2.$$

- (b) Suppose $p(r) = (r - r_1)(r - r_2)$ where r_1 and r_2 are real and distinct, and let ρ be the smaller of the two numbers $\{1/|r_1|, 1/|r_2|\}$. Show that if c_1 and c_2 are constants then the sequence

$$a_n = c_1 r_1^n + c_2 r_2^n, \quad n \geq 0$$

satisfies (B). Conclude from this that any function of the form

$$y = \sum_{n=0}^{\infty} (c_1 r_1^n + c_2 r_2^n) x^n$$

is a solution of (A) on $(-\rho, \rho)$.

- (c) Use (b) and the formula for the sum of a geometric series to show that the functions

$$y_1 = \frac{1}{1 - r_1 x} \quad \text{and} \quad y_2 = \frac{1}{1 - r_2 x}$$

form a fundamental set of solutions of (A) on $(-\rho, \rho)$.

- (d) Show that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on any interval that doesn't contain either $1/r_1$ or $1/r_2$.
 (e) Suppose $p(r) = (r - r_1)^2$, and let $\rho = 1/|r_1|$. Show that if c_1 and c_2 are constants then the sequence

$$a_n = (c_1 + c_2 n) r_1^n, \quad n \geq 0$$

satisfies (B). Conclude from this that any function of the form

$$y = \sum_{n=0}^{\infty} (c_1 + c_2 n) r_1^n x^n$$

is a solution of (A) on $(-\rho, \rho)$.

- (f) Use (e) and the formula for the sum of a geometric series to show that the functions

$$y_1 = \frac{1}{1 - r_1 x} \quad \text{and} \quad y_2 = \frac{x}{(1 - r_1 x)^2}$$

form a fundamental set of solutions of (A) on $(-\rho, \rho)$.

- (g) Show that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on any interval that does not contain $1/r_1$.

31. Use the results of Exercise 9.3.30 to find the general solution of the given equation on any interval on which polynomial multiplying y'' has no zeros.

(a) $(1 + 3x + 2x^2)y'' + (6 + 8x)y' + 4y = 0$

(b) $(1 - 5x + 6x^2)y'' - (10 - 24x)y' + 12y = 0$

(c) $(1 - 4x + 4x^2)y'' - (8 - 16x)y' + 8y = 0$

(d) $(4 + 4x + x^2)y'' + (8 + 4x)y' + 2y = 0$

(e) $(4 + 8x + 3x^2)y'' + (16 + 12x)y' + 6y = 0$

In Exercises 9.3.32–9.3.38 find the coefficients $a_0, \dots, a_N$ for N at least 7 in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

32. $\boxed{\text{C}}$ $y'' + 2xy' + (3 + 2x^2)y = 0, \quad y(0) = 1, \quad y'(0) = -2$

33. $\boxed{\text{C}}$ $y'' - 3xy' + (5 + 2x^2)y = 0, \quad y(0) = 1, \quad y'(0) = -2$

34. $\boxed{\text{C}}$ $y'' + 5xy' - (3 - x^2)y = 0, \quad y(0) = 6, \quad y'(0) = -2$

35. $\boxed{\text{C}}$ $y'' - 2xy' - (2 + 3x^2)y = 0, \quad y(0) = 2, \quad y'(0) = -5$

36. $\boxed{\text{C}}$ $y'' - 3xy' + (2 + 4x^2)y = 0, \quad y(0) = 3, \quad y'(0) = 6$

37. $\boxed{\text{C}}$ $2y'' + 5xy' + (4 + 2x^2)y = 0, \quad y(0) = 3, \quad y'(0) = -2$

38. $\boxed{\text{C}}$ $3y'' + 2xy' + (4 - x^2)y = 0, \quad y(0) = -2, \quad y'(0) = 3$

39. Find power series in x for the solutions y_1 and y_2 of

$$y'' + 4xy' + (2 + 4x^2)y = 0$$

such that $y_1(0) = 1, y_1'(0) = 0, y_2(0) = 0, y_2'(0) = 1$, and identify y_1 and y_2 in terms of familiar elementary functions.

In Exercises 9.3.40–9.3.49 find the coefficients $a_0, \dots, a_N$ for N at least 7 in the series solution

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

40. $\boxed{\text{C}}$ $(1 + x)y'' + x^2y' + (1 + 2x)y = 0, \quad y(0) = 2, \quad y'(0) = 3$

41. $\boxed{\text{C}}$ $y'' + (1 + 2x + x^2)y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 3$

42. $\boxed{\text{C}}$ $(1 + x^2)y'' + (2 + x^2)y' + xy = 0, \quad y(0) = -3, \quad y'(0) = 5$

43. $\boxed{\text{C}}$ $(1 + x)y'' + (1 - 3x + 2x^2)y' - (x - 4)y = 0, \quad y(1) = -2, \quad y'(1) = 3$

44. $\boxed{\text{C}}$ $y'' + (13 + 12x + 3x^2)y' + (5 + 2x)y = 0, \quad y(-2) = 2, \quad y'(-2) = -3$

45. $\boxed{\text{C}}$ $(1 + 2x + 3x^2)y'' + (2 - x^2)y' + (1 + x)y = 0, \quad y(0) = 1, \quad y'(0) = -2$

46. $\boxed{\text{C}}$ $(3 + 4x + x^2)y'' - (5 + 4x - x^2)y' - (2 + x)y = 0, \quad y(-2) = 2, \quad y'(-2) = -1$

47. $\boxed{\text{C}}$ $(1 + 2x + x^2)y'' + (1 - x)y = 0, \quad y(0) = 2, \quad y'(0) = -1$

48. $\boxed{\text{C}}$ $(x - 2x^2)y'' + (1 + 3x - x^2)y' + (2 + x)y = 0, \quad y(1) = 1, \quad y'(1) = 0$

49. $\boxed{\text{C}}$ $(16 - 11x + 2x^2)y'' + (10 - 6x + x^2)y' - (2 - x)y = 0, \quad y(3) = 1, \quad y'(3) = -2$

9.4 Series Solutions of Linear Second Order Equations

This section sets the stage for Sections 1.5, 1.6, and 1.7. If you're not interested in those sections, but wish to learn about Euler equations, omit the introductory paragraphs and start reading at Definition 9.4.2.

In the next three sections we'll continue to study equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \quad (9.66)$$

where P_0 , P_1 , and P_2 are polynomials, but the emphasis will be different from that of Sections 7.2 and 7.3, where we obtained solutions of (9.66) near an ordinary point x_0 in the form of power series in $x - x_0$. If x_0 is a singular point of (9.66) (that is, if $P_0(x_0) = 0$), the solutions can't in general be represented by power series in $x - x_0$. Nevertheless, it's often necessary in physical applications to study the behavior of solutions of (9.66) near a singular point. Although this can be difficult in the absence of some sort of assumption on the nature of the singular point, equations that satisfy the requirements of the next definition can be solved by series methods discussed in the next three sections. Fortunately, many equations arising in applications satisfy these requirements.

Definition 9.4.1 Let P_0 , P_1 , and P_2 be polynomials with no common factor and suppose $P_0(x_0) = 0$. Then x_0 is a *regular singular point* of the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \quad (9.67)$$

if (9.67) can be written as

$$(x - x_0)^2 A(x)y'' + (x - x_0)B(x)y' + C(x)y = 0 \quad (9.68)$$

where A , B , and C are polynomials and $A(x_0) \neq 0$; otherwise, x_0 is an *irregular* singular point of (9.67).

Example 9.4.1: Bessel's equation,

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \quad (9.69)$$

has the singular point $x_0 = 0$. Since this equation is of the form (9.68) with $x_0 = 0$, $A(x) = 1$, $B(x) = 1$, and $C(x) = x^2 - \nu^2$, it follows that $x_0 = 0$ is a regular singular point of (9.69).

Example 9.4.2: Legendre's equation,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad (9.70)$$

has the singular points $x_0 = \pm 1$. Multiplying through by $1 - x$ yields

$$(x - 1)^2(x + 1)y'' + 2x(x - 1)y' - \alpha(\alpha + 1)(x - 1)y = 0,$$

which is of the form (9.68) with $x_0 = 1$, $A(x) = x + 1$, $B(x) = 2x$, and $C(x) = -\alpha(\alpha + 1)(x - 1)$. Therefore $x_0 = 1$ is a regular singular point of (9.70). We leave it to you to show that $x_0 = -1$ is also a regular singular point of (9.70).

Example 9.4.3: The equation

$$x^3 y'' + x y' + y = 0$$

has an irregular singular point at $x_0 = 0$. (Verify.)

For convenience we restrict our attention to the case where $x_0 = 0$ is a regular singular point of (9.67). This isn't really a restriction, since if $x_0 \neq 0$ is a regular singular point of (9.67) then introducing the new independent variable $t = x - x_0$ and the new unknown $Y(t) = y(t + x_0)$ leads to a differential equation with polynomial coefficients that has a regular singular point at $t_0 = 0$. This is illustrated in Exercise 9.4.22 for Legendre's equation, and in Exercise 9.4.23 for the general case.

9.4.1 Euler Equations

The simplest kind of equation with a regular singular point at $x_0 = 0$ is the Euler equation, defined as follows.

Definition 9.4.2 An *Euler equation* is an equation that can be written in the form

$$ax^2 y'' + bxy' + cy = 0, \quad (9.71)$$

where a, b , and c are real constants and $a \neq 0$.

Theorem 6.1.1 implies that (9.71) has solutions defined on $(0, \infty)$ and $(-\infty, 0)$, since (9.71) can be rewritten as

$$ay'' + \frac{b}{x}y' + \frac{c}{x^2}y = 0.$$

For convenience we'll restrict our attention to the interval $(0, \infty)$. (Exercise 9.4.19 deals with solutions of (9.71) on $(-\infty, 0)$.) The key to finding solutions on $(0, \infty)$ is that if $x > 0$ then x^r is defined as a real-valued function on $(0, \infty)$ for all values of r , and substituting $y = x^r$ into (9.71) produces

$$\begin{aligned} ax^2(x^r)'' + bx(x^r)' + cx^r &= ax^2r(r-1)x^{r-2} + bxxrx^{r-1} + cx^r \\ &= [ar(r-1) + br + c]x^r. \end{aligned} \quad (9.72)$$

The polynomial

$$p(r) = ar(r-1) + br + c$$

is called the *indicial polynomial* of (9.71), and $p(r) = 0$ is its *indicial equation*. From (9.72) we can see that $y = x^r$ is a solution of (9.71) on $(0, \infty)$ if and only if $p(r) = 0$. Therefore, if the indicial equation has distinct real roots r_1 and r_2 then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ form a fundamental set of solutions of (9.71) on $(0, \infty)$, since $y_2/y_1 = x^{r_2-r_1}$ is nonconstant. In this case

$$y = c_1 x^{r_1} + c_2 x^{r_2}$$

is the general solution of (9.71) on $(0, \infty)$.

Example 9.4.4: Find the general solution of

$$x^2 y'' - xy' - 8y = 0 \quad (9.73)$$

on $(0, \infty)$.

Solution: The indicial polynomial of (9.73) is

$$p(r) = r(r-1) - r - 8 = (r-4)(r+2).$$

Therefore $y_1 = x^4$ and $y_2 = x^{-2}$ are solutions of (9.73) on $(0, \infty)$, and its general solution on $(0, \infty)$ is

$$y = c_1 x^4 + \frac{c_2}{x^2}.$$

Example 9.4.5: Find the general solution of

$$6x^2y'' + 5xy' - y = 0 \quad (9.74)$$

on $(0, \infty)$.

Solution: The indicial polynomial of (9.74) is

$$p(r) = 6r(r-1) + 5r - 1 = (2r-1)(3r+1).$$

Therefore the general solution of (9.74) on $(0, \infty)$ is

$$y = c_1x^{1/2} + c_2x^{-1/3}. \blacksquare$$

If the indicial equation has a repeated root r_1 , then $y_1 = x^{r_1}$ is a solution of

$$ax^2y'' + bxy' + cy = 0, \quad (9.75)$$

on $(0, \infty)$, but (9.75) has no other solution of the form $y = x^r$. If the indicial equation has complex conjugate zeros then (9.75) has no real-valued solutions of the form $y = x^r$. Fortunately we can use the results of Section 5.2 for constant coefficient equations to solve (9.75) in any case.

Theorem 9.4.3. *Suppose the roots of the indicial equation*

$$ar(r-1) + br + c = 0 \quad (9.76)$$

are r_1 and r_2 . Then the general solution of the Euler equation

$$ax^2y'' + bxy' + cy = 0 \quad (9.77)$$

on $(0, \infty)$ is

$$\begin{aligned} y &= c_1x^{r_1} + c_2x^{r_2} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;} \\ y &= x^{r_1}(c_1 + c_2 \ln x) \text{ if } r_1 = r_2; \\ y &= x^\lambda [c_1 \cos(\omega \ln x) + c_2 \sin(\omega \ln x)] \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega > 0. \end{aligned}$$

Proof. We first show that $y = y(x)$ satisfies (9.77) on $(0, \infty)$ if and only if $Y(t) = y(e^t)$ satisfies the constant coefficient equation

$$a \frac{d^2Y}{dt^2} + (b-a) \frac{dY}{dt} + cY = 0 \quad (9.78)$$

on $(-\infty, \infty)$. To do this, it's convenient to write $x = e^t$, or, equivalently, $t = \ln x$; thus, $Y(t) = y(x)$, where $x = e^t$. From the chain rule,

$$\frac{dY}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

and, since

$$\frac{dx}{dt} = e^t = x,$$

it follows that

$$\frac{dY}{dt} = x \frac{dy}{dx}. \quad (9.79)$$

Differentiating this with respect to t and using the chain rule again yields

$$\begin{aligned} \frac{d^2Y}{dt^2} &= \frac{d}{dt} \left(\frac{dY}{dt} \right) = \frac{d}{dt} \left(x \frac{dy}{dx} \right) \\ &= \frac{dx}{dt} \frac{dy}{dx} + x \frac{d^2y}{dx^2} \frac{dx}{dt} \\ &= x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} \left(\text{since } \frac{dx}{dt} = x \right). \end{aligned}$$

From this and (9.79),

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 Y}{dt^2} - \frac{dY}{dt}.$$

Substituting this and (9.79) into (9.77) yields (9.78). Since (9.76) is the characteristic equation of (9.78), Theorem 5.2.1 implies that the general solution of (9.78) on $(-\infty, \infty)$ is

$$\begin{aligned} Y(t) &= c_1 e^{r_1 t} + c_2 e^{r_2 t} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;} \\ Y(t) &= e^{r_1 t} (c_1 + c_2 t) \text{ if } r_1 = r_2; \\ Y(t) &= e^{\lambda t} (c_1 \cos \omega t + c_2 \sin \omega t) \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega \neq 0. \end{aligned}$$

Since $Y(t) = y(e^t)$, substituting $t = \ln x$ in the last three equations shows that the general solution of (9.77) on $(0, \infty)$ has the form stated in the theorem.

Example 9.4.6: Find the general solution of

$$x^2 y'' - 5xy' + 9y = 0 \tag{9.80}$$

on $(0, \infty)$.

Solution: The indicial polynomial of (9.80) is

$$p(r) = r(r-1) - 5r + 9 = (r-3)^2.$$

Therefore the general solution of (9.80) on $(0, \infty)$ is

$$y = x^3(c_1 + c_2 \ln x).$$

Example 9.4.7: Find the general solution of

$$x^2 y'' + 3xy' + 2y = 0 \tag{9.81}$$

on $(0, \infty)$.

Solution: The indicial polynomial of (9.81) is

$$p(r) = r(r-1) + 3r + 2 = (r+1)^2 + 1.$$

The roots of the indicial equation are $r = -1 \pm i$ and the general solution of (9.81) on $(0, \infty)$ is

$$y = \frac{1}{x} [c_1 \cos(\ln x) + c_2 \sin(\ln x)].$$

9.4.2 Exercises

In Exercises 9.4.1–9.4.18 find the general solution of the given Euler equation on $(0, \infty)$.

1. $x^2 y'' + 7xy' + 8y = 0$

2. $x^2 y'' - 7xy' + 7y = 0$

3. $x^2 y'' - xy' + y = 0$

4. $x^2 y'' + 5xy' + 4y = 0$

5. $x^2 y'' + xy' + y = 0$

6. $x^2 y'' - 3xy' + 13y = 0$

7. $x^2 y'' + 3xy' - 3y = 0$

8. $12x^2 y'' - 5xy'' + 6y = 0$

9. $4x^2y'' + 8xy' + y = 0$ 10. $3x^2y'' - xy' + y = 0$
 11. $2x^2y'' - 3xy' + 2y = 0$ 12. $x^2y'' + 3xy' + 5y = 0$
 13. $9x^2y'' + 15xy' + y = 0$ 14. $x^2y'' - xy' + 10y = 0$
 15. $x^2y'' - 6y = 0$ 16. $2x^2y'' + 3xy' - y = 0$
 17. $x^2y'' - 3xy' + 4y = 0$ 18. $2x^2y'' + 10xy' + 9y = 0$
 19. (a) Adapt the proof of Theorem 9.4.3 to show that $y = y(x)$ satisfies the Euler equation

$$ax^2y'' + bxy' + cy = 0 \quad (9.1)$$

on $(-\infty, 0)$ if and only if $Y(t) = y(-e^t)$

$$a \frac{d^2Y}{dt^2} + (b-a) \frac{dY}{dt} + cY = 0.$$

on $(-\infty, \infty)$.

(b) Use (a) to show that the general solution of (9.1) on $(-\infty, 0)$ is

$$\begin{aligned} y &= c_1|x|^{r_1} + c_2|x|^{r_2} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;} \\ y &= |x|^{r_1}(c_1 + c_2 \ln|x|) \text{ if } r_1 = r_2; \\ y &= |x|^\lambda [c_1 \cos(\omega \ln|x|) + c_2 \sin(\omega \ln|x|)] \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega > 0. \end{aligned}$$

20. Use reduction of order to show that if

$$ar(r-1) + br + c = 0$$

has a repeated root r_1 then $y = x^{r_1}(c_1 + c_2 \ln x)$ is the general solution of

$$ax^2y'' + bxy' + cy = 0$$

on $(0, \infty)$.

21. A nontrivial solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

is said to be *oscillatory* on an interval (a, b) if it has infinitely many zeros on (a, b) . Otherwise y is said to be *nonoscillatory* on (a, b) . Show that the equation

$$x^2y'' + ky = 0 \quad (k = \text{constant})$$

has oscillatory solutions on $(0, \infty)$ if and only if $k > 1/4$.

22. In Example 9.4.2 we saw that $x_0 = 1$ and $x_0 = -1$ are regular singular points of Legendre's equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0. \quad (\text{A})$$

(a) Introduce the new variables $t = x - 1$ and $Y(t) = y(t+1)$, and show that y is a solution of (A) if and only if Y is a solution of

$$t(2+t) \frac{d^2Y}{dt^2} + 2(1+t) \frac{dY}{dt} - \alpha(\alpha+1)Y = 0,$$

which has a regular singular point at $t_0 = 0$.

- (b) Introduce the new variables $t = x + 1$ and $Y(t) = y(t - 1)$, and show that y is a solution of (A) if and only if Y is a solution of

$$t(2-t)\frac{d^2Y}{dt^2} + 2(1-t)\frac{dY}{dt} + \alpha(\alpha+1)Y = 0,$$

which has a regular singular point at $t_0 = 0$.

23. Let P_0, P_1 , and P_2 be polynomials with no common factor, and suppose $x_0 \neq 0$ is a singular point of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \quad (\text{A})$$

Let $t = x - x_0$ and $Y(t) = y(t + x_0)$.

- (a) Show that y is a solution of (A) if and only if Y is a solution of

$$R_0(t)\frac{d^2Y}{dt^2} + R_1(t)\frac{dY}{dt} + R_2(t)Y = 0. \quad (\text{B})$$

where

$$R_i(t) = P_i(t + x_0), \quad i = 0, 1, 2.$$

- (b) Show that R_0, R_1 , and R_2 are polynomials in t with no common factors, and $R_0(0) = 0$; thus, $t_0 = 0$ is a singular point of (B).

9.5 The Method of Frobenius I

In this section we begin to study series solutions of a homogeneous linear second order differential equation with a regular singular point at $x_0 = 0$, so it can be written as

$$x^2 A(x)y'' + xB(x)y' + C(x)y = 0, \quad (9.2)$$

where A, B, C are polynomials and $A(0) \neq 0$.

We'll see that (9.2) always has at least one solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

where $a_0 \neq 0$ and r is a suitably chosen number. The method we will use to find solutions of this form and other forms that we'll encounter in the next two sections is called *the method of Frobenius*, and we'll call them *Frobenius solutions*.

It can be shown that the power series $\sum_{n=0}^{\infty} a_n x^n$ in a Frobenius solution of (9.2) converges on some open interval $(-\rho, \rho)$, where $0 < \rho \leq \infty$. However, since x^r may be complex for negative x or undefined if $x = 0$, we'll consider solutions defined for positive values of x . Easy modifications of our results yield solutions defined for negative values of x . (Exercise 9.5.54).

We'll restrict our attention to the case where A, B , and C are polynomials of degree not greater than two, so (9.2) becomes

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0, \quad (9.3)$$

where α_i, β_i , and γ_i are real constants and $\alpha_0 \neq 0$. Most equations that arise in applications can be written this way. Some examples are

$$\begin{aligned} \alpha x^2 y'' + \beta x y' + \gamma y &= 0 & (\text{Euler's equation}), \\ x^2 y'' + x y' + (x^2 - \nu^2)y &= 0 & (\text{Bessel's equation}), \end{aligned}$$

and

$$x y'' + (1 - x)y' + \lambda y = 0, \quad (\text{Laguerre's equation}),$$

where we would multiply the last equation through by x to put it in the form (9.3). However, the method of Frobenius can be extended to the case where A, B , and C are functions that can be represented by power series in x on some interval that contains zero, and $A_0(0) \neq 0$ (Exercises 9.5.57 and 9.5.58).

The next two theorems will enable us to develop systematic methods for finding Frobenius solutions of (9.3).

Theorem 9.5.1. *Let*

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y,$$

and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_1(r) &= \alpha_1 r(r-1) + \beta_1 r + \gamma_1, \\ p_2(r) &= \alpha_2 r(r-1) + \beta_2 r + \gamma_2. \end{aligned}$$

Suppose the series

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (9.4)$$

converges on $(0, \rho)$. Then

$$Ly = \sum_{n=0}^{\infty} b_n x^{n+r} \quad (9.5)$$

on $(0, \rho)$, where

$$\begin{aligned} b_0 &= p_0(r)a_0, \\ b_1 &= p_0(r+1)a_1 + p_1(r)a_0, \\ b_n &= p_0(n+r)a_n + p_1(n+r-1)a_{n-1} + p_2(n+r-2)a_{n-2}, \quad n \geq 2. \end{aligned} \quad (9.6)$$

Proof. We begin by showing that if y is given by (9.4) and α, β , and γ are constants, then

$$\alpha x^2 y'' + \beta x y' + \gamma y = \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r}, \quad (9.7)$$

where

$$p(r) = \alpha r(r-1) + \beta r + \gamma.$$

Differentiating (3) twice yields

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \quad (9.8)$$

and

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}. \quad (9.9)$$

Multiplying (9.8) by x and (9.9) by x^2 yields

$$x y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r}$$

and

$$x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r}.$$

Therefore

$$\begin{aligned} \alpha x^2 y'' + \beta x y' + \gamma y &= \sum_{n=0}^{\infty} [\alpha(n+r)(n+r-1) + \beta(n+r) + \gamma] a_n x^{n+r} \\ &= \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r}, \end{aligned}$$

which proves (9.7).

Multiplying (9.7) by x yields

$$x(\alpha x^2 y'' + \beta x y' + \gamma y) = \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r+1} = \sum_{n=1}^{\infty} p(n+r-1)a_{n-1} x^{n+r}. \quad (9.10)$$

Multiplying (9.7) by x^2 yields

$$x^2(\alpha x^2 y'' + \beta x y' + \gamma y) = \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r+2} = \sum_{n=2}^{\infty} p(n+r-2)a_{n-2} x^{n+r}. \quad (9.11)$$

To use these results, we rewrite

$$Ly = x^2(\alpha_0 + \alpha_1x + \alpha_2x^2)y'' + x(\beta_0 + \beta_1x + \beta_2x^2)y' + (\gamma_0 + \gamma_1x + \gamma_2x^2)y$$

as

$$\begin{aligned} Ly &= (\alpha_0x^2y'' + \beta_0xy' + \gamma_0y) + x(\alpha_1x^2y'' + \beta_1xy' + \gamma_1y) \\ &\quad + x^2(\alpha_2x^2y'' + \beta_2xy' + \gamma_2y). \end{aligned} \quad (9.12)$$

From (9.7) with $p = p_0$,

$$\alpha_0x^2y'' + \beta_0xy' + \gamma_0y = \sum_{n=0}^{\infty} p_0(n+r)a_nx^{n+r}.$$

From (9.10) with $p = p_1$,

$$x(\alpha_1x^2y'' + \beta_1xy' + \gamma_1y) = \sum_{n=1}^{\infty} p_1(n+r-1)a_{n-1}x^{n+r}.$$

From (9.11) with $p = p_2$,

$$x^2(\alpha_2x^2y'' + \beta_2xy' + \gamma_2y) = \sum_{n=2}^{\infty} p_2(n+r-2)a_{n-2}x^{n+r}.$$

Therefore we can rewrite (9.12) as

$$\begin{aligned} Ly &= \sum_{n=0}^{\infty} p_0(n+r)a_nx^{n+r} + \sum_{n=1}^{\infty} p_1(n+r-1)a_{n-1}x^{n+r} \\ &\quad + \sum_{n=2}^{\infty} p_2(n+r-2)a_{n-2}x^{n+r}, \end{aligned}$$

or

$$\begin{aligned} Ly &= p_0(r)a_0x^r + [p_0(r+1)a_1 + p_1(r)a_2]x^{r+1} \\ &\quad + \sum_{n=2}^{\infty} [p_0(n+r)a_n + p_1(n+r-1)a_{n-1} + p_2(n+r-2)a_{n-2}]x^{n+r}, \end{aligned}$$

which implies (9.5) with $\{b_n\}$ defined as in (9.6).

Theorem 9.5.2. *Let*

$$Ly = x^2(\alpha_0 + \alpha_1x + \alpha_2x^2)y'' + x(\beta_0 + \beta_1x + \beta_2x^2)y' + (\gamma_0 + \gamma_1x + \gamma_2x^2)y,$$

where $\alpha_0 \neq 0$, and define

$$\begin{aligned} p_0(r) &= \alpha_0r(r-1) + \beta_0r + \gamma_0, \\ p_1(r) &= \alpha_1r(r-1) + \beta_1r + \gamma_1, \\ p_2(r) &= \alpha_2r(r-1) + \beta_2r + \gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n . Define

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r) + p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}, \quad n \geq 2. \end{aligned} \quad (9.13)$$

Then the Frobenius series

$$y(x, r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n \quad (9.14)$$

converges and satisfies

$$Ly(x, r) = p_0(r)x^r \quad (9.15)$$

on the interval $(0, \rho)$, where ρ is the distance from the origin to the nearest zero of $A(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ in the complex plane. (If A is constant, then $\rho = \infty$.)

If $\{a_n(r)\}$ is determined by the recurrence relation (9.13) then substituting $a_n = a_n(r)$ into (9.6) yields $b_0 = p_0(r)$ and $b_n = 0$ for $n \geq 1$, so (9.5) reduces to (9.15). We omit the proof that the series (9.14) converges on $(0, \rho)$. ■

If $\alpha_i = \beta_i = \gamma_i = 0$ for $i = 1, 2$, then $Ly = 0$ reduces to the Euler equation

$$\alpha_0 x^2 y'' + \beta_0 x y' + \gamma_0 y = 0.$$

Theorem 9.4.3 shows that the solutions of this equation are determined by the zeros of the indicial polynomial

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Since (9.15) implies that this is also true for the solutions of $Ly = 0$, we'll also say that p_0 is the *indicial polynomial* of (9.3), and that $p_0(r) = 0$ is the *indicial equation* of $Ly = 0$. We'll consider only cases where the indicial equation has real roots r_1 and r_2 , with $r_1 \geq r_2$.

Theorem 9.5.3. *Let L and $\{a_n(r)\}$ be as in Theorem 9.5.2, and suppose the indicial equation $p_0(r) = 0$ of $Ly = 0$ has real roots r_1 and r_2 , where $r_1 \geq r_2$. Then*

$$y_1(x) = y(x, r_1) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

is a Frobenius solution of $Ly = 0$. Moreover, if $r_1 - r_2$ isn't an integer then

$$y_2(x) = y(x, r_2) = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n$$

is also a Frobenius solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Proof. Since r_1 and r_2 are roots of $p_0(r) = 0$, the indicial polynomial can be factored as

$$p_0(r) = \alpha_0(r - r_1)(r - r_2). \quad (9.16)$$

Therefore

$$p_0(n + r_1) = n\alpha_0(n + r_1 - r_2),$$

which is nonzero if $n > 0$, since $r_1 - r_2 \geq 0$. Therefore the assumptions of Theorem 9.5.2 hold with $r = r_1$, and (9.15) implies that $Ly_1 = p_0(r_1)x^{r_1} = 0$.

Now suppose $r_1 - r_2$ isn't an integer. From (9.16),

$$p_0(n + r_2) = n\alpha_0(n - r_1 + r_2) \neq 0 \quad \text{if } n = 1, 2, \dots.$$

Hence, the assumptions of Theorem 9.5.2 hold with $r = r_2$, and (9.15) implies that $Ly_2 = p_0(r_2)x^{r_2} = 0$. We leave the proof that $\{y_1, y_2\}$ is a fundamental set of solutions as an exercise (Exercise 9.5.52). ■

It isn't always possible to obtain explicit formulas for the coefficients in Frobenius solutions. However, we can always set up the recurrence relations and use them to compute as many coefficients as we want. The next example illustrates this.

Example 9.5.1: Find a fundamental set of Frobenius solutions of

$$2x^2(1+x+x^2)y'' + x(9+11x+11x^2)y' + (6+10x+7x^2)y = 0. \quad (9.17)$$

Compute just the first six coefficients $a_0, \dots, a_5$ in each solution.

Solution: For the given equation, the polynomials defined in Theorem 9.5.2 are

$$\begin{aligned} p_0(r) &= 2r(r-1) + 9r + 6 = (2r+3)(r+2), \\ p_1(r) &= 2r(r-1) + 11r + 10 = (2r+5)(r+2), \\ p_2(r) &= 2r(r-1) + 11r + 7 = (2r+7)(r+1). \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1 = -3/2$ and $r_2 = -2$, so $r_1 - r_2 = 1/2$. Therefore Theorem 9.5.3 implies that

$$y_1 = x^{-3/2} \sum_{n=0}^{\infty} a_n(-3/2)x^n \quad \text{and} \quad y_2 = x^{-2} \sum_{n=0}^{\infty} a_n(-2)x^n \quad (9.18)$$

form a fundamental set of Frobenius solutions of (9.17). To find the coefficients in these series, we use the recurrence relation of Theorem 9.5.2; thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)} = -\frac{(2r+5)(r+2)}{(2r+5)(r+3)} = -\frac{r+2}{r+3}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1} + p_2(n+r-2)a_{n-2}}{p_0(n+r)} \\ &= -\frac{(n+r+1)(2n+2r+3)a_{n-1}(r) + (n+r-1)(2n+2r+3)a_{n-2}(r)}{(n+r+2)(2n+2r+3)} \\ &= -\frac{(n+r+1)a_{n-1}(r) + (n+r-1)a_{n-2}(r)}{n+r+2}, \quad n \geq 2. \end{aligned}$$

Setting $r = -3/2$ in these equations yields

$$\begin{aligned} a_0(-3/2) &= 1, \\ a_1(-3/2) &= -1/3, \\ a_n(-3/2) &= -\frac{(2n-1)a_{n-1}(-3/2) + (2n-5)a_{n-2}(-3/2)}{2n+1}, \quad n \geq 2, \end{aligned} \quad (9.19)$$

and setting $r = -2$ yields

$$\begin{aligned} a_0(-2) &= 1, \\ a_1(-2) &= 0, \\ a_n(-2) &= -\frac{(n-1)a_{n-1}(-2) + (n-3)a_{n-2}(-2)}{n}, \quad n \geq 2. \end{aligned} \quad (9.20)$$

Calculating with (9.19) and (9.20) and substituting the results into (9.18) yields the fundamental set of Frobenius solutions

$$\begin{aligned} y_1 &= x^{-3/2} \left(1 - \frac{1}{3}x + \frac{2}{5}x^2 - \frac{5}{21}x^3 + \frac{7}{135}x^4 + \frac{76}{1155}x^5 + \cdots \right), \\ y_2 &= x^{-2} \left(1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5 + \cdots \right). \end{aligned}$$

9.5.1 Special Cases With Two Term Recurrence Relations

For $n \geq 2$, the recurrence relation (9.13) of Theorem 9.5.2 involves the three coefficients $a_n(r)$, $a_{n-1}(r)$, and $a_{n-2}(r)$. We'll now consider some special cases where (9.13) reduces to a two term recurrence relation; that is, a relation involving only $a_n(r)$ and $a_{n-1}(r)$ or only $a_n(r)$ and $a_{n-2}(r)$. This simplification often makes it possible to obtain explicit formulas for the coefficients of Frobenius solutions.

We first consider equations of the form

$$x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y = 0$$

with $\alpha_0 \neq 0$. For this equation, $\alpha_2 = \beta_2 = \gamma_2 = 0$, so $p_2 \equiv 0$ and the recurrence relations in Theorem 9.5.2 simplify to

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r), \quad n \geq 1. \end{aligned} \quad (9.21)$$

Example 9.5.2: Find a fundamental set of Frobenius solutions of

$$x^2(3+x)y'' + 5x(1+x)y' - (1-4x)y = 0. \quad (9.22)$$

Give explicit formulas for the coefficients in the solutions.

Solution: For this equation, the polynomials defined in Theorem 9.5.2 are

$$\begin{aligned} p_0(r) &= 3r(r-1) + 5r - 1 = (3r-1)(r+1), \\ p_1(r) &= r(r-1) + 5r + 4 = (r+2)^2, \\ p_2(r) &= 0. \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1 = 1/3$ and $r_2 = -1$, so $r_1 - r_2 = 4/3$. Therefore Theorem 9.5.3 implies that

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} a_n(1/3)x^n \quad \text{and} \quad y_2 = x^{-1} \sum_{n=0}^{\infty} a_n(-1)x^n$$

form a fundamental set of Frobenius solutions of (9.22). To find the coefficients in these series, we use the recurrence relations (9.21); thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r) \\ &= -\frac{(n+r+1)^2}{(3n+3r-1)(n+r+1)}a_{n-1}(r) \\ &= -\frac{n+r+1}{3n+3r-1}a_{n-1}(r), \quad n \geq 1. \end{aligned} \quad (9.23)$$

Setting $r = 1/3$ in (9.23) yields

$$\begin{aligned} a_0(1/3) &= 1, \\ a_n(1/3) &= -\frac{3n+4}{9n}a_{n-1}(1/3), \quad n \geq 1. \end{aligned}$$

By using the product notation introduced in Section 7.2 and proceeding as we did in the examples in that section yields

$$a_n(1/3) = \frac{(-1)^n \prod_{j=1}^n (3j+4)}{9^n n!}, \quad n \geq 0.$$

Therefore

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^n (3j+4)}{9^n n!} x^n$$

is a Frobenius solution of (9.22).

Setting $r = -1$ in (9.23) yields

$$\begin{aligned} a_0(-1) &= 1, \\ a_n(-1) &= -\frac{n}{3n-4} a_{n-1}(-1), \quad n \geq 1, \end{aligned}$$

so

$$a_n(-1) = \frac{(-1)^n n!}{\prod_{j=1}^n (3j-4)}.$$

Therefore

$$y_2 = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\prod_{j=1}^n (3j-4)} x^n$$

is a Frobenius solution of (9.22), and $\{y_1, y_2\}$ is a fundamental set of solutions. ■

We now consider equations of the form

$$x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0 \quad (9.24)$$

with $\alpha_0 \neq 0$. For this equation, $\alpha_1 = \beta_1 = \gamma_1 = 0$, so $p_1 \equiv 0$ and the recurrence relations in Theorem 9.5.2 simplify to

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= 0, \\ a_n(r) &= -\frac{p_2(n+r-2)}{p_0(n+r)} a_{n-2}(r), \quad n \geq 2. \end{aligned}$$

Since $a_1(r) = 0$, the last equation implies that $a_n(r) = 0$ if n is odd, so the Frobenius solutions are of the form

$$y(x, r) = x^r \sum_{m=0}^{\infty} a_{2m}(r) x^{2m},$$

where

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r), \quad m \geq 1. \end{aligned} \quad (9.25)$$

Example 9.5.3: Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y'' - x(3+4x^2)y' + (2-2x^2)y = 0. \quad (9.26)$$

Give explicit formulas for the coefficients in the solutions.

Solution: For this equation, the polynomials defined in Theorem 9.5.2 are

$$\begin{aligned} p_0(r) &= 2r(r-1) - 3r + 2 = (r-2)(2r-1), \\ p_1(r) &= 0 \\ p_2(r) &= -[r(r-1) + 4r + 2] = -(r+1)(r+2). \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1 = 2$ and $r_2 = 1/2$, so $r_1 - r_2 = 3/2$. Therefore Theorem 9.5.3 implies that

$$y_1 = x^2 \sum_{m=0}^{\infty} a_{2m}(1/3)x^{2m} \quad \text{and} \quad y_2 = x^{1/2} \sum_{m=0}^{\infty} a_{2m}(1/2)x^{2m}$$

form a fundamental set of Frobenius solutions of (9.26). To find the coefficients in these series, we use the recurrence relation (9.25); thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r) \\ &= \frac{(2m+r)(2m+r-1)}{(2m+r-2)(4m+2r-1)} a_{2m-2}(r), \quad m \geq 1. \end{aligned} \tag{9.27}$$

Setting $r = 2$ in (9.27) yields

$$\begin{aligned} a_0(2) &= 1, \\ a_{2m}(2) &= \frac{(m+1)(2m+1)}{m(4m+3)} a_{2m-2}(2), \quad m \geq 1, \end{aligned}$$

so

$$a_{2m}(2) = (m+1) \prod_{j=1}^m \frac{2j+1}{4j+3}.$$

Therefore

$$y_1 = x^2 \sum_{m=0}^{\infty} (m+1) \left(\prod_{j=1}^m \frac{2j+1}{4j+3} \right) x^{2m}$$

is a Frobenius solution of (9.26).

Setting $r = 1/2$ in (9.27) yields

$$\begin{aligned} a_0(1/2) &= 1, \\ a_{2m}(1/2) &= \frac{(4m-1)(4m+1)}{8m(4m-3)} a_{2m-2}(1/2), \quad m \geq 1, \end{aligned}$$

so

$$a_{2m}(1/2) = \frac{1}{8^m m!} \prod_{j=1}^m \frac{(4j-1)(4j+1)}{4j-3}.$$

Therefore

$$y_2 = x^{1/2} \sum_{m=0}^{\infty} \frac{1}{8^m m!} \left(\prod_{j=1}^m \frac{(4j-1)(4j+1)}{4j-3} \right) x^{2m}$$

is a Frobenius solution of (9.26) and $\{y_1, y_2\}$ is a fundamental set of solutions.

Remark 9.5.1: Thus far, we considered only the case where the indicial equation has real roots that don't differ by an integer, which allows us to apply Theorem 9.5.3. However, for equations of the form (9.24), the sequence $\{a_{2m}(r)\}$ in (9.25) is defined for $r = r_2$ if $r_1 - r_2$ isn't an *even* integer. It can be shown (Exercise 9.5.56) that in this case

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{2m}(r_1) x^{2m} \quad \text{and} \quad y_2 = x^{r_2} \sum_{m=0}^{\infty} a_{2m}(r_2) x^{2m}$$

form a fundamental set Frobenius solutions of (9.24).

As we said at the end of Section 7.2, if you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficients is essentially irrelevant; recursive computation is usually more efficient. Since it's also laborious, we encourage you to write short programs to implement recurrence relations on a calculator or computer, even in exercises where this is not specifically required.

In practical use of the method of Frobenius when $x_0 = 0$ is a regular singular point, we're interested in how well the functions

$$y_N(x, r_i) = x^{r_i} \sum_{n=0}^N a_n(r_i) x^n, \quad i = 1, 2,$$

approximate solutions to a given equation when r_i is a zero of the indicial polynomial. In dealing with the corresponding problem for the case where $x_0 = 0$ is an ordinary point, we used numerical integration to solve the differential equation subject to initial conditions $y(0) = a_0$, $y'(0) = a_1$, and compared the result with values of the Taylor polynomial

$$T_N(x) = \sum_{n=0}^N a_n x^n.$$

We can't do that here, since in general we can't prescribe arbitrary initial values for solutions of a differential equation at a singular point. Therefore, motivated by Theorem 9.5.2 (specifically, (9.15)), we suggest the following procedure.

Verification Procedure

Let L and $Y_n(x; r_i)$ be defined by

$$Ly = x^2(\alpha_0 + \alpha_1x + \alpha_2x^2)y'' + x(\beta_0 + \beta_1x + \beta_2x^2)y' + (\gamma_0 + \gamma_1x + \gamma_2x^2)y$$

and

$$y_N(x; r_i) = x^{r_i} \sum_{n=0}^N a_n(r_i)x^n,$$

where the coefficients $\{a_n(r_i)\}_{n=0}^N$ are computed as in (9.13), Theorem 9.5.2. Compute the error

$$E_N(x; r_i) = x^{-r_i}Ly_N(x; r_i)/\alpha_0 \quad (9.28)$$

for various values of N and various values of x in the interval $(0, \rho)$, with ρ as defined in Theorem 9.5.2.

The multiplier x^{-r_i}/α_0 on the right of (9.28) eliminates the effects of small or large values of x^{r_i} near $x = 0$, and of multiplication by an arbitrary constant. In some exercises you will be asked to estimate the maximum value of $E_N(x; r_i)$ on an interval $(0, \delta]$ by computing $E_N(x_m; r_i)$ at the M points $x_m = m\delta/M$, $m = 1, 2, \dots, M$, and finding the maximum of the absolute values:

$$\sigma_N(\delta) = \max\{|E_N(x_m; r_i)|, m = 1, 2, \dots, M\}. \quad (9.29)$$

(For simplicity, this notation ignores the dependence of the right side of the equation on i and M .)

To implement this procedure, you'll have to write a computer program to calculate $\{a_n(r_i)\}$ from the applicable recurrence relation, and to evaluate $E_N(x; r_i)$.

The next exercise set contains five exercises specifically identified by L that ask you to implement the verification procedure. These particular exercises were chosen arbitrarily you can just as well formulate such laboratory problems for any of the equations in any of the Exercises 9.5.1–9.5.10, 9.5.14–9.5.25, and 9.5.28–9.5.51

9.5.2 Exercises

This set contains exercises specifically identified by L that ask you to implement the verification procedure. These particular exercises were chosen arbitrarily you can just as well formulate such laboratory problems for any of the equations in Exercises 9.5.1–9.5.10, 9.5.14–9.5.25, and 9.5.28–9.5.51.

In Exercises 9.5.1–9.5.10 find a fundamental set of Frobenius solutions. Compute $a_0, a_1, \dots, a_N$ for N at least 7 in each solution.

1. C $2x^2(1 + x + x^2)y'' + x(3 + 3x + 5x^2)y' - y = 0$
2. C $3x^2y'' + 2x(1 + x - 2x^2)y' + (2x - 8x^2)y = 0$
3. C $x^2(3 + 3x + x^2)y'' + x(5 + 8x + 7x^2)y' - (1 - 2x - 9x^2)y = 0$
4. C $4x^2y'' + x(7 + 2x + 4x^2)y' - (1 - 4x - 7x^2)y = 0$
5. C $12x^2(1 + x)y'' + x(11 + 35x + 3x^2)y' - (1 - 10x - 5x^2)y = 0$
6. C $x^2(5 + x + 10x^2)y'' + x(4 + 3x + 48x^2)y' + (x + 36x^2)y = 0$
7. C $8x^2y'' - 2x(3 - 4x - x^2)y' + (3 + 6x + x^2)y = 0$
8. C $18x^2(1 + x)y'' + 3x(5 + 11x + x^2)y' - (1 - 2x - 5x^2)y = 0$
9. C $x(3 + x + x^2)y'' + (4 + x - x^2)y' + xy = 0$
10. C $10x^2(1 + x + 2x^2)y'' + x(13 + 13x + 66x^2)y' - (1 + 4x + 10x^2)y = 0$

11. L The Frobenius solutions of

$$2x^2(1+x+x^2)y'' + x(9+11x+11x^2)y' + (6+10x+7x^2)y = 0$$

obtained in Example 9.5.1 are defined on $(0, \rho)$, where ρ is defined in Theorem 9.5.2. Find ρ . Then do the following experiments for each Frobenius solution, with $M = 20$ and $\delta = .5\rho, .7\rho$, and $.9\rho$ in the verification procedure described at the end of this section.

- (a) Compute $\sigma_N(\delta)$ (see Eqn. (9.29)) for $N = 5, 10, 15, \dots, 50$.
 (b) Find N such that $\sigma_N(\delta) < 10^{-5}$.
 (c) Find N such that $\sigma_N(\delta) < 10^{-10}$.
12. L By Theorem 9.5.2 the Frobenius solutions of the equation in Exercise 9.5.4 are defined on $(0, \infty)$. Do experiments (a), (b), and (c) of Exercise 9.5.11 for each Frobenius solution, with $M = 20$ and $\delta = 1, 2$, and 3 in the verification procedure described at the end of this section.
13. L The Frobenius solutions of the equation in Exercise 9.5.6 are defined on $(0, \rho)$, where ρ is defined in Theorem 9.5.2. Find ρ and do experiments (a), (b), and (c) of Exercise 9.5.11 for each Frobenius solution, with $M = 20$ and $\delta = .3\rho, .4\rho$, and $.5\rho$, in the verification procedure described at the end of this section.

In Exercises 9.5.14–9.5.25 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients in each solution.

14. $2x^2y'' + x(3+2x)y' - (1-x)y = 0$
 15. $x^2(3+x)y'' + x(5+4x)y' - (1-2x)y = 0$
 16. $2x^2y'' + x(5+x)y' - (2-3x)y = 0$
 17. $3x^2y'' + x(1+x)y' - y = 0$
 18. $2x^2y'' - xy' + (1-2x)y = 0$
 19. $9x^2y'' + 9xy' - (1+3x)y = 0$
 20. $3x^2y'' + x(1+x)y' - (1+3x)y = 0$
 21. $2x^2(3+x)y'' + x(1+5x)y' + (1+x)y = 0$
 22. $x^2(4+x)y'' - x(1-3x)y' + y = 0$
 23. $2x^2y'' + 5xy' + (1+x)y = 0$
 24. $x^2(3+4x)y'' + x(5+18x)y' - (1-12x)y = 0$
 25. $6x^2y'' + x(10-x)y' - (2+x)y = 0$
26. L By Theorem 9.5.2 the Frobenius solutions of the equation in Exercise 9.5.17 are defined on $(0, \infty)$. Do experiments (a), (b), and (c) of Exercise 9.5.11 for each Frobenius solution, with $M = 20$ and $\delta = 3, 6, 9$, and 12 in the verification procedure described at the end of this section.
27. L The Frobenius solutions of the equation in Exercise 9.5.22 are defined on $(0, \rho)$, where ρ is defined in Theorem 9.5.2. Find ρ and do experiments (a), (b), and (c) of Exercise 9.5.11 for each Frobenius solution, with $M = 20$ and $\delta = .25\rho, .5\rho$, and $.75\rho$ in the verification procedure described at the end of this section.

In Exercises 9.5.28–9.5.32 find a fundamental set of Frobenius solutions. Compute coefficients $a_0, \dots, a_N$ for N at least 7 in each solution.

28. $\boxed{\text{C}}$ $x^2(8+x)y'' + x(2+3x)y' + (1+x)y = 0$

29. $\boxed{\text{C}}$ $x^2(3+4x)y'' + x(11+4x)y' - (3+4x)y = 0$

30. $\boxed{\text{C}}$ $2x^2(2+3x)y'' + x(4+11x)y' - (1-x)y = 0$

31. $\boxed{\text{C}}$ $x^2(2+x)y'' + 5x(1-x)y' - (2-8x)y$

32. $\boxed{\text{C}}$ $x^2(6+x)y'' + x(11+4x)y' + (1+2x)y = 0$

In Exercises 9.5.33–9.5.46 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients in each solution.

33. $8x^2y'' + x(2+x^2)y' + y = 0$

34. $8x^2(1-x^2)y'' + 2x(1-13x^2)y' + (1-9x^2)y = 0$

35. $x^2(1+x^2)y'' - 2x(2-x^2)y' + 4y = 0$

36. $x(3+x^2)y'' + (2-x^2)y' - 8xy = 0$

37. $4x^2(1-x^2)y'' + x(7-19x^2)y' - (1+14x^2)y = 0$

38. $3x^2(2-x^2)y'' + x(1-11x^2)y' + (1-5x^2)y = 0$

39. $2x^2(2+x^2)y'' - x(12-7x^2)y' + (7+3x^2)y = 0$

40. $2x^2(2+x^2)y'' + x(4+7x^2)y' - (1-3x^2)y = 0$

41. $2x^2(1+2x^2)y'' + 5x(1+6x^2)y' - (2-40x^2)y = 0$

42. $3x^2(1+x^2)y'' + 5x(1+x^2)y' - (1+5x^2)y = 0$

43. $x(1+x^2)y'' + (4+7x^2)y' + 8xy = 0$

44. $x^2(2+x^2)y'' + x(3+x^2)y' - y = 0$

45. $2x^2(1+x^2)y'' + x(3+8x^2)y' - (3-4x^2)y = 0$

46. $9x^2y'' + 3x(3+x^2)y' - (1-5x^2)y = 0$

In Exercises 9.5.47–9.5.51 find a fundamental set of Frobenius solutions. Compute the coefficients $a_0, \dots, a_{2M}$ for M at least 7 in each solution.

47. $\boxed{\text{C}}$ $6x^2y'' + x(1+6x^2)y' + (1+9x^2)y = 0$

48. $\boxed{\text{C}}$ $x^2(8+x^2)y'' + 7x(2+x^2)y' - (2-9x^2)y = 0$

49. $\boxed{\text{C}}$ $9x^2(1+x^2)y'' + 3x(3+13x^2)y' - (1-25x^2)y = 0$

50. $\boxed{\text{C}}$ $4x^2(1+x^2)y'' + 4x(1+6x^2)y' - (1-25x^2)y = 0$

51. $\boxed{\text{C}}$ $8x^2(1+2x^2)y'' + 2x(5+34x^2)y' - (1-30x^2)y = 0$

52. Suppose $r_1 > r_2$, $a_0 = b_0 = 1$, and the Frobenius series

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

both converge on an interval $(0, \rho)$.

- (a) Show that y_1 and y_2 are linearly independent on $(0, \rho)$. HINT: Show that if c_1 and c_2 are constants such that $c_1 y_1 + c_2 y_2 \equiv 0$ on $(0, \rho)$, then

$$c_1 x^{r_1-r_2} \sum_{n=0}^{\infty} a_n x^n + c_2 \sum_{n=0}^{\infty} b_n x^n = 0, \quad 0 < x < \rho.$$

Then let $x \rightarrow 0+$ to conclude that $c_2 = 0$.

- (b) Use the result of (b) to complete the proof of Theorem 9.5.3.

53. The equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \quad (9.1)$$

is *Bessel's equation of order ν* . (Here ν is a parameter, and this use of "order" should not be confused with its usual use as in "the order of the equation.") The solutions of (9.1) are *Bessel functions of order ν* .

- (a) Assuming that ν isn't an integer, find a fundamental set of Frobenius solutions of (9.1).
 (b) If $\nu = 1/2$, the solutions of (9.1) reduce to familiar elementary functions. Identify these functions.

54. (a) Verify that

$$\frac{d}{dx} (|x|^r x^n) = (n+r)|x|^r x^{n-1} \quad \text{and} \quad \frac{d^2}{dx^2} (|x|^r x^n) = (n+r)(n+r-1)|x|^r x^{n-2}$$

if $x \neq 0$.

- (b) Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0.$$

Show that if $x^r \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ on $(0, \rho)$ then $|x|^r \sum_{n=0}^{\infty} a_n x^n$ is a solution on $(-\rho, 0)$ and $(0, \rho)$.

55. (a) Deduce from Eqn. (9.21) that

$$a_n(r) = (-1)^n \prod_{j=1}^n \frac{p_1(j+r-1)}{p_0(j+r)}.$$

- (b) Conclude that if $p_0(r) = \alpha_0(r-r_1)(r-r_2)$ where $r_1 - r_2$ is not an integer, then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n \quad \text{and} \quad y_2 = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n$$

form a fundamental set of Frobenius solutions of

$$x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y = 0.$$

- (c) Show that if p_0 satisfies the hypotheses of (b) then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^n (j+r_1-r_2)} \left(\frac{\gamma_1}{\alpha_0} \right)^n x^n$$

and

$$y_2 = x^{r_2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^n (j+r_2-r_1)} \left(\frac{\gamma_1}{\alpha_0} \right)^n x^n$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x) y = 0.$$

56. Let

$$Ly = x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_2(r) = \alpha_2 r(r-1) + \beta_2 r + \gamma_2.$$

(a) Use Theorem 9.5.2 to show that if

$$\begin{aligned} a_0(r) &= 1, \\ p_0(2m+r)a_{2m}(r) + p_2(2m+r-2)a_{2m-2}(r) &= 0, \quad m \geq 1, \end{aligned} \tag{9.1}$$

then the Frobenius series $y(x, r) = x^r \sum_{m=0}^{\infty} a_{2m} x^{2m}$ satisfies $Ly(x, r) = p_0(r)x^r$.

(b) Deduce from (9.1) that if $p_0(2m+r)$ is nonzero for every positive integer m then

$$a_{2m}(r) = (-1)^m \prod_{j=1}^m \frac{p_2(2j+r-2)}{p_0(2j+r)}.$$

(c) Conclude that if $p_0(r) = \alpha_0(r-r_1)(r-r_2)$ where $r_1 - r_2$ is not an even integer, then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{2m}(r_1)x^{2m} \quad \text{and} \quad y_2 = x^{r_2} \sum_{m=0}^{\infty} a_{2m}(r_2)x^{2m}$$

form a fundamental set of Frobenius solutions of $Ly = 0$.

(d) Show that if p_0 satisfies the hypotheses of (c) then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m! \prod_{j=1}^m (2j+r_1-r_2)} \left(\frac{\gamma_2}{\alpha_0} \right)^m x^{2m}$$

and

$$y_2 = x^{r_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m! \prod_{j=1}^m (2j+r_2-r_1)} \left(\frac{\gamma_2}{\alpha_0} \right)^m x^{2m}$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_2 x^2)y = 0.$$

57. Let

$$Ly = x^2 q_0(x)y'' + x q_1(x)y' + q_2(x)y,$$

where

$$q_0(x) = \sum_{j=0}^{\infty} \alpha_j x^j, \quad q_1(x) = \sum_{j=0}^{\infty} \beta_j x^j, \quad q_2(x) = \sum_{j=0}^{\infty} \gamma_j x^j,$$

and define

$$p_j(r) = \alpha_j r(r-1) + \beta_j r + \gamma_j, \quad j = 0, 1, \dots$$

Let $y = x^r \sum_{n=0}^{\infty} a_n x^n$. Show that

$$Ly = x^r \sum_{n=0}^{\infty} b_n x^n,$$

where

$$b_n = \sum_{j=0}^n p_j(n+r-j)a_{n-j}.$$

58. (a) Let L be as in Exercise 9.5.57. Show that if

$$y(x, r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n$$

where

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{1}{p_0(n+r)} \sum_{j=1}^n p_j(n+r-j) a_{n-j}(r), \quad n \geq 1, \end{aligned}$$

then

$$Ly(x, r) = p_0(r) x^r.$$

- (b) Conclude that if

$$p_0(r) = \alpha_0(r - r_1)(r - r_2)$$

where $r_1 - r_2$ isn't an integer then $y_1 = y(x, r_1)$ and $y_2 = y(x, r_2)$ are solutions of $Ly = 0$.

59. Let

$$Ly = x^2(\alpha_0 + \alpha_q x^q) y'' + x(\beta_0 + \beta_q x^q) y' + (\gamma_0 + \gamma_q x^q) y$$

where q is a positive integer, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_q(r) = \alpha_q r(r-1) + \beta_q r + \gamma_q.$$

- (a) Show that if

$$y(x, r) = x^r \sum_{m=0}^{\infty} a_{qm}(r) x^{qm}$$

where

$$\begin{aligned} a_0(r) &= 1, \\ a_{qm}(r) &= -\frac{p_q(q(m-1) + r)}{p_0(qm + r)} a_{q(m-1)}(r), \quad m \geq 1, \end{aligned} \tag{9.1}$$

then

$$Ly(x, r) = p_0(r) x^r.$$

- (b) Deduce from (9.1) that

$$a_{qm}(r) = (-1)^m \prod_{j=1}^m \frac{p_q(q(j-1) + r)}{p_0(qj + r)}.$$

- (c) Conclude that if $p_0(r) = \alpha_0(r - r_1)(r - r_2)$ where $r_1 - r_2$ is not an integer multiple of q , then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{qm}(r_1) x^{qm} \quad \text{and} \quad y_2 = x^{r_2} \sum_{m=0}^{\infty} a_{qm}(r_2) x^{qm}$$

form a fundamental set of Frobenius solutions of $Ly = 0$.

- (d) Show that if p_0 satisfies the hypotheses of (c) then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{q^m m! \prod_{j=1}^m (qj + r_1 - r_2)} \left(\frac{\gamma_q}{\alpha_0} \right)^m x^{qm}$$

and

$$y_2 = x^{r_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{q^m m! \prod_{j=1}^m (qj + r_2 - r_1)} \left(\frac{\gamma_q}{\alpha_0} \right)^m x^{qm}$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_q x^q) y = 0.$$

60. (a) Suppose α_0, α_1 , and α_2 are real numbers with $\alpha_0 \neq 0$, and $\{a_n\}_{n=0}^{\infty}$ is defined by

$$\alpha_0 a_1 + \alpha_1 a_0 = 0$$

and

$$\alpha_0 a_n + \alpha_1 a_{n-1} + \alpha_2 a_{n-2} = 0, \quad n \geq 2.$$

Show that

$$(\alpha_0 + \alpha_1 x + \alpha_2 x^2) \sum_{n=0}^{\infty} a_n x^n = \alpha_0 a_0,$$

and infer that

$$\sum_{n=0}^{\infty} a_n x^n = \frac{\alpha_0 a_0}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}.$$

- (b) With α_0, α_1 , and α_2 as in (a), consider the equation

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0, \quad (9.1)$$

and define

$$p_j(r) = \alpha_j r(r-1) + \beta_j r + \gamma_j, \quad j = 0, 1, 2.$$

Suppose

$$\frac{p_1(r-1)}{p_0(r)} = \frac{\alpha_1}{\alpha_0}, \quad \frac{p_2(r-2)}{p_0(r)} = \frac{\alpha_2}{\alpha_0},$$

and

$$p_0(r) = \alpha_0(r-r_1)(r-r_2),$$

where $r_1 > r_2$. Show that

$$y_1 = \frac{x^{r_1}}{\alpha_0 + \alpha_1 x + \alpha_2 x^2} \quad \text{and} \quad y_2 = \frac{x^{r_2}}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}$$

form a fundamental set of Frobenius solutions of (9.1) on any interval $(0, \rho)$ on which $\alpha_0 + \alpha_1 x + \alpha_2 x^2$ has no zeros.

In Exercises 9.5.61–9.5.68 use the method suggested by Exercise 9.5.60 to find the general solution on some interval $(0, \rho)$.

61. $2x^2(1+x)y'' - x(1-3x)y' + y = 0$
62. $6x^2(1+2x^2)y'' + x(1+50x^2)y' + (1+30x^2)y = 0$
63. $28x^2(1-3x)y'' - 7x(5+9x)y' + 7(2+9x)y = 0$
64. $9x^2(5+x)y'' + 9x(5+3x)y' - (5-8x)y = 0$
65. $8x^2(2-x^2)y'' + 2x(10-21x^2)y' - (2+35x^2)y = 0$
66. $4x^2(1+3x+x^2)y'' - 4x(1-3x-3x^2)y' + 3(1-x+x^2)y = 0$
67. $3x^2(1+x^2)y'' - x(1-10x-11x^2)y' + (1+5x^2)y = 0$
68. $4x^2(3+2x+x^2)y'' - x(3-14x-15x^2)y' + (3+7x^2)y = 0$

9.6 The Method of Frobenius II

In this section we discuss a method for finding two linearly independent Frobenius solutions of a homogeneous linear second order equation near a regular singular point in the case where the indicial equation has a repeated real root. As in the preceding section, we consider equations that can be written as

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0 \quad (9.2)$$

where $\alpha_0 \neq 0$. We assume that the indicial equation $p_0(r) = 0$ has a repeated real root r_1 . In this case Theorem 9.5.3 implies that (9.2) has one solution of the form

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n,$$

but does not provide a second solution y_2 such that $\{y_1, y_2\}$ is a fundamental set of solutions. The following extension of Theorem 9.5.2 provides a way to find a second solution.

Theorem 9.6.1. *Let*

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y, \quad (9.3)$$

where $\alpha_0 \neq 0$ and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_1(r) &= \alpha_1 r(r-1) + \beta_1 r + \gamma_1, \\ p_2(r) &= \alpha_2 r(r-1) + \beta_2 r + \gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n , and define

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r) + p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}, \quad n \geq 2. \end{aligned}$$

Then the Frobenius series

$$y(x, r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n \quad (9.4)$$

satisfies

$$Ly(x, r) = p_0(r)x^r. \quad (9.5)$$

Moreover,

$$\frac{\partial y}{\partial r}(x, r) = y(x, r) \ln x + x^r \sum_{n=1}^{\infty} a'_n(r) x^n, \quad (9.6)$$

and

$$L\left(\frac{\partial y}{\partial r}(x, r)\right) = p'_0(r)x^r + x^r p_0(r) \ln x. \quad (9.7)$$

Proof. Theorem 9.5.2 implies (9.5). Differentiating formally with respect to r in (9.4) yields

$$\begin{aligned}\frac{\partial y}{\partial r}(x, r) &= \frac{\partial}{\partial r}(x^r) \sum_{n=0}^{\infty} a_n(r)x^n + x^r \sum_{n=1}^{\infty} a'_n(r)x^n \\ &= x^r \ln x \sum_{n=0}^{\infty} a_n(r)x^n + x^r \sum_{n=1}^{\infty} a'_n(r)x^n \\ &= y(x, r) \ln x + x^r \sum_{n=1}^{\infty} a'_n(r)x^n,\end{aligned}$$

which proves (9.6).

To prove that $\partial y(x, r)/\partial r$ satisfies (9.7), we view y in (9.3) as a function $y = y(x, r)$ of two variables, where the prime indicates partial differentiation with respect to x ; thus,

$$y' = y'(x, r) = \frac{\partial y}{\partial x}(x, r) \quad \text{and} \quad y'' = y''(x, r) = \frac{\partial^2 y}{\partial x^2}(x, r).$$

With this notation we can use (9.3) to rewrite (9.5) as

$$x^2 q_0(x) \frac{\partial^2 y}{\partial x^2}(x, r) + x q_1(x) \frac{\partial y}{\partial x}(x, r) + q_2(x) y(x, r) = p_0(r) x^r, \quad (9.8)$$

where

$$\begin{aligned}q_0(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2, \\ q_1(x) &= \beta_0 + \beta_1 x + \beta_2 x^2, \\ q_2(x) &= \gamma_0 + \gamma_1 x + \gamma_2 x^2.\end{aligned}$$

Differentiating both sides of (9.8) with respect to r yields

$$x^2 q_0(x) \frac{\partial^3 y}{\partial r \partial x^2}(x, r) + x q_1(x) \frac{\partial^2 y}{\partial r \partial x}(x, r) + q_2(x) \frac{\partial y}{\partial r}(x, r) = p'_0(r) x^r + p_0(r) x^r \ln x.$$

By changing the order of differentiation in the first two terms on the left we can rewrite this as

$$x^2 q_0(x) \frac{\partial^3 y}{\partial x^2 \partial r}(x, r) + x q_1(x) \frac{\partial^2 y}{\partial x \partial r}(x, r) + q_2(x) \frac{\partial y}{\partial r}(x, r) = p'_0(r) x^r + p_0(r) x^r \ln x,$$

or

$$x^2 q_0(x) \frac{\partial^2}{\partial x^2} \left(\frac{\partial y}{\partial r}(x, r) \right) + x q_1(x) \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial r}(x, r) \right) + q_2(x) \frac{\partial y}{\partial r}(x, r) = p'_0(r) x^r + p_0(r) x^r \ln x,$$

which is equivalent to (9.7).

Theorem 9.6.2. *Let L be as in Theorem 9.6.1 and suppose the indicial equation $p_0(r) = 0$ has a repeated real root r_1 . Then*

$$y_1(x) = y(x, r_1) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

and

$$y_2(x) = \frac{\partial y}{\partial r}(x, r_1) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n \quad (9.9)$$

form a fundamental set of solutions of $Ly = 0$.

Proof. Since r_1 is a repeated root of $p_0(r) = 0$, the indicial polynomial can be factored as

$$p_0(r) = \alpha_0(r - r_1)^2,$$

so

$$p_0(n + r_1) = \alpha_0 n^2,$$

which is nonzero if $n > 0$. Therefore the assumptions of Theorem 9.6.1 hold with $r = r_1$, and (9.5) implies that $Ly_1 = p_0(r_1)x^{r_1} = 0$. Since

$$p'_0(r) = 2\alpha(r - r_1)$$

it follows that $p'_0(r_1) = 0$, so (9.7) implies that

$$Ly_2 = p'_0(r_1)x^{r_1} + x^{r_1}p_0(r_1)\ln x = 0.$$

This proves that y_1 and y_2 are both solutions of $Ly = 0$. We leave the proof that $\{y_1, y_2\}$ is a fundamental set as an exercise (Exercise 9.6.53).

Example 9.6.1: Find a fundamental set of solutions of

$$x^2(1 - 2x + x^2)y'' - x(3 + x)y' + (4 + x)y = 0. \quad (9.10)$$

Compute just the terms involving x^{n+r_1} , where $0 \leq n \leq 4$ and r_1 is the root of the indicial equation.

Solution: For the given equation, the polynomials defined in Theorem 9.6.1 are

$$\begin{aligned} p_0(r) &= r(r-1) - 3r + 4 = (r-2)^2, \\ p_1(r) &= -2r(r-1) - r + 1 = -(r-1)(2r+1), \\ p_2(r) &= r(r-1). \end{aligned}$$

Since $r_1 = 2$ is a repeated root of the indicial polynomial p_0 , Theorem 9.6.2 implies that

$$y_1 = x^2 \sum_{n=0}^{\infty} a_n(2)x^n \quad \text{and} \quad y_2 = y_1 \ln x + x^2 \sum_{n=1}^{\infty} a'_n(2)x^n \quad (9.11)$$

form a fundamental set of Frobenius solutions of (9.10). To find the coefficients in these series, we use the recurrence formulas from Theorem 9.6.1:

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)} = -\frac{(r-1)(2r+1)}{(r-1)^2} = \frac{2r+1}{r-1}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r) + p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)} \\ &= \frac{(n+r-2)[(2n+2r-1)a_{n-1}(r) - (n+r-3)a_{n-2}(r)]}{(n+r-2)^2} \\ &= \frac{(2n+2r-1)}{(n+r-2)}a_{n-1}(r) - \frac{(n+r-3)}{(n+r-2)}a_{n-2}(r), \quad n \geq 2. \end{aligned} \quad (9.12)$$

Differentiating yields

$$\begin{aligned} a'_1(r) &= -\frac{3}{(r-1)^2}, \\ a'_n(r) &= \frac{2n+2r-1}{n+r-2}a'_{n-1}(r) - \frac{n+r-3}{n+r-2}a'_{n-2}(r) \\ &\quad - \frac{3}{(n+r-2)^2}a_{n-1}(r) - \frac{1}{(n+r-2)^2}a_{n-2}(r), \quad n \geq 2. \end{aligned} \quad (9.13)$$

Setting $r = 2$ in (9.12) and (9.13) yields

$$\begin{aligned} a_0(2) &= 1, \\ a_1(2) &= 5, \\ a_n(2) &= \frac{(2n+3)}{n}a_{n-1}(2) - \frac{(n-1)}{n}a_{n-2}(2), \quad n \geq 2 \end{aligned} \tag{9.14}$$

and

$$\begin{aligned} a'_1(2) &= -3, \\ a'_n(2) &= \frac{2n+3}{n}a'_{n-1}(2) - \frac{n-1}{n}a'_{n-2}(2) - \frac{3}{n^2}a_{n-1}(2) - \frac{1}{n^2}a_{n-2}(2), \quad n \geq 2. \end{aligned} \tag{9.15}$$

Computing recursively with (9.14) and (9.15) yields

$$a_0(2) = 1, a_1(2) = 5, a_2(2) = 17, a_3(2) = \frac{143}{3}, a_4(2) = \frac{355}{3},$$

and

$$a'_1(2) = -3, a'_2(2) = -\frac{29}{2}, a'_3(2) = -\frac{859}{18}, a'_4(2) = -\frac{4693}{36}.$$

Substituting these coefficients into (9.11) yields

$$y_1 = x^2 \left(1 + 5x + 17x^2 + \frac{143}{3}x^3 + \frac{355}{3}x^4 + \cdots \right)$$

and

$$y_2 = y_1 \ln x - x^3 \left(3 + \frac{29}{2}x + \frac{859}{18}x^2 + \frac{4693}{36}x^3 + \cdots \right). \blacksquare$$

Since the recurrence formula (9.12) involves three terms, it's not possible to obtain a simple explicit formula for the coefficients in the Frobenius solutions of (9.10). However, as we saw in the preceding sections, the recurrence formula for $\{a_n(r)\}$ involves only two terms if either $\alpha_1 = \beta_1 = \gamma_1 = 0$ or $\alpha_2 = \beta_2 = \gamma_2 = 0$ in (9.2). In this case, it's often possible to find explicit formulas for the coefficients. The next two examples illustrate this.

Example 9.6.2: Find a fundamental set of Frobenius solutions of

$$2x^2(2+x)y'' + 5x^2y' + (1+x)y = 0. \tag{9.16}$$

Give explicit formulas for the coefficients in the solutions.

Solution: For the given equation, the polynomials defined in Theorem 9.6.1 are

$$\begin{aligned} p_0(r) &= 4r(r-1) + 1 = (2r-1)^2, \\ p_1(r) &= 2r(r-1) + 5r + 1 = (r+1)(2r+1), \\ p_2(r) &= 0. \end{aligned}$$

Since $r_1 = 1/2$ is a repeated zero of the indicial polynomial p_0 , Theorem 9.6.2 implies that

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} a_n(1/2)x^n \tag{9.17}$$

and

$$y_2 = y_1 \ln x + x^{1/2} \sum_{n=1}^{\infty} a'_n(1/2)x^n \tag{9.18}$$

form a fundamental set of Frobenius solutions of (9.16). Since $p_2 \equiv 0$, the recurrence formulas in Theorem 9.6.1 reduce to

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r), \\ &= -\frac{(n+r)(2n+2r-1)}{(2n+2r-1)^2}a_{n-1}(r), \\ &= -\frac{n+r}{2n+2r-1}a_{n-1}(r), \quad n \geq 0. \end{aligned}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^n \frac{j+r}{2j+2r-1}, \quad n \geq 0. \quad (9.19)$$

Setting $r = 1/2$ yields

$$\begin{aligned} a_n(1/2) &= (-1)^n \prod_{j=1}^n \frac{j+1/2}{2j} = (-1)^n \prod_{j=1}^n \frac{2j+1}{4j}, \\ &= \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!}, \quad n \geq 0. \end{aligned} \quad (9.20)$$

Substituting this into (9.17) yields

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} x^n.$$

To obtain y_2 in (9.18), we must compute $a'_n(1/2)$ for $n = 1, 2, \dots$. We'll do this by logarithmic differentiation. From (9.19),

$$|a_n(r)| = \prod_{j=1}^n \frac{|j+r|}{|2j+2r-1|}, \quad n \geq 1.$$

Therefore

$$\ln |a_n(r)| = \sum_{j=1}^n (\ln |j+r| - \ln |2j+2r-1|).$$

Differentiating with respect to r yields

$$\frac{a'_n(r)}{a_n(r)} = \sum_{j=1}^n \left(\frac{1}{j+r} - \frac{2}{2j+2r-1} \right).$$

Therefore

$$a'_n(r) = a_n(r) \sum_{j=1}^n \left(\frac{1}{j+r} - \frac{2}{2j+2r-1} \right).$$

Setting $r = 1/2$ here and recalling (9.20) yields

$$a'_n(1/2) = \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j+1/2} - \sum_{j=1}^n \frac{1}{j} \right). \quad (9.21)$$

Since

$$\frac{1}{j+1/2} - \frac{1}{j} = \frac{j-j-1/2}{j(j+1/2)} = -\frac{1}{j(2j+1)},$$

(9.21) can be rewritten as

$$a'_n(1/2) = -\frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} \sum_{j=1}^n \frac{1}{j(2j+1)}.$$

Therefore, from (9.18),

$$y_2 = y_1 \ln x - x^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j(2j+1)} \right) x^n.$$

Example 9.6.3: Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y'' - 2x(1+2x^2)y' + (2-2x^2)y = 0. \quad (9.22)$$

Give explicit formulas for the coefficients in the solutions.

Solution: For (9.22), the polynomials defined in Theorem 9.6.1 are

$$\begin{aligned} p_0(r) &= 2r(r-1) - 2r + 2 = 2(r-1)^2, \\ p_1(r) &= 0, \\ p_2(r) &= -r(r-1) - 4r - 2 = -(r+1)(r+2). \end{aligned}$$

As in Section 7.5, since $p_1 \equiv 0$, the recurrence formulas of Theorem 9.6.1 imply that $a_n(r) = 0$ if n is odd, and

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r) \\ &= \frac{(2m+r-1)(2m+r)}{2(2m+r-1)^2} a_{2m-2}(r) \\ &= \frac{2m+r}{2(2m+r-1)} a_{2m-2}(r), \quad m \geq 1. \end{aligned}$$

Since $r_1 = 1$ is a repeated root of the indicial polynomial p_0 , Theorem 9.6.2 implies that

$$y_1 = x \sum_{m=0}^{\infty} a_{2m}(1) x^{2m} \quad (9.23)$$

and

$$y_2 = y_1 \ln x + x \sum_{m=1}^{\infty} a'_{2m}(1) x^{2m} \quad (9.24)$$

form a fundamental set of Frobenius solutions of (9.22). We leave it to you to show that

$$a_{2m}(r) = \frac{1}{2^m} \prod_{j=1}^m \frac{2j+r}{2j+r-1}. \quad (9.25)$$

Setting $r = 1$ yields

$$a_{2m}(1) = \frac{1}{2^m} \prod_{j=1}^m \frac{2j+1}{2j} = \frac{\prod_{j=1}^m (2j+1)}{4^m m!}, \quad (9.26)$$

and substituting this into (9.23) yields

$$y_1 = x \sum_{m=0}^{\infty} \frac{\prod_{j=1}^m (2j+1)}{4^m m!} x^{2m}.$$

To obtain y_2 in (9.24), we must compute $a'_{2m}(1)$ for $m = 1, 2, \dots$. Again we use logarithmic differentiation. From (9.25),

$$|a_{2m}(r)| = \frac{1}{2^m} \prod_{j=1}^m \frac{|2j+r|}{|2j+r-1|}.$$

Taking logarithms yields

$$\ln |a_{2m}(r)| = -m \ln 2 + \sum_{j=1}^m (\ln |2j+r| - \ln |2j+r-1|).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = \sum_{j=1}^m \left(\frac{1}{2j+r} - \frac{1}{2j+r-1} \right).$$

Therefore

$$a'_{2m}(r) = a_{2m}(r) \sum_{j=1}^m \left(\frac{1}{2j+r} - \frac{1}{2j+r-1} \right).$$

Setting $r = 1$ and recalling (9.26) yields

$$a'_{2m}(1) = \frac{\prod_{j=1}^m (2j+1)}{4^m m!} \sum_{j=1}^m \left(\frac{1}{2j+1} - \frac{1}{2j} \right). \quad (9.27)$$

Since

$$\frac{1}{2j+1} - \frac{1}{2j} = -\frac{1}{2j(2j+1)},$$

(9.27) can be rewritten as

$$a'_{2m}(1) = -\frac{\prod_{j=1}^m (2j+1)}{2 \cdot 4^m m!} \sum_{j=1}^m \frac{1}{j(2j+1)}.$$

Substituting this into (9.24) yields

$$y_2 = y_1 \ln x - \frac{x}{2} \sum_{m=1}^{\infty} \frac{\prod_{j=1}^m (2j+1)}{4^m m!} \left(\sum_{j=1}^m \frac{1}{j(2j+1)} \right) x^{2m}. \blacksquare$$

If the solution $y_1 = y(x, r_1)$ of $Ly = 0$ reduces to a finite sum, then there's a difficulty in using logarithmic differentiation to obtain the coefficients $\{a'_n(r_1)\}$ in the second solution. The next example illustrates this difficulty and shows how to overcome it.

Example 9.6.4: Find a fundamental set of Frobenius solutions of

$$x^2 y'' - x(5-x)y' + (9-4x)y = 0. \quad (9.28)$$

Give explicit formulas for the coefficients in the solutions.

Solution: For (9.28) the polynomials defined in Theorem 9.6.1 are

$$\begin{aligned} p_0(r) &= r(r-1) - 5r + 9 = (r-3)^2, \\ p_1(r) &= r-4, \\ p_2(r) &= 0. \end{aligned}$$

Since $r_1 = 3$ is a repeated zero of the indicial polynomial p_0 , Theorem 9.6.2 implies that

$$y_1 = x^3 \sum_{n=0}^{\infty} a_n(3)x^n \quad (9.29)$$

and

$$y_2 = y_1 \ln x + x^3 \sum_{n=1}^{\infty} a'_n(3)x^n \quad (9.30)$$

are linearly independent Frobenius solutions of (9.28). To find the coefficients in (9.29) we use the recurrence formulas

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= -\frac{n+r-5}{(n+r-3)^2} a_{n-1}(r), \quad n \geq 1. \end{aligned}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^n \frac{j+r-5}{(j+r-3)^2}. \quad (9.31)$$

Setting $r = 3$ here yields

$$a_n(3) = (-1)^n \prod_{j=1}^n \frac{j-2}{j^2},$$

so $a_1(3) = 1$ and $a_n(3) = 0$ if $n \geq 2$. Substituting these coefficients into (9.29) yields

$$y_1 = x^3(1+x).$$

To obtain y_2 in (9.30) we must compute $a'_n(3)$ for $n = 1, 2, \dots$. Let's first try logarithmic differentiation. From (9.31),

$$|a_n(r)| = \prod_{j=1}^n \frac{|j+r-5|}{|j+r-3|^2}, \quad n \geq 1,$$

so

$$\ln |a_n(r)| = \sum_{j=1}^n (\ln |j+r-5| - 2 \ln |j+r-3|).$$

Differentiating with respect to r yields

$$\frac{a'_n(r)}{a_n(r)} = \sum_{j=1}^n \left(\frac{1}{j+r-5} - \frac{2}{j+r-3} \right).$$

Therefore

$$a'_n(r) = a_n(r) \sum_{j=1}^n \left(\frac{1}{j+r-5} - \frac{2}{j+r-3} \right). \quad (9.32)$$

However, we can't simply set $r = 3$ here if $n \geq 2$, since the bracketed expression in the sum corresponding to $j = 2$ contains the term $1/(r-3)$. In fact, since $a_n(3) = 0$ for $n \geq 2$, the formula (9.32) for $a'_n(r)$ is actually an indeterminate form at $r = 3$.

We overcome this difficulty as follows. From (9.31) with $n = 1$,

$$a_1(r) = -\frac{r-4}{(r-2)^2}.$$

Therefore

$$a_1'(r) = \frac{r-6}{(r-2)^3},$$

so

$$a_1'(3) = -3. \quad (9.33)$$

From (9.31) with $n \geq 2$,

$$a_n(r) = (-1)^n(r-4)(r-3) \frac{\prod_{j=3}^n(j+r-5)}{\prod_{j=1}^n(j+r-3)^2} = (r-3)c_n(r),$$

where

$$c_n(r) = (-1)^n(r-4) \frac{\prod_{j=3}^n(j+r-5)}{\prod_{j=1}^n(j+r-3)^2}, \quad n \geq 2.$$

Therefore

$$a_n'(r) = c_n(r) + (r-3)c_n'(r), \quad n \geq 2,$$

which implies that $a_n'(3) = c_n(3)$ if $n \geq 3$. We leave it to you to verify that

$$a_n'(3) = c_n(3) = \frac{(-1)^{n+1}}{n(n-1)n!}, \quad n \geq 2.$$

Substituting this and (9.33) into (9.30) yields

$$y_2 = x^3(1+x) \ln x - 3x^4 - x^3 \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)n!} x^n.$$

9.6.1 Exercises

In Exercises 9.6.1–9.6.11 find a fundamental set of Frobenius solutions. Compute the terms involving x^{n+r_1} , where $0 \leq n \leq N$ (N at least 7) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $N > 7$.

1. ☐ $x^2 y'' - x(1-x)y' + (1-x^2)y = 0$
2. ☐ $x^2(1+x+2x^2)y' + x(3+6x+7x^2)y' + (1+6x-3x^2)y = 0$
3. ☐ $x^2(1+2x+x^2)y'' + x(1+3x+4x^2)y' - x(1-2x)y = 0$
4. ☐ $4x^2(1+x+x^2)y'' + 12x^2(1+x)y' + (1+3x+3x^2)y = 0$
5. ☐ $x^2(1+x+x^2)y'' - x(1-4x-2x^2)y' + y = 0$
6. ☐ $9x^2 y'' + 3x(5+3x-2x^2)y' + (1+12x-14x^2)y = 0$
7. ☐ $x^2 y'' + x(1+x+x^2)y' + x(2-x)y = 0$
8. ☐ $x^2(1+2x)y'' + x(5+14x+3x^2)y' + (4+18x+12x^2)y = 0$
9. ☐ $4x^2 y'' + 2x(4+x+x^2)y' + (1+5x+3x^2)y = 0$
10. ☐ $16x^2 y'' + 4x(6+x+2x^2)y' + (1+5x+18x^2)y = 0$
11. ☐ $9x^2(1+x)y'' + 3x(5+11x-x^2)y' + (1+16x-7x^2)y = 0$

In Exercises 9.6.12–9.6.22 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

12. $4x^2 y'' + (1+4x)y = 0$

13. $36x^2(1-2x)y'' + 24x(1-9x)y' + (1-70x)y = 0$
14. $x^2(1+x)y'' - x(3-x)y' + 4y = 0$
15. $x^2(1-2x)y'' - x(5-4x)y' + (9-4x)y = 0$
16. $25x^2y'' + x(15+x)y' + (1+x)y = 0$
17. $2x^2(2+x)y'' + x^2y' + (1-x)y = 0$
18. $x^2(9+4x)y'' + 3xy' + (1+x)y = 0$
19. $x^2y'' - x(3-2x)y' + (4+3x)y = 0$
20. $x^2(1-4x)y'' + 3x(1-6x)y' + (1-12x)y = 0$
21. $x^2(1+2x)y'' + x(3+5x)y' + (1-2x)y = 0$
22. $2x^2(1+x)y'' - x(6-x)y' + (8-x)y = 0$

In Exercises 9.6.23–9.6.27 find a fundamental set of Frobenius solutions. Compute the terms involving x^{n+r_1} , where $0 \leq n \leq N$ (N at least 7) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $N > 7$.

23. C $x^2(1+2x)y'' + x(5+9x)y' + (4+3x)y = 0$
24. C $x^2(1-2x)y'' - x(5+4x)y' + (9+4x)y = 0$
25. C $x^2(1+4x)y'' - x(1-4x)y' + (1+x)y = 0$
26. C $x^2(1+x)y'' + x(1+2x)y' + xy = 0$
27. C $x^2(1-x)y'' + x(7+x)y' + (9-x)y = 0$

In Exercises 9.6.28–9.6.38 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

28. $x^2y'' - x(1-x^2)y' + (1+x^2)y = 0$
29. $x^2(1+x^2)y'' - 3x(1-x^2)y' + 4y = 0$
30. $4x^2y'' + 2x^3y' + (1+3x^2)y = 0$
31. $x^2(1+x^2)y'' - x(1-2x^2)y' + y = 0$
32. $2x^2(2+x^2)y'' + 7x^3y' + (1+3x^2)y = 0$
33. $x^2(1+x^2)y'' - x(1-4x^2)y' + (1+2x^2)y = 0$
34. $4x^2(4+x^2)y'' + 3x(8+3x^2)y' + (1-9x^2)y = 0$
35. $3x^2(3+x^2)y'' + x(3+11x^2)y' + (1+5x^2)y = 0$
36. $4x^2(1+4x^2)y'' + 32x^3y' + y = 0$
37. $9x^2y'' - 3x(7-2x^2)y' + (25+2x^2)y = 0$
38. $x^2(1+2x^2)y'' + x(3+7x^2)y' + (1-3x^2)y = 0$

In Exercises 9.6.39–9.6.43 find a fundamental set of Frobenius solutions. Compute the terms involving x^{2m+r_1} , where $0 \leq m \leq M$ (M at least 3) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $M > 3$.

39. C $x^2(1+x^2)y'' + x(3+8x^2)y' + (1+12x^2)y$
40. C $x^2y'' - x(1-x^2)y' + (1+x^2)y = 0$
41. C $x^2(1-2x^2)y'' + x(5-9x^2)y' + (4-3x^2)y = 0$
42. C $x^2(2+x^2)y'' + x(14-x^2)y' + 2(9+x^2)y = 0$

43. C $x^2(1+x^2)y'' + x(3+7x^2)y' + (1+8x^2)y = 0$

In Exercises 9.6.44–9.6.52 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

44. $x^2(1-2x)y'' + 3xy' + (1+4x)y = 0$

45. $x(1+x)y'' + (1-x)y' + y = 0$

46. $x^2(1-x)y'' + x(3-2x)y' + (1+2x)y = 0$

47. $4x^2(1+x)y'' - 4x^2y' + (1-5x)y = 0$

48. $x^2(1-x)y'' - x(3-5x)y' + (4-5x)y = 0$

49. $x^2(1+x^2)y'' - x(1+9x^2)y' + (1+25x^2)y = 0$

50. $9x^2y'' + 3x(1-x^2)y' + (1+7x^2)y = 0$

51. $x(1+x^2)y'' + (1-x^2)y' - 8xy = 0$

52. $4x^2y'' + 2x(4-x^2)y' + (1+7x^2)y = 0$

53. Under the assumptions of Theorem 9.6.2, suppose the power series

$$\sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad \sum_{n=1}^{\infty} a'_n(r_1)x^n$$

converge on $(-\rho, \rho)$.

(a) Show that

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad y_2 = y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n$$

are linearly independent on $(0, \rho)$. HINT: Show that if c_1 and c_2 are constants such that $c_1y_1 + c_2y_2 \equiv 0$ on $(0, \rho)$, then

$$(c_1 + c_2 \ln x) \sum_{n=0}^{\infty} a_n(r_1)x^n + c_2 \sum_{n=1}^{\infty} a'_n(r_1)x^n = 0, \quad 0 < x < \rho.$$

Then let $x \rightarrow 0+$ to conclude that $c_2 = 0$.

(b) Use the result of (a) to complete the proof of Theorem 9.6.2.

54. Let

$$Ly = x^2(\alpha_0 + \alpha_1x)y'' + x(\beta_0 + \beta_1x)y' + (\gamma_0 + \gamma_1x)y$$

and define

$$p_0(r) = \alpha_0r(r-1) + \beta_0r + \gamma_0 \quad \text{and} \quad p_1(r) = \alpha_1r(r-1) + \beta_1r + \gamma_1.$$

Theorem 9.6.1 and Exercise 9.5.55(a) imply that if

$$y(x, r) = x^r \sum_{n=0}^{\infty} a_n(r)x^n$$

where

$$a_n(r) = (-1)^n \prod_{j=1}^n \frac{p_1(j+r-1)}{p_0(j+r)},$$

then

$$Ly(x, r) = p_0(r)x^r.$$

Now suppose $p_0(r) = \alpha_0(r-r_1)^2$ and $p_1(k+r_1) \neq 0$ if k is a nonnegative integer.

(a) Show that $Ly = 0$ has the solution

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n,$$

where

$$a_n(r_1) = \frac{(-1)^n}{\alpha_0^n (n!)^2} \prod_{j=1}^n p_1(j + r_1 - 1).$$

(b) Show that $Ly = 0$ has the second solution

$$y_2 = y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n(r_1) J_n x^n,$$

where

$$J_n = \sum_{j=1}^n \frac{p_1'(j + r_1 - 1)}{p_1(j + r_1 - 1)} - 2 \sum_{j=1}^n \frac{1}{j}.$$

(c) Conclude from (a) and (b) that if $\gamma_1 \neq 0$ then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\gamma_1}{\alpha_0} \right)^n x^n$$

and

$$y_2 = y_1 \ln x - 2x^{r_1} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\gamma_1}{\alpha_0} \right)^n \left(\sum_{j=1}^n \frac{1}{j} \right) x^n$$

are solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x) y = 0.$$

(The conclusion is also valid if $\gamma_1 = 0$. Why?)

55. Let

$$Ly = x^2(\alpha_0 + \alpha_q x^q) y'' + x(\beta_0 + \beta_q x^q) y' + (\gamma_0 + \gamma_q x^q) y$$

where q is a positive integer, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_q(r) = \alpha_q r(r-1) + \beta_q r + \gamma_q.$$

Suppose

$$p_0(r) = \alpha_0(r - r_1)^2 \quad \text{and} \quad p_q(r) \not\equiv 0.$$

(a) Recall from Exercise 9.5.59 that $Ly = 0$ has the solution

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{qm}(r_1) x^{qm},$$

where

$$a_{qm}(r_1) = \frac{(-1)^m}{(q^2 \alpha_0)^m (m!)^2} \prod_{j=1}^m p_q(q(j-1) + r_1).$$

(b) Show that $Ly = 0$ has the second solution

$$y_2 = y_1 \ln x + x^{r_1} \sum_{m=1}^{\infty} a'_{qm}(r_1) J_m x^{qm},$$

where

$$J_m = \sum_{j=1}^m \frac{p'_q(q(j-1) + r_1)}{p_q(q(j-1) + r_1)} - \frac{2}{q} \sum_{j=1}^m \frac{1}{j}.$$

(c) Conclude from (a) and (b) that if $\gamma_q \neq 0$ then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\gamma_q}{q^2 \alpha_0} \right)^m x^{qm}$$

and

$$y_2 = y_1 \ln x - \frac{2}{q} x^{r_1} \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\gamma_q}{q^2 \alpha_0} \right)^m \left(\sum_{j=1}^m \frac{1}{j} \right) x^{qm}$$

are solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_q x^q) y = 0.$$

56. The equation

$$x y'' + y' + x y = 0$$

is *Bessel's equation of order zero*. (See Exercise 9.5.53.) Find two linearly independent Frobenius solutions of this equation.

57. Suppose the assumptions of Exercise 9.5.53 hold, except that

$$p_0(r) = \alpha_0(r - r_1)^2.$$

Show that

$$y_1 = \frac{x^{r_1}}{\alpha_0 + \alpha_1 x + \alpha_2 x^2} \quad \text{and} \quad y_2 = \frac{x^{r_1} \ln x}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}$$

are linearly independent Frobenius solutions of

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2) y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2) y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2) y = 0$$

on any interval $(0, \rho)$ on which $\alpha_0 + \alpha_1 x + \alpha_2 x^2$ has no zeros.

In Exercises 9.6.58–9.6.65 use the method suggested by Exercise 9.6.57 to find the general solution on some interval $(0, \rho)$.

58. $4x^2(1+x)y'' + 8x^2y' + (1+x)y = 0$

59. $9x^2(3+x)y'' + 3x(3+7x)y' + (3+4x)y = 0$

60. $x^2(2-x^2)y'' - x(2+3x^2)y' + (2-x^2)y = 0$

61. $16x^2(1+x^2)y'' + 8x(1+9x^2)y' + (1+49x^2)y = 0$

62. $x^2(4+3x)y'' - x(4-3x)y' + 4y = 0$

63. $4x^2(1+3x+x^2)y'' + 8x^2(3+2x)y' + (1+3x+9x^2)y = 0$

64. $x^2(1-x)^2y'' - x(1+2x-3x^2)y' + (1+x^2)y = 0$

65. $9x^2(1+x+x^2)y'' + 3x(1+7x+13x^2)y' + (1+4x+25x^2)y = 0$

66. (a) Let L and $y(x, r)$ be as in Exercises 9.5.57 and 9.5.58. Extend Theorem 9.6.1 by showing that

$$L \left(\frac{\partial y}{\partial r}(x, r) \right) = p'_0(r)x^r + x^r p_0(r) \ln x.$$

(b) Show that if

$$p_0(r) = \alpha_0(r - r_1)^2$$

then

$$y_1 = y(x, r_1) \quad \text{and} \quad y_2 = \frac{\partial y}{\partial r}(x, r_1)$$

are solutions of $Ly = 0$.

9.7 The Method of Frobenius III

In Sections 7.5 and 7.6 we discussed methods for finding Frobenius solutions of a homogeneous linear second order equation near a regular singular point in the case where the indicial equation has a repeated root or distinct real roots that don't differ by an integer. In this section we consider the case where the indicial equation has distinct real roots that differ by an integer. We'll limit our discussion to equations that can be written as

$$x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y = 0 \quad (9.34)$$

or

$$x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0,$$

where the roots of the indicial equation differ by a positive integer.

We begin with a theorem that provides a fundamental set of solutions of equations of the form (9.34).

Theorem 9.7.1. *Let*

$$Ly = x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y,$$

where $\alpha_0 \neq 0$, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0,$$

$$p_1(r) = \alpha_1 r(r-1) + \beta_1 r + \gamma_1.$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n , and define

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r), \quad n \geq 1. \end{aligned} \quad (9.35)$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r) = 0$, and suppose $r_1 = r_2 + k$, where k is a positive integer. Then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

is a Frobenius solution of $Ly = 0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1, \\ a_n(r_2) &= -\frac{p_1(n+r_2-1)}{p_0(n+r_2)} a_{n-1}(r_2), \quad 1 \leq n \leq k-1, \end{aligned} \quad (9.36)$$

and

$$C = -\frac{p_1(r_1-1)}{k\alpha_0} a_{k-1}(r_2), \quad (9.37)$$

then

$$y_2 = x^{r_2} \sum_{n=0}^{k-1} a_n(r_2) x^n + C \left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n \right) \quad (9.38)$$

is also a solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Proof. Theorem 9.5.3 implies that $Ly_1 = 0$. We'll now show that $Ly_2 = 0$. Since L is a linear operator, this is equivalent to showing that

$$L \left(x^{r_2} \sum_{n=0}^{k-1} a_n(r_2) x^n \right) + CL \left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n \right) = 0. \quad (9.39)$$

To verify this, we'll show that

$$L\left(x^{r_2} \sum_{n=0}^{k-1} a_n(r_2)x^n\right) = p_1(r_1 - 1)a_{k-1}(r_2)x^{r_1} \quad (9.40)$$

and

$$L\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n\right) = k\alpha_0 x^{r_1}. \quad (9.41)$$

This will imply that $Ly_2 = 0$, since substituting (9.40) and (9.41) into (9.39) and using (9.37) yields

$$\begin{aligned} Ly_2 &= [p_1(r_1 - 1)a_{k-1}(r_2) + Ck\alpha_0] x^{r_1} \\ &= [p_1(r_1 - 1)a_{k-1}(r_2) - p_1(r_1 - 1)a_{k-1}(r_2)] x^{r_1} = 0. \end{aligned}$$

We'll prove (9.41) first. From Theorem 9.6.1,

$$L\left(y(x, r) \ln x + x^r \sum_{n=1}^{\infty} a'_n(r)x^n\right) = p'_0(r)x^r + x^r p_0(r) \ln x.$$

Setting $r = r_1$ and recalling that $p_0(r_1) = 0$ and $y_1 = y(x, r_1)$ yields

$$L\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n\right) = p'_0(r_1)x^{r_1}. \quad (9.42)$$

Since r_1 and r_2 are the roots of the indicial equation, the indicial polynomial can be written as

$$p_0(r) = \alpha_0(r - r_1)(r - r_2) = \alpha_0[r^2 - (r_1 + r_2)r + r_1r_2].$$

Differentiating this yields

$$p'_0(r) = \alpha_0(2r - r_1 - r_2).$$

Therefore $p'_0(r_1) = \alpha_0(r_1 - r_2) = k\alpha_0$, so (9.42) implies (9.41).

Before proving (9.40), we first note $a_n(r_2)$ is well defined by (9.36) for $1 \leq n \leq k - 1$, since $p_0(n + r_2) \neq 0$ for these values of n . However, we can't define $a_n(r_2)$ for $n \geq k$ with (9.36), since $p_0(k + r_2) = p_0(r_1) = 0$. For convenience, we define $a_n(r_2) = 0$ for $n \geq k$. Then, from Theorem 9.5.1,

$$L\left(x^{r_2} \sum_{n=0}^{k-1} a_n(r_2)x^n\right) = L\left(x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)x^n\right) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n, \quad (9.43)$$

where $b_0 = p_0(r_2) = 0$ and

$$b_n = p_0(n + r_2)a_n(r_2) + p_1(n + r_2 - 1)a_{n-1}(r_2), \quad n \geq 1.$$

If $1 \leq n \leq k - 1$, then (9.36) implies that $b_n = 0$. If $n \geq k + 1$, then $b_n = 0$ because $a_{n-1}(r_2) = a_n(r_2) = 0$. Therefore (9.43) reduces to

$$L\left(x^{r_2} \sum_{n=0}^{k-1} a_n(r_2)x^n\right) = [p_0(k + r_2)a_k(r_2) + p_1(k + r_2 - 1)a_{k-1}(r_2)] x^{k+r_2}.$$

Since $a_k(r_2) = 0$ and $k + r_2 = r_1$, this implies (9.40).

We leave the proof that $\{y_1, y_2\}$ is a fundamental set as an exercise (Exercise 9.7.41).

Example 9.7.1: Find a fundamental set of Frobenius solutions of

$$2x^2(2 + x)y'' - x(4 - 7x)y' - (5 - 3x)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution: For the given equation, the polynomials defined in Theorem 9.7.1 are

$$\begin{aligned} p_0(r) &= 4r(r-1) - 4r - 5 = (2r+1)(2r-5), \\ p_1(r) &= 2r(r-1) + 7r + 3 = (r+1)(2r+3). \end{aligned}$$

The roots of the indicial equation are $r_1 = 5/2$ and $r_2 = -1/2$, so $k = r_1 - r_2 = 3$. Therefore Theorem 9.7.1 implies that

$$y_1 = x^{5/2} \sum_{n=0}^{\infty} a_n(5/2)x^n \quad (9.44)$$

and

$$y_2 = x^{-1/2} \sum_{n=0}^2 a_n(-1/2) + C \left(y_1 \ln x + x^{5/2} \sum_{n=1}^{\infty} a'_n(5/2)x^n \right) \quad (9.45)$$

(with C as in (9.37)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (9.35) is

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= -\frac{(n+r)(2n+2r+1)}{(2n+2r+1)(2n+2r-5)} a_{n-1}(r), \\ &= -\frac{n+r}{2n+2r-5} a_{n-1}(r), \quad n \geq 1, \end{aligned} \quad (9.46)$$

which implies that

$$a_n(r) = (-1)^n \prod_{j=1}^n \frac{j+r}{2j+2r-5}, \quad n \geq 0. \quad (9.47)$$

Therefore

$$a_n(5/2) = \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!}. \quad (9.48)$$

Substituting this into (9.44) yields

$$y_1 = x^{5/2} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!} x^n.$$

To compute the coefficients $a_0(-1/2)$, $a_1(-1/2)$ and $a_2(-1/2)$ in y_2 , we set $r = -1/2$ in (9.46) and apply the resulting recurrence formula for $n = 1, 2$; thus,

$$\begin{aligned} a_0(-1/2) &= 1, \\ a_n(-1/2) &= -\frac{2n-1}{4(n-3)} a_{n-1}(-1/2), \quad n = 1, 2. \end{aligned}$$

The last formula yields

$$a_1(-1/2) = 1/8 \quad \text{and} \quad a_2(-1/2) = 3/32.$$

Substituting $r_1 = 5/2$, $r_2 = -1/2$, $k = 3$, and $\alpha_0 = 4$ into (9.37) yields $C = -15/128$. Therefore, from (9.45),

$$y_2 = x^{-1/2} \left(1 + \frac{1}{8}x + \frac{3}{32}x^2 \right) - \frac{15}{128} \left(y_1 \ln x + x^{5/2} \sum_{n=1}^{\infty} a'_n(5/2)x^n \right). \quad (9.49)$$

We use logarithmic differentiation to obtain $a'_n(r)$. From (9.47),

$$|a_n(r)| = \prod_{j=1}^n \frac{|j+r|}{|2j+2r-5|}, \quad n \geq 1.$$

Therefore

$$\ln |a_n(r)| = \sum_{j=1}^n (\ln |j+r| - \ln |2j+2r-5|).$$

Differentiating with respect to r yields

$$\frac{a'_n(r)}{a_n(r)} = \sum_{j=1}^n \left(\frac{1}{j+r} - \frac{2}{2j+2r-5} \right).$$

Therefore

$$a'_n(r) = a_n(r) \sum_{j=1}^n \left(\frac{1}{j+r} - \frac{2}{2j+2r-5} \right).$$

Setting $r = 5/2$ here and recalling (9.48) yields

$$a'_n(5/2) = \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!} \sum_{j=1}^n \left(\frac{1}{j+5/2} - \frac{1}{j} \right). \quad (9.50)$$

Since

$$\frac{1}{j+5/2} - \frac{1}{j} = -\frac{5}{j(2j+5)},$$

we can rewrite (9.50) as

$$a'_n(5/2) = -5 \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j(2j+5)} \right).$$

Substituting this into (9.49) yields

$$\begin{aligned} y_2 &= x^{-1/2} \left(1 + \frac{1}{8}x + \frac{3}{32}x^2 \right) - \frac{15}{128}y_1 \ln x \\ &\quad + \frac{75}{128}x^{5/2} \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j(2j+5)} \right) x^n. \blacksquare \end{aligned}$$

If $C = 0$ in (9.37), there's no need to compute

$$y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n$$

in the formula (9.38) for y_2 . Therefore it's best to compute C before computing $\{a'_n(r_1)\}_{n=1}^{\infty}$. This is illustrated in the next example. (See also Exercises 9.7.44 and 9.7.45.)

Example 9.7.2: Find a fundamental set of Frobenius solutions of

$$x^2(1-2x)y'' + x(8-9x)y' + (6-3x)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution: For the given equation, the polynomials defined in Theorem 9.7.1 are

$$\begin{aligned} p_0(r) &= r(r-1) + 8r + 6 = (r+1)(r+6) \\ p_1(r) &= -2r(r-1) - 9r - 3 = -(r+3)(2r+1). \end{aligned}$$

The roots of the indicial equation are $r_1 = -1$ and $r_2 = -6$, so $k = r_1 - r_2 = 5$. Therefore Theorem 9.7.1 implies that

$$y_1 = x^{-1} \sum_{n=0}^{\infty} a_n(-1)x^n \quad (9.51)$$

and

$$y_2 = x^{-6} \sum_{n=0}^4 a_n(-6) + C \left(y_1 \ln x + x^{-1} \sum_{n=1}^{\infty} a'_n(-1)x^n \right) \quad (9.52)$$

(with C as in (9.37)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (9.35) is

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= \frac{(n+r+2)(2n+2r-1)}{(n+r+1)(n+r+6)} a_{n-1}(r), \quad n \geq 1, \end{aligned} \quad (9.53)$$

which implies that

$$\begin{aligned} a_n(r) &= \prod_{j=1}^n \frac{(j+r+2)(2j+2r-1)}{(j+r+1)(j+r+6)} \\ &= \left(\prod_{j=1}^n \frac{j+r+2}{j+r+1} \right) \left(\prod_{j=1}^n \frac{2j+2r-1}{j+r+6} \right). \end{aligned} \quad (9.54)$$

Since

$$\prod_{j=1}^n \frac{j+r+2}{j+r+1} = \frac{(r+3)(r+4) \cdots (n+r+2)}{(r+2)(r+3) \cdots (n+r+1)} = \frac{n+r+2}{r+2}$$

because of cancellations, (9.54) simplifies to

$$a_n(r) = \frac{n+r+2}{r+2} \prod_{j=1}^n \frac{2j+2r-1}{j+r+6}.$$

Therefore

$$a_n(-1) = (n+1) \prod_{j=1}^n \frac{2j-3}{j+5}.$$

Substituting this into (9.51) yields

$$y_1 = x^{-1} \sum_{n=0}^{\infty} (n+1) \left(\prod_{j=1}^n \frac{2j-3}{j+5} \right) x^n.$$

To compute the coefficients $a_0(-6), \dots, a_4(-6)$ in y_2 , we set $r = -6$ in (9.53) and apply the resulting recurrence formula for $n = 1, 2, 3, 4$; thus,

$$\begin{aligned} a_0(-6) &= 1, \\ a_n(-6) &= \frac{(n-4)(2n-13)}{n(n-5)} a_{n-1}(-6), \quad n = 1, 2, 3, 4. \end{aligned}$$

The last formula yields

$$a_1(-6) = -\frac{33}{4}, a_2(-6) = \frac{99}{4}, a_3(-6) = -\frac{231}{8}, a_4(-6) = 0.$$

Since $a_4(-6) = 0$, (9.37) implies that the constant C in (9.52) is zero. Therefore (9.52) reduces to

$$y_2 = x^{-6} \left(1 - \frac{33}{4}x + \frac{99}{4}x^2 - \frac{231}{8}x^3 \right). \blacksquare$$

We now consider equations of the form

$$x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0,$$

where the roots of the indicial equation are real and differ by an even integer. The case where the roots are real and differ by an odd integer can be handled by the method discussed in 9.5.56.

The proof of the next theorem is similar to the proof of Theorem 9.7.1 (Exercise 9.7.43).

Theorem 9.7.2. *Let*

$$Ly = x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y,$$

where $\alpha_0 \neq 0$, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0,$$

$$p_2(r) = \alpha_2 r(r-1) + \beta_2 r + \gamma_2.$$

Suppose r is a real number such that $p_0(2m+r)$ is nonzero for all positive integers m , and define

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r), \quad m \geq 1. \end{aligned} \tag{9.55}$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r) = 0$, and suppose $r_1 = r_2 + 2k$, where k is a positive integer. Then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{2m}(r_1) x^{2m}$$

is a Frobenius solution of $Ly = 0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1, \\ a_{2m}(r_2) &= -\frac{p_2(2m+r_2-2)}{p_0(2m+r_2)} a_{2m-2}(r_2), \quad 1 \leq m \leq k-1 \end{aligned}$$

and

$$C = -\frac{p_2(r_1-2)}{2k\alpha_0} a_{2k-2}(r_2), \tag{9.56}$$

then

$$y_2 = x^{r_2} \sum_{m=0}^{k-1} a_{2m}(r_2) x^{2m} + C \left(y_1 \ln x + x^{r_1} \sum_{m=1}^{\infty} a'_{2m}(r_1) x^{2m} \right) \tag{9.57}$$

is also a solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Example 9.7.3: Find a fundamental set of Frobenius solutions of

$$x^2(1+x^2)y'' + x(3+10x^2)y' - (15-14x^2)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution: For the given equation, the polynomials defined in Theorem 9.7.2 are

$$\begin{aligned} p_0(r) &= r(r-1) + 3r - 15 = (r-3)(r+5) \\ p_2(r) &= r(r-1) + 10r + 14 = (r+2)(r+7). \end{aligned}$$

The roots of the indicial equation are $r_1 = 3$ and $r_2 = -5$, so $k = (r_1 - r_2)/2 = 4$. Therefore Theorem 9.7.2 implies that

$$y_1 = x^3 \sum_{m=0}^{\infty} a_{2m}(3)x^{2m} \quad (9.58)$$

and

$$y_2 = x^{-5} \sum_{m=0}^3 a_{2m}(-5)x^{2m} + C \left(y_1 \ln x + x^3 \sum_{m=1}^{\infty} a'_{2m}(3)x^{2m} \right)$$

(with C as in (9.56)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (9.55) is

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r) \\ &= -\frac{(2m+r)(2m+r+5)}{(2m+r-3)(2m+r+5)} a_{2m-2}(r) \\ &= -\frac{2m+r}{2m+r-3} a_{2m-2}(r), \quad m \geq 1, \end{aligned} \quad (9.59)$$

which implies that

$$a_{2m}(r) = (-1)^m \prod_{j=1}^m \frac{2j+r}{2j+r-3}, \quad m \geq 0. \quad (9.60)$$

Therefore

$$a_{2m}(3) = \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!}. \quad (9.61)$$

Substituting this into (9.58) yields

$$y_1 = x^3 \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!} x^{2m}.$$

To compute the coefficients $a_2(-5)$, $a_4(-5)$, and $a_6(-5)$ in y_2 , we set $r = -5$ in (9.59) and apply the resulting recurrence formula for $m = 1, 2, 3$; thus,

$$a_{2m}(-5) = -\frac{2m-5}{2(m-4)} a_{2m-2}(-5), \quad m = 1, 2, 3.$$

This yields

$$a_2(-5) = -\frac{1}{2}, \quad a_4(-5) = \frac{1}{8}, \quad a_6(-5) = \frac{1}{16}.$$

Substituting $r_1 = 3$, $r_2 = -5$, $k = 4$, and $\alpha_0 = 1$ into (9.56) yields $C = -3/16$. Therefore, from (9.57),

$$y_2 = x^{-5} \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 \right) - \frac{3}{16} \left(y_1 \ln x + x^3 \sum_{m=1}^{\infty} a'_{2m}(3)x^{2m} \right). \quad (9.62)$$

To obtain $a'_{2m}(r)$ we use logarithmic differentiation. From (9.60),

$$|a_{2m}(r)| = \prod_{j=1}^m \frac{|2j+r|}{|2j+r-3|}, \quad m \geq 1.$$

Therefore

$$\ln |a_{2m}(r)| = \sum_{j=1}^n (\ln |2j+r| - \ln |2j+r-3|).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = \sum_{j=1}^m \left(\frac{1}{2j+r} - \frac{1}{2j+r-3} \right).$$

Therefore

$$a'_{2m}(r) = a_{2m}(r) \sum_{j=1}^n \left(\frac{1}{2j+r} - \frac{1}{2j+r-3} \right).$$

Setting $r = 3$ here and recalling (9.61) yields

$$a'_{2m}(3) = \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!} \sum_{j=1}^m \left(\frac{1}{2j+3} - \frac{1}{2j} \right). \quad (9.63)$$

Since

$$\frac{1}{2j+3} - \frac{1}{2j} = -\frac{3}{2j(2j+3)},$$

we can rewrite (9.63) as

$$a'_{2m}(3) = -\frac{3}{2} \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!} \left(\sum_{j=1}^m \frac{1}{j(2j+3)} \right).$$

Substituting this into (9.62) yields

$$\begin{aligned} y_2 &= x^{-5} \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 \right) - \frac{3}{16}y_1 \ln x \\ &\quad + \frac{9}{32}x^3 \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!} \left(\sum_{j=1}^m \frac{1}{j(2j+3)} \right) x^{2m}. \end{aligned}$$

Example 9.7.4: Find a fundamental set of Frobenius solutions of

$$x^2(1-2x^2)y'' + x(7-13x^2)y' - 14x^2y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution: For the given equation, the polynomials defined in Theorem 9.7.2 are

$$\begin{aligned} p_0(r) &= r(r-1) + 7r = r(r+6), \\ p_2(r) &= -2r(r-1) - 13r - 14 = -(r+2)(2r+7). \end{aligned}$$

The roots of the indicial equation are $r_1 = 0$ and $r_2 = -6$, so $k = (r_1 - r_2)/2 = 3$. Therefore Theorem 9.7.2 implies that

$$y_1 = \sum_{m=0}^{\infty} a_{2m}(0)x^{2m}, \quad (9.64)$$

and

$$y_2 = x^{-6} \sum_{m=0}^2 a_{2m}(-6)x^{2m} + C \left(y_1 \ln x + \sum_{m=1}^{\infty} a'_{2m}(0)x^{2m} \right) \quad (9.65)$$

(with C as in (9.56)) form a fundamental set of solutions of $Ly = 0$. The recurrence formulas (9.55) are

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\ &= \frac{(2m+r)(4m+2r+3)}{(2m+r)(2m+r+6)}a_{2m-2}(r) \\ &= \frac{4m+2r+3}{2m+r+6}a_{2m-2}(r), \quad m \geq 1, \end{aligned} \tag{9.66}$$

which implies that

$$a_{2m}(r) = \prod_{j=1}^m \frac{4j+2r+3}{2j+r+6}.$$

Setting $r = 0$ yields

$$a_{2m}(0) = 6 \frac{\prod_{j=1}^m (4j+3)}{2^m(m+3)!}.$$

Substituting this into (9.64) yields

$$y_1 = 6 \sum_{m=0}^{\infty} \frac{\prod_{j=1}^m (4j+3)}{2^m(m+3)!} x^{2m}.$$

To compute the coefficients $a_0(-6)$, $a_2(-6)$, and $a_4(-6)$ in y_2 , we set $r = -6$ in (9.66) and apply the resulting recurrence formula for $m = 1, 2$; thus,

$$\begin{aligned} a_0(-6) &= 1, \\ a_{2m}(-6) &= \frac{4m-9}{2m}a_{2m-2}(-6), \quad m = 1, 2. \end{aligned}$$

The last formula yields

$$a_2(-6) = -\frac{5}{2} \quad \text{and} \quad a_4(-6) = \frac{5}{8}.$$

Since $p_2(-2) = 0$, the constant C in (9.56) is zero. Therefore (9.65) reduces to

$$y_2 = x^{-6} \left(1 - \frac{5}{2}x^2 + \frac{5}{8}x^4 \right).$$

9.7.1 Exercises

In Exercises 9.7.1–9.7.40 find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

1. $x^2y'' - 3xy' + (3+4x)y = 0$
2. $xy'' + y = 0$
3. $4x^2(1+x)y'' + 4x(1+2x)y' - (1+3x)y = 0$
4. $xy'' + xy' + y = 0$
5. $2x^2(2+3x)y'' + x(4+21x)y' - (1-9x)y = 0$
6. $x^2y'' + x(2+x)y' - (2-3x)y = 0$
7. $4x^2y'' + 4xy' - (9-x)y = 0$
8. $x^2y'' + 10xy' + (14+x)y = 0$
9. $4x^2(1+x)y'' + 4x(3+8x)y' - (5-49x)y = 0$

10. $x^2(1+x)y'' - x(3+10x)y' + 30xy = 0$
11. $x^2y'' + x(1+x)y' - 3(3+x)y = 0$
12. $x^2y'' + x(1-2x)y' - (4+x)y = 0$
13. $x(1+x)y'' - 4y' - 2y = 0$
14. $x^2(1+2x)y'' + x(9+13x)y' + (7+5x)y = 0$
15. $4x^2y'' - 2x(4-x)y' - (7+5x)y = 0$
16. $3x^2(3+x)y'' - x(15+x)y' - 20y = 0$
17. $x^2(1+x)y'' + x(1-10x)y' - (9-10x)y = 0$
18. $x^2(1+x)y'' + 3x^2y' - (6-x)y = 0$
19. $x^2(1+2x)y'' - 2x(3+14x)y' + (6+100x)y = 0$
20. $x^2(1+x)y'' - x(6+11x)y' + (6+32x)y = 0$
21. $4x^2(1+x)y'' + 4x(1+4x)y' - (49+27x)y = 0$
22. $x^2(1+2x)y'' - x(9+8x)y' - 12xy = 0$
23. $x^2(1+x^2)y'' - x(7-2x^2)y' + 12y = 0$
24. $x^2y'' - x(7-x^2)y' + 12y = 0$
25. $xy'' - 5y' + xy = 0$
26. $x^2y'' + x(1+2x^2)y' - (1-10x^2)y = 0$
27. $x^2y'' - xy' - (3-x^2)y = 0$
28. $4x^2y'' + 2x(8+x^2)y' + (5+3x^2)y = 0$
29. $x^2y'' + x(1+x^2)y' - (1-3x^2)y = 0$
30. $x^2y'' + x(1-2x^2)y' - 4(1+2x^2)y = 0$
31. $4x^2y'' + 8xy' - (35-x^2)y = 0$
32. $9x^2y'' - 3x(11+2x^2)y' + (13+10x^2)y = 0$
33. $x^2y'' + x(1-2x^2)y' - 4(1-x^2)y = 0$
34. $x^2y'' + x(1-3x^2)y' - 4(1-3x^2)y = 0$
35. $x^2(1+x^2)y'' + x(5+11x^2)y' + 24x^2y = 0$
36. $4x^2(1+x^2)y'' + 8xy' - (35-x^2)y = 0$
37. $x^2(1+x^2)y'' - x(5-x^2)y' - (7+25x^2)y = 0$
38. $x^2(1+x^2)y'' + x(5+2x^2)y' - 21y = 0$
39. $x^2(1+2x^2)y'' - x(3+x^2)y' - 2x^2y = 0$
40. $4x^2(1+x^2)y'' + 4x(2+x^2)y' - (15+x^2)y = 0$
41. (a) Under the assumptions of Theorem 9.7.1, show that

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n$$

and

$$y_2 = x^{r_2} \sum_{n=0}^{k-1} a_n(r_2)x^n + C \left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n \right)$$

are linearly independent. HINT: Show that if c_1 and c_2 are constants such that $c_1y_1 + c_2y_2 \equiv 0$ on an interval $(0, \rho)$, then

$$x^{-r_2}(c_1y_1(x) + c_2y_2(x)) = 0, \quad 0 < x < \rho.$$

Then let $x \rightarrow 0+$ to conclude that $c_2=0$.

(b) Use the result of (a) to complete the proof of Theorem 9.7.1.

42. Find a fundamental set of Frobenius solutions of Bessel's equation

$$x^2 y'' + x y' + (x^2 - v^2)y = 0$$

in the case where v is a positive integer.

43. Prove Theorem 9.7.2.

44. Under the assumptions of Theorem 9.7.1, show that $C = 0$ if and only if $p_1(r_2 + 1) = 0$ for some integer in $\{0, 1, \dots, k-1\}$.

45. Under the assumptions of Theorem 9.7.2, show that $C = 0$ if and only if $p_2(r_2 + 2) = 0$ for some integer ℓ in $\{0, 1, \dots, k-1\}$.

46. Let

$$Ly = \alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x)y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Show that if

$$p_0(r) = \alpha_0(r-r_1)(r-r_2)$$

where $r_1 - r_2 = k$, a positive integer, then $Ly = 0$ has the solutions

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^n (j+k)} \left(\frac{\gamma_1}{\alpha_0} \right)^n x^n$$

and

$$y_2 = x^{r_2} \sum_{n=0}^{k-1} \frac{(-1)^n}{n! \prod_{j=1}^n (j-k)} \left(\frac{\gamma_1}{\alpha_0} \right)^n x^n - \frac{1}{k!(k-1)!} \left(\frac{\gamma_1}{\alpha_0} \right)^k \left(y_1 \ln x - x^{r_1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^n (j+k)} \left(\frac{\gamma_1}{\alpha_0} \right)^n \left(\sum_{j=1}^n \frac{2j+k}{j(j+k)} \right) x^n \right).$$

47. Let

$$Ly = \alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_2 x^2)y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Show that if

$$p_0(r) = \alpha_0(r-r_1)(r-r_2)$$

where $r_1 - r_2 = 2k$, an even positive integer, then $Ly = 0$ has the solutions

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^m (j+k)} \left(\frac{\gamma_2}{\alpha_0} \right)^m x^{2m}$$

and

$$y_2 = x^{r_2} \sum_{m=0}^{k-1} \frac{(-1)^m}{4^m m! \prod_{j=1}^m (j-k)} \left(\frac{\gamma_2}{\alpha_0} \right)^m x^{2m} - \frac{2}{4^k k!(k-1)!} \left(\frac{\gamma_2}{\alpha_0} \right)^k \left(y_1 \ln x - \frac{x^{r_1}}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^m (j+k)} \left(\frac{\gamma_2}{\alpha_0} \right)^m \left(\sum_{j=1}^m \frac{2j+k}{j(j+k)} \right) x^{2m} \right).$$

48. Let L be as in Exercises 9.5.57 and 9.5.58, and suppose the indicial polynomial of $Ly = 0$ is

$$p_0(r) = \alpha_0(r - r_1)(r - r_2),$$

with $k = r_1 - r_2$, where k is a positive integer. Define $a_0(r) = 1$ for all r . If r is a real number such that $p_0(n + r)$ is nonzero for all positive integers n , define

$$a_n(r) = -\frac{1}{p_0(n + r)} \sum_{j=1}^n p_j(n + r - j)a_{n-j}(r), \quad n \geq 1,$$

and let

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n.$$

Define

$$a_n(r_2) = -\frac{1}{p_0(n + r_2)} \sum_{j=1}^n p_j(n + r_2 - j)a_{n-j}(r_2) \text{ if } n \geq 1 \text{ and } n \neq k,$$

and let $a_k(r_2)$ be arbitrary.

- (a) Conclude from Exercise 7.6..9.6.66 that

$$L \left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n \right) = k\alpha_0 x^{r_1}.$$

- (b) Conclude from Exercise 7.5..9.5.57 that

$$L \left(x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)x^n \right) = Ax^{r_1},$$

where

$$A = \sum_{j=1}^k p_j(r_1 - j)a_{k-j}(r_2).$$

- (c) Show that y_1 and

$$y_2 = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)x^n - \frac{A}{k\alpha_0} \left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n \right)$$

form a fundamental set of Frobenius solutions of $Ly = 0$.

- (d) Show that choosing the arbitrary quantity $a_k(r_2)$ to be nonzero merely adds a multiple of y_1 to y_2 . Conclude that we may as well take $a_k(r_2) = 0$.

Chapter 10

More about Power Series Methods

10.1 Power Series

Many functions can be written in terms of a power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

If we assume that a solution of a differential equation is written as a power series, then perhaps we can use a method reminiscent of undetermined coefficients. That is, we will try to solve for the numbers a_k . Before we carry out this process, we review some results and concepts about power series.

10.1.1 Definition

As we said, a *power series* is an expression such as

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots, \quad (10.1)$$

where $a_0, a_1, a_2, \dots, a_k, \dots$ and x_0 are constants. Let

$$S_n(x) = \sum_{k=0}^n a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n,$$

denote the so-called *partial sum*. If for some x , the limit

$$\lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k (x - x_0)^k$$

exists, we say the series (10.1) *converges* at x . At $x = x_0$, the series always converges to a_0 . When (10.1) converges at any other $x \neq x_0$, we say (10.1) is a *convergent power series*, and we write

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k (x - x_0)^k.$$

If the series does not converge for any point $x \neq x_0$, we say that the series is *divergent*.

Example 10.1.1: The series

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

is convergent for any x . Recall that $k! = 1 \cdot 2 \cdot 3 \cdots k$ is the factorial. By convention we define $0! = 1$. You may recall that this series converges to e^x .

We say that (10.1) *converges absolutely* at x whenever the limit

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n |a_k| |x - x_0|^k$$

exists. That is, the series $\sum_{k=0}^{\infty} |a_k| |x - x_0|^k$ is convergent. If (10.1) converges absolutely at x , then it converges at x . However, the opposite implication is not true.

Example 10.1.2: The series

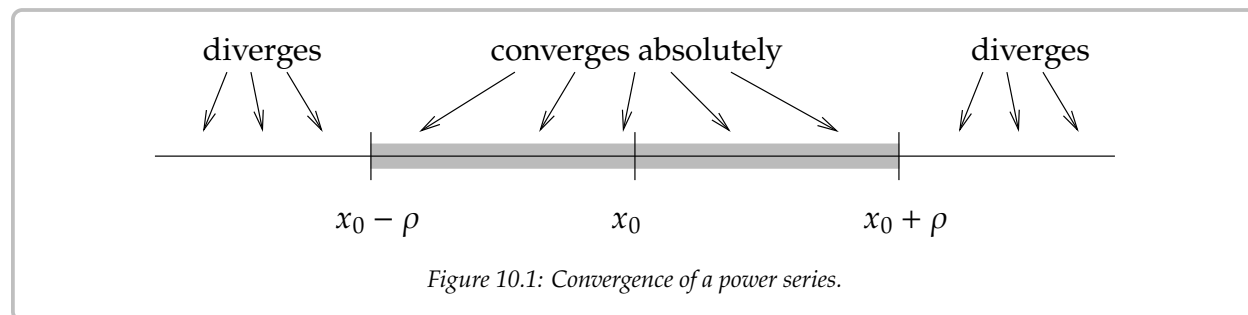
$$\sum_{k=1}^{\infty} \frac{1}{k} x^k$$

converges absolutely for all x in the interval $(-1, 1)$. It converges at $x = -1$, as $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges (conditionally) by the alternating series test. The power series does not converge absolutely at $x = -1$, because $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. The series diverges at $x = 1$.

10.1.2 Radius of Convergence

If a power series converges absolutely at some x_1 , then for all x such that $|x - x_0| \leq |x_1 - x_0|$ (that is, x is closer than x_1 to x_0) we have $|a_k(x - x_0)^k| \leq |a_k(x_1 - x_0)^k|$ for all k . As the numbers $|a_k(x_1 - x_0)^k|$ sum to some finite limit, summing smaller positive numbers $|a_k(x - x_0)^k|$ must also have a finite limit. Hence, the series must converge absolutely at x .

Theorem 10.1.1. For a power series (10.1), there exists a number ρ (we allow $\rho = \infty$) called the radius of convergence such that the series converges absolutely on the interval $(x_0 - \rho, x_0 + \rho)$ and diverges for $x < x_0 - \rho$ and $x > x_0 + \rho$. We write $\rho = \infty$ if the series converges for all x .



See Figure 10.1. In Example 10.1.1, the radius of convergence is $\rho = \infty$ as the series converges everywhere. In Example 10.1.2, the radius of convergence is $\rho = 1$. We note that $\rho = 0$ is another way of saying that the series is divergent.

A useful test for convergence of a series is the *ratio test*. Suppose that

$$\sum_{k=0}^{\infty} c_k$$

is a series and the limit

$$L = \lim_{k \rightarrow \infty} \left| \frac{c_{k+1}}{c_k} \right|$$

exists. Then the series converges absolutely if $L < 1$ and diverges if $L > 1$.

We apply this test to the series (10.1). Let $c_k = a_k(x - x_0)^k$ in the test. Compute

$$L = \lim_{k \rightarrow \infty} \left| \frac{c_{k+1}}{c_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}(x - x_0)^{k+1}}{a_k(x - x_0)^k} \right| = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| |x - x_0|.$$

Define A by

$$A = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|.$$

Then the series (10.1) converges absolutely if $1 > L = A|x - x_0|$. If $A > 0$, then the series converges absolutely if $|x - x_0| < 1/A$, and diverges if $|x - x_0| > 1/A$. That is, the radius of convergence is $1/A$. If $A = 0$, then the series always converges.

A similar test is the *root test*. Suppose

$$L = \lim_{k \rightarrow \infty} \sqrt[k]{|c_k|}$$

exists. Then $\sum_{k=0}^{\infty} c_k$ converges absolutely if $L < 1$ and diverges if $L > 1$. We can use the same calculation as above to find A . Let us summarize.

Theorem 10.1.2 (Ratio and root tests for power series). *Consider a power series*

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k$$

such that

$$A = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| \quad \text{or} \quad A = \lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}$$

exists. If $A = 0$, then the radius of convergence of the series is ∞ . Otherwise, the radius of convergence is $1/A$.

Example 10.1.3: Suppose we have the series

$$\sum_{k=0}^{\infty} 2^{-k}(x - 1)^k.$$

First we compute the limit in the ratio test,

$$A = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{2^{-k-1}}{2^{-k}} \right| = \lim_{k \rightarrow \infty} 2^{-1} = 1/2.$$

Therefore the radius of convergence is 2, and the series converges absolutely on the interval $(-1, 3)$. And we could just as well have used the root test:

$$A = \lim_{k \rightarrow \infty} \lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \sqrt[k]{|2^{-k}|} = \lim_{k \rightarrow \infty} 2^{-1} = 1/2.$$

Example 10.1.4: Consider

$$\sum_{k=0}^{\infty} \frac{1}{k^k} x^k.$$

Compute the limit for the root test,

$$A = \lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \sqrt[k]{\left| \frac{1}{k^k} \right|} = \lim_{k \rightarrow \infty} \sqrt[k]{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{1}{k} = 0.$$

So the radius of convergence is ∞ : the series converges everywhere. The ratio test would also work here.

The root or the ratio test does not always apply. That is, the limit of $\left| \frac{a_{k+1}}{a_k} \right|$ or $\sqrt[k]{|a_k|}$ might not exist. There exist more sophisticated ways of finding the radius of convergence, but those would be beyond the scope of this chapter. The two methods above cover many of the series that arise in practice. Often if the root test applies, so does the ratio test, and vice versa, though the limit might be easier to compute in one way than the other.

10.1.3 Analytic Functions

Functions represented by power series are called *analytic functions*. Not every function is analytic, although the majority of the functions you have seen in calculus are. An analytic function $f(x)$ is equal to its *Taylor series** (a power series computed from f) near a point x_0 . That is, for x near x_0 ,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots, \quad (10.2)$$

where $f^{(k)}(x_0)$ denotes the k^{th} derivative of $f(x)$ at the point x_0 .

For example, sine is an analytic function and its Taylor series around $x_0 = 0$ is given by

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

In Figure 10.2 we plot $\sin(x)$ and the truncations of the series up to degree 5 and 9. You can see that the approximation is very good for x near 0, but gets worse for x further away from 0. This is what happens in general. To get a good approximation far away from x_0 you need to take more and more terms of the Taylor series.

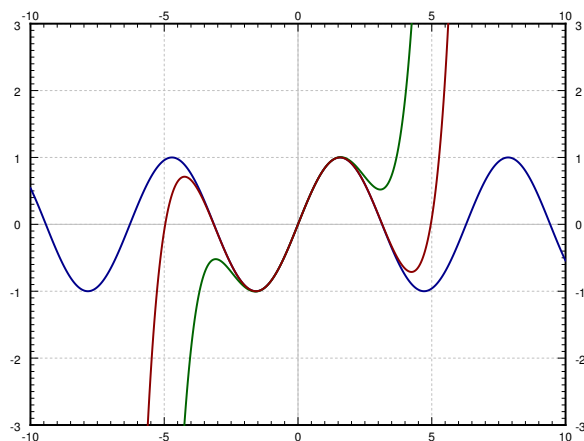


Figure 10.2: The sine function and its Taylor approximations around $x_0 = 0$ of 5th and 9th degree.

10.1.4 Manipulating Power Series

One of the main properties of power series that we will use is that we can differentiate them term by term. That is, suppose that $\sum a_k(x - x_0)^k$ is a convergent power series. Then for x in the radius of convergence we have

$$\frac{d}{dx} \left[\sum_{k=0}^{\infty} a_k(x - x_0)^k \right] = \sum_{k=1}^{\infty} k a_k(x - x_0)^{k-1} = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots.$$

Notice that the term corresponding to $k = 0$ disappeared as it was constant. The radius of convergence of the differentiated series is the same as that of the original.

Example 10.1.5: Let us show that the exponential $y = e^x$ solves $y' = y$. Suppose we didn't know that. Write

$$y = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

*Named after the English mathematician Sir Brook Taylor (1685–1731).

Now differentiate

$$y' = \sum_{k=1}^{\infty} k \frac{1}{k!} x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1}.$$

We *reindex* the series by simply replacing k with $k + 1$. The series does not change, what changes is simply how we write it. After reindexing the series starts at $k = 0$ again.

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{k+1=1}^{\infty} \frac{1}{((k+1)-1)!} x^{(k+1)-1} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

That was precisely the power series for e^x we started with, so we showed that $\frac{d}{dx}[e^x] = e^x$.

Convergent power series can be added and multiplied together, and multiplied by constants using the following rules. First, we can add series by adding term by term,

$$\left(\sum_{k=0}^{\infty} a_k (x - x_0)^k \right) + \left(\sum_{k=0}^{\infty} b_k (x - x_0)^k \right) = \sum_{k=0}^{\infty} (a_k + b_k) (x - x_0)^k.$$

We can multiply by constants,

$$\alpha \left(\sum_{k=0}^{\infty} a_k (x - x_0)^k \right) = \sum_{k=0}^{\infty} \alpha a_k (x - x_0)^k.$$

We can also multiply series together,

$$\left(\sum_{k=0}^{\infty} a_k (x - x_0)^k \right) \left(\sum_{k=0}^{\infty} b_k (x - x_0)^k \right) = \sum_{k=0}^{\infty} c_k (x - x_0)^k,$$

where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$. The radius of convergence of the sum or the product is at least the minimum of the radii of convergence of the two series involved.

10.1.5 Power Series for Rational Functions

Polynomials are simply finite power series. That is, a polynomial is a power series where the a_k are zero for all k large enough. We can always expand a polynomial as a power series about any point x_0 by writing the polynomial as a polynomial in $(x - x_0)$. For example, let us write $2x^2 - 3x + 4$ as a power series around $x_0 = 1$:

$$2x^2 - 3x + 4 = 3 + (x - 1) + 2(x - 1)^2.$$

In other words $a_0 = 3$, $a_1 = 1$, $a_2 = 2$, and all other $a_k = 0$. To do this, we know that $a_k = 0$ for all $k \geq 3$ as the polynomial is of degree 2. We write $a_0 + a_1(x - 1) + a_2(x - 1)^2$, we expand, and we solve for a_0 , a_1 , and a_2 . We could have also differentiated at $x = 1$ and used the Taylor series formula (10.2).

Let us look at rational functions, that is, ratios of polynomials. An important fact is that a series for a function only defines the function on an interval even if the function is defined elsewhere. For example, for $-1 < x < 1$,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

This series is called the *geometric series*. The ratio test tells us that the radius of convergence is 1. The series diverges for $x \leq -1$ and $x \geq 1$, even though $\frac{1}{1-x}$ is defined for all $x \neq 1$.

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions around a point, as long as the denominator is not zero at x_0 . Note that as for polynomials, we could equivalently use the Taylor series expansion (10.2).

Example 10.1.6: Expand $\frac{x}{1+2x+x^2}$ as a power series around the origin ($x_0 = 0$) and find the radius of convergence.

First, write $1 + 2x + x^2 = (1 + x)^2 = (1 - (-x))^2$. Compute

$$\begin{aligned}\frac{x}{1+2x+x^2} &= x \left(\frac{1}{1-(-x)} \right)^2 \\ &= x \left(\sum_{k=0}^{\infty} (-1)^k x^k \right)^2 \\ &= x \left(\sum_{k=0}^{\infty} c_k x^k \right) \\ &= \sum_{k=0}^{\infty} c_k x^{k+1},\end{aligned}$$

where to get c_k , we use the formula for the product of series. We obtain, $c_0 = 1$, $c_1 = -1 - 1 = -2$, $c_2 = 1 + 1 + 1 = 3$, etc. Therefore

$$\frac{x}{1+2x+x^2} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^k = x - 2x^2 + 3x^3 - 4x^4 + \dots$$

The radius of convergence is at least 1. We use the ratio test

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+2}(k+1)}{(-1)^{k+1}k} \right| = \lim_{k \rightarrow \infty} \frac{k+1}{k} = 1.$$

So the radius of convergence is actually equal to 1.

When the rational function is more complicated, it is also possible to use method of partial fractions. For example, to find the Taylor series for $\frac{x^3+x}{x^2-1}$, we write

$$\frac{x^3+x}{x^2-1} = x + \frac{1}{1+x} - \frac{1}{1-x} = x + \sum_{k=0}^{\infty} (-1)^k x^k - \sum_{k=0}^{\infty} x^k = -x + \sum_{\substack{k=3 \\ k \text{ odd}}}^{\infty} (-2)x^k.$$

10.1.6 Exercises

Exercise 10.1.1: Is the power series $\sum_{k=0}^{\infty} e^k x^k$ convergent? If so, what is the radius of convergence?

Exercise 10.1.2: Is the power series $\sum_{k=0}^{\infty} k x^k$ convergent? If so, what is the radius of convergence?

Exercise 10.1.3: Is the power series $\sum_{k=0}^{\infty} k! x^k$ convergent? If so, what is the radius of convergence?

Exercise 10.1.4: Is the power series $\sum_{k=0}^{\infty} \frac{1}{(2k)!} (x-10)^k$ convergent? If so, what is the radius of convergence?

Exercise 10.1.5: Determine the Taylor series for $\sin x$ around the point $x_0 = \pi$.

Exercise 10.1.6: Determine the Taylor series for $\ln x$ around the point $x_0 = 1$, and find the radius of convergence.

Exercise 10.1.7: Determine the Taylor series and its radius of convergence of $\frac{1}{1+x}$ around $x_0 = 0$.

Exercise 10.1.8: Determine the Taylor series and its radius of convergence of $\frac{x}{4-x^2}$ around $x_0 = 0$. Hint: You will not be able to use the ratio test.

Exercise 10.1.9: Expand $x^5 + 5x + 1$ as a power series around $x_0 = 5$.

Exercise 10.1.10: Suppose that the ratio test applies to a series $\sum_{k=0}^{\infty} a_k x^k$. Show, using the ratio test, that the radius of convergence of the differentiated series is the same as that of the original series.

Exercise 10.1.11: Suppose that f is an analytic function such that $f^{(n)}(0) = n$. Find $f(1)$.

Exercise 10.1.101: Is the power series $\sum_{n=1}^{\infty} (0.1)^n x^n$ convergent? If so, what is the radius of convergence?

Exercise 10.1.102 (challenging): Is the power series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ convergent? If so, what is the radius of convergence?

Exercise 10.1.103: Using the geometric series, expand $\frac{1}{1-x}$ around $x_0 = 2$. For what x does the series converge?

Exercise 10.1.104 (challenging): Find the Taylor series for $x^7 e^x$ around $x_0 = 0$.

Exercise 10.1.105 (challenging): Imagine f and g are analytic functions such that $f^{(k)}(0) = g^{(k)}(0)$ for all large enough k . What can you say about $f(x) - g(x)$?

10.2 Series Solutions of Linear Second Order ODEs

Suppose we have a linear second order homogeneous ODE of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

Suppose that $P(x)$, $Q(x)$, and $R(x)$ are polynomials. We will try a solution of the form

$$y = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

and solve for the a_k to try to obtain a solution defined in some interval around x_0 .

The point x_0 is called an *ordinary point* if $P(x_0) \neq 0$. That is, the functions

$$\frac{Q(x)}{P(x)} \quad \text{and} \quad \frac{R(x)}{P(x)}$$

are defined for x near x_0 . If $P(x_0) = 0$, then we say x_0 is a *singular point*. Handling singular points is harder than ordinary points and so we now focus only on ordinary points.

Example 10.2.1: We start with a very simple example

$$y'' - y = 0.$$

Let us try a power series solution near $x_0 = 0$, which is an ordinary point. Every point is an ordinary point in fact, as the equation is constant coefficient. We already know we should obtain exponentials or the hyperbolic sine and cosine, but let us pretend we do not know this fact.

We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

If we differentiate, the $k = 0$ term is a constant and hence disappears. We get

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

We differentiate yet again to obtain (now the $k = 1$ term disappears)

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

We reindex the series (replace k with $k + 2$) to obtain

$$y'' = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

Now we plug y and y'' into the differential equation

$$\begin{aligned} 0 = y'' - y &= \left(\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1) a_{k+2} - a_k \right) x^k \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1) a_{k+2} - a_k \right) x^k. \end{aligned}$$

As $y'' - y$ is supposed to be equal to 0, we know that the coefficients of the resulting series must be equal to 0. Therefore,

$$(k+2)(k+1)a_{k+2} - a_k = 0, \quad \text{or} \quad a_{k+2} = \frac{a_k}{(k+2)(k+1)}.$$

The equation above is called a *recurrence relation* for the coefficients of the power series. It did not matter what a_0 or a_1 was. They can be arbitrary. But once we pick a_0 and a_1 , all other coefficients are determined by the recurrence relation.

Let us see what the coefficients must be. First, a_0 and a_1 are arbitrary. Then,

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{(3)(2)}, \quad a_4 = \frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)}, \quad a_5 = \frac{a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)}, \quad \dots$$

So for even k , that is $k = 2n$, we have

$$a_k = a_{2n} = \frac{a_0}{(2n)!},$$

and for odd k , that is $k = 2n + 1$, we have

$$a_k = a_{2n+1} = \frac{a_1}{(2n+1)!}.$$

Let us write down the series

$$y = \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} \left(\frac{a_0}{(2n)!} x^{2n} + \frac{a_1}{(2n+1)!} x^{2n+1} \right) = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

We recognize the two series as the hyperbolic sine and cosine. Therefore,

$$y = a_0 \cosh x + a_1 \sinh x.$$

Of course, in general we will not be able to recognize the series that appears, since usually there will not be any elementary function that matches it. In that case we will be content with the series.

Example 10.2.2: Let us do a more complex example. Consider *Airy's equation*^{*}:

$$y'' - xy = 0,$$

near the point $x_0 = 0$. Note that $x_0 = 0$ is an ordinary point.

We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate twice (as above) to obtain

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

We plug y into the equation

$$\begin{aligned} 0 = y'' - xy &= \left(\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \right) - x \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= \left(\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \right) - \left(\sum_{k=0}^{\infty} a_k x^{k+1} \right). \end{aligned}$$

^{*}Named after the English mathematician [Sir George Biddell Airy](#) (1801–1892).

We reindex to make things easier to sum

$$\begin{aligned} 0 &= y'' - xy = \left(2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2}x^k \right) - \left(\sum_{k=1}^{\infty} a_{k-1}x^k \right) \\ &= 2a_2 + \sum_{k=1}^{\infty} \left((k+2)(k+1)a_{k+2} - a_{k-1} \right) x^k. \end{aligned}$$

Again $y'' - xy$ is supposed to be 0, so $a_2 = 0$, and

$$(k+2)(k+1)a_{k+2} - a_{k-1} = 0, \quad \text{or} \quad a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}.$$

We jump in steps of three. First, since $a_2 = 0$, we must have $a_5 = 0$, $a_8 = 0$, $a_{11} = 0$, etc. In general, $a_{3n+2} = 0$.

The constants a_0 and a_1 are arbitrary and we obtain

$$a_3 = \frac{a_0}{(3)(2)}, \quad a_4 = \frac{a_1}{(4)(3)}, \quad a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)}, \quad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)}, \quad \dots$$

For a_k where k is a multiple of 3, that is, $k = 3n$, we notice that

$$a_{3n} = \frac{a_0}{(2)(3)(5)(6) \cdots (3n-1)(3n)}.$$

For a_k where $k = 3n + 1$, we notice

$$a_{3n+1} = \frac{a_1}{(3)(4)(6)(7) \cdots (3n)(3n+1)}.$$

In other words, if we write down the series for y , it has two parts

$$\begin{aligned} y &= \left(a_0 + \frac{a_0}{6}x^3 + \frac{a_0}{180}x^6 + \cdots + \frac{a_0}{(2)(3)(5)(6) \cdots (3n-1)(3n)}x^{3n} + \cdots \right) \\ &\quad + \left(a_1x + \frac{a_1}{12}x^4 + \frac{a_1}{504}x^7 + \cdots + \frac{a_1}{(3)(4)(6)(7) \cdots (3n)(3n+1)}x^{3n+1} + \cdots \right) \\ &= a_0 \left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1}{(2)(3)(5)(6) \cdots (3n-1)(3n)}x^{3n} + \cdots \right) \\ &\quad + a_1 \left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots + \frac{1}{(3)(4)(6)(7) \cdots (3n)(3n+1)}x^{3n+1} + \cdots \right). \end{aligned}$$

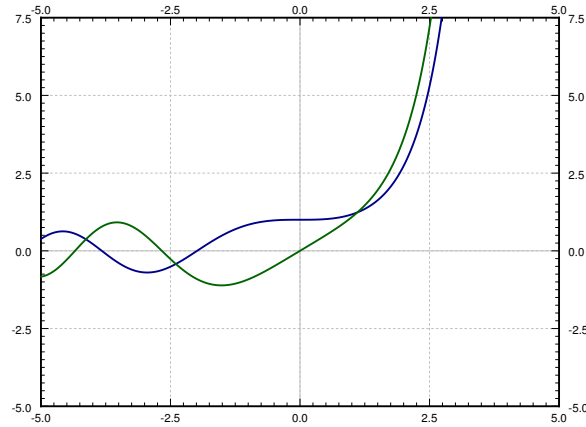
We define

$$\begin{aligned} y_1(x) &= 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1}{(2)(3)(5)(6) \cdots (3n-1)(3n)}x^{3n} + \cdots, \\ y_2(x) &= x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots + \frac{1}{(3)(4)(6)(7) \cdots (3n)(3n+1)}x^{3n+1} + \cdots, \end{aligned}$$

and write the general solution to the equation as $y(x) = a_0y_1(x) + a_1y_2(x)$. If we plug in $x = 0$ into the power series for y_1 and y_2 , we find $y_1(0) = 1$ and $y_2(0) = 0$. Similarly, $y_1'(0) = 0$ and $y_2'(0) = 1$. Therefore $y = a_0y_1 + a_1y_2$ is a solution that satisfies the initial conditions $y(0) = a_0$ and $y'(0) = a_1$.

The functions y_1 and y_2 cannot be written in terms of the elementary functions that you know. See [Figure 10.3](#) on the next page for the plot of the solutions y_1 and y_2 . These functions have many interesting properties. For example, they are oscillatory for negative x (like solutions to $y'' + y = 0$) and for positive x they grow without bound (like solutions to $y'' - y = 0$).

Sometimes a solution may turn out to be a polynomial.

Figure 10.3: The two solutions y_1 and y_2 to Airy's equation.

Example 10.2.3: Let us find a solution to the so-called *Hermite's equation of order n^** :

$$y'' - 2xy' + 2ny = 0.$$

Let us find a solution around the point $x_0 = 0$. We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate (as above) to obtain

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1},$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

Now we plug into the equation

$$\begin{aligned} 0 &= y'' - 2xy' + 2ny \\ &= \left(\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \right) - 2x \left(\sum_{k=1}^{\infty} k a_k x^{k-1} \right) + 2n \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= \left(\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k \right) + \left(\sum_{k=0}^{\infty} 2n a_k x^k \right) \\ &= \left(2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} x^k \right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k \right) + \left(2n a_0 + \sum_{k=1}^{\infty} 2n a_k x^k \right) \\ &= 2a_2 + 2n a_0 + \sum_{k=1}^{\infty} ((k+2)(k+1) a_{k+2} - 2k a_k + 2n a_k) x^k. \end{aligned}$$

As $y'' - 2xy' + 2ny = 0$, we have

$$(k+2)(k+1) a_{k+2} + (-2k+2n) a_k = 0, \quad \text{or} \quad a_{k+2} = \frac{(2k-2n)}{(k+2)(k+1)} a_k.$$

*Named after the French mathematician [Charles Hermite](#) (1822–1901).

This recurrence relation actually includes $a_2 = -na_0$ (which comes about from the constant term $2a_2 + 2na_0 = 0$). Again a_0 and a_1 are arbitrary.

$$\begin{aligned} a_2 &= \frac{-2n}{(2)(1)}a_0, & a_3 &= \frac{2(1-n)}{(3)(2)}a_1, \\ a_4 &= \frac{2(2-n)}{(4)(3)}a_2 = \frac{2^2(2-n)(-n)}{(4)(3)(2)(1)}a_0, \\ a_5 &= \frac{2(3-n)}{(5)(4)}a_3 = \frac{2^2(3-n)(1-n)}{(5)(4)(3)(2)}a_1, \quad \dots \end{aligned}$$

Let us separate the even and odd coefficients. We find that

$$\begin{aligned} a_{2m} &= \frac{2^m(-n)(2-n)\cdots(2m-2-n)}{(2m)!}, \\ a_{2m+1} &= \frac{2^m(1-n)(3-n)\cdots(2m-1-n)}{(2m+1)!}. \end{aligned}$$

Let us write down the two series, one with the even powers and one with the odd.

$$\begin{aligned} y_1(x) &= 1 + \frac{2(-n)}{2!}x^2 + \frac{2^2(-n)(2-n)}{4!}x^4 + \frac{2^3(-n)(2-n)(4-n)}{6!}x^6 + \dots, \\ y_2(x) &= x + \frac{2(1-n)}{3!}x^3 + \frac{2^2(1-n)(3-n)}{5!}x^5 + \frac{2^3(1-n)(3-n)(5-n)}{7!}x^7 + \dots. \end{aligned}$$

We then write

$$y(x) = a_0y_1(x) + a_1y_2(x).$$

We remark that if n is a positive even integer, then $y_1(x)$ is a polynomial as all the coefficients in the series beyond degree n are zero. If n is a positive odd integer, then $y_2(x)$ is a polynomial. For example, if $n = 4$, then

$$y_1(x) = 1 + \frac{2(-4)}{2!}x^2 + \frac{2^2(-4)(2-4)}{4!}x^4 = 1 - 4x^2 + \frac{4}{3}x^4.$$

10.2.1 Exercises

In the following exercises, when asked to solve an equation using power series methods, you should find the first few terms of the series, and if possible find a general formula for the k^{th} coefficient.

Exercise 10.2.1: Use power series methods to solve $y'' + y = 0$ at the point $x_0 = 1$.

Exercise 10.2.2: Use power series methods to solve $y'' + 4xy = 0$ at the point $x_0 = 0$.

Exercise 10.2.3: Use power series methods to solve $y'' - xy = 0$ at the point $x_0 = 1$.

Exercise 10.2.4: Use power series methods to solve $y'' + x^2y = 0$ at the point $x_0 = 0$.

Exercise 10.2.5: The methods work for other orders than second order. Try the methods of this section to solve the first order system $y' - xy = 0$ at the point $x_0 = 0$.

Exercise 10.2.6 (Chebyshev's equation of order p):

- Solve $(1 - x^2)y'' - xy' + p^2y = 0$ using power series methods at $x_0 = 0$.
- For what p is there a polynomial solution?

Exercise 10.2.7: Find a polynomial solution to $(x^2 + 1)y'' - 2xy' + 2y = 0$ using power series methods.

Exercise 10.2.8:

- Use power series methods to solve $(1 - x)y'' + y = 0$ at the point $x_0 = 0$.
- Use the solution to part a) to find a solution for $xy'' + y = 0$ around the point $x_0 = 1$.

Exercise 10.2.101: Use power series methods to solve $y'' + 2x^3y = 0$ at the point $x_0 = 0$.

Exercise 10.2.102 (challenging): *Power series methods also work for nonhomogeneous equations.*

- a) Use power series methods to solve $y'' - xy = \frac{1}{1-x}$ at the point $x_0 = 0$. Hint: Recall the geometric series.
- b) Now solve for the initial condition $y(0) = 0, y'(0) = 0$.

Exercise 10.2.103: *Attempt to solve $x^2y'' - y = 0$ at $x_0 = 0$ using the power series method of this section (x_0 is a singular point). Can you find at least one solution? Can you find more than one solution?*

10.3 Singular Points and the Method of Frobenius

The behavior of ODEs at singular points can be complicated, for certain singular points, we can find a solution on at least one side of the singular point using a modification of the power series. Let us look at some examples before giving a general method.

10.3.1 Examples

Example 10.3.1: Let us first look at a simple first order equation

$$2xy' - y = 0.$$

Note that $x = 0$ is a singular point. Setting $x = 0$ in the equation, we find that any solution defined near zero satisfies $y(0) = 0$, but it is even worse. If we try to plug in

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

we obtain

$$\begin{aligned} 0 = 2xy' - y &= 2x \left(\sum_{k=1}^{\infty} k a_k x^{k-1} \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= a_0 + \sum_{k=1}^{\infty} (2k a_k - a_k) x^k. \end{aligned}$$

First, $a_0 = 0$. Next, the only way to solve $0 = 2k a_k - a_k = (2k - 1) a_k$ for $k = 1, 2, 3, \dots$ is for $a_k = 0$ for all k . Therefore, in this manner we only get the trivial solution $y = 0$. We need a nonzero solution to get the general solution to the equation.

Let us try $y = x^r$ for some real number r . Consequently our solution—if we can find one—may only make sense for positive x . Then $y' = r x^{r-1}$. So

$$0 = 2xy' - y = 2xr x^{r-1} - x^r = (2r - 1)x^r.$$

Thus $r = 1/2$ and so $y = x^{1/2}$. As the equation is linear, the general solution for positive x is

$$y = C x^{1/2}.$$

If $C \neq 0$, then the derivative of the solution “blows up” at $x = 0$ (the singular point). There is only one solution that is differentiable at $x = 0$ and that’s the trivial solution $y = 0$.

Not every problem with a singular point has a solution of the form $y = x^r$, of course. But perhaps we can combine the methods. What we will do is to try a solution of the form

$$y = x^r f(x)$$

for positive x , where $f(x)$ is an analytic function (a power series).

Example 10.3.2: Consider the equation

$$4x^2 y'' - 4x^2 y' + (1 - 2x)y = 0,$$

and again note that $x = 0$ is a singular point.

Let us try

$$y = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r},$$

where r is a real number, not necessarily an integer. Again if such a solution exists, it may only exist for positive x . First we find the derivatives

$$y' = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1},$$

$$y'' = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}.$$

We plug those into our equation:

$$\begin{aligned} 0 &= 4x^2 y'' - 4x^2 y' + (1-2x)y \\ &= 4x^2 \left(\sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2} \right) - 4x^2 \left(\sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1} \right) + (1-2x) \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) \\ &= \left(\sum_{k=0}^{\infty} 4(k+r)(k+r-1) a_k x^{k+r} \right) \\ &\quad - \left(\sum_{k=0}^{\infty} 4(k+r) a_k x^{k+r+1} \right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) - \left(\sum_{k=0}^{\infty} 2a_k x^{k+r+1} \right) \\ &= \left(\sum_{k=0}^{\infty} 4(k+r)(k+r-1) a_k x^{k+r} \right) \\ &\quad - \left(\sum_{k=1}^{\infty} 4(k+r-1) a_{k-1} x^{k+r} \right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) - \left(\sum_{k=1}^{\infty} 2a_{k-1} x^{k+r} \right) \\ &= 4r(r-1) a_0 x^r + a_0 x^r + \sum_{k=1}^{\infty} \left(4(k+r)(k+r-1) a_k - 4(k+r-1) a_{k-1} + a_k - 2a_{k-1} \right) x^{k+r} \\ &= (4r(r-1) + 1) a_0 x^r + \sum_{k=1}^{\infty} \left((4(k+r)(k+r-1) + 1) a_k - (4(k+r-1) + 2) a_{k-1} \right) x^{k+r}. \end{aligned}$$

First, to have a solution we must have $(4r(r-1) + 1) a_0 = 0$. Supposing $a_0 \neq 0$,

$$4r(r-1) + 1 = 0.$$

This equation is called the *indicial equation*. This particular indicial equation has a double root at $r = 1/2$.

OK, so we know what r has to be. That knowledge we obtained simply by looking at the coefficient of x^r . All other coefficients of x^{k+r} also have to be zero so

$$(4(k+r)(k+r-1) + 1) a_k - (4(k+r-1) + 2) a_{k-1} = 0.$$

If we plug in $r = 1/2$ and solve for a_k , we get

$$a_k = \frac{4(k + 1/2 - 1) + 2}{4(k + 1/2)(k + 1/2 - 1) + 1} a_{k-1} = \frac{1}{k} a_{k-1}.$$

Let us set $a_0 = 1$. Then

$$a_1 = \frac{1}{1} a_0 = 1, \quad a_2 = \frac{1}{2} a_1 = \frac{1}{2}, \quad a_3 = \frac{1}{3} a_2 = \frac{1}{3 \cdot 2}, \quad a_4 = \frac{1}{4} a_3 = \frac{1}{4 \cdot 3 \cdot 2}, \quad \dots$$

Extrapolating, we notice that

$$a_k = \frac{1}{k(k-1)(k-2) \cdots 3 \cdot 2} = \frac{1}{k!}.$$

In other words,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1/2} = x^{1/2} \sum_{k=0}^{\infty} \frac{1}{k!} x^k = x^{1/2} e^x.$$

That was lucky! In general, we will not be able to write the series in terms of elementary functions.

We have one solution, let us call it $y_1 = x^{1/2} e^x$. But what about a second solution? If we want a general solution, we need two linearly independent solutions. Picking a_0 to be a different constant only gets us a constant multiple of y_1 , and we do not have any other r to try; we only have one solution to the indicial equation. Well, there are powers of x floating around and we are taking derivatives, perhaps the logarithm (the antiderivative of x^{-1}) is around as well. It turns out we want to try for another solution of the form

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+r} + (\ln x) y_1,$$

which in our case is

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+1/2} + (\ln x) x^{1/2} e^x.$$

We now differentiate this equation, substitute into the differential equation and solve for b_k . A long computation ensues and we obtain some recursion relation for b_k . The reader can (and should) try this to obtain for example the first three terms

$$b_1 = b_0 - 1, \quad b_2 = \frac{2b_1 - 1}{4}, \quad b_3 = \frac{6b_2 - 1}{18}, \quad \dots$$

We then fix b_0 and obtain a solution y_2 . Then we write the general solution as $y = Ay_1 + By_2$.

10.3.2 The Method of Frobenius

Before giving the general method, let us clarify when the method applies. Let

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

be an ODE. As before, if $P(x_0) = 0$, then x_0 is a singular point. If we divide by $P(x)$ to put the equation in standard form $y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$, perhaps the singularities introduced are not too bad. More specifically, if the limits

$$\lim_{x \rightarrow x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \rightarrow x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

both exist and are finite, then we say that x_0 is a *regular singular point*.

Example 10.3.3: Often, and for the rest of this section, $x_0 = 0$. Consider

$$x^2 y'' + x(1+x)y' + (\pi + x^2)y = 0.$$

Write

$$\begin{aligned} \lim_{x \rightarrow 0} x \frac{Q(x)}{P(x)} &= \lim_{x \rightarrow 0} x \frac{x(1+x)}{x^2} = \lim_{x \rightarrow 0} (1+x) = 1, \\ \lim_{x \rightarrow 0} x^2 \frac{R(x)}{P(x)} &= \lim_{x \rightarrow 0} x^2 \frac{(\pi + x^2)}{x^2} = \lim_{x \rightarrow 0} (\pi + x^2) = \pi. \end{aligned}$$

So $x = 0$ is a regular singular point.

On the other hand, if we make the slight change

$$x^2 y'' + (1+x)y' + (\pi + x^2)y = 0,$$

then

$$\lim_{x \rightarrow 0} x \frac{Q(x)}{P(x)} = \lim_{x \rightarrow 0} x \frac{(1+x)}{x^2} = \lim_{x \rightarrow 0} \frac{1+x}{x} = \text{DNE}.$$

Here DNE stands for *does not exist*. The point 0 is singular, but not a regular singular point.

We now discuss the general *Method of Frobenius**. We only consider the method at the point $x = 0$ for simplicity. If $x_0 \neq 0$, then in the solution, we would replace every x with $(x - x_0)$. The main idea is the following theorem.

Theorem 10.3.1 (Method of Frobenius). *Suppose that*

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (10.3)$$

has a regular singular point at $x = 0$, then there exists at least one solution of the form

$$y = x^r \sum_{k=0}^{\infty} a_k x^k.$$

A solution of this form is called a Frobenius-type solution.

The method usually breaks down like this:

- (i) We seek a Frobenius-type solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}.$$

We plug this y into equation (10.3). We collect terms and write everything as a single series.

- (ii) The obtained series must be zero. Setting the first coefficient (usually the coefficient of x^r) in the series to zero we obtain the *indicial equation*, which is a quadratic polynomial in r .
- (iii) If the indicial equation has two real roots r_1 and r_2 such that $r_1 - r_2$ is not an integer, then we have two linearly independent Frobenius-type solutions. Using the first root, we plug in

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and we solve for all a_k to obtain the first solution. Then using the second root, we plug in

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k,$$

and solve for all b_k to obtain the second solution.

- (iv) If the indicial equation has a doubled root r , then there we find one solution

$$y_1 = x^r \sum_{k=0}^{\infty} a_k x^k,$$

and then we obtain a new solution by plugging

$$y_2 = x^r \sum_{k=0}^{\infty} b_k x^k + (\ln x)y_1,$$

into equation (10.3) and solving for the constants b_k .

*Named after the German mathematician [Ferdinand Georg Frobenius](#) (1849–1917).

(v) If the indicial equation has two real roots such that $r_1 - r_2$ is an integer, then one solution is

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and the second linearly independent solution is of the form

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k + C(\ln x)y_1,$$

where we plug y_2 into (10.3) and solve for the constants b_k and C .

(vi) Finally, if the indicial equation has complex roots, then solving for a_k in the solution

$$y = x^{r_1} \sum_{k=0}^{\infty} a_k x^k$$

results in a complex-valued function—all the a_k are complex numbers. We obtain our two linearly independent solutions* by taking the real and imaginary parts of y .

The main idea is to find at least one Frobenius-type solution. If we are lucky and find two, we are done. If we only get one, we either use the ideas above or even a different method such as reduction of order (see §7.1) to obtain a second solution.

10.3.3 Bessel Functions

An important class of functions that arise commonly in physics are the *Bessel functions*[†]. For example, these functions appear when solving the wave equation in two and three dimensions. First consider *Bessel's equation* of order p :

$$x^2 y'' + x y' + (x^2 - p^2) y = 0.$$

We allow p to be any number, not just an integer, although integers and multiples of $1/2$ are most important in applications.

When we plug

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

into Bessel's equation of order p , we obtain the indicial equation

$$r(r-1) + r - p^2 = (r-p)(r+p) = 0.$$

We obtain two roots, $r_1 = p$ and $r_2 = -p$. If p is not an integer, then following the method of Frobenius and setting $a_0 = 1$, we find linearly independent solutions of the form

$$y_1 = x^p \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k! (k+p)(k-1+p) \cdots (2+p)(1+p)},$$

$$y_2 = x^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k! (k-p)(k-1-p) \cdots (2-p)(1-p)}.$$

Exercise 10.3.1:

*See Joseph L. Neuringer, *The Frobenius method for complex roots of the indicial equation*, International Journal of Mathematical Education in Science and Technology, Volume 9, Issue 1, 1978, 71–77.

[†]Named after the German astronomer and mathematician [Friedrich Wilhelm Bessel](#) (1784–1846).

- a) Verify that the indicial equation of Bessel's equation of order p is $(r - p)(r + p) = 0$.
 b) Suppose p is not an integer. Carry out the computation to obtain the solutions y_1 and y_2 above.

Bessel functions are convenient constant multiples of y_1 and y_2 . First we must define the *gamma function*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

Notice that $\Gamma(1) = 1$. The gamma function also has a wonderful property

$$\Gamma(x + 1) = x\Gamma(x).$$

From this property, it follows that $\Gamma(n) = (n-1)!$ when n is an integer. So the gamma function is a continuous version of the factorial. We compute:

$$\begin{aligned}\Gamma(k + p + 1) &= (k + p)(k - 1 + p) \cdots (2 + p)(1 + p)\Gamma(1 + p), \\ \Gamma(k - p + 1) &= (k - p)(k - 1 - p) \cdots (2 - p)(1 - p)\Gamma(1 - p).\end{aligned}$$

Exercise 10.3.2: Verify the identities above using $\Gamma(x + 1) = x\Gamma(x)$.

We define the *Bessel functions of the first kind* of order p and $-p$ as

$$\begin{aligned}J_p(x) &= \frac{1}{2^p \Gamma(1 + p)} y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + p + 1)} \left(\frac{x}{2}\right)^{2k+p}, \\ J_{-p}(x) &= \frac{1}{2^{-p} \Gamma(1 - p)} y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k - p + 1)} \left(\frac{x}{2}\right)^{2k-p}.\end{aligned}$$

As these are constant multiples of the solutions we found above, these are both solutions to Bessel's equation of order p . The constants are picked for convenience.

When p is not an integer, J_p and J_{-p} are linearly independent. When n is an integer we obtain

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k + n)!} \left(\frac{x}{2}\right)^{2k+n}.$$

In this case

$$J_n(x) = (-1)^n J_{-n}(x),$$

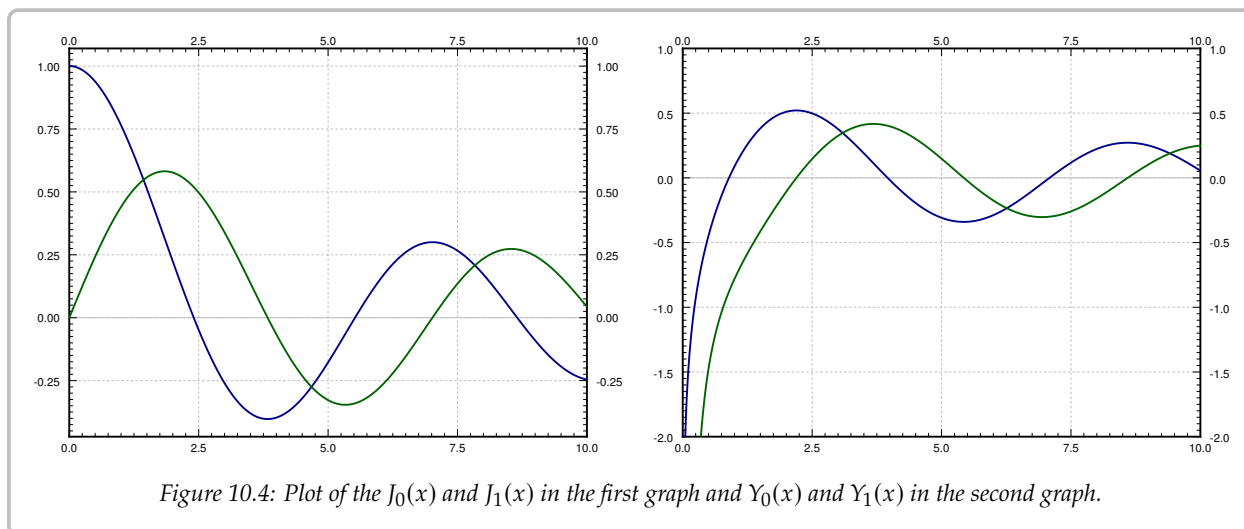
and so J_{-n} is not a second linearly independent solution. The other solution is the so-called *Bessel function of second kind*. These make sense only for integer orders n and are defined as limits of linear combinations of $J_p(x)$ and $J_{-p}(x)$, as p approaches n in the following way:

$$Y_n(x) = \lim_{p \rightarrow n} \frac{\cos(p\pi)J_p(x) - J_{-p}(x)}{\sin(p\pi)}.$$

Each linear combination of $J_p(x)$ and $J_{-p}(x)$ is a solution to Bessel's equation of order p . Then as we take the limit as p goes to n , we see that $Y_n(x)$ is a solution to Bessel's equation of order n . It also turns out that $Y_n(x)$ and $J_n(x)$ are linearly independent. Therefore when n is an integer, we have the general solution to Bessel's equation of order n :

$$y = AJ_n(x) + BY_n(x),$$

for arbitrary constants A and B . Note that $Y_n(x)$ goes to negative infinity at $x = 0$. Many mathematical software packages have these functions $J_n(x)$ and $Y_n(x)$ defined, so they can be used just like say $\sin(x)$ and $\cos(x)$. In fact, Bessel functions have some similar properties. For example, $-J_1(x)$ is a derivative of $J_0(x)$, and in general the derivative of $J_n(x)$ can be written as a linear combination of $J_{n-1}(x)$ and $J_{n+1}(x)$. Furthermore,



these functions oscillate, although they are not periodic. See Figure 10.4 on the following page for graphs of Bessel functions.

Example 10.3.4: Other equations can sometimes be solved in terms of the Bessel functions. For example, given a positive constant λ ,

$$xy'' + y' + \lambda^2 xy = 0,$$

can be changed to $x^2 y'' + xy' + \lambda^2 x^2 y = 0$. Changing variables $t = \lambda x$, we obtain, via the chain rule, the equation in y and t :

$$t^2 y'' + ty' + t^2 y = 0,$$

which we recognize as Bessel's equation of order 0. Therefore the general solution is $y(t) = AJ_0(t) + BY_0(t)$, or in terms of x :

$$y = AJ_0(\lambda x) + BY_0(\lambda x).$$

This equation comes up, for example, when finding the fundamental modes of vibration of a circular drum, but we digress.

10.3.4 Exercises

Exercise 10.3.3: Find a particular (Frobenius-type) solution of $x^2 y'' + xy' + (1+x)y = 0$.

Exercise 10.3.4: Find a particular (Frobenius-type) solution of $xy'' - y = 0$.

Exercise 10.3.5: Find a particular (Frobenius-type) solution of $y'' + \frac{1}{x}y' - xy = 0$.

Exercise 10.3.6: Find the general solution of $2xy'' + y' - x^2 y = 0$.

Exercise 10.3.7: Find the general solution of $x^2 y'' - xy' - y = 0$.

Exercise 10.3.8: In the following equations classify the point $x = 0$ as ordinary, regular singular, or singular but not regular singular.

a) $x^2(1+x^2)y'' + xy = 0$

b) $x^2 y'' + y' + y = 0$

c) $xy'' + x^3 y' + y = 0$

d) $xy'' + xy' - e^x y = 0$

e) $x^2 y'' + x^2 y' + x^2 y = 0$

Exercise 10.3.101: In the following equations classify the point $x = 0$ as ordinary, regular singular, or singular but not regular singular.

a) $y'' + y = 0$

b) $x^3 y'' + (1 + x)y = 0$

c) $xy'' + x^5 y' + y = 0$

d) $\sin(x)y'' - y = 0$

e) $\cos(x)y'' - \sin(x)y = 0$

Exercise 10.3.102: Find the general solution of $x^2 y'' - y = 0$.

Exercise 10.3.103: Find a particular solution of $x^2 y'' + (x - 3/4)y = 0$.

Exercise 10.3.104 (tricky): Find the general solution of $x^2 y'' - xy' + y = 0$.

Chapter 11

Laplace Transforms

11.1 Introduction to the Laplace Transform

11.1.1 Definition of the Laplace Transform

To define the Laplace transform, we first recall the definition of an improper integral. If g is integrable over the interval $[a, T]$ for every $T > a$, then the *improper integral of g over $[a, \infty)$* is defined as

$$\int_a^\infty g(t) dt = \lim_{T \rightarrow \infty} \int_a^T g(t) dt. \quad (11.1)$$

We say that the improper integral *converges* if the limit in (11.1) exists; otherwise, we say that the improper integral *diverges* or *does not exist*. Here's the definition of the Laplace transform of a function f .

Definition 11.1.1 Let f be defined for $t \geq 0$ and let s be a real number. Then the *Laplace transform of f* is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt, \quad (11.2)$$

for those values of s for which the improper integral converges.

It is important to keep in mind that the variable of integration in (11.2) is t , while s is a parameter independent of t . We use t as the independent variable for f because in applications the Laplace transform is usually applied to functions of time.

The Laplace transform can be viewed as an operator $\mathcal{L}$ that transforms the function $f = f(t)$ into the function $F = F(s)$. Thus, (11.2) can be expressed as

$$F = \mathcal{L}(f).$$

The functions f and F form a *transform pair*, which we'll sometimes denote by

$$f(t) \leftrightarrow F(s).$$

It can be shown that if $F(s)$ is defined for $s = s_0$ then it's defined for all $s > s_0$ (Exercise 11.1.14(b)).

11.1.2 Computation of Some Simple Laplace Transforms

Example 11.1.1: Find the Laplace transform of $f(t) = 1$.

Solution: From (11.2) with $f(t) = 1$,

$$F(s) = \int_0^{\infty} e^{-st} dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} dt.$$

If $s \neq 0$ then

$$\int_0^T e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^T = \frac{1 - e^{-sT}}{s}. \quad (11.3)$$

Therefore

$$\lim_{T \rightarrow \infty} \int_0^T e^{-st} dt = \begin{cases} \frac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases} \quad (11.4)$$

If $s = 0$ the integrand reduces to the constant 1, and

$$\lim_{T \rightarrow \infty} \int_0^T 1 dt = \lim_{T \rightarrow \infty} \int_0^T 1 dt = \lim_{T \rightarrow \infty} T = \infty.$$

Therefore $F(0)$ is undefined, and

$$F(s) = \int_0^{\infty} e^{-st} dt = \frac{1}{s}, \quad s > 0.$$

This result can be written in operator notation as

$$\mathcal{L}(1) = \frac{1}{s}, \quad s > 0,$$

or as the transform pair

$$1 \leftrightarrow \frac{1}{s}, \quad s > 0.$$

Remark 11.1.1: It is convenient to combine the steps of integrating from 0 to T and letting $T \rightarrow \infty$. Therefore, instead of writing (11.3) and (11.4) as separate steps we write

$$\int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \begin{cases} \frac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases}$$

We'll follow this practice throughout this chapter.

Example 11.1.2: Find the Laplace transform of $f(t) = t$.

Solution: From (11.2) with $f(t) = t$,

$$F(s) = \int_0^{\infty} e^{-st} t dt. \quad (11.5)$$

If $s \neq 0$, integrating by parts yields

$$\begin{aligned} \int_0^{\infty} e^{-st} t dt &= -\frac{te^{-st}}{s} \Big|_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt = -\left[\frac{t}{s} + \frac{1}{s^2} \right] e^{-st} \Big|_0^{\infty} \\ &= \begin{cases} \frac{1}{s^2}, & s > 0, \\ \infty, & s < 0. \end{cases} \end{aligned}$$

If $s = 0$, the integral in (11.5) becomes

$$\int_0^{\infty} t \, dt = \left. \frac{t^2}{2} \right|_0^{\infty} = \infty.$$

Therefore $F(0)$ is undefined and

$$F(s) = \frac{1}{s^2}, \quad s > 0.$$

This result can also be written as

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0,$$

or as the transform pair

$$t \leftrightarrow \frac{1}{s^2}, \quad s > 0.$$

Example 11.1.3: Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution: From (11.2) with $f(t) = e^{at}$,

$$F(s) = \int_0^{\infty} e^{-st} e^{at} \, dt.$$

Combining the exponentials yields

$$F(s) = \int_0^{\infty} e^{-(s-a)t} \, dt.$$

However, we know from Example 11.1.1 that

$$\int_0^{\infty} e^{-st} \, dt = \frac{1}{s}, \quad s > 0.$$

Replacing s by $s - a$ here shows that

$$F(s) = \frac{1}{s - a}, \quad s > a.$$

This can also be written as

$$\mathcal{L}(e^{at}) = \frac{1}{s - a}, \quad s > a, \quad \text{or} \quad e^{at} \leftrightarrow \frac{1}{s - a}, \quad s > a.$$

Example 11.1.4: Find the Laplace transforms of $f(t) = \sin \omega t$ and $g(t) = \cos \omega t$, where ω is a constant.

Solution: Define

$$F(s) = \int_0^{\infty} e^{-st} \sin \omega t \, dt \tag{11.6}$$

and

$$G(s) = \int_0^{\infty} e^{-st} \cos \omega t \, dt. \tag{11.7}$$

If $s > 0$, integrating (11.6) by parts yields

$$F(s) = -\frac{e^{-st}}{s} \sin \omega t \Big|_0^{\infty} + \frac{\omega}{s} \int_0^{\infty} e^{-st} \cos \omega t \, dt,$$

so

$$F(s) = \frac{\omega}{s} G(s). \tag{11.8}$$

If $s > 0$, integrating (11.7) by parts yields

$$G(s) = -\frac{e^{-st} \cos \omega t}{s} \Big|_0^{\infty} - \frac{\omega}{s} \int_0^{\infty} e^{-st} \sin \omega t \, dt,$$

so

$$G(s) = \frac{1}{s} - \frac{\omega}{s} F(s).$$

Now substitute from (11.8) into this to obtain

$$G(s) = \frac{1}{s} - \frac{\omega^2}{s^2} G(s).$$

Solving this for $G(s)$ yields

$$G(s) = \frac{s}{s^2 + \omega^2}, \quad s > 0.$$

This and (11.8) imply that

$$F(s) = \frac{\omega}{s^2 + \omega^2}, \quad s > 0.$$

11.1.3 Tables of Laplace transforms

Extensive tables of Laplace transforms have been compiled and are commonly used in applications. The brief table of Laplace transforms in the Appendix will be adequate for our purposes.

Example 11.1.5: Use the table of Laplace transforms to find $\mathcal{L}(t^3 e^{4t})$.

Solution: The table includes the transform pair

$$t^n e^{at} \leftrightarrow \frac{n!}{(s-a)^{n+1}}.$$

Setting $n = 3$ and $a = 4$ here yields

$$\mathcal{L}(t^3 e^{4t}) = \frac{3!}{(s-4)^4} = \frac{6}{(s-4)^4}. \quad \blacksquare$$

We'll sometimes write Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms.

11.1.4 Linearity of the Laplace Transform

The next theorem presents an important property of the Laplace transform.

Theorem 11.1.2. [*Linearity Property*] Suppose $\mathcal{L}(f_i)$ is defined for $s > s_i$, $1 \leq i \leq n$. Let s_0 be the largest of the numbers $s_1, s_2, \dots, s_n$, and let $c_1, c_2, \dots, c_n$ be constants. Then

$$\mathcal{L}(c_1 f_1 + c_2 f_2 + \dots + c_n f_n) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2) + \dots + c_n \mathcal{L}(f_n) \text{ for } s > s_0.$$

Proof. We give the proof for the case where $n = 2$. If $s > s_0$ then

$$\begin{aligned} \mathcal{L}(c_1 f_1 + c_2 f_2) &= \int_0^\infty e^{-st} (c_1 f_1(t) + c_2 f_2(t)) dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t) dt + c_2 \int_0^\infty e^{-st} f_2(t) dt \\ &= c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2). \end{aligned}$$

Example 11.1.6: Use Theorem 11.1.2 and the known Laplace transform

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

to find $\mathcal{L}(\cosh bt)$ ($b \neq 0$).

Solution: By definition,

$$\cosh bt = \frac{e^{bt} + e^{-bt}}{2}.$$

Therefore

$$\begin{aligned} \mathcal{L}(\cosh bt) &= \mathcal{L}\left(\frac{1}{2}e^{bt} + \frac{1}{2}e^{-bt}\right) \\ &= \frac{1}{2}\mathcal{L}(e^{bt}) + \frac{1}{2}\mathcal{L}(e^{-bt}) \quad (\text{linearity property}) \\ &= \frac{1}{2} \frac{1}{s-b} + \frac{1}{2} \frac{1}{s+b}, \end{aligned} \quad (11.9)$$

where the first transform on the right is defined for $s > b$ and the second for $s > -b$; hence, both are defined for $s > |b|$. Simplifying the last expression in (11.9) yields

$$\mathcal{L}(\cosh bt) = \frac{s}{s^2 - b^2}, \quad s > |b|.$$

11.1.5 The First Shifting Theorem

The next theorem enables us to start with known transform pairs and derive others. (For other results of this kind, see Exercises 11.1.6 and 11.1.13.)

Theorem 11.1.3. [First Shifting Theorem] If

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad (11.10)$$

is the Laplace transform of $f(t)$ for $s > s_0$, then $F(s-a)$ is the Laplace transform of $e^{at} f(t)$ for $s > s_0 + a$.

PROOF. Replacing s by $s-a$ in (11.10) yields

$$F(s-a) = \int_0^{\infty} e^{-(s-a)t} f(t) dt \quad (11.11)$$

if $s-a > s_0$; that is, if $s > s_0 + a$. However, (11.11) can be rewritten as

$$F(s-a) = \int_0^{\infty} e^{-st} (e^{at} f(t)) dt,$$

which implies the conclusion.

Example 11.1.7: Use Theorem 11.1.3 and the known Laplace transforms of 1, t , $\cos \omega t$, and $\sin \omega t$ to find

$$\mathcal{L}(e^{at}), \quad \mathcal{L}(te^{at}), \quad \mathcal{L}(e^{\lambda t} \sin \omega t), \text{ and } \mathcal{L}(e^{\lambda t} \cos \omega t).$$

Solution: In the following table the known transform pairs are listed on the left and the required transform pairs listed on the right are obtained by applying Theorem 11.1.3.

$f(t) \leftrightarrow F(s)$	$e^{at} f(t) \leftrightarrow F(s-a)$
$1 \leftrightarrow \frac{1}{s}, \quad s > 0$	$e^{at} \leftrightarrow \frac{1}{(s-a)}, \quad s > a$
$t \leftrightarrow \frac{1}{s^2}, \quad s > 0$	$te^{at} \leftrightarrow \frac{1}{(s-a)^2}, \quad s > a$
$\sin \omega t \leftrightarrow \frac{\omega}{s^2 + \omega^2}, \quad s > 0$	$e^{\lambda t} \sin \omega t \leftrightarrow \frac{\omega}{(s-\lambda)^2 + \omega^2}, \quad s > \lambda$
$\cos \omega t \leftrightarrow \frac{s}{s^2 + \omega^2}, \quad s > 0$	$e^{\lambda t} \cos \omega t \leftrightarrow \frac{s-\lambda}{(s-\lambda)^2 + \omega^2}, \quad s > \lambda$

11.1.6 Existence of Laplace Transforms

Not every function has a Laplace transform. For example, it can be shown (Exercise 11.1.3) that

$$\int_0^{\infty} e^{-st} e^{t^2} dt = \infty$$

for every real number s . Hence, the function $f(t) = e^{t^2}$ does not have a Laplace transform.

Our next objective is to establish conditions that ensure the existence of the Laplace transform of a function. We first review some relevant definitions from calculus.

Recall that a limit

$$\lim_{t \rightarrow t_0} f(t)$$

exists if and only if the one-sided limits

$$\lim_{t \rightarrow t_0^-} f(t) \quad \text{and} \quad \lim_{t \rightarrow t_0^+} f(t)$$

both exist and are equal; in this case,

$$\lim_{t \rightarrow t_0} f(t) = \lim_{t \rightarrow t_0^-} f(t) = \lim_{t \rightarrow t_0^+} f(t).$$

Recall also that f is continuous at a point t_0 in an open interval (a, b) if and only if

$$\lim_{t \rightarrow t_0} f(t) = f(t_0),$$

which is equivalent to

$$\lim_{t \rightarrow t_0^+} f(t) = \lim_{t \rightarrow t_0^-} f(t) = f(t_0). \quad (11.12)$$

For simplicity, we define

$$f(t_0+) = \lim_{t \rightarrow t_0^+} f(t) \quad \text{and} \quad f(t_0-) = \lim_{t \rightarrow t_0^-} f(t),$$

so (11.12) can be expressed as

$$f(t_0+) = f(t_0-) = f(t_0).$$

If $f(t_0+)$ and $f(t_0-)$ have finite but distinct values, we say that f has a *jump discontinuity* at t_0 , and

$$f(t_0+) - f(t_0-)$$

is called the *jump* in f at t_0 (Figure 11.1).

If $f(t_0+)$ and $f(t_0-)$ are finite and equal, but either f isn't defined at t_0 or it's defined but

$$f(t_0) \neq f(t_0+) = f(t_0-),$$

we say that f has a *removable discontinuity* at t_0 (Figure 11.2). This terminology is appropriate since a function f with a removable discontinuity at t_0 can be made continuous at t_0 by defining (or redefining)

$$f(t_0) = f(t_0+) = f(t_0-).$$

Remark 11.1.2: We know from calculus that a definite integral isn't affected by changing the values of its integrand at isolated points. Therefore, redefining a function f to make it continuous at removable discontinuities does not change $\mathcal{L}(f)$.

Definition 11.1.4

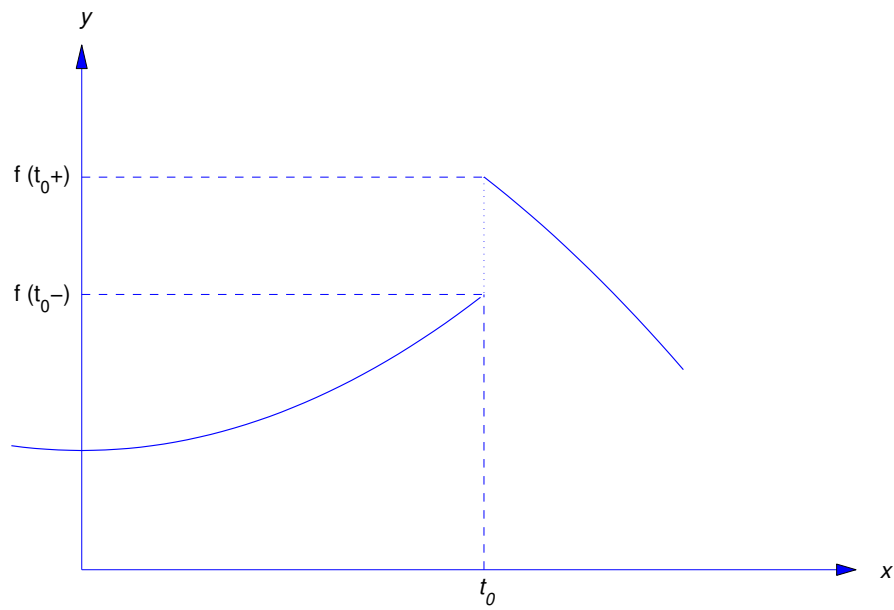


Figure 11.1: A jump discontinuity

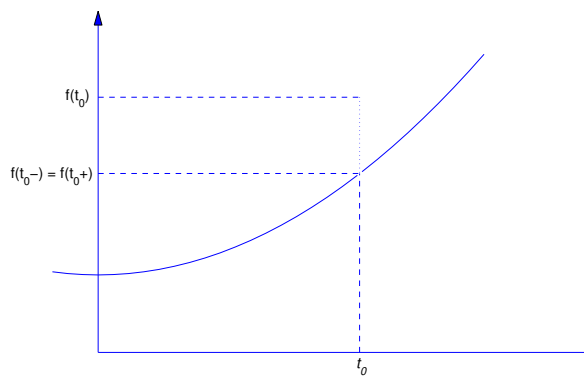
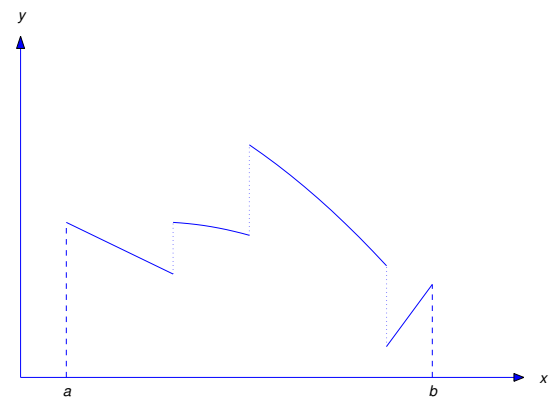


Figure 11.2

Figure 11.3: A piecewise continuous function on $[a, b]$

- (i) A function f is said to be *piecewise continuous* on a finite closed interval $[0, T]$ if $f(0+)$ and $f(T-)$ are finite and f is continuous on the open interval $(0, T)$ except possibly at finitely many points, where f may have jump discontinuities or removable discontinuities.
- (ii) A function f is said to be *piecewise continuous* on the infinite interval $[0, \infty)$ if it's piecewise continuous on $[0, T]$ for every $T > 0$.

Figure 11.3 shows the graph of a typical piecewise continuous function.

It is shown in calculus that if a function is piecewise continuous on a finite closed interval then it's integrable on that interval. But if f is piecewise continuous on $[0, \infty)$, then so is $e^{-st}f(t)$, and therefore

$$\int_0^T e^{-st} f(t) dt$$

exists for every $T > 0$. However, piecewise continuity alone does not guarantee that the improper integral

$$\int_0^\infty e^{-st} f(t) dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} f(t) dt \quad (11.13)$$

converges for s in some interval (s_0, ∞) . For example, we noted earlier that (11.13) diverges for all s if $f(t) = e^{t^2}$. Stated informally, this occurs because e^{t^2} increases too rapidly as $t \rightarrow \infty$. The next definition provides a constraint on the growth of a function that guarantees convergence of its Laplace transform for s in some interval (s_0, ∞) .

Definition 11.1.5 A function f is said to be of *exponential order* s_0 if there are constants M and t_0 such that

$$|f(t)| \leq Me^{s_0 t}, \quad t \geq t_0. \quad (11.14)$$

In situations where the specific value of s_0 is irrelevant we say simply that f is of *exponential order*.

The next theorem gives useful sufficient conditions for a function f to have a Laplace transform. The proof is sketched in Exercise 11.1.10.

Theorem 11.1.6. If f is piecewise continuous on $[0, \infty)$ and of exponential order s_0 , then $\mathcal{L}(f)$ is defined for $s > s_0$.

Remark 11.1.3: We emphasize that the conditions of Theorem 11.1.6 are sufficient, but *not necessary*, for f to have a Laplace transform. For example, Exercise 11.1.14(c) shows that f may have a Laplace transform even though f isn't of exponential order.

Example 11.1.8: If f is bounded on some interval $[t_0, \infty)$, say

$$|f(t)| \leq M, \quad t \geq t_0,$$

then (11.14) holds with $s_0 = 0$, so f is of exponential order zero. Thus, for example, $\sin \omega t$ and $\cos \omega t$ are of exponential order zero, and Theorem 11.1.6 implies that $\mathcal{L}(\sin \omega t)$ and $\mathcal{L}(\cos \omega t)$ exist for $s > 0$. This is consistent with the conclusion of Example 11.1.4.

Example 11.1.9: It can be shown that if $\lim_{t \rightarrow \infty} e^{-s_0 t} f(t)$ exists and is finite then f is of exponential order s_0 (Exercise 11.1.9). If α is any real number and $s_0 > 0$ then $f(t) = t^\alpha$ is of exponential order s_0 , since

$$\lim_{t \rightarrow \infty} e^{-s_0 t} t^\alpha = 0,$$

by L'Hôpital's rule. If $\alpha \geq 0$, f is also continuous on $[0, \infty)$. Therefore Exercise 11.1.9 and Theorem 11.1.6 imply that $\mathcal{L}(t^\alpha)$ exists for $s \geq s_0$. However, since s_0 is an arbitrary positive number, this really implies that $\mathcal{L}(t^\alpha)$ exists for all $s > 0$. This is consistent with the results of Example 11.1.2 and Exercises 11.1.6 and 11.1.8.

Example 11.1.10: Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -3e^{-t}, & t \geq 1. \end{cases}$$

Solution: Since f is defined by different formulas on $[0, 1)$ and $[1, \infty)$, we write

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt = \int_0^1 e^{-st}(1) dt + \int_1^{\infty} e^{-st}(-3e^{-t}) dt.$$

Since

$$\int_0^1 e^{-st} dt = \begin{cases} \frac{1 - e^{-s}}{s}, & s \neq 0, \\ 1, & s = 0, \end{cases}$$

and

$$\int_1^{\infty} e^{-st}(-3e^{-t}) dt = -3 \int_1^{\infty} e^{-(s+1)t} dt = -\frac{3e^{-(s+1)}}{s+1}, \quad s > -1,$$

it follows that

$$F(s) = \begin{cases} \frac{1 - e^{-s}}{s} - \frac{3e^{-(s+1)}}{s+1}, & s > -1, s \neq 0, \\ 1 - \frac{3}{e}, & s = 0. \end{cases}$$

This is consistent with Theorem 11.1.6, since

$$|f(t)| \leq 3e^{-t}, \quad t \geq 1,$$

and therefore f is of exponential order $s_0 = -1$.

Remark 11.1.4: In Section 8.4 we'll develop a more efficient method for finding Laplace transforms of piecewise continuous functions.

Example 11.1.11: We stated earlier that

$$\int_0^{\infty} e^{-st} e^{t^2} dt = \infty$$

for all s , so Theorem 11.1.6 implies that $f(t) = e^{t^2}$ is not of exponential order, since

$$\lim_{t \rightarrow \infty} \frac{e^{t^2}}{Me^{s_0 t}} = \lim_{t \rightarrow \infty} \frac{1}{M} e^{t^2 - s_0 t} = \infty,$$

so

$$e^{t^2} > Me^{s_0 t}$$

for sufficiently large values of t , for any choice of M and s_0 (Exercise 11.1.3).

11.1.7 Exercises

- Find the Laplace transforms of the following functions by evaluating the integral $F(s) = \int_0^{\infty} e^{-st} f(t) dt$.

(a) t	(b) te^{-t}	(c) $\sinh bt$
(d) $e^{2t} - 3e^t$	(e) t^2	
- Use the table of Laplace transforms to find the Laplace transforms of the following functions.

(a) $\cosh t \sin t$	(b) $\sin^2 t$	(c) $\cos^2 2t$
(d) $\cosh^2 t$	(e) $t \sinh 2t$	(f) $\sin t \cos t$
(g) $\sin\left(t + \frac{\pi}{4}\right)$	(h) $\cos 2t - \cos 3t$	(i) $\sin 2t + \cos 4t$

3. Show that

$$\int_0^{\infty} e^{-st} e^{t^2} dt = \infty$$

for every real number s .

4. Graph the following piecewise continuous functions and evaluate $f(t+)$, $f(t-)$, and $f(t)$ at each point of discontinuity.

$$(a) f(t) = \begin{cases} -t, & 0 \leq t < 2, \\ t-4, & 2 \leq t < 3, \\ 1, & t \geq 3. \end{cases} \quad (b) f(t) = \begin{cases} t^2 + 2, & 0 \leq t < 1, \\ 4, & t = 1, \\ t, & t > 1. \end{cases}$$

$$(c) f(t) = \begin{cases} \sin t, & 0 \leq t < \pi/2, \\ 2 \sin t, & \pi/2 \leq t < \pi, \\ \cos t, & t \geq \pi. \end{cases} \quad (d) f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 2, & t = 1, \\ 2-t, & 1 \leq t < 2, \\ 3, & t = 2, \\ 6, & t > 2. \end{cases}$$

5. Find the Laplace transform:

$$(a) f(t) = \begin{cases} e^{-t}, & 0 \leq t < 1, \\ e^{-2t}, & t \geq 1. \end{cases} \quad (b) f(t) = \begin{cases} 1, & 0 \leq t < 4, \\ t, & t \geq 4. \end{cases}$$

$$(c) f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 1, & t \geq 1. \end{cases} \quad (d) f(t) = \begin{cases} te^t, & 0 \leq t < 1, \\ e^t, & t \geq 1. \end{cases}$$

6. Prove that if $f(t) \leftrightarrow F(s)$ then $t^k f(t) \leftrightarrow (-1)^k F^{(k)}(s)$. HINT: Assume that it's permissible to differentiate the integral $\int_0^{\infty} e^{-st} f(t) dt$ with respect to s under the integral sign.
7. Use the known Laplace transforms

$$\mathcal{L}(e^{\lambda t} \sin \omega t) = \frac{\omega}{(s - \lambda)^2 + \omega^2} \quad \text{and} \quad \mathcal{L}(e^{\lambda t} \cos \omega t) = \frac{s - \lambda}{(s - \lambda)^2 + \omega^2}$$

and the result of Exercise 11.1.6 to find $\mathcal{L}(te^{\lambda t} \cos \omega t)$ and $\mathcal{L}(te^{\lambda t} \sin \omega t)$.

8. Use the known Laplace transform $\mathcal{L}(1) = 1/s$ and the result of Exercise 11.1.6 to show that

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad n = \text{integer}.$$

9. (a) Show that if $\lim_{t \rightarrow \infty} e^{-s_0 t} f(t)$ exists and is finite then f is of exponential order s_0 .
 (b) Show that if f is of exponential order s_0 then $\lim_{t \rightarrow \infty} e^{-st} f(t) = 0$ for all $s > s_0$.
 (c) Show that if f is of exponential order s_0 and $g(t) = f(t + \tau)$ where $\tau > 0$, then g is also of exponential order s_0 .
10. Recall the next theorem from calculus.

THEOREM A. Let g be integrable on $[0, T]$ for every $T > 0$. Suppose there's a function w defined on some interval $[\tau, \infty)$ (with $\tau \geq 0$) such that $|g(t)| \leq w(t)$ for $t \geq \tau$ and $\int_{\tau}^{\infty} w(t) dt$ converges. Then $\int_0^{\infty} g(t) dt$ converges.

Use Theorem A to show that if f is piecewise continuous on $[0, \infty)$ and of exponential order s_0 , then f has a Laplace transform $F(s)$ defined for $s > s_0$.

11. Prove: If f is piecewise continuous and of exponential order then $\lim_{s \rightarrow \infty} F(s) = 0$.
 12. Prove: If f is continuous on $[0, \infty)$ and of exponential order $s_0 > 0$, then

$$\mathcal{L}\left(\int_0^t f(\tau) d\tau\right) = \frac{1}{s} \mathcal{L}(f), \quad s > s_0.$$

HINT: Use integration by parts to evaluate the transform on the left.

13. Suppose f is piecewise continuous and of exponential order, and that $\lim_{t \rightarrow 0^+} f(t)/t$ exists. Show that

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(r) dr.$$

HINT: Use the results of Exercises 6 and 11.

14. Suppose f is piecewise continuous on $[0, \infty)$.

- (a) Prove: If the integral $g(t) = \int_0^t e^{-s_0\tau} f(\tau) d\tau$ satisfies the inequality $|g(t)| \leq M$ ($t \geq 0$), then f has a Laplace transform $F(s)$ defined for $s > s_0$. HINT: Use integration by parts to show that

$$\int_0^T e^{-st} f(t) dt = e^{-(s-s_0)T} g(T) + (s - s_0) \int_0^T e^{-(s-s_0)t} g(t) dt.$$

- (b) Show that if $\mathcal{L}(f)$ exists for $s = s_0$ then it exists for $s > s_0$. Show that the function

$$f(t) = te^{t^2} \cos(e^{t^2})$$

has a Laplace transform defined for $s > 0$, even though f isn't of exponential order.

- (c) Show that the function

$$f(t) = te^{t^2} \cos(e^{t^2})$$

has a Laplace transform defined for $s > 0$, even though f isn't of exponential order.

15. Use the table of Laplace transforms and the result of Exercise 11.1.13 to find the Laplace transforms of the following functions.

$$\begin{array}{lll} \text{(a)} \frac{\sin \omega t}{t} \quad (\omega > 0) & \text{(b)} \frac{\cos \omega t - 1}{t} \quad (\omega > 0) & \text{(c)} \frac{e^{at} - e^{bt}}{t} \\ \text{(d)} \frac{\cosh t - 1}{t} & \text{(e)} \frac{\sinh^2 t}{t} & \end{array}$$

16. The *gamma function* is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx,$$

which can be shown to converge if $\alpha > 0$.

- (a) Use integration by parts to show that

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha), \quad \alpha > 0.$$

- (b) Show that $\Gamma(n + 1) = n!$ if $n = 1, 2, 3, \dots$

- (c) From (b) and the table of Laplace transforms,

$$\mathcal{L}(t^\alpha) = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}, \quad s > 0,$$

if α is a nonnegative integer. Show that this formula is valid for any $\alpha > -1$. HINT: Change the variable of integration in the integral for $\Gamma(\alpha + 1)$.

17. Suppose f is continuous on $[0, T]$ and $f(t + T) = f(t)$ for all $t \geq 0$. (We say in this case that f is *periodic with period* T .)

- (a) Conclude from Theorem 11.1.6 that the Laplace transform of f is defined for $s > 0$. HINT: Since f is continuous on $[0, T]$ and periodic with period T , it's bounded on $[0, \infty)$.

(b) (b) Show that

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt, \quad s > 0.$$

HINT: Write

$$F(s) = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t) dt.$$

Then show that

$$\int_{nT}^{(n+1)T} e^{-st} f(t) dt = e^{-nsT} \int_0^T e^{-st} f(t) dt,$$

and recall the formula for the sum of a geometric series.

18. Use the formula given in Exercise 11.1.17(b) to find the Laplace transforms of the given periodic functions:

(a) $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 2 - t, & 1 \leq t < 2, \end{cases} \quad f(t+2) = f(t), \quad t \geq 0$

(b) $f(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1, \end{cases} \quad f(t+1) = f(t), \quad t \geq 0$

(c) $f(t) = |\sin t|$

(d) $f(t) = \begin{cases} \sin t, & 0 \leq t < \pi, \\ 0, & \pi \leq t < 2\pi, \end{cases} \quad f(t+2\pi) = f(t)$

11.2 The Inverse Laplace Transform

11.2.1 Definition of the Inverse Laplace Transform

In Section 8.1 we defined the Laplace transform of f by

$$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t) dt.$$

We'll also say that f is an *inverse Laplace Transform* of F , and write

$$f = \mathcal{L}^{-1}(F).$$

To solve differential equations with the Laplace transform, we must be able to obtain f from its transform F . There's a formula for doing this, but we can't use it because it requires the theory of functions of a complex variable. Fortunately, we can use the table of Laplace transforms to find inverse transforms that we'll need.

Example 11.2.1: Use the table of Laplace transforms to find

$$\text{(a) } \mathcal{L}^{-1}\left(\frac{1}{s^2 - 1}\right) \quad \text{and} \quad \text{(b) } \mathcal{L}^{-1}\left(\frac{s}{s^2 + 9}\right).$$

Solution (a): Setting $b = 1$ in the transform pair

$$\sinh bt \leftrightarrow \frac{b}{s^2 - b^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2 - 1}\right) = \sinh t.$$

Solution (b): Setting $\omega = 3$ in the transform pair

$$\cos \omega t \leftrightarrow \frac{s}{s^2 + \omega^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{s}{s^2 + 9}\right) = \cos 3t. \blacksquare$$

The next theorem enables us to find inverse transforms of linear combinations of transforms in the table. We omit the proof.

Theorem 11.2.1. [*Linearity Property*] If $F_1, F_2, \dots, F_n$ are Laplace transforms and $c_1, c_2, \dots, c_n$ are constants, then

$$\mathcal{L}^{-1}(c_1 F_1 + c_2 F_2 + \dots + c_n F_n) = c_1 \mathcal{L}^{-1}(F_1) + c_2 \mathcal{L}^{-1}(F_2) + \dots + c_n \mathcal{L}^{-1}(F_n).$$

Example 11.2.2: Find

$$\mathcal{L}^{-1}\left(\frac{8}{s+5} + \frac{7}{s^2+3}\right).$$

Solution: From the table of Laplace transforms in Section 8.8,,

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad \sin \omega t \leftrightarrow \frac{\omega}{s^2 + \omega^2}.$$

Theorem 11.2.1 with $a = -5$ and $\omega = \sqrt{3}$ yields

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{8}{s+5} + \frac{7}{s^2+3}\right) &= 8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) + 7\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right) \\ &= 8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) + \frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right) \\ &= 8e^{-5t} + \frac{7}{\sqrt{3}}\sin\sqrt{3}t.\end{aligned}$$

Example 11.2.3: Find

$$\mathcal{L}^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).$$

Solution: Completing the square in the denominator yields

$$\frac{3s+8}{s^2+2s+5} = \frac{3s+8}{(s+1)^2+4}.$$

Because of the form of the denominator, we consider the transform pairs

$$e^{-t}\cos 2t \leftrightarrow \frac{s+1}{(s+1)^2+4} \quad \text{and} \quad e^{-t}\sin 2t \leftrightarrow \frac{2}{(s+1)^2+4},$$

and write

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{3s+8}{(s+1)^2+4}\right) &= \mathcal{L}^{-1}\left(\frac{3s+3}{(s+1)^2+4}\right) + \mathcal{L}^{-1}\left(\frac{5}{(s+1)^2+4}\right) \\ &= 3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) + \frac{5}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right) \\ &= e^{-t}(3\cos 2t + \frac{5}{2}\sin 2t).\end{aligned}$$

Remark 11.2.1: We'll often write inverse Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms in Section 8.8.

11.2.2 Inverse Laplace Transforms of Rational Functions

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function

$$F(s) = \frac{P(s)}{Q(s)},$$

where P and Q are polynomials in s with no common factors. Since it can be shown that $\lim_{s \rightarrow \infty} F(s) = 0$ if F is a Laplace transform, we need only consider the case where $\text{degree}(P) < \text{degree}(Q)$. To obtain $\mathcal{L}^{-1}(F)$, we find the partial fraction expansion of F , obtain inverse transforms of the individual terms in the expansion from the table of Laplace transforms, and use the linearity property of the inverse transform. The next two examples illustrate this.

Example 11.2.4: Find the inverse Laplace transform of

$$F(s) = \frac{3s+2}{s^2-3s+2}. \quad (11.15)$$

Solution: (METHOD 1) Factoring the denominator in (11.15) yields

$$F(s) = \frac{3s + 2}{(s - 1)(s - 2)}. \quad (11.16)$$

The form for the partial fraction expansion is

$$\frac{3s + 2}{(s - 1)(s - 2)} = \frac{A}{s - 1} + \frac{B}{s - 2}. \quad (11.17)$$

Multiplying this by $(s - 1)(s - 2)$ yields

$$3s + 2 = (s - 2)A + (s - 1)B.$$

Setting $s = 2$ yields $B = 8$ and setting $s = 1$ yields $A = -5$. Therefore

$$F(s) = -\frac{5}{s - 1} + \frac{8}{s - 2}$$

and

$$\mathcal{L}^{-1}(F) = -5\mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s - 2}\right) = -5e^t + 8e^{2t}.$$

Solution: (METHOD 2) We don't really have to multiply (11.17) by $(s - 1)(s - 2)$ to compute A and B . We can obtain A by simply ignoring the factor $s - 1$ in the denominator of (11.16) and setting $s = 1$ elsewhere; thus,

$$A = \left. \frac{3s + 2}{s - 2} \right|_{s=1} = \frac{3 \cdot 1 + 2}{1 - 2} = -5. \quad (11.18)$$

Similarly, we can obtain B by ignoring the factor $s - 2$ in the denominator of (11.16) and setting $s = 2$ elsewhere; thus,

$$B = \left. \frac{3s + 2}{s - 1} \right|_{s=2} = \frac{3 \cdot 2 + 2}{2 - 1} = 8. \quad (11.19)$$

To justify this, we observe that multiplying (11.17) by $s - 1$ yields

$$\frac{3s + 2}{s - 2} = A + (s - 1)\frac{B}{s - 2},$$

and setting $s = 1$ leads to (11.18). Similarly, multiplying (11.17) by $s - 2$ yields

$$\frac{3s + 2}{s - 1} = (s - 2)\frac{A}{s - 2} + B$$

and setting $s = 2$ leads to (11.19). (It isn't necessary to write the last two equations. We wrote them only to justify the shortcut procedure indicated in (11.18) and (11.19).) ■

The shortcut employed in the second solution of Example 11.2.4 is *Heaviside's method*. The next theorem states this method formally. For a proof and an extension of this theorem, see Exercise 11.2.10.

Theorem 11.2.2. *Suppose*

$$F(s) = \frac{P(s)}{(s - s_1)(s - s_2) \cdots (s - s_n)}, \quad (11.20)$$

where $s_1, s_2, \dots, s_n$ are distinct and P is a polynomial of degree less than n . Then

$$F(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \cdots + \frac{A_n}{s - s_n},$$

where A_i can be computed from (11.20) by ignoring the factor $s - s_i$ and setting $s = s_i$ elsewhere.

Example 11.2.5: Find the inverse Laplace transform of

$$F(s) = \frac{6 + (s+1)(s^2 - 5s + 11)}{s(s-1)(s-2)(s+1)}. \quad (11.21)$$

Solution: The partial fraction expansion of (11.21) is of the form

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{s+1}. \quad (11.22)$$

To find A , we ignore the factor s in the denominator of (11.21) and set $s = 0$ elsewhere. This yields

$$A = \frac{6 + (1)(11)}{(-1)(-2)(1)} = \frac{17}{2}.$$

Similarly, the other coefficients are given by

$$B = \frac{6 + (2)(7)}{(1)(-1)(2)} = -10,$$

$$C = \frac{6 + 3(5)}{2(1)(3)} = \frac{7}{2},$$

and

$$D = \frac{6}{(-1)(-2)(-3)} = -1.$$

Therefore

$$F(s) = \frac{17}{2} \frac{1}{s} - \frac{10}{s-1} + \frac{7}{2} \frac{1}{s-2} - \frac{1}{s+1}$$

and

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \frac{17}{2} \mathcal{L}^{-1}\left(\frac{1}{s}\right) - 10 \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{7}{2} \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) \\ &= \frac{17}{2} - 10e^t + \frac{7}{2}e^{2t} - e^{-t}. \end{aligned}$$

Remark 11.2.2: We didn't "multiply out" the numerator in (11.21) before computing the coefficients in (11.22), since it wouldn't simplify the computations.

Example 11.2.6: Find the inverse Laplace transform of

$$F(s) = \frac{8 - (s+2)(4s+10)}{(s+1)(s+2)^2}. \quad (11.23)$$

Solution: The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}. \quad (11.24)$$

Because of the repeated factor $(s+2)^2$ in (11.23), Heaviside's method doesn't work. Instead, we find a common denominator in (11.24). This yields

$$F(s) = \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2}. \quad (11.25)$$

If (11.23) and (11.25) are to be equivalent, then

$$A(s+2)^2 + B(s+1)(s+2) + C(s+1) = 8 - (s+2)(4s+10). \quad (11.26)$$

The two sides of this equation are polynomials of degree two. From a theorem of algebra, they will be equal for all s if they are equal for any three distinct values of s . We may determine A , B and C by choosing convenient values of s .

The left side of (11.26) suggests that we take $s = -2$ to obtain $C = -8$, and $s = -1$ to obtain $A = 2$. We can now choose any third value of s to determine B . Taking $s = 0$ yields $4A + 2B + C = -12$. Since $A = 2$ and $C = -8$ this implies that $B = -6$. Therefore

$$F(s) = \frac{2}{s+1} - \frac{6}{s+2} - \frac{8}{(s+2)^2}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &= 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - 6\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 8\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) \\ &= 2e^{-t} - 6e^{-2t} - 8te^{-2t}.\end{aligned}$$

Example 11.2.7: Find the inverse Laplace transform of

$$F(s) = \frac{s^2 - 5s + 7}{(s+2)^3}.$$

Solution: The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}.$$

The easiest way to obtain A , B , and C is to expand the numerator in powers of $s+2$. This yields

$$s^2 - 5s + 7 = [(s+2) - 2]^2 - 5[(s+2) - 2] + 7 = (s+2)^2 - 9(s+2) + 21.$$

Therefore

$$\begin{aligned}F(s) &= \frac{(s+2)^2 - 9(s+2) + 21}{(s+2)^3} \\ &= \frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 9\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{21}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+2)^3}\right) \\ &= e^{-2t} \left(1 - 9t + \frac{21}{2}t^2\right).\end{aligned}$$

Example 11.2.8: Find the inverse Laplace transform of

$$F(s) = \frac{1 - s(5 + 3s)}{s[(s+1)^2 + 1]}. \quad (11.27)$$

Solution: One form for the partial fraction expansion of F is

$$F(s) = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 1}. \quad (11.28)$$

However, we see from the table of Laplace transforms that the inverse transform of the second fraction on the right of (11.28) will be a linear combination of the inverse transforms

$$e^{-t} \cos t \quad \text{and} \quad e^{-t} \sin t$$

of

$$\frac{s+1}{(s+1)^2+1} \quad \text{and} \quad \frac{1}{(s+1)^2+1}$$

respectively. Therefore, instead of (11.28) we write

$$F(s) = \frac{A}{s} + \frac{B(s+1) + C}{(s+1)^2+1}. \quad (11.29)$$

Finding a common denominator yields

$$F(s) = \frac{A[(s+1)^2+1] + B(s+1)s + Cs}{s[(s+1)^2+1]}. \quad (11.30)$$

If (11.27) and (11.30) are to be equivalent, then

$$A[(s+1)^2+1] + B(s+1)s + Cs = 1 - s(5+3s).$$

This is true for all s if it's true for three distinct values of s . Choosing $s = 0, -1$, and 1 yields the system

$$\begin{aligned} 2A &= 1 \\ A - C &= 3 \\ 5A + 2B + C &= -7. \end{aligned}$$

Solving this system yields

$$A = \frac{1}{2}, \quad B = -\frac{7}{2}, \quad C = -\frac{5}{2}.$$

Hence, from (11.29),

$$F(s) = \frac{1}{2s} - \frac{7}{2} \frac{s+1}{(s+1)^2+1} - \frac{5}{2} \frac{1}{(s+1)^2+1}.$$

Therefore

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \frac{1}{2} \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{7}{2} \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - \frac{5}{2} \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) \\ &= \frac{1}{2} - \frac{7}{2} e^{-t} \cos t - \frac{5}{2} e^{-t} \sin t. \end{aligned}$$

Example 11.2.9: Find the inverse Laplace transform of

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)}. \quad (11.31)$$

Solution: The form for the partial fraction expansion is

$$F(s) = \frac{A+Bs}{s^2+1} + \frac{C+Ds}{s^2+4}.$$

The coefficients A, B, C and D can be obtained by finding a common denominator and equating the resulting numerator to the numerator in (11.31). However, since there's no first power of s in the denominator of (11.31), there's an easier way: the expansion of

$$F_1(s) = \frac{1}{(s^2+1)(s^2+4)}$$

can be obtained quickly by using Heaviside's method to expand

$$\frac{1}{(x+1)(x+4)} = \frac{1}{3} \left(\frac{1}{x+1} - \frac{1}{x+4} \right)$$

and then setting $x = s^2$ to obtain

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3} \left(\frac{1}{s^2+1} - \frac{1}{s^2+4} \right).$$

Multiplying this by $8 + 3s$ yields

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)} = \frac{1}{3} \left(\frac{8+3s}{s^2+1} - \frac{8+3s}{s^2+4} \right).$$

Therefore

$$\mathcal{L}^{-1}(F) = \frac{8}{3} \sin t + \cos t - \frac{4}{3} \sin 2t - \cos 2t.$$

Some software packages that do symbolic algebra can find partial fraction expansions very easily. We recommend that you use such a package if one is available to you, but only after you've done enough partial fraction expansions on your own to master the technique.

11.2.3 Exercises

1. Use the table of Laplace transforms to find the inverse Laplace transform.

(a) $\frac{3}{(s-7)^4}$

(b) $\frac{2s-4}{s^2-4s+13}$

(c) $\frac{1}{s^2+4s+20}$

(d) $\frac{2}{s^2+9}$

(e) $\frac{s^2-1}{(s^2+1)^2}$

(f) $\frac{1}{(s-2)^2-4}$

(g) $\frac{12s-24}{(s^2-4s+85)^2}$

(h) $\frac{2}{(s-3)^2-9}$

(i) $\frac{s^2-4s+3}{(s^2-4s+5)^2}$

2. Use Theorem 11.2.1 and the table of Laplace transforms to find the inverse Laplace transform.

(a) $\frac{2s+3}{(s-7)^4}$

(b) $\frac{s^2-1}{(s-2)^6}$

(c) $\frac{s+5}{s^2+6s+18}$

(d) $\frac{2s+1}{s^2+9}$

(e) $\frac{s}{s^2+2s+1}$

(f) $\frac{s+1}{s^2-9}$

(g) $\frac{s^3+2s^2-s-3}{(s+1)^4}$

(h) $\frac{2s+3}{(s-1)^2+4}$

(i) $\frac{1}{s} - \frac{s}{s^2+1}$

(j) $\frac{3s+4}{s^2-1}$

(k) $\frac{3}{s-1} + \frac{4s+1}{s^2+9}$

(l) $\frac{3}{(s+2)^2} - \frac{2s+6}{s^2+4}$

3. Use Heaviside's method to find the inverse Laplace transform.

(a) $\frac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)}$

(b) $\frac{7+(s+4)(18-3s)}{(s-3)(s-1)(s+4)}$

(c) $\frac{2+(s-2)(3-2s)}{(s-2)(s+2)(s-3)}$

(d) $\frac{3-(s-1)(s+1)}{(s+4)(s-2)(s-1)}$

(e) $\frac{3+(s-2)(10-2s-s^2)}{(s-2)(s+2)(s-1)(s+3)}$

(f) $\frac{3+(s-3)(2s^2+s-21)}{(s-3)(s-1)(s+4)(s-2)}$

4. Find the inverse Laplace transform.

$$(a) \frac{2+3s}{(s^2+1)(s+2)(s+1)}$$

$$(b) \frac{3s^2+2s+1}{(s^2+1)(s^2+2s+2)}$$

$$(c) \frac{3s+2}{(s-2)(s^2+2s+5)}$$

$$(d) \frac{3s^2+2s+1}{(s-1)^2(s+2)(s+3)}$$

$$(e) \frac{2s^2+s+3}{(s-1)^2(s+2)^2}$$

$$(f) \frac{3s+2}{(s^2+1)(s-1)^2}$$

5. Use the method of Example 11.2.9 to find the inverse Laplace transform.

$$(a) \frac{3s+2}{(s^2+4)(s^2+9)}$$

$$(b) \frac{-4s+1}{(s^2+1)(s^2+16)}$$

$$(c) \frac{5s+3}{(s^2+1)(s^2+4)}$$

$$(d) \frac{-s+1}{(4s^2+1)(s^2+1)}$$

$$(e) \frac{17s-34}{(s^2+16)(16s^2+1)}$$

$$(f) \frac{2s-1}{(4s^2+1)(9s^2+1)}$$

6. Find the inverse Laplace transform.

$$(a) \frac{17s-15}{(s^2-2s+5)(s^2+2s+10)}$$

$$(b) \frac{8s+56}{(s^2-6s+13)(s^2+2s+5)}$$

$$(c) \frac{s+9}{(s^2+4s+5)(s^2-4s+13)}$$

$$(d) \frac{3s-2}{(s^2-4s+5)(s^2-6s+13)}$$

$$(e) \frac{3s-1}{(s^2-2s+2)(s^2+2s+5)}$$

$$(f) \frac{20s+40}{(4s^2-4s+5)(4s^2+4s+5)}$$

7. Find the inverse Laplace transform.

$$(a) \frac{1}{s(s^2+1)}$$

$$(b) \frac{1}{(s-1)(s^2-2s+17)}$$

$$(c) \frac{3s+2}{(s-2)(s^2+2s+10)}$$

$$(d) \frac{34-17s}{(2s-1)(s^2-2s+5)}$$

$$(e) \frac{s+2}{(s-3)(s^2+2s+5)}$$

$$(f) \frac{2s-2}{(s-2)(s^2+2s+10)}$$

8. Find the inverse Laplace transform.

$$(a) \frac{2s+1}{(s^2+1)(s-1)(s-3)}$$

$$(b) \frac{s+2}{(s^2+2s+2)(s^2-1)}$$

$$(c) \frac{2s-1}{(s^2-2s+2)(s+1)(s-2)}$$

$$(d) \frac{s-6}{(s^2-1)(s^2+4)}$$

$$(e) \frac{2s-3}{s(s-2)(s^2-2s+5)}$$

$$(f) \frac{5s-15}{(s^2-4s+13)(s-2)(s-1)}$$

9. Given that $f(t) \leftrightarrow F(s)$, find the inverse Laplace transform of $F(as-b)$, where $a > 0$.

10. (a) If $s_1, s_2, \dots, s_n$ are distinct and P is a polynomial of degree less than n , then

$$\frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)} = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n}.$$

Multiply through by $s-s_i$ to show that A_i can be obtained by ignoring the factor $s-s_i$ on the left and setting $s=s_i$ elsewhere.

(b) Suppose P and Q_1 are polynomials such that $\text{degree}(P) \leq \text{degree}(Q_1)$ and $Q_1(s_1) \neq 0$. Show that the coefficient of $1/(s-s_1)$ in the partial fraction expansion of

$$F(s) = \frac{P(s)}{(s-s_1)Q_1(s)}$$

is $P(s_1)/Q_1(s_1)$.

(c) Explain how the results of (a) and (b) are related.

11.3 Solution of Initial Value Problems

11.3.1 Laplace Transforms of Derivatives

In the rest of this chapter we'll use the Laplace transform to solve initial value problems for constant coefficient second order equations. To do this, we must know how the Laplace transform of f' is related to the Laplace transform of f . The next theorem answers this question.

Theorem 11.3.1. *Suppose f is continuous on $[0, \infty)$ and of exponential order s_0 , and f' is piecewise continuous on $[0, \infty)$. Then f and f' have Laplace transforms for $s > s_0$, and*

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0). \quad (11.32)$$

Proof. We know from Theorem 8.1.6 that $\mathcal{L}(f)$ is defined for $s > s_0$. We first consider the case where f' is continuous on $[0, \infty)$. Integration by parts yields

$$\begin{aligned} \int_0^T e^{-st} f'(t) dt &= e^{-st} f(t) \Big|_0^T + s \int_0^T e^{-st} f(t) dt \\ &= e^{-sT} f(T) - f(0) + s \int_0^T e^{-st} f(t) dt \end{aligned} \quad (11.33)$$

for any $T > 0$. Since f is of exponential order s_0 , $\lim_{T \rightarrow \infty} e^{-sT} f(T) = 0$ and the last integral in (11.33) converges as $T \rightarrow \infty$ if $s > s_0$. Therefore

$$\begin{aligned} \int_0^\infty e^{-st} f'(t) dt &= -f(0) + s \int_0^\infty e^{-st} f(t) dt \\ &= -f(0) + s\mathcal{L}(f), \end{aligned}$$

which proves (11.32). Now suppose $T > 0$ and f' is only piecewise continuous on $[0, T]$, with discontinuities at $t_1 < t_2 < \cdots < t_{n-1}$. For convenience, let $t_0 = 0$ and $t_n = T$. Integrating by parts yields

$$\begin{aligned} \int_{t_{i-1}}^{t_i} e^{-st} f'(t) dt &= e^{-st} f(t) \Big|_{t_{i-1}}^{t_i} + s \int_{t_{i-1}}^{t_i} e^{-st} f(t) dt \\ &= e^{-st_i} f(t_i) - e^{-st_{i-1}} f(t_{i-1}) + s \int_{t_{i-1}}^{t_i} e^{-st} f(t) dt. \end{aligned}$$

Summing both sides of this equation from $i = 1$ to n and noting that

$$\begin{aligned} &(e^{-st_1} f(t_1) - e^{-st_0} f(t_0)) + (e^{-st_2} f(t_2) - e^{-st_1} f(t_1)) + \cdots + (e^{-st_N} f(t_N) - e^{-st_{N-1}} f(t_{N-1})) \\ &= e^{-st_N} f(t_N) - e^{-st_0} f(t_0) = e^{-sT} f(T) - f(0) \end{aligned}$$

yields (11.33), so (11.32) follows as before.

Example 11.3.1: In Example 11.1.4 we saw that

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}.$$

Applying (11.32) with $f(t) = \cos \omega t$ shows that

$$\mathcal{L}(-\omega \sin \omega t) = s \frac{s}{s^2 + \omega^2} - 1 = -\frac{\omega^2}{s^2 + \omega^2}.$$

Therefore

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2},$$

which agrees with the corresponding result obtained in 11.1.4. ■

In Section 2.1 we showed that the solution of the initial value problem

$$y' = ay, \quad y(0) = y_0, \quad (11.34)$$

is $y = y_0 e^{at}$. We'll now obtain this result by using the Laplace transform.

Let $Y(s) = \mathcal{L}(y)$ be the Laplace transform of the unknown solution of (11.34). Taking Laplace transforms of both sides of (11.34) yields

$$\mathcal{L}(y') = \mathcal{L}(ay),$$

which, by Theorem 11.3.1, can be rewritten as

$$s\mathcal{L}(y) - y(0) = a\mathcal{L}(y),$$

or

$$sY(s) - y_0 = aY(s).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{y_0}{s - a},$$

so

$$y = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{y_0}{s - a}\right) = y_0 \mathcal{L}^{-1}\left(\frac{1}{s - a}\right) = y_0 e^{at},$$

which agrees with the known result.

We need the next theorem to solve second order differential equations using the Laplace transform.

Theorem 11.3.2. *Suppose f and f' are continuous on $[0, \infty)$ and of exponential order s_0 , and that f'' is piecewise continuous on $[0, \infty)$. Then f , f' , and f'' have Laplace transforms for $s > s_0$,*

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0), \quad (11.35)$$

and

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - f'(0) - sf(0). \quad (11.36)$$

Proof. Theorem 11.3.1 implies that $\mathcal{L}(f')$ exists and satisfies (11.35) for $s > s_0$. To prove that $\mathcal{L}(f'')$ exists and satisfies (11.36) for $s > s_0$, we first apply Theorem 11.3.1 to $g = f'$. Since g satisfies the hypotheses of Theorem 11.3.1, we conclude that $\mathcal{L}(g')$ is defined and satisfies

$$\mathcal{L}(g') = s\mathcal{L}(g) - g(0)$$

for $s > s_0$. However, since $g' = f''$, this can be rewritten as

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0).$$

Substituting (11.35) into this yields (11.36).

11.3.2 Solving Second Order Equations with the Laplace Transform

We'll now use the Laplace transform to solve initial value problems for second order equations.

Example 11.3.2: Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \quad y'(0) = 3. \quad (11.37)$$

Solution: Taking Laplace transforms of both sides of the differential equation in (11.37) yields

$$\mathcal{L}(y'' - 6y' + 5y) = \mathcal{L}(3e^{2t}) = \frac{3}{s-2},$$

which we rewrite as

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 5\mathcal{L}(y) = \frac{3}{s-2}. \quad (11.38)$$

Now denote $\mathcal{L}(y) = Y(s)$. Theorem 11.3.2 and the initial conditions in (11.37) imply that

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 2$$

and

$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0) = s^2Y(s) - 3 - 2s.$$

Substituting from the last two equations into (11.38) yields

$$(s^2Y(s) - 3 - 2s) - 6(sY(s) - 2) + 5Y(s) = \frac{3}{s-2}.$$

Therefore

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + (3 + 2s) + 6(-2), \quad (11.39)$$

so

$$(s-5)(s-1)Y(s) = \frac{3 + (s-2)(2s-9)}{s-2},$$

and

$$Y(s) = \frac{3 + (s-2)(2s-9)}{(s-2)(s-5)(s-1)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = -\frac{1}{s-2} + \frac{1}{2} \frac{1}{s-5} + \frac{5}{2} \frac{1}{s-1},$$

and taking the inverse transform of this yields

$$y = -e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^t$$

as the solution of (11.37). ■

It isn't necessary to write all the steps that we used to obtain (11.39). To see how to avoid this, let's apply the method of Example 11.3.2 to the general initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \quad (11.40)$$

Taking Laplace transforms of both sides of the differential equation in (11.40) yields

$$a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = F(s). \quad (11.41)$$

Now let $Y(s) = \mathcal{L}(y)$. Theorem 11.3.2 and the initial conditions in (11.40) imply that

$$\mathcal{L}(y') = sY(s) - k_0 \quad \text{and} \quad \mathcal{L}(y'') = s^2Y(s) - k_1 - k_0s.$$

Substituting these into (11.41) yields

$$a(s^2Y(s) - k_1 - k_0s) + b(sY(s) - k_0) + cY(s) = F(s). \quad (11.42)$$

The coefficient of $Y(s)$ on the left is the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation for (11.40). Using this and moving the terms involving k_0 and k_1 to the right side of (11.42) yields

$$p(s)Y(s) = F(s) + a(k_1 + k_0s) + bk_0. \quad (11.43)$$

This equation corresponds to (11.39) of Example 11.3.2. Having established the form of this equation in the general case, it is preferable to go directly from the initial value problem to this equation. You may find it easier to remember (11.43) rewritten as

$$p(s)Y(s) = F(s) + a(y'(0) + sy(0)) + by(0). \quad (11.44)$$

Example 11.3.3: Use the Laplace transform to solve the initial value problem

$$2y'' + 3y' + y = 8e^{-2t}, \quad y(0) = -4, \quad y'(0) = 2. \quad (11.45)$$

Solution: The characteristic polynomial is

$$p(s) = 2s^2 + 3s + 1 = (2s + 1)(s + 1)$$

and

$$F(s) = \mathcal{L}(8e^{-2t}) = \frac{8}{s + 2},$$

so (11.44) becomes

$$(2s + 1)(s + 1)Y(s) = \frac{8}{s + 2} + 2(2 - 4s) + 3(-4).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{4(1 - (s + 2)(s + 1))}{(s + 1/2)(s + 1)(s + 2)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = \frac{4}{3} \frac{1}{s + 1/2} - \frac{8}{s + 1} + \frac{8}{3} \frac{1}{s + 2},$$

so the solution of (11.45) is

$$y = \mathcal{L}^{-1}(Y(s)) = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$$

(Figure 11.4).

Example 11.3.4: Solve the initial value problem

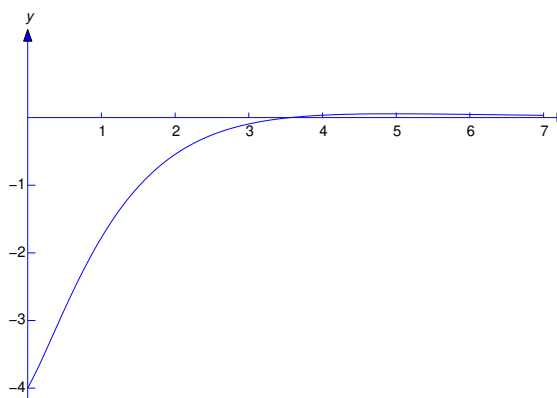
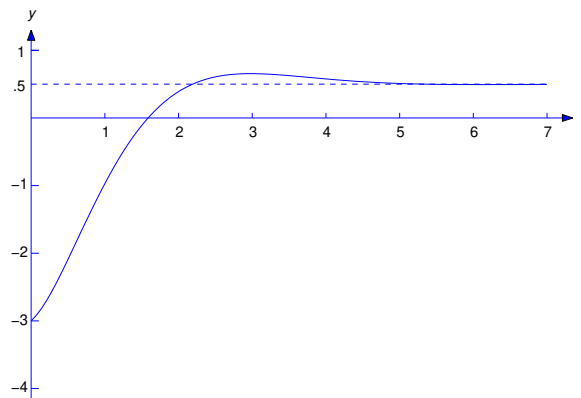
$$y'' + 2y' + 2y = 1, \quad y(0) = -3, \quad y'(0) = 1. \quad (11.46)$$

Solution: The characteristic polynomial is

$$p(s) = s^2 + 2s + 2 = (s + 1)^2 + 1$$

and

$$F(s) = \mathcal{L}(1) = \frac{1}{s},$$

Figure 11.4: $y = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$ Figure 11.5: $y = \frac{1}{2} - \frac{7}{2}e^{-t} \cos t - \frac{5}{2}e^{-t} \sin t$

so (11.44) becomes

$$[(s+1)^2 + 1] Y(s) = \frac{1}{s} + 1 \cdot (1 - 3s) + 2(-3).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{1 - s(5 + 3s)}{s[(s+1)^2 + 1]}.$$

In Example 11.2.8 we found the inverse transform of this function to be

$$y = \frac{1}{2} - \frac{7}{2}e^{-t} \cos t - \frac{5}{2}e^{-t} \sin t$$

(Figure 11.5), which is therefore the solution of (11.46).

Remark 11.3.1: In our examples we applied Theorems 11.3.1 and 11.3.2 without verifying that the unknown function y satisfies their hypotheses. This is characteristic of the formal manipulative way in which the Laplace transform is used to solve differential equations. Any doubts about the validity of the method for solving a given equation can be resolved by verifying that the resulting function y is the solution of the given problem.

11.3.3 Exercises

In Exercises 11.3.1–11.3.31 use the Laplace transform to solve the initial value problem.

1. $y'' + 3y' + 2y = e^t$, $y(0) = 1$, $y'(0) = -6$
2. $y'' - y' - 6y = 2$, $y(0) = 1$, $y'(0) = 0$
3. $y'' + y' - 2y = 2e^{3t}$, $y(0) = -1$, $y'(0) = 4$
4. $y'' - 4y = 2e^{3t}$, $y(0) = 1$, $y'(0) = -1$
5. $y'' + y' - 2y = e^{3t}$, $y(0) = 1$, $y'(0) = -1$
6. $y'' + 3y' + 2y = 6e^t$, $y(0) = 1$, $y'(0) = -1$
7. $y'' + y = \sin 2t$, $y(0) = 0$, $y'(0) = 1$
8. $y'' - 3y' + 2y = 2e^{3t}$, $y(0) = 1$, $y'(0) = -1$
9. $y'' - 3y' + 2y = e^{4t}$, $y(0) = 1$, $y'(0) = -2$
10. $y'' - 3y' + 2y = e^{3t}$, $y(0) = -1$, $y'(0) = -4$

11. $y'' + 3y' + 2y = 2e^t$, $y(0) = 0$, $y'(0) = -1$
12. $y'' + y' - 2y = -4$, $y(0) = 2$, $y'(0) = 3$
13. $y'' + 4y = 4$, $y(0) = 0$, $y'(0) = 1$
14. $y'' - y' - 6y = 2$, $y(0) = 1$, $y'(0) = 0$
15. $y'' + 3y' + 2y = e^t$, $y(0) = 0$, $y'(0) = 1$
16. $y'' - y = 1$, $y(0) = 1$, $y'(0) = 0$
17. $y'' + 4y = 3 \sin t$, $y(0) = 1$, $y'(0) = -1$
18. $y'' + y' = 2e^{3t}$, $y(0) = -1$, $y'(0) = 4$
19. $y'' + y = 1$, $y(0) = 2$, $y'(0) = 0$
20. $y'' + y = t$, $y(0) = 0$, $y'(0) = 2$
21. $y'' + y = t - 3 \sin 2t$, $y(0) = 1$, $y'(0) = -3$
22. $y'' + 5y' + 6y = 2e^{-t}$, $y(0) = 1$, $y'(0) = 3$
23. $y'' + 2y' + y = 6 \sin t - 4 \cos t$, $y(0) = -1$, $y'(0) = 1$
24. $y'' - 2y' - 3y = 10 \cos t$, $y(0) = 2$, $y'(0) = 7$
25. $y'' + y = 4 \sin t + 6 \cos t$, $y(0) = -6$, $y'(0) = 2$
26. $y'' + 4y = 8 \sin 2t + 9 \cos t$, $y(0) = 1$, $y'(0) = 0$
27. $y'' - 5y' + 6y = 10e^t \cos t$, $y(0) = 2$, $y'(0) = 1$
28. $y'' + 2y' + 2y = 2t$, $y(0) = 2$, $y'(0) = -7$
29. $y'' - 2y' + 2y = 5 \sin t + 10 \cos t$, $y(0) = 1$, $y'(0) = 2$
30. $y'' + 4y' + 13y = 10e^{-t} - 36e^t$, $y(0) = 0$, $y'(0) = -16$
31. $y'' + 4y' + 5y = e^{-t}(\cos t + 3 \sin t)$, $y(0) = 0$, $y'(0) = 4$
32. $2y'' - 3y' - 2y = 4e^t$, $y(0) = 1$, $y'(0) = -2$
33. $6y'' - y' - y = 3e^{2t}$, $y(0) = 0$, $y'(0) = 0$
34. $2y'' + 2y' + y = 2t$, $y(0) = 1$, $y'(0) = -1$
35. $4y'' - 4y' + 5y = 4 \sin t - 4 \cos t$, $y(0) = 0$, $y'(0) = 11/17$
36. $4y'' + 4y' + y = 3 \sin t + \cos t$, $y(0) = 2$, $y'(0) = -1$
37. $9y'' + 6y' + y = 3e^{3t}$, $y(0) = 0$, $y'(0) = -3$
38. Suppose a, b , and c are constants and $a \neq 0$. Let

$$y_1 = \mathcal{L}^{-1} \left(\frac{as + b}{as^2 + bs + c} \right) \quad \text{and} \quad y_2 = \mathcal{L}^{-1} \left(\frac{a}{as^2 + bs + c} \right).$$

Show that

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

HINT: Use the Laplace transform to solve the initial value problems

$$\begin{aligned} ay'' + by' + cy &= 0, & y(0) &= 1, & y'(0) &= 0 \\ ay'' + by' + cy &= 0, & y(0) &= 0, & y'(0) &= 1. \end{aligned}$$

11.4 The Unit Step Function

In the next section we'll consider initial value problems

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

where a , b , and c are constants and f is piecewise continuous. In this section we'll develop procedures for using the table of Laplace transforms to find Laplace transforms of piecewise continuous functions, and to find the piecewise continuous inverses of Laplace transforms.

Example 11.4.1: Use the table of Laplace transforms to find the Laplace transform of

$$f(t) = \begin{cases} 2t + 1, & 0 \leq t < 2, \\ 3t, & t \geq 2 \end{cases} \quad (11.47)$$

(Figure 11.6).

Solution: Since the formula for f changes at $t = 2$, we write

$$\begin{aligned} \mathcal{L}(f) &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^2 e^{-st} (2t + 1) dt + \int_2^{\infty} e^{-st} (3t) dt. \end{aligned} \quad (11.48)$$

To relate the first term to a Laplace transform, we add and subtract

$$\int_2^{\infty} e^{-st} (2t + 1) dt$$

in (11.48) to obtain

$$\begin{aligned} \mathcal{L}(f) &= \int_0^{\infty} e^{-st} (2t + 1) dt + \int_2^{\infty} e^{-st} (3t - 2t - 1) dt \\ &= \int_0^{\infty} e^{-st} (2t + 1) dt + \int_2^{\infty} e^{-st} (t - 1) dt \\ &= \mathcal{L}(2t + 1) + \int_2^{\infty} e^{-st} (t - 1) dt. \end{aligned} \quad (11.49)$$

To relate the last integral to a Laplace transform, we make the change of variable $x = t - 2$ and rewrite the integral as

$$\begin{aligned} \int_2^{\infty} e^{-st} (t - 1) dt &= \int_0^{\infty} e^{-s(x+2)} (x + 1) dx \\ &= e^{-2s} \int_0^{\infty} e^{-sx} (x + 1) dx. \end{aligned}$$

Since the symbol used for the variable of integration has no effect on the value of a definite integral, we can now replace x by the more standard t and write

$$\int_2^{\infty} e^{-st} (t - 1) dt = e^{-2s} \int_0^{\infty} e^{-st} (t + 1) dt = e^{-2s} \mathcal{L}(t + 1).$$

This and (11.49) imply that

$$\mathcal{L}(f) = \mathcal{L}(2t + 1) + e^{-2s} \mathcal{L}(t + 1).$$

Now we can use the table of Laplace transforms to find that

$$\mathcal{L}(f) = \frac{2}{s^2} + \frac{1}{s} + e^{-2s} \left(\frac{1}{s^2} + \frac{1}{s} \right). \blacksquare$$

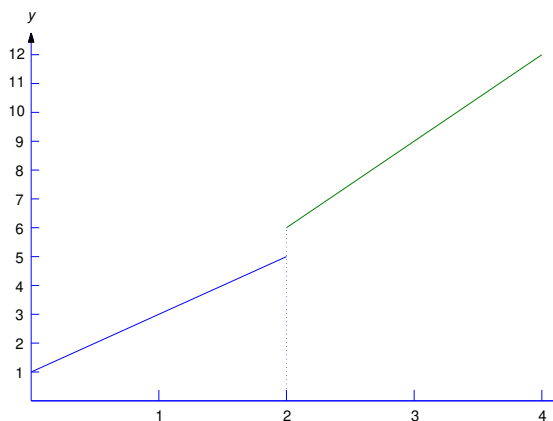
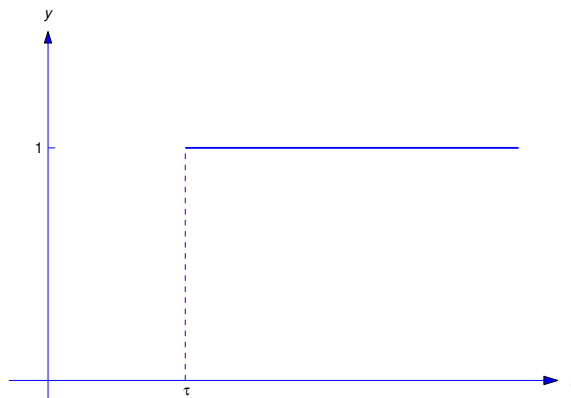


Figure 11.6: The piecewise continuous function (11.47)

Figure 11.7: $y = u(t - \tau)$

11.4.1 Laplace Transforms of Piecewise Continuous Functions

We'll now develop the method of Example 11.4.1 into a systematic way to find the Laplace transform of a piecewise continuous function. It is convenient to introduce the *unit step function*, defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases} \quad (11.50)$$

Thus, $u(t)$ “steps” from the constant value 0 to the constant value 1 at $t = 0$. If we replace t by $t - \tau$ in (11.50), then

$$u(t - \tau) = \begin{cases} 0, & t < \tau, \\ 1, & t \geq \tau \end{cases};$$

that is, the step now occurs at $t = \tau$ (Figure 11.7).

The step function enables us to represent piecewise continuous functions conveniently. For example, consider the function

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t \geq t_1, \end{cases} \quad (11.51)$$

where we assume that f_0 and f_1 are defined on $[0, \infty)$, even though they equal f only on the indicated intervals. This assumption enables us to rewrite (11.51) as

$$f(t) = f_0(t) + u(t - t_1) (f_1(t) - f_0(t)). \quad (11.52)$$

To verify this, note that if $t < t_1$ then $u(t - t_1) = 0$ and (11.52) becomes

$$f(t) = f_0(t) + (0) (f_1(t) - f_0(t)) = f_0(t).$$

If $t \geq t_1$ then $u(t - t_1) = 1$ and (11.52) becomes

$$f(t) = f_0(t) + (1) (f_1(t) - f_0(t)) = f_1(t).$$

We need the next theorem to show how (11.52) can be used to find $\mathcal{L}(f)$.

Theorem 11.4.1. *Let g be defined on $[0, \infty)$. Suppose $\tau \geq 0$ and $\mathcal{L}(g(t + \tau))$ exists for $s > s_0$. Then $\mathcal{L}(u(t - \tau)g(t))$ exists for $s > s_0$, and*

$$\mathcal{L}(u(t - \tau)g(t)) = e^{-s\tau} \mathcal{L}(g(t + \tau)).$$

Proof. By definition,

$$\mathcal{L}(u(t-\tau)g(t)) = \int_0^{\infty} e^{-st} u(t-\tau)g(t) dt.$$

From this and the definition of $u(t-\tau)$,

$$\mathcal{L}(u(t-\tau)g(t)) = \int_0^{\tau} e^{-st}(0) dt + \int_{\tau}^{\infty} e^{-st} g(t) dt.$$

The first integral on the right equals zero. Introducing the new variable of integration $x = t - \tau$ in the second integral yields

$$\mathcal{L}(u(t-\tau)g(t)) = \int_0^{\infty} e^{-s(x+\tau)} g(x+\tau) dx = e^{-s\tau} \int_0^{\infty} e^{-sx} g(x+\tau) dx.$$

Changing the name of the variable of integration in the last integral from x to t yields

$$\mathcal{L}(u(t-\tau)g(t)) = e^{-s\tau} \int_0^{\infty} e^{-st} g(t+\tau) dt = e^{-s\tau} \mathcal{L}(g(t+\tau)). \blacksquare$$

Example 11.4.2: Find

$$\mathcal{L}(u(t-1)(t^2+1)).$$

Solution: Here $\tau = 1$ and $g(t) = t^2 + 1$, so

$$g(t+1) = (t+1)^2 + 1 = t^2 + 2t + 2.$$

Since

$$\mathcal{L}(g(t+1)) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s},$$

Theorem 11.4.1 implies that

$$\mathcal{L}(u(t-1)(t^2+1)) = e^{-s} \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s} \right).$$

Example 11.4.3: Use Theorem 11.4.1 to find the Laplace transform of the function

$$f(t) = \begin{cases} 2t+1, & 0 \leq t < 2, \\ 3t, & t \geq 2, \end{cases}$$

from Example 11.4.1.

Solution: We first write f in the form (11.52) as

$$f(t) = 2t+1 + u(t-2)(t-1).$$

Therefore

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(2t+1) + \mathcal{L}(u(t-2)(t-1)) \\ &= \mathcal{L}(2t+1) + e^{-2s} \mathcal{L}(t+1) \quad (\text{from Theorem 11.4.1}) \\ &= \frac{2}{s^2} + \frac{1}{s} + e^{-2s} \left(\frac{1}{s^2} + \frac{1}{s} \right), \end{aligned}$$

which is the result obtained in Example 11.4.1. ■

Formula (11.52) can be extended to more general piecewise continuous functions. For example, we can write

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t_1 \leq t < t_2, \\ f_2(t), & t \geq t_2, \end{cases}$$

as

$$f(t) = f_0(t) + u(t - t_1)(f_1(t) - f_0(t)) + u(t - t_2)(f_2(t) - f_1(t))$$

if f_0 , f_1 , and f_2 are all defined on $[0, \infty)$.

Example 11.4.4: Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ -2t + 1, & 2 \leq t < 3, \\ 3t, & 3 \leq t < 5, \\ t - 1, & t \geq 5 \end{cases} \quad (11.53)$$

(Figure 11.8).

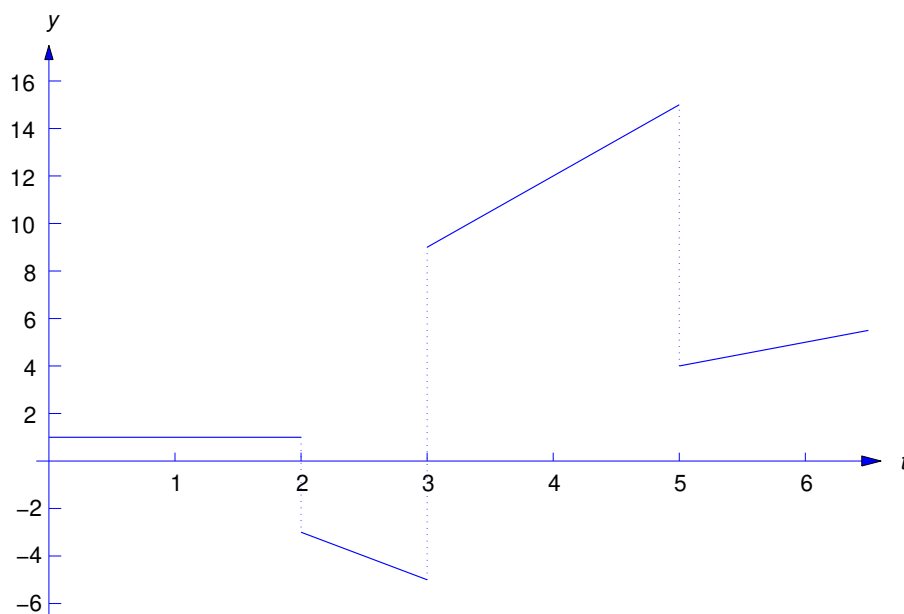


Figure 11.8: The piecewise continuous function (11.53)

Solution: In terms of step functions,

$$\begin{aligned} f(t) &= 1 + u(t - 2)(-2t + 1 - 1) + u(t - 3)(3t + 2t - 1) \\ &\quad + u(t - 5)(t - 1 - 3t), \end{aligned}$$

or

$$f(t) = 1 - 2u(t - 2)t + u(t - 3)(5t - 1) - u(t - 5)(2t + 1).$$

Now Theorem 11.4.1 implies that

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(1) - 2e^{-2s} \mathcal{L}(t + 2) + e^{-3s} \mathcal{L}(5(t + 3) - 1) - e^{-5s} \mathcal{L}(2(t + 5) + 1) \\ &= \mathcal{L}(1) - 2e^{-2s} \mathcal{L}(t + 2) + e^{-3s} \mathcal{L}(5t + 14) - e^{-5s} \mathcal{L}(2t + 11) \\ &= \frac{1}{s} - 2e^{-2s} \left(\frac{1}{s^2} + \frac{2}{s} \right) + e^{-3s} \left(\frac{5}{s^2} + \frac{14}{s} \right) - e^{-5s} \left(\frac{2}{s^2} + \frac{11}{s} \right). \blacksquare \end{aligned}$$

The trigonometric identities

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad (11.54)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad (11.55)$$

are useful in problems that involve shifting the arguments of trigonometric functions. We'll use these identities in the next example.

Example 11.4.5: Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \leq t < \frac{\pi}{2}, \\ \cos t - 3 \sin t, & \frac{\pi}{2} \leq t < \pi, \\ 3 \cos t, & t \geq \pi \end{cases} \quad (11.56)$$

(Figure 11.9).

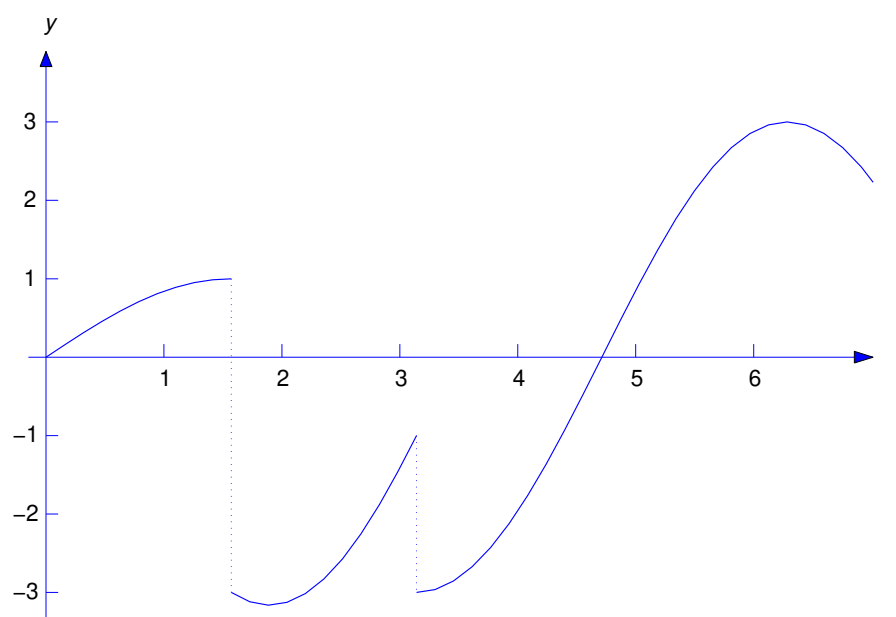


Figure 11.9: The piecewise continuous function (11.56)

Solution: In terms of step functions,

$$f(t) = \sin t + u(t - \pi/2)(\cos t - 4 \sin t) + u(t - \pi)(2 \cos t + 3 \sin t).$$

Now Theorem 11.4.1 implies that

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(\sin t) + e^{-\frac{\pi}{2}s} \mathcal{L}(\cos(t + \frac{\pi}{2}) - 4 \sin(t + \frac{\pi}{2})) \\ &\quad + e^{-\pi s} \mathcal{L}(2 \cos(t + \pi) + 3 \sin(t + \pi)). \end{aligned} \quad (11.57)$$

Since

$$\cos\left(t + \frac{\pi}{2}\right) - 4 \sin\left(t + \frac{\pi}{2}\right) = -\sin t - 4 \cos t$$

and

$$2 \cos(t + \pi) + 3 \sin(t + \pi) = -2 \cos t - 3 \sin t,$$

we see from (11.57) that

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(\sin t) - e^{-\pi s/2} \mathcal{L}(\sin t + 4 \cos t) - e^{-\pi s} \mathcal{L}(2 \cos t + 3 \sin t) \\ &= \frac{1}{s^2 + 1} - e^{-\frac{\pi}{2}s} \left(\frac{1 + 4s}{s^2 + 1} \right) - e^{-\pi s} \left(\frac{3 + 2s}{s^2 + 1} \right). \blacksquare \end{aligned}$$

11.4.2 The Second Shifting Theorem

Replacing $g(t)$ by $g(t - \tau)$ in Theorem 11.4.1 yields the next theorem.

Theorem 11.4.2. [Second Shifting Theorem] If $\tau \geq 0$ and $\mathcal{L}(g)$ exists for $s > s_0$ then $\mathcal{L}(u(t - \tau)g(t - \tau))$ exists for $s > s_0$ and

$$\mathcal{L}(u(t - \tau)g(t - \tau)) = e^{-s\tau} \mathcal{L}(g(t)),$$

or, equivalently,

$$\text{if } g(t) \leftrightarrow G(s), \text{ then } u(t - \tau)g(t - \tau) \leftrightarrow e^{-s\tau} G(s). \quad (11.58)$$

Remark 11.4.1: Recall that the First Shifting Theorem (Theorem 11.1.3 states that multiplying a function by e^{at} corresponds to shifting the argument of its transform by a units. Theorem 11.4.2 states that multiplying a Laplace transform by the exponential $e^{-s\tau}$ corresponds to shifting the argument of the inverse transform by τ units.

Example 11.4.6: Use (11.58) to find

$$\mathcal{L}^{-1} \left(\frac{e^{-2s}}{s^2} \right).$$

Solution: To apply (11.58) we let $\tau = 2$ and $G(s) = 1/s^2$. Then $g(t) = t$ and (11.58) implies that

$$\mathcal{L}^{-1} \left(\frac{e^{-2s}}{s^2} \right) = u(t - 2)(t - 2). \blacksquare$$

Example 11.4.7: Find the inverse Laplace transform h of

$$H(s) = \frac{1}{s^2} - e^{-s} \left(\frac{1}{s^2} + \frac{2}{s} \right) + e^{-4s} \left(\frac{4}{s^3} + \frac{1}{s} \right),$$

and find distinct formulas for h on appropriate intervals.

Solution: Let

$$G_0(s) = \frac{1}{s^2}, \quad G_1(s) = \frac{1}{s^2} + \frac{2}{s}, \quad G_2(s) = \frac{4}{s^3} + \frac{1}{s}.$$

Then

$$g_0(t) = t, \quad g_1(t) = t + 2, \quad g_2(t) = 2t^2 + 1.$$

Hence, (11.58) and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}(G_0(s)) - \mathcal{L}^{-1}(e^{-s}G_1(s)) + \mathcal{L}^{-1}(e^{-4s}G_2(s)) \\ &= t - u(t - 1)[(t - 1) + 2] + u(t - 4)[2(t - 4)^2 + 1] \\ &= t - u(t - 1)(t + 1) + u(t - 4)(2t^2 - 16t + 33), \end{aligned}$$

which can also be written as

$$h(t) = \begin{cases} t, & 0 \leq t < 1, \\ -1, & 1 \leq t < 4, \\ 2t^2 - 16t + 32, & t \geq 4. \end{cases} \quad \blacksquare$$

Example 11.4.8: Find the inverse transform of

$$H(s) = \frac{2s}{s^2 + 4} - e^{-\frac{\pi}{2}s} \frac{3s + 1}{s^2 + 9} + e^{-\pi s} \frac{s + 1}{s^2 + 6s + 10}.$$

Solution: Let

$$G_0(s) = \frac{2s}{s^2 + 4}, \quad G_1(s) = -\frac{(3s + 1)}{s^2 + 9},$$

and

$$G_2(s) = \frac{s + 1}{s^2 + 6s + 10} = \frac{(s + 3) - 2}{(s + 3)^2 + 1}.$$

Then

$$g_0(t) = 2 \cos 2t, \quad g_1(t) = -3 \cos 3t - \frac{1}{3} \sin 3t,$$

and

$$g_2(t) = e^{-3t}(\cos t - 2 \sin t).$$

Therefore (11.58) and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= 2 \cos 2t - u(t - \pi/2) \left[3 \cos 3(t - \pi/2) + \frac{1}{3} \sin 3 \left(t - \frac{\pi}{2} \right) \right] \\ &\quad + u(t - \pi) e^{-3(t-\pi)} [\cos(t - \pi) - 2 \sin(t - \pi)]. \end{aligned}$$

Using the trigonometric identities (11.54) and (11.55), we can rewrite this as

$$\begin{aligned} h(t) &= 2 \cos 2t + u(t - \pi/2) \left(3 \sin 3t - \frac{1}{3} \cos 3t \right) \\ &\quad - u(t - \pi) e^{-3(t-\pi)} (\cos t - 2 \sin t) \end{aligned} \quad (11.59)$$

(Figure 11.10).

11.4.3 Exercises

In Exercises 11.4.1–11.4.6 find the Laplace transform by the method of Example 11.4.1. Then express the given function f in terms of unit step functions as in Eqn. (11.52), and use Theorem 11.4.1 to find $\mathcal{L}(f)$. Where indicated by C/G, graph f .

1. $f(t) = \begin{cases} 1, & 0 \leq t < 4, \\ t, & t \geq 4. \end{cases}$
2. $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 1, & t \geq 1. \end{cases}$
3. C/G $f(t) = \begin{cases} 2t - 1, & 0 \leq t < 2, \\ t, & t \geq 2. \end{cases}$
4. C/G $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ t + 2, & t \geq 1. \end{cases}$
5. $f(t) = \begin{cases} t - 1, & 0 \leq t < 2, \\ 4, & t \geq 2. \end{cases}$
6. $f(t) = \begin{cases} t^2, & 0 \leq t < 1, \\ 0, & t \geq 1. \end{cases}$

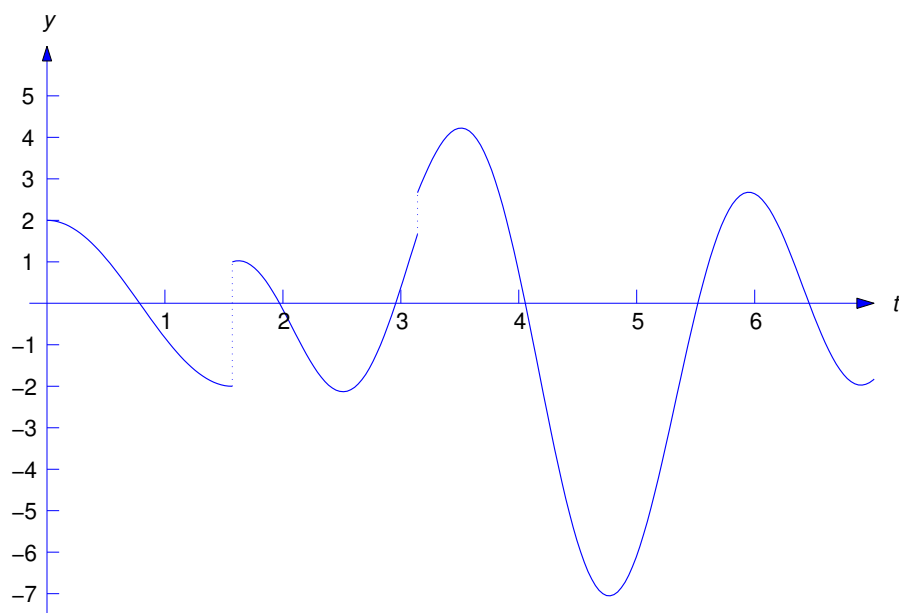


Figure 11.10: The piecewise continuous function (11.59)

In Exercises 11.4.7–11.4.18 express the given function f in terms of unit step functions and use Theorem 11.4.1 to find $\mathcal{L}(f)$. Where indicated by C/G, graph f .

$$7. \quad f(t) = \begin{cases} 0, & 0 \leq t < 2, \\ t^2 + 3t, & t \geq 2. \end{cases} \quad 8. \quad f(t) = \begin{cases} t^2 + 2, & 0 \leq t < 1, \\ t, & t \geq 1. \end{cases}$$

$$9. \quad f(t) = \begin{cases} te^t, & 0 \leq t < 1, \\ e^t, & t \geq 1. \end{cases} \quad 10. \quad f(t) = \begin{cases} e^{-t}, & 0 \leq t < 1, \\ e^{-2t}, & t \geq 1. \end{cases}$$

$$11. \quad f(t) = \begin{cases} -t, & 0 \leq t < 2, \\ t - 4, & 2 \leq t < 3, \\ 1, & t \geq 3. \end{cases} \quad 12. \quad f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

$$13. \quad f(t) = \begin{cases} t, & 0 \leq t < 1, \\ t^2, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases} \quad 14. \quad f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 2 - t, & 1 \leq t < 2, \\ 6, & t \geq 2. \end{cases}$$

$$15. \quad \text{C/G} \quad f(t) = \begin{cases} \sin t, & 0 \leq t < \frac{\pi}{2}, \\ 2 \sin t, & \frac{\pi}{2} \leq t < \pi, \\ \cos t, & t \geq \pi. \end{cases}$$

$$16. \quad \boxed{\text{C/G}} \quad f(t) = \begin{cases} 2, & 0 \leq t < 1, \\ -2t + 2, & 1 \leq t < 3, \\ 3t, & t \geq 3. \end{cases}$$

$$17. \quad \boxed{\text{C/G}} \quad f(t) = \begin{cases} 3, & 0 \leq t < 2, \\ 3t + 2, & 2 \leq t < 4, \\ 4t, & t \geq 4. \end{cases}$$

$$18. \quad \boxed{\text{C/G}} \quad f(t) = \begin{cases} (t+1)^2, & 0 \leq t < 1, \\ (t+2)^2, & t \geq 1. \end{cases}$$

In Exercises 11.4.19–11.4.28 use Theorem 11.4.2 to express the inverse transforms in terms of step functions, and then find distinct formulas for the inverse transforms on the appropriate intervals, as in Example 11.4.7. Where indicated by $\boxed{\text{C/G}}$, graph the inverse transform.

$$19. \quad H(s) = \frac{e^{-2s}}{s-2} \qquad 20. \quad H(s) = \frac{e^{-s}}{s(s+1)}$$

$$21. \quad \boxed{\text{C/G}} \quad H(s) = \frac{e^{-s}}{s^3} + \frac{e^{-2s}}{s^2}$$

$$22. \quad \boxed{\text{C/G}} \quad H(s) = \left(\frac{2}{s} + \frac{1}{s^2}\right) + e^{-s} \left(\frac{3}{s} - \frac{1}{s^2}\right) + e^{-3s} \left(\frac{1}{s} + \frac{1}{s^2}\right)$$

$$23. \quad H(s) = \left(\frac{5}{s} - \frac{1}{s^2}\right) + e^{-3s} \left(\frac{6}{s} + \frac{7}{s^2}\right) + \frac{3e^{-6s}}{s^3}$$

$$24. \quad H(s) = \frac{e^{-\pi s}(1-2s)}{s^2+4s+5}$$

$$25. \quad \boxed{\text{C/G}} \quad H(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) + e^{-\frac{\pi}{2}s} \left(\frac{3s-1}{s^2+1}\right)$$

$$26. \quad H(s) = e^{-2s} \left[\frac{3(s-3)}{(s+1)(s-2)} - \frac{s+1}{(s-1)(s-2)} \right]$$

$$27. \quad H(s) = \frac{1}{s} + \frac{1}{s^2} + e^{-s} \left(\frac{3}{s} + \frac{2}{s^2}\right) + e^{-3s} \left(\frac{4}{s} + \frac{3}{s^2}\right)$$

$$28. \quad H(s) = \frac{1}{s} - \frac{2}{s^3} + e^{-2s} \left(\frac{3}{s} - \frac{1}{s^3}\right) + \frac{e^{-4s}}{s^2}$$

$$29. \quad \text{Find } \mathcal{L}(u(t-\tau)).$$

$$30. \quad \text{Let } \{t_m\}_{m=0}^{\infty} \text{ be a sequence of points such that } t_0 = 0, t_{m+1} > t_m, \text{ and } \lim_{m \rightarrow \infty} t_m = \infty. \text{ For each nonnegative integer } m, \text{ let } f_m \text{ be continuous on } [t_m, \infty), \text{ and let } f \text{ be defined on } [0, \infty) \text{ by}$$

$$f(t) = f_m(t), \quad t_m \leq t < t_{m+1} \quad (m = 0, 1, \dots).$$

Show that f is piecewise continuous on $[0, \infty)$ and that it has the step function representation

$$f(t) = f_0(t) + \sum_{m=1}^{\infty} u(t-t_m) (f_m(t) - f_{m-1}(t)), \quad 0 \leq t < \infty.$$

How do we know that the series on the right converges for all t in $[0, \infty)$?

31. In addition to the assumptions of Exercise 11.4.30, assume that

$$|f_m(t)| \leq M e^{s_0 t}, \quad t \geq t_m, \quad m = 0, 1, \dots, \quad (\text{A})$$

and that the series

$$\sum_{m=0}^{\infty} e^{-\rho t_m} \quad (\text{B})$$

converges for some $\rho > 0$. Using the steps listed below, show that $\mathcal{L}(f)$ is defined for $s > s_0$ and

$$\mathcal{L}(f) = \mathcal{L}(f_0) + \sum_{m=1}^{\infty} e^{-s t_m} \mathcal{L}(g_m) \quad (\text{C})$$

for $s > s_0 + \rho$, where

$$g_m(t) = f_m(t + t_m) - f_{m-1}(t + t_m).$$

- (a) Use (A) and Theorem 8.1.6 to show that

$$\mathcal{L}(f) = \sum_{m=0}^{\infty} \int_{t_m}^{t_{m+1}} e^{-st} f_m(t) dt \quad (\text{D})$$

is defined for $s > s_0$.

- (b) Show that (D) can be rewritten as

$$\mathcal{L}(f) = \sum_{m=0}^{\infty} \left(\int_{t_m}^{\infty} e^{-st} f_m(t) dt - \int_{t_{m+1}}^{\infty} e^{-st} f_m(t) dt \right). \quad (\text{E})$$

- (c) Use (A), the assumed convergence of (B), and the comparison test to show that the series

$$\sum_{m=0}^{\infty} \int_{t_m}^{\infty} e^{-st} f_m(t) dt \quad \text{and} \quad \sum_{m=0}^{\infty} \int_{t_{m+1}}^{\infty} e^{-st} f_m(t) dt$$

both converge (absolutely) if $s > s_0 + \rho$.

- (d) Show that (E) can be rewritten as

$$\mathcal{L}(f) = \mathcal{L}(f_0) + \sum_{m=1}^{\infty} \int_{t_m}^{\infty} e^{-st} (f_m(t) - f_{m-1}(t)) dt$$

if $s > s_0 + \rho$.

- (e) Complete the proof of (C).

32. Suppose $\{t_m\}_{m=0}^{\infty}$ and $\{f_m\}_{m=0}^{\infty}$ satisfy the assumptions of Exercises 11.4.30 and 11.4.31, and there's a positive constant K such that $t_m \geq Km$ for m sufficiently large. Show that the series (B) of Exercise 11.4.31 converges for any $\rho > 0$, and conclude from this that (C) of Exercise 11.4.31 holds for $s > s_0$.

In Exercises 11.4.33–11.4.36 find the step function representation of f and use the result of Exercise 11.4.32 to find $\mathcal{L}(f)$. HINT: You will need formulas related to the formula for the sum of a geometric series.

33. $f(t) = m + 1, \quad m \leq t < m + 1 \quad (m = 0, 1, 2, \dots)$
 34. $f(t) = (-1)^m, \quad m \leq t < m + 1 \quad (m = 0, 1, 2, \dots)$
 35. $f(t) = (m + 1)^2, \quad m \leq t < m + 1 \quad (m = 0, 1, 2, \dots)$
 36. $f(t) = (-1)^m m, \quad m \leq t < m + 1 \quad (m = 0, 1, 2, \dots)$

11.5 Constant Coefficient Equations with Piecewise Continuous Forcing Functions

We'll now consider initial value problems of the form

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1, \quad (11.60)$$

where a , b , and c are constants ($a \neq 0$) and f is piecewise continuous on $[0, \infty)$. Problems of this kind occur in situations where the input to a physical system undergoes instantaneous changes, as when a switch is turned on or off or the forces acting on the system change abruptly.

It can be shown (Exercises 11.5.23 and 11.5.24) that the differential equation in (11.60) has no solutions on an open interval that contains a jump discontinuity of f . Therefore we must define what we mean by a solution of (11.60) on $[0, \infty)$ in the case where f has jump discontinuities. The next theorem motivates our definition. We omit the proof.

Theorem 11.5.1. *Suppose a , b , and c are constants ($a \neq 0$), and f is piecewise continuous on $[0, \infty)$ with jump discontinuities at $t_1, \dots, t_n$, where*

$$0 < t_1 < \dots < t_n.$$

Let k_0 and k_1 be arbitrary real numbers. Then there is a unique function y defined on $[0, \infty)$ with these properties:

- (a) $y(0) = k_0$ and $y'(0) = k_1$.
- (b) y and y' are continuous on $[0, \infty)$.
- (c) y'' is defined on every open subinterval of $[0, \infty)$ that does not contain any of the points $t_1, \dots, t_n$, and

$$ay'' + by' + cy = f(t)$$

on every such subinterval.

- (d) y'' has limits from the right and left at $t_1, \dots, t_n$.

We define the function y of Theorem 11.5.1 to be the solution of the initial value problem (11.60).

We begin by considering initial value problems of the form

$$ay'' + by' + cy = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t \geq t_1, \end{cases} \quad y(0) = k_0, \quad y'(0) = k_1, \quad (11.61)$$

where the forcing function has a single jump discontinuity at t_1 .

We can solve (11.61) by the these steps:

Step 1. Find the solution y_0 of the initial value problem

$$ay'' + by' + cy = f_0(t), \quad y(0) = k_0, \quad y'(0) = k_1.$$

Step 2. Compute $c_0 = y_0(t_1)$ and $c_1 = y'_0(t_1)$.

Step 3. Find the solution y_1 of the initial value problem

$$ay'' + by' + cy = f_1(t), \quad y(t_1) = c_0, \quad y'(t_1) = c_1.$$

Step 4. Obtain the solution y of (11.61) as

$$y = \begin{cases} y_0(t), & 0 \leq t < t_1 \\ y_1(t), & t \geq t_1. \end{cases}$$

It is shown in Exercise 11.5.23 that y' exists and is continuous at t_1 . The next example illustrates this procedure.

Example 11.5.1: Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2, \quad y'(0) = -1, \quad (11.62)$$

where

$$f(t) = \begin{cases} 1, & 0 \leq t < \frac{\pi}{2}, \\ -1, & t \geq \frac{\pi}{2}. \end{cases}$$

Solution: The initial value problem in Step 1 is

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = -1.$$

We leave it to you to verify that its solution is

$$y_0 = 1 + \cos t - \sin t.$$

Doing Step 2 yields $y_0(\pi/2) = 0$ and $y'_0(\pi/2) = -1$, so the second initial value problem is

$$y'' + y = -1, \quad y\left(\frac{\pi}{2}\right) = 0, \quad y'\left(\frac{\pi}{2}\right) = -1.$$

We leave it to you to verify that the solution of this problem is

$$y_1 = -1 + \cos t + \sin t.$$

Hence, the solution of (11.62) is

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \leq t < \frac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \geq \frac{\pi}{2} \end{cases} \quad (11.63)$$

(Figure:8.5.1).

If f_0 and f_1 are defined on $[0, \infty)$, we can rewrite (11.61) as

$$ay'' + by' + cy = f_0(t) + u(t - t_1)(f_1(t) - f_0(t)), \quad y(0) = k_0, \quad y'(0) = k_1,$$

and apply the method of Laplace transforms. We'll now solve the problem considered in Example 11.5.1 by this method.

Example 11.5.2: Use the Laplace transform to solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2, \quad y'(0) = -1, \quad (11.64)$$

where

$$f(t) = \begin{cases} 1, & 0 \leq t < \frac{\pi}{2}, \\ -1, & t \geq \frac{\pi}{2}. \end{cases}$$

Solution: Here

$$f(t) = 1 - 2u\left(t - \frac{\pi}{2}\right),$$

so Theorem 11.4.1 (with $g(t) = 1$) implies that

$$\mathcal{L}(f) = \frac{1 - 2e^{-\pi s/2}}{s}.$$

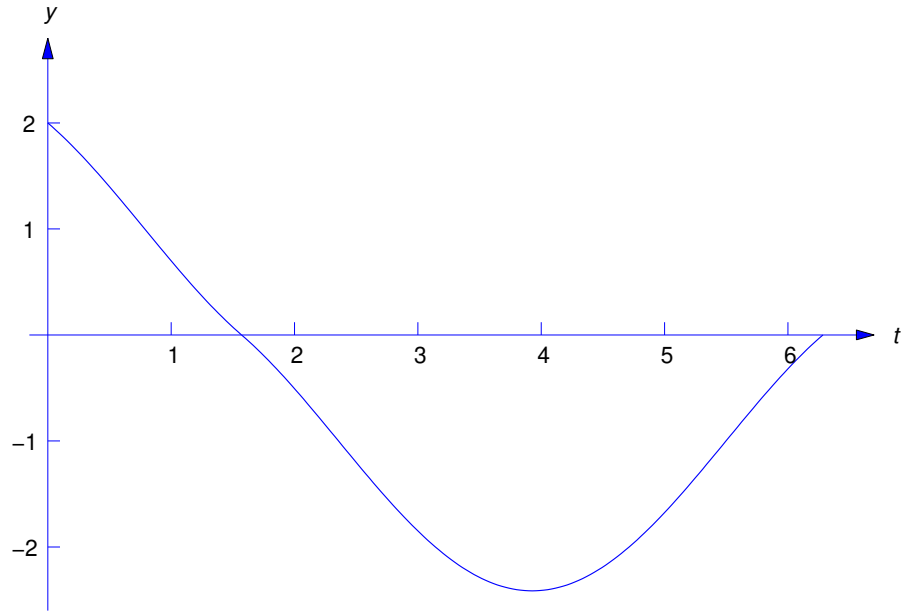


Figure 11.11: Graph of (11.63)

Therefore, transforming (11.64) yields

$$(s^2 + 1)Y(s) = \frac{1 - 2e^{-\pi s/2}}{s} - 1 + 2s,$$

so

$$Y(s) = (1 - 2e^{-\pi s/2})G(s) + \frac{2s - 1}{s^2 + 1}, \quad (11.65)$$

with

$$G(s) = \frac{1}{s(s^2 + 1)}.$$

The form for the partial fraction expansion of G is

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}. \quad (11.66)$$

Multiplying through by $s(s^2 + 1)$ yields

$$A(s^2 + 1) + (Bs + C)s = 1,$$

or

$$(A + B)s^2 + Cs + A = 1.$$

Equating coefficients of like powers of s on the two sides of this equation shows that $A = 1$, $B = -A = -1$ and $C = 0$. Hence, from (11.66),

$$G(s) = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Therefore

$$g(t) = 1 - \cos t.$$

From this, (11.65), and Theorem 11.4.2,

$$y = 1 - \cos t - 2u\left(t - \frac{\pi}{2}\right)\left(1 - \cos\left(t - \frac{\pi}{2}\right)\right) + 2\cos t - \sin t.$$

Simplifying this (recalling that $\cos(t - \pi/2) = \sin t$) yields

$$y = 1 + \cos t - \sin t - 2u\left(t - \frac{\pi}{2}\right)(1 - \sin t),$$

or

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \leq t < \frac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \geq \frac{\pi}{2}, \end{cases}$$

which is the result obtained in Example 11.5.1.

Remark 11.5.1: It isn't obvious that using the Laplace transform to solve (11.61) as we did in Example 11.5.2 yields a function y with the properties stated in Theorem 11.5.1; that is, such that y and y' are continuous on $[0, \infty)$ and y'' has limits from the right and left at t_1 . However, this is true if f_0 and f_1 are continuous and of exponential order on $[0, \infty)$. A proof is sketched in Exercises 11.6.11–11.6.13.

Example 11.5.3: Solve the initial value problem

$$y'' - y = f(t), \quad y(0) = -1, \quad y'(0) = 2, \quad (11.67)$$

where

$$f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 1, & t \geq 1. \end{cases}$$

Solution: Here

$$f(t) = t - u(t-1)(t-1),$$

so

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(t) - \mathcal{L}(u(t-1)(t-1)) \\ &= \mathcal{L}(t) - e^{-s}\mathcal{L}(t) \quad (\text{from Theorem 11.4.1}) \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}. \end{aligned}$$

Since transforming (11.67) yields

$$(s^2 - 1)Y(s) = \mathcal{L}(f) + 2 - s,$$

we see that

$$Y(s) = (1 - e^{-s})H(s) + \frac{2-s}{s^2-1}, \quad (11.68)$$

where

$$H(s) = \frac{1}{s^2(s^2-1)} = \frac{1}{s^2-1} - \frac{1}{s^2};$$

therefore

$$h(t) = \sinh t - t. \quad (11.69)$$

Since

$$\mathcal{L}^{-1}\left(\frac{2-s}{s^2-1}\right) = 2\sinh t - \cosh t,$$

we conclude from (11.68), (11.69), and Theorem 11.4.1 that

$$y = \sinh t - t - u(t-1)(\sinh(t-1) - t + 1) + 2\sinh t - \cosh t,$$

or

$$y = 3 \sinh t - \cosh t - t - u(t-1)(\sinh(t-1) - t + 1) \quad (11.70)$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = 1$.

Example 11.5.4: Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \quad (11.71)$$

where

$$f(t) = \begin{cases} 0, & 0 \leq t < \frac{\pi}{4}, \\ \cos 2t, & \frac{\pi}{4} \leq t < \pi, \\ 0, & t \geq \pi. \end{cases}$$

Solution: Here

$$f(t) = u(t - \pi/4) \cos 2t - u(t - \pi) \cos 2t,$$

so

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(u(t - \pi/4) \cos 2t) - \mathcal{L}(u(t - \pi) \cos 2t) \\ &= e^{-\pi s/4} \mathcal{L}(\cos 2(t + \pi/4)) - e^{-\pi s} \mathcal{L}(\cos 2(t + \pi)) \\ &= -e^{-\pi s/4} \mathcal{L}(\sin 2t) - e^{-\pi s} \mathcal{L}(\cos 2t) \\ &= -\frac{2e^{-\pi s/4}}{s^2 + 4} - \frac{se^{-\pi s}}{s^2 + 4}. \end{aligned}$$

Since transforming (11.71) yields

$$(s^2 + 1)Y(s) = \mathcal{L}(f),$$

we see that

$$Y(s) = e^{-\pi s/4} H_1(s) + e^{-\pi s} H_2(s), \quad (11.72)$$

where

$$H_1(s) = -\frac{2}{(s^2 + 1)(s^2 + 4)} \quad \text{and} \quad H_2(s) = -\frac{s}{(s^2 + 1)(s^2 + 4)}. \quad (11.73)$$

To simplify the required partial fraction expansions, we first write

$$\frac{1}{(x+1)(x+4)} = \frac{1}{3} \left[\frac{1}{x+1} - \frac{1}{x+4} \right].$$

Setting $x = s^2$ and substituting the result in (11.73) yields

$$H_1(s) = -\frac{2}{3} \left[\frac{1}{s^2 + 1} - \frac{1}{s^2 + 4} \right] \quad \text{and} \quad H_2(s) = -\frac{1}{3} \left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + 4} \right].$$

The inverse transforms are

$$h_1(t) = -\frac{2}{3} \sin t + \frac{1}{3} \sin 2t \quad \text{and} \quad h_2(t) = -\frac{1}{3} \cos t + \frac{1}{3} \cos 2t.$$

From (11.72) and Theorem 8.4.2,

$$y = u\left(t - \frac{\pi}{4}\right) h_1\left(t - \frac{\pi}{4}\right) + u(t - \pi) h_2(t - \pi). \quad (11.74)$$

Since

$$\begin{aligned} h_1\left(t - \frac{\pi}{4}\right) &= -\frac{2}{3}\sin\left(t - \frac{\pi}{4}\right) + \frac{1}{3}\sin 2\left(t - \frac{\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t \end{aligned}$$

and

$$\begin{aligned} h_2(t - \pi) &= -\frac{1}{3}\cos(t - \pi) + \frac{1}{3}\cos 2(t - \pi) \\ &= \frac{1}{3}\cos t + \frac{1}{3}\cos 2t, \end{aligned}$$

(11.74) can be rewritten as

$$y = -\frac{1}{3}u\left(t - \frac{\pi}{4}\right)\left(\sqrt{2}(\sin t - \cos t) + \cos 2t\right) + \frac{1}{3}u(t - \pi)(\cos t + \cos 2t)$$

or

$$y = \begin{cases} 0, & 0 \leq t < \frac{\pi}{4}, \\ -\frac{\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t, & \frac{\pi}{4} \leq t < \pi, \\ -\frac{\sqrt{2}}{3}\sin t + \frac{1 + \sqrt{2}}{3}\cos t, & t \geq \pi. \end{cases} \quad (11.75)$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = \pi/4$ and $t_2 = \pi$ (Figure 11.12).

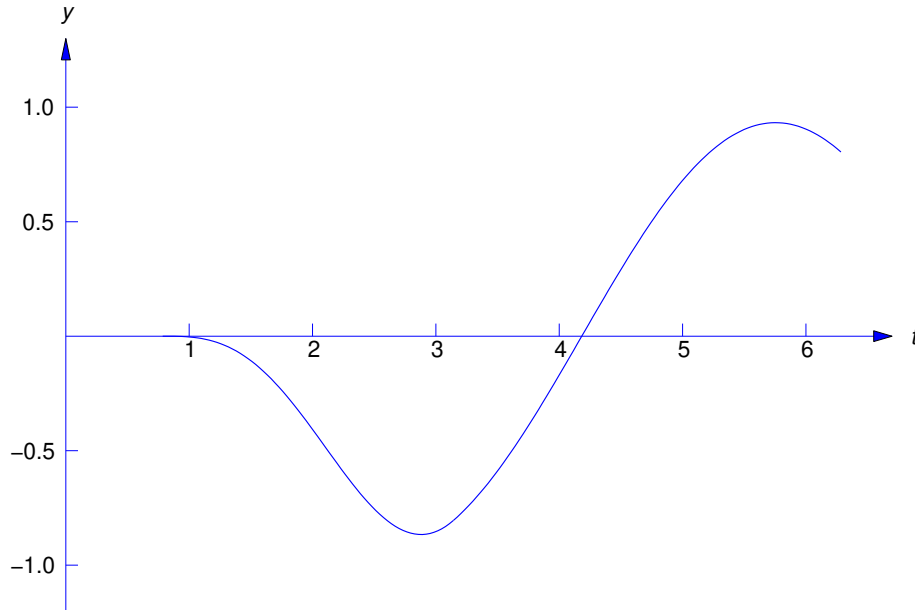


Figure 11.12: Graph of (11.75)

11.5.1 Exercises

In Exercises 11.5.1–11.5.20 use the Laplace transform to solve the initial value problem. Where indicated by C/G, graph the solution.

$$1. \quad y'' + y = \begin{cases} 3, & 0 \leq t < \pi, \\ 0, & t \geq \pi, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$$

$$2. \quad y'' + y = \begin{cases} 3, & 0 \leq t < 4, \\ 2t - 5, & t \geq 4, \end{cases} \quad y(0) = 1, \quad y'(0) = 0$$

$$3. \quad y'' - 2y' = \begin{cases} 4, & 0 \leq t < 1, \\ 6, & t \geq 1, \end{cases} \quad y(0) = -6, \quad y'(0) = 1$$

$$4. \quad y'' - y = \begin{cases} e^{2t}, & 0 \leq t < 2, \\ 1, & t \geq 2, \end{cases} \quad y(0) = 3, \quad y'(0) = -1$$

$$5. \quad y'' - 3y' + 2y = \begin{cases} 0, & 0 \leq t < 1, \\ 1, & 1 \leq t < 2, \\ -1, & t \geq 2, \end{cases} \quad y(0) = -3, \quad y'(0) = 1$$

$$6. \quad \text{C/G} \quad y'' + 4y = \begin{cases} |\sin t|, & 0 \leq t < 2\pi, \\ 0, & t \geq 2\pi, \end{cases} \quad y(0) = -3, \quad y'(0) = 1$$

$$7. \quad y'' - 5y' + 4y = \begin{cases} 1, & 0 \leq t < 1 \\ -1, & 1 \leq t < 2, \\ 0, & t \geq 2, \end{cases} \quad y(0) = 3, \quad y'(0) = -5$$

$$8. \quad y'' + 9y = \begin{cases} \cos t, & 0 \leq t < \frac{3\pi}{2}, \\ \sin t, & t \geq \frac{3\pi}{2}, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$$

$$9. \quad \text{C/G} \quad y'' + 4y = \begin{cases} t, & 0 \leq t < \frac{\pi}{2}, \\ \pi, & t \geq \frac{\pi}{2}, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$$

$$10. \quad y'' + y = \begin{cases} t, & 0 \leq t < \pi, \\ -t, & t \geq \pi, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$$

$$11. \quad y'' - 3y' + 2y = \begin{cases} 0, & 0 \leq t < 2, \\ 2t - 4, & t \geq 2, \end{cases}, \quad y(0) = 0, \quad y'(0) = 0$$

$$12. \quad y'' + y = \begin{cases} t, & 0 \leq t < 2\pi, \\ -2t, & t \geq 2\pi, \end{cases} \quad y(0) = 1, \quad y'(0) = 2$$

$$13. \quad \text{C/G} \quad y'' + 3y' + 2y = \begin{cases} 1, & 0 \leq t < 2, \\ -1, & t \geq 2, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$$

$$14. \quad y'' - 4y' + 3y = \begin{cases} -1, & 0 \leq t < 1, \\ 1, & t \geq 1, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$$

$$15. \quad y'' + 2y' + y = \begin{cases} e^t, & 0 \leq t < 1, \\ e^t - 1, & t \geq 1, \end{cases} \quad y(0) = 3, \quad y'(0) = -1$$

$$16. \quad y'' + 2y' + y = \begin{cases} 4e^t, & 0 \leq t < 1, \\ 0, & t \geq 1, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$$

$$17. \quad y'' + 3y' + 2y = \begin{cases} e^{-t}, & 0 \leq t < 1, \\ 0, & t \geq 1, \end{cases} \quad y(0) = 1, \quad y'(0) = -1$$

$$18. \quad y'' - 4y' + 4y = \begin{cases} e^{2t}, & 0 \leq t < 2, \\ -e^{2t}, & t \geq 2, \end{cases} \quad y(0) = 0, \quad y'(0) = -1$$

$$19. \quad \boxed{\text{C/G}} \quad y'' = \begin{cases} t^2, & 0 \leq t < 1, \\ -t, & 1 \leq t < 2, \\ t + 1, & t \geq 2, \end{cases} \quad y(0) = 1, \quad y'(0) = 0$$

$$20. \quad y'' + 2y' + 2y = \begin{cases} 1, & 0 \leq t < 2\pi, \\ t, & 2\pi \leq t < 3\pi, \\ -1, & t \geq 3\pi, \end{cases} \quad y(0) = 2, \quad y'(0) = -1$$

21. Solve the initial value problem

$$y'' = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = m + 1, \quad m \leq t < m + 1, \quad m = 0, 1, 2, \dots$$

22. Solve the given initial value problem and find a formula that does not involve step functions and represents y on each interval of continuity of f .

(a) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$

$$f(t) = m + 1, \quad m\pi \leq t < (m + 1)\pi, \quad m = 0, 1, 2, \dots$$

(b) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$

$$f(t) = (m + 1)t, \quad 2m\pi \leq t < 2(m + 1)\pi, \quad m = 0, 1, 2, \dots \quad \text{HINT: You'll need the formula}$$

$$1 + 2 + \dots + m = \frac{m(m + 1)}{2}.$$

(c) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$

$$f(t) = (-1)^m, \quad m\pi \leq t < (m + 1)\pi, \quad m = 0, 1, 2, \dots$$

(d) $y'' - y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$

$$f(t) = m + 1, \quad m \leq t < (m + 1), \quad m = 0, 1, 2, \dots$$

HINT: You will need the formula

$$1 + r + \dots + r^m = \frac{1 - r^{m+1}}{1 - r} \quad (r \neq 1).$$

(e) $y'' + 2y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$

$$f(t) = (m + 1)(\sin t + 2 \cos t), \quad 2m\pi \leq t < 2(m + 1)\pi, \quad m = 0, 1, 2, \dots$$

(See the hint in (d).)

(f) $y'' - 3y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$

$$f(t) = m + 1, \quad m \leq t < m + 1, \quad m = 0, 1, 2, \dots$$

(See the hints in (b) and (d).)

23. (a) Let g be continuous on (α, β) and differentiable on the (α, t_0) and (t_0, β) . Suppose $A = \lim_{t \rightarrow t_0^-} g'(t)$ and $B = \lim_{t \rightarrow t_0^+} g'(t)$ both exist. Use the mean value theorem to show that

$$\lim_{t \rightarrow t_0^-} \frac{g(t) - g(t_0)}{t - t_0} = A \quad \text{and} \quad \lim_{t \rightarrow t_0^+} \frac{g(t) - g(t_0)}{t - t_0} = B.$$

(b) Conclude from (a) that $g'(t_0)$ exists and g' is continuous at t_0 if $A = B$.

(c) Conclude from (a) that if g is differentiable on (α, β) then g' can't have a jump discontinuity on (α, β) .

24. (a) Let a, b , and c be constants, with $a \neq 0$. Let f be piecewise continuous on an interval (α, β) , with a single jump discontinuity at a point t_0 in (α, β) . Suppose y and y' are continuous on (α, β) and y'' on (α, t_0) and (t_0, β) . Suppose also that

$$ay'' + by' + cy = f(t) \quad (\text{A})$$

on (α, t_0) and (t_0, β) . Show that

$$y''(t_0+) - y''(t_0-) = \frac{f(t_0+) - f(t_0-)}{a} \neq 0.$$

- (b) Use (a) and Exercise 11.5.23(c) to show that (A) does not have solutions on any interval (α, β) that contains a jump discontinuity of f .
25. Suppose P_0, P_1 , and P_2 are continuous and P_0 has no zeros on an open interval (a, b) , and that F has a jump discontinuity at a point t_0 in (a, b) . Show that the differential equation

$$P_0(t)y'' + P_1(t)y' + P_2(t)y = F(t)$$

has no solutions on (a, b) . HINT: Generalize the result of Exercise 11.5.24 and use Exercise 11.5.23(c).

26. Let $0 = t_0 < t_1 < \cdots < t_n$. Suppose f_m is continuous on $[t_m, \infty)$ for $m = 1, \dots, n$. Let

$$f(t) = \begin{cases} f_m(t), & t_m \leq t < t_{m+1}, & m = 1, \dots, n-1, \\ f_n(t), & t \geq t_n. \end{cases}$$

Show that the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

as defined following Theorem 8.5.1, is given by

$$y = \begin{cases} z_0(t), & 0 \leq t < t_1, \\ z_0(t) + z_1(t), & t_1 \leq t < t_2, \\ \vdots & \\ z_0 + \cdots + z_{n-1}(t), & t_{n-1} \leq t < t_n, \\ z_0 + \cdots + z_n(t), & t \geq t_n, \end{cases}$$

where z_0 is the solution of

$$az'' + bz' + cz = f_0(t), \quad z(0) = k_0, \quad z'(0) = k_1$$

and z_m is the solution of

$$az'' + bz' + cz = f_m(t) - f_{m-1}(t), \quad z(t_m) = 0, \quad z'(t_m) = 0$$

for $m = 1, \dots, n$.

11.6 Convolution

In this section we consider the problem of finding the inverse Laplace transform of a product $H(s) = F(s)G(s)$, where F and G are the Laplace transforms of known functions f and g . To motivate our interest in this problem, consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Taking Laplace transforms yields

$$(as^2 + bs + c)Y(s) = F(s),$$

so

$$Y(s) = F(s)G(s), \tag{11.76}$$

where

$$G(s) = \frac{1}{as^2 + bs + c}.$$

Until now we've been interested in the factorization indicated in (11.76), since we dealt only with differential equations with specific forcing functions. Hence, we could simply do the indicated multiplication in (11.76) and use the table of Laplace transforms to find $y = \mathcal{L}^{-1}(Y)$. However, this isn't possible if we want a formula for y in terms of f , which may be unspecified.

To motivate the formula for $\mathcal{L}^{-1}(FG)$, consider the initial value problem

$$y' - ay = f(t), \quad y(0) = 0, \tag{11.77}$$

which we first solve without using the Laplace transform. The solution of the differential equation in (11.77) is of the form $y = ue^{at}$ where

$$u' = e^{-at}f(t).$$

Integrating this from 0 to t and imposing the initial condition $u(0) = y(0) = 0$ yields

$$u = \int_0^t e^{-a\tau} f(\tau) d\tau.$$

Therefore

$$y(t) = e^{at} \int_0^t e^{-a\tau} f(\tau) d\tau = \int_0^t e^{a(t-\tau)} f(\tau) d\tau. \tag{11.78}$$

Now we'll use the Laplace transform to solve (11.77) and compare the result to (11.78). Taking Laplace transforms in (11.77) yields

$$(s - a)Y(s) = F(s),$$

so

$$Y(s) = F(s) \frac{1}{s - a},$$

which implies that

$$y(t) = \mathcal{L}^{-1} \left(F(s) \frac{1}{s - a} \right). \tag{11.79}$$

If we now let $g(t) = e^{at}$, so that

$$G(s) = \frac{1}{s - a},$$

then (11.78) and (11.79) can be written as

$$y(t) = \int_0^t f(\tau)g(t - \tau) d\tau$$

and

$$y = \mathcal{L}^{-1}(FG),$$

respectively. Therefore

$$\mathcal{L}^{-1}(FG) = \int_0^t f(\tau)g(t-\tau) d\tau \quad (11.80)$$

in this case.

This motivates the next definition.

Definition 11.6.1 The *convolution* $f * g$ of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau) d\tau.$$

It can be shown (Exercise 11.6.6) that $f * g = g * f$; that is,

$$\int_0^t f(t-\tau)g(\tau) d\tau = \int_0^t f(\tau)g(t-\tau) d\tau.$$

Eqn. (11.80) shows that $\mathcal{L}^{-1}(FG) = f * g$ in the special case where $g(t) = e^{at}$. This next theorem states that this is true in general.

Theorem 11.6.2. [The Convolution Theorem] If $\mathcal{L}(f) = F$ and $\mathcal{L}(g) = G$, then

$$\mathcal{L}(f * g) = FG.$$

A complete proof of the convolution theorem is beyond the scope of this book. However, we'll assume that $f * g$ has a Laplace transform and verify the conclusion of the theorem in a purely computational way. By the definition of the Laplace transform,

$$\mathcal{L}(f * g) = \int_0^\infty e^{-st}(f * g)(t) dt = \int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau) d\tau dt.$$

This iterated integral equals a double integral over the region shown in Figure 11.13. Reversing the order of integration yields

$$\mathcal{L}(f * g) = \int_0^\infty f(\tau) \int_\tau^\infty e^{-st}g(t-\tau) dt d\tau. \quad (11.81)$$

However, the substitution $x = t - \tau$ shows that

$$\begin{aligned} \int_\tau^\infty e^{-st}g(t-\tau) dt &= \int_0^\infty e^{-s(x+\tau)}g(x) dx \\ &= e^{-s\tau} \int_0^\infty e^{-sx}g(x) dx = e^{-s\tau}G(s). \end{aligned}$$

Substituting this into (11.81) and noting that $G(s)$ is independent of τ yields

$$\begin{aligned} \mathcal{L}(f * g) &= \int_0^\infty e^{-s\tau}f(\tau)G(s) d\tau \\ &= G(s) \int_0^\infty e^{-st}f(\tau) d\tau = F(s)G(s). \end{aligned}$$

Example 11.6.1: Let

$$f(t) = e^{at} \quad \text{and} \quad g(t) = e^{bt} \quad (a \neq b).$$

Verify that $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$, as implied by the convolution theorem.

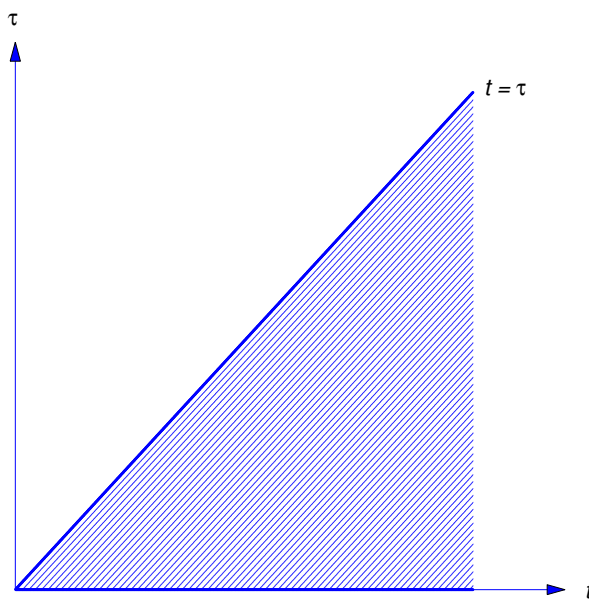


Figure 11.13

Solution: We first compute

$$\begin{aligned}
 (f * g)(t) &= \int_0^t e^{a\tau} e^{b(t-\tau)} d\tau = e^{bt} \int_0^t e^{(a-b)\tau} d\tau \\
 &= e^{bt} \left. \frac{e^{(a-b)\tau}}{a-b} \right|_0^t = \frac{e^{bt} [e^{(a-b)t} - 1]}{a-b} \\
 &= \frac{e^{at} - e^{bt}}{a-b}.
 \end{aligned}$$

Since

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad e^{bt} \leftrightarrow \frac{1}{s-b},$$

it follows that

$$\begin{aligned}
 \mathcal{L}(f * g) &= \frac{1}{a-b} \left[\frac{1}{s-a} - \frac{1}{s-b} \right] \\
 &= \frac{1}{(s-a)(s-b)} \\
 &= \mathcal{L}(e^{at}) \mathcal{L}(e^{bt}) = \mathcal{L}(f) \mathcal{L}(g).
 \end{aligned}$$

11.6.1 A Formula for the Solution of an Initial Value Problem

The convolution theorem provides a formula for the solution of an initial value problem for a linear constant coefficient second order equation with an unspecified. The next three examples illustrate this.

Example 11.6.2: Find a formula for the solution of the initial value problem

$$y'' - 2y' + y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \quad (11.82)$$

Solution: Taking Laplace transforms in (11.82) yields

$$(s^2 - 2s + 1)Y(s) = F(s) + (k_1 + k_0s) - 2k_0.$$

Therefore

$$\begin{aligned} Y(s) &= \frac{1}{(s-1)^2}F(s) + \frac{k_1 + k_0s - 2k_0}{(s-1)^2} \\ &= \frac{1}{(s-1)^2}F(s) + \frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2}. \end{aligned}$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2}\right) = e^t(k_0 + (k_1 - k_0)t).$$

Since

$$\frac{1}{(s-1)^2} \leftrightarrow te^t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}F(s)\right) = \int_0^t \tau e^\tau f(t-\tau) d\tau.$$

Therefore the solution of (11.82) is

$$y(t) = e^t(k_0 + (k_1 - k_0)t) + \int_0^t \tau e^\tau f(t-\tau) d\tau.$$

Example 11.6.3: Find a formula for the solution of the initial value problem

$$y'' + 4y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \quad (11.83)$$

Solution: Taking Laplace transforms in (11.83) yields

$$(s^2 + 4)Y(s) = F(s) + k_1 + k_0s.$$

Therefore

$$Y(s) = \frac{1}{(s^2 + 4)}F(s) + \frac{k_1 + k_0s}{s^2 + 4}.$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{k_1 + k_0s}{s^2 + 4}\right) = k_0 \cos 2t + \frac{k_1}{2} \sin 2t.$$

Since

$$\frac{1}{(s^2 + 4)} \leftrightarrow \frac{1}{2} \sin 2t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2 + 4)}F(s)\right) = \frac{1}{2} \int_0^t f(t-\tau) \sin 2\tau d\tau.$$

Therefore the solution of (11.83) is

$$y(t) = k_0 \cos 2t + \frac{k_1}{2} \sin 2t + \frac{1}{2} \int_0^t f(t-\tau) \sin 2\tau d\tau.$$

Example 11.6.4: Find a formula for the solution of the initial value problem

$$y'' + 2y' + 2y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \quad (11.84)$$

Solution: Taking Laplace transforms in (11.84) yields

$$(s^2 + 2s + 2)Y(s) = F(s) + k_1 + k_0s + 2k_0.$$

Therefore

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)^2 + 1} F(s) + \frac{k_1 + k_0s + 2k_0}{(s+1)^2 + 1} \\ &= \frac{1}{(s+1)^2 + 1} F(s) + \frac{(k_1 + k_0) + k_0(s+1)}{(s+1)^2 + 1}. \end{aligned}$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1} \left(\frac{(k_1 + k_0) + k_0(s+1)}{(s+1)^2 + 1} \right) = e^{-t} ((k_1 + k_0) \sin t + k_0 \cos t).$$

Since

$$\frac{1}{(s+1)^2 + 1} \leftrightarrow e^{-t} \sin t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1} \left(\frac{1}{(s+1)^2 + 1} F(s) \right) = \int_0^t f(t-\tau) e^{-\tau} \sin \tau \, d\tau.$$

Therefore the solution of (11.84) is

$$y(t) = e^{-t} ((k_1 + k_0) \sin t + k_0 \cos t) + \int_0^t f(t-\tau) e^{-\tau} \sin \tau \, d\tau. \quad (11.85)$$

11.6.2 Evaluating Convolution Integrals

We'll say that an integral of the form $\int_0^t u(\tau)v(t-\tau) \, d\tau$ is a *convolution integral*. The convolution theorem provides a convenient way to evaluate convolution integrals.

Example 11.6.5: Evaluate the convolution integral

$$h(t) = \int_0^t (t-\tau)^5 \tau^7 \, d\tau.$$

Solution: We could evaluate this integral by expanding $(t-\tau)^5$ in powers of τ and then integrating. However, the convolution theorem provides an easier way. The integral is the convolution of $f(t) = t^5$ and $g(t) = t^7$. Since

$$t^5 \leftrightarrow \frac{5!}{s^6} \quad \text{and} \quad t^7 \leftrightarrow \frac{7!}{s^8},$$

the convolution theorem implies that

$$h(t) \leftrightarrow \frac{5!7!}{s^{14}} = \frac{5!7!}{13!} \frac{13!}{s^{14}},$$

where we have written the second equality because

$$\frac{13!}{s^{14}} \leftrightarrow t^{13}.$$

Hence,

$$h(t) = \frac{5!7!}{13!} t^{13}.$$

Example 11.6.6: Use the convolution theorem and a partial fraction expansion to evaluate the convolution integral

$$h(t) = \int_0^t \sin a(t - \tau) \cos b\tau \, d\tau \quad (|a| \neq |b|).$$

Solution: Since

$$\sin at \leftrightarrow \frac{a}{s^2 + a^2} \quad \text{and} \quad \cos bt \leftrightarrow \frac{s}{s^2 + b^2},$$

the convolution theorem implies that

$$H(s) = \frac{a}{s^2 + a^2} \frac{s}{s^2 + b^2}.$$

Expanding this in a partial fraction expansion yields

$$H(s) = \frac{a}{b^2 - a^2} \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right].$$

Therefore

$$h(t) = \frac{a}{b^2 - a^2} (\cos at - \cos bt).$$

11.6.3 Volterra Integral Equations

An equation of the form

$$y(t) = f(t) + \int_0^t k(t - \tau)y(\tau) \, d\tau \tag{11.86}$$

is a *Volterra integral equation*. Here f and k are given functions and y is unknown. Since the integral on the right is a convolution integral, the convolution theorem provides a convenient formula for solving (11.86). Taking Laplace transforms in (11.86) yields

$$Y(s) = F(s) + K(s)Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s) = \frac{F(s)}{1 - K(s)}.$$

We then obtain the solution of (11.86) as $y = \mathcal{L}^{-1}(Y)$.

Example 11.6.7: Solve the integral equation

$$y(t) = 1 + 2 \int_0^t e^{-2(t-\tau)} y(\tau) \, d\tau. \tag{11.87}$$

Solution: Taking Laplace transforms in (11.87) yields

$$Y(s) = \frac{1}{s} + \frac{2}{s+2} Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s) = \frac{1}{s} + \frac{2}{s^2}.$$

Hence,

$$y(t) = 1 + 2t.$$

11.6.4 Transfer Functions

The next theorem presents a formula for the solution of the general initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

where we assume for simplicity that f is continuous on $[0, \infty)$ and that $\mathcal{L}(f)$ exists. In Exercises 11.6.11–11.6.14 it's shown that the formula is valid under much weaker conditions on f .

Theorem 11.6.3. *Suppose f is continuous on $[0, \infty)$ and has a Laplace transform. Then the solution of the initial value problem*

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1, \quad (11.88)$$

is

$$y(t) = k_0 y_1(t) + k_1 y_2(t) + \int_0^t w(\tau) f(t - \tau) d\tau, \quad (11.89)$$

where y_1 and y_2 satisfy

$$ay_1'' + by_1' + cy_1 = 0, \quad y_1(0) = 1, \quad y_1'(0) = 0, \quad (11.90)$$

and

$$ay_2'' + by_2' + cy_2 = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1, \quad (11.91)$$

and

$$w(t) = \frac{1}{a} y_2(t). \quad (11.92)$$

Proof. Taking Laplace transforms in (11.88) yields

$$p(s)Y(s) = F(s) + a(k_1 + k_0s) + bk_0,$$

where

$$p(s) = as^2 + bs + c.$$

Hence,

$$Y(s) = W(s)F(s) + V(s) \quad (11.93)$$

with

$$W(s) = \frac{1}{p(s)} \quad (11.94)$$

and

$$V(s) = \frac{a(k_1 + k_0s) + bk_0}{p(s)}. \quad (11.95)$$

Taking Laplace transforms in (11.90) and (11.91) shows that

$$p(s)Y_1(s) = as + b \quad \text{and} \quad p(s)Y_2(s) = a.$$

Therefore

$$Y_1(s) = \frac{as + b}{p(s)}$$

and

$$Y_2(s) = \frac{a}{p(s)}. \quad (11.96)$$

Hence, (11.95) can be rewritten as

$$V(s) = k_0 Y_1(s) + k_1 Y_2(s).$$

Substituting this into (11.93) yields

$$Y(s) = k_0 Y_1(s) + k_1 Y_2(s) + \frac{1}{a} Y_2(s) F(s).$$

Taking inverse transforms and invoking the convolution theorem yields (11.89). Finally, (11.94) and (11.96) imply (11.92). ■

It is useful to note from (11.89) that y is of the form

$$y = v + h,$$

where

$$v(t) = k_0 y_1(t) + k_1 y_2(t)$$

depends on the initial conditions and is independent of the forcing function, while

$$h(t) = \int_0^t w(\tau) f(t - \tau) d\tau$$

depends on the forcing function and is independent of the initial conditions. If the zeros of the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation have negative real parts, then y_1 and y_2 both approach zero as $t \rightarrow \infty$, so $\lim_{t \rightarrow \infty} v(t) = 0$ for any choice of initial conditions. Moreover, the value of $h(t)$ is essentially independent of the values of $f(t - \tau)$ for large τ , since $\lim_{\tau \rightarrow \infty} w(\tau) = 0$. In this case we say that v and h are *transient* and *steady state components*, respectively, of the solution y of (11.88). These definitions apply to the initial value problem of Example 11.6.4, where the zeros of

$$p(s) = s^2 + 2s + 2 = (s + 1)^2 + 1$$

are $-1 \pm i$. From (11.85), we see that the solution of the general initial value problem of Example 11.6.4 is $y = v + h$, where

$$v(t) = e^{-t} ((k_1 + k_0) \sin t + k_0 \cos t)$$

is the transient component of the solution and

$$h(t) = \int_0^t f(t - \tau) e^{-\tau} \sin \tau d\tau$$

is the steady state component. The definitions don't apply to the initial value problems considered in Examples 11.6.2 and 11.6.3, since the zeros of the characteristic polynomials in these two examples don't have negative real parts.

In physical applications where the input f and the output y of a device are related by (11.88), the zeros of the characteristic polynomial usually do have negative real parts. Then $W = \mathcal{L}(w)$ is called the *transfer function* of the device. Since

$$H(s) = W(s)F(s),$$

we see that

$$W(s) = \frac{H(s)}{F(s)}$$

is the ratio of the transform of the steady state output to the transform of the input.

Because of the form of

$$h(t) = \int_0^t w(\tau) f(t - \tau) d\tau,$$

w is sometimes called the *weighting function* of the device, since it assigns weights to past values of the input f . It is also called the *impulse response* of the device, for reasons discussed in the next section.

Formula (11.89) is given in more detail in Exercises 11.6.8–11.6.10 for the three possible cases where the zeros of $p(s)$ are real and distinct, real and repeated, or complex conjugates, respectively.

11.6.5 Exercises

1. Express the inverse transform as an integral.

(a) $\frac{1}{s^2(s^2 + 4)}$

(b) $\frac{s}{(s + 2)(s^2 + 9)}$

(c) $\frac{s}{(s^2 + 4)(s^2 + 9)}$

(d) $\frac{s}{(s^2 + 1)^2}$

(e) $\frac{1}{s(s - a)}$

(f) $\frac{1}{(s + 1)(s^2 + 2s + 2)}$

(g) $\frac{1}{(s + 1)^2(s^2 + 4s + 5)}$

(h) $\frac{1}{(s - 1)^3(s + 2)^2}$

(i) $\frac{s - 1}{s^2(s^2 - 2s + 2)}$

(j) $\frac{s(s + 3)}{(s^2 + 4)(s^2 + 6s + 10)}$

(k) $\frac{1}{(s - 3)^5 s^6}$

(l) $\frac{1}{(s - 1)^3(s^2 + 4)}$

(m) $\frac{1}{s^2(s - 2)^3}$

(n) $\frac{1}{s^7(s - 2)^6}$

2. Find the Laplace transform.

(a) $\int_0^t \sin a\tau \cos b(t - \tau) d\tau$

(b) $\int_0^t e^\tau \sin a(t - \tau) d\tau$

(c) $\int_0^t \sinh a\tau \cosh a(t - \tau) d\tau$

(d) $\int_0^t \tau(t - \tau) \sin \omega\tau \cos \omega(t - \tau) d\tau$

(e) $e^t \int_0^t \sin \omega\tau \cos \omega(t - \tau) d\tau$

(f) $e^t \int_0^t \tau^2(t - \tau)e^\tau d\tau$

(g) $e^{-t} \int_0^t e^{-\tau} \cos \omega(t - \tau) d\tau$

(h) $e^t \int_0^t e^{2\tau} \sinh(t - \tau) d\tau$

(i) $\int_0^t \tau e^{2\tau} \sin 2(t - \tau) d\tau$

(j) $\int_0^t (t - \tau)^3 e^\tau d\tau$

(k) $\int_0^t \tau^6 e^{-(t-\tau)} \sin 3(t - \tau) d\tau$

(l) $\int_0^t \tau^2(t - \tau)^3 d\tau$

(m) $\int_0^t (t - \tau)^7 e^{-\tau} \sin 2\tau d\tau$

(n) $\int_0^t (t - \tau)^4 \sin 2\tau d\tau$

3. Find a formula for the solution of the initial value problem.

(a) $y'' + 3y' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$

(b) $y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 0$

(c) $y'' + 2y' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$

(d) $y'' + k^2 y = f(t), \quad y(0) = 1, \quad y'(0) = -1$

(e) $y'' + 6y' + 9y = f(t), \quad y(0) = 0, \quad y'(0) = -2$

(f) $y'' - 4y = f(t), \quad y(0) = 0, \quad y'(0) = 3$

(g) $y'' - 5y' + 6y = f(t), \quad y(0) = 1, \quad y'(0) = 3$

(h) $y'' + \omega^2 y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$

4. Solve the integral equation.

$$(a) y(t) = t - \int_0^t (t - \tau)y(\tau) d\tau$$

$$(b) y(t) = \sin t - 2 \int_0^t \cos(t - \tau)y(\tau) d\tau$$

$$(c) y(t) = 1 + 2 \int_0^t y(\tau) \cos(t - \tau) d\tau \quad (d) y(t) = t + \int_0^t y(\tau)e^{-(t-\tau)} d\tau$$

$$(e) y'(t) = t + \int_0^t y(\tau) \cos(t - \tau) d\tau, y(0) = 4$$

$$(f) y(t) = \cos t - \sin t + \int_0^t y(\tau) \sin(t - \tau) d\tau$$

5. Use the convolution theorem to evaluate the integral.

$$(a) \int_0^t (t - \tau)^7 \tau^8 d\tau$$

$$(b) \int_0^t (t - \tau)^{13} \tau^7 d\tau$$

$$(c) \int_0^t (t - \tau)^6 \tau^7 d\tau$$

$$(d) \int_0^t e^{-\tau} \sin(t - \tau) d\tau$$

$$(e) \int_0^t \sin \tau \cos 2(t - \tau) d\tau$$

6. Show that

$$\int_0^t f(t - \tau)g(\tau) d\tau = \int_0^t f(\tau)g(t - \tau) d\tau$$

by introducing the new variable of integration $x = t - \tau$ in the first integral.

7. Use the convolution theorem to show that if $f(t) \leftrightarrow F(s)$ then

$$\int_0^t f(\tau) d\tau \leftrightarrow \frac{F(s)}{s}.$$

8. Show that if $p(s) = as^2 + bs + c$ has distinct real zeros r_1 and r_2 then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

is

$$\begin{aligned} y(t) = & k_0 \frac{r_2 e^{r_1 t} - r_1 e^{r_2 t}}{r_2 - r_1} + k_1 \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} \\ & + \frac{1}{a(r_2 - r_1)} \int_0^t (e^{r_2 \tau} - e^{r_1 \tau}) f(t - \tau) d\tau. \end{aligned}$$

9. Show that if $p(s) = as^2 + bs + c$ has a repeated real zero r_1 then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

is

$$y(t) = k_0(1 - r_1 t)e^{r_1 t} + k_1 t e^{r_1 t} + \frac{1}{a} \int_0^t \tau e^{r_1 \tau} f(t - \tau) d\tau.$$

10. Show that if $p(s) = as^2 + bs + c$ has complex conjugate zeros $\lambda \pm i\omega$ then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

is

$$y(t) = e^{\lambda t} \left[k_0(\cos \omega t - \frac{\lambda}{\omega} \sin \omega t) + \frac{k_1}{\omega} \sin \omega t \right] + \frac{1}{a\omega} \int_0^t e^{\lambda t} f(t - \tau) \sin \omega \tau d\tau.$$

11. Let

$$w = \mathcal{L}^{-1} \left(\frac{1}{as^2 + bs + c} \right),$$

where a, b , and c are constants and $a \neq 0$.

- (a) Show that w is the solution of

$$aw'' + bw' + cw = 0, \quad w(0) = 0, \quad w'(0) = \frac{1}{a}.$$

- (b) Let f be continuous on $[0, \infty)$ and define

$$h(t) = \int_0^t w(t - \tau) f(\tau) d\tau.$$

Use *Leibniz's rule* for differentiating an integral with respect to a parameter to show that h is the solution of

$$ah'' + bh' + ch = f, \quad h(0) = 0, \quad h'(0) = 0.$$

- (c) Show that the function y in Eqn. (11.89) is the solution of Eqn. (11.88) provided that f is continuous on $[0, \infty)$; thus, it's not necessary to assume that f has a Laplace transform.

12. Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0, \tag{A}$$

where a, b , and c are constants, $a \neq 0$, and

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t \geq t_1. \end{cases}$$

Assume that f_0 is continuous and of exponential order on $[0, \infty)$ and f_1 is continuous and of exponential order on $[t_1, \infty)$. Let

$$p(s) = as^2 + bs + c.$$

- (a) Show that the Laplace transform of the solution of (A) is

$$Y(s) = \frac{F_0(s) + e^{-st_1} G(s)}{p(s)}$$

where $g(t) = f_1(t + t_1) - f_0(t + t_1)$.

- (b) Let w be as in Exercise 11.6.11. Use Theorem 11.4.2 and the convolution theorem to show that the solution of (A) is

$$y(t) = \int_0^t w(t - \tau) f_0(\tau) d\tau + u(t - t_1) \int_0^{t-t_1} w(t - t_1 - \tau) g(\tau) d\tau$$

for $t > 0$.

- (c) Henceforth, assume only that f_0 is continuous on $[0, \infty)$ and f_1 is continuous on $[t_1, \infty)$. Use Exercise 11.6.11 (a) and (b) to show that

$$y'(t) = \int_0^t w'(t-\tau)f_0(\tau) d\tau + u(t-t_1) \int_0^{t-t_1} w'(t-t_1-\tau)g(\tau) d\tau$$

for $t > 0$, and

$$y''(t) = \frac{f(t)}{a} + \int_0^t w''(t-\tau)f_0(\tau) d\tau + u(t-t_1) \int_0^{t-t_1} w''(t-t_1-\tau)g(\tau) d\tau$$

for $0 < t < t_1$ and $t > t_1$. Also, show y satisfies the differential equation in (A) on $(0, t_1)$ and (t_1, ∞) .

- (d) Show that y and y' are continuous on $[0, \infty)$.

13. Suppose

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t_1 \leq t < t_2, \\ \vdots & \\ f_{k-1}(t), & t_{k-1} \leq t < t_k, \\ f_k(t), & t \geq t_k, \end{cases}$$

where f_m is continuous on $[t_m, \infty)$ for $m = 0, \dots, k$ (let $t_0 = 0$), and define

$$g_m(t) = f_m(t+t_m) - f_{m-1}(t+t_m), \quad m = 1, \dots, k.$$

Extend the results of Exercise 11.6.12 to show that the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

is

$$y(t) = \int_0^t w(t-\tau)f_0(\tau) d\tau + \sum_{m=1}^k u(t-t_m) \int_0^{t-t_m} w(t-t_m-\tau)g_m(\tau) d\tau.$$

14. Let $\{t_m\}_{m=0}^\infty$ be a sequence of points such that $t_0 = 0$, $t_{m+1} > t_m$, and $\lim_{m \rightarrow \infty} t_m = \infty$. For each nonnegative integer m let f_m be continuous on $[t_m, \infty)$, and let f be defined on $[0, \infty)$ by

$$f(t) = f_m(t), \quad t_m \leq t < t_{m+1} \quad m = 0, 1, 2, \dots$$

Let

$$g_m(t) = f_m(t+t_m) - f_{m-1}(t+t_m), \quad m = 1, \dots, k.$$

Extend the results of Exercise 11.6.13 to show that the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

is

$$y(t) = \int_0^t w(t-\tau)f_0(\tau) d\tau + \sum_{m=1}^\infty u(t-t_m) \int_0^{t-t_m} w(t-t_m-\tau)g_m(\tau) d\tau.$$

HINT: See Exercise 11.4.30.

11.7 Constant Coefficient Equations with Impulses & the Dirac Delta

So far in this chapter, we've considered initial value problems for the constant coefficient equation

$$ay'' + by' + cy = f(t),$$

where f is continuous or piecewise continuous on $[0, \infty)$. In this section we consider initial value problems where f represents a force that's very large for a short time and zero otherwise. We say that such forces are *impulsive*. Impulsive forces occur, for example, when two objects collide. Since it isn't feasible to represent such forces as continuous or piecewise continuous functions, we must construct a different mathematical model to deal with them.

If f is an integrable function and $f(t) = 0$ for t outside of the interval $[t_0, t_0 + h]$, then $\int_{t_0}^{t_0+h} f(t) dt$ is called the *total impulse* of f . We're interested in the idealized situation where h is so small that the total impulse can be assumed to be applied instantaneously at $t = t_0$. We say in this case that f is an *impulse function*. In particular, we denote by $\delta(t - t_0)$ the impulse function with total impulse equal to one, applied at $t = t_0$. (The impulse function $\delta(t)$ obtained by setting $t_0 = 0$ is the *Dirac δ function*.) It must be understood, however, that $\delta(t - t_0)$ isn't a function in the standard sense, since our "definition" implies that $\delta(t - t_0) = 0$ if $t \neq t_0$, while

$$\int_{t_0}^{t_0} \delta(t - t_0) dt = 1.$$

From calculus we know that no function can have these properties; nevertheless, there's a branch of mathematics known as the *theory of distributions* where the definition can be made rigorous. Since the theory of distributions is beyond the scope of this book, we'll take an intuitive approach to impulse functions.

Our first task is to define what we mean by the solution of the initial value problem

$$ay'' + by' + cy = \delta(t - t_0), \quad y(0) = 0, \quad y'(0) = 0,$$

where t_0 is a fixed nonnegative number. The next theorem will motivate our definition.

Theorem 11.7.1. *Suppose $t_0 \geq 0$. For each positive number h , let y_h be the solution of the initial value problem*

$$ay_h'' + by_h' + cy_h = f_h(t), \quad y_h(0) = 0, \quad y_h'(0) = 0, \quad (11.97)$$

where

$$f_h(t) = \begin{cases} 0, & 0 \leq t < t_0, \\ 1/h, & t_0 \leq t < t_0 + h, \\ 0, & t \geq t_0 + h, \end{cases} \quad (11.98)$$

so f_h has unit total impulse equal to the area of the shaded rectangle in Figure 11.14. Then

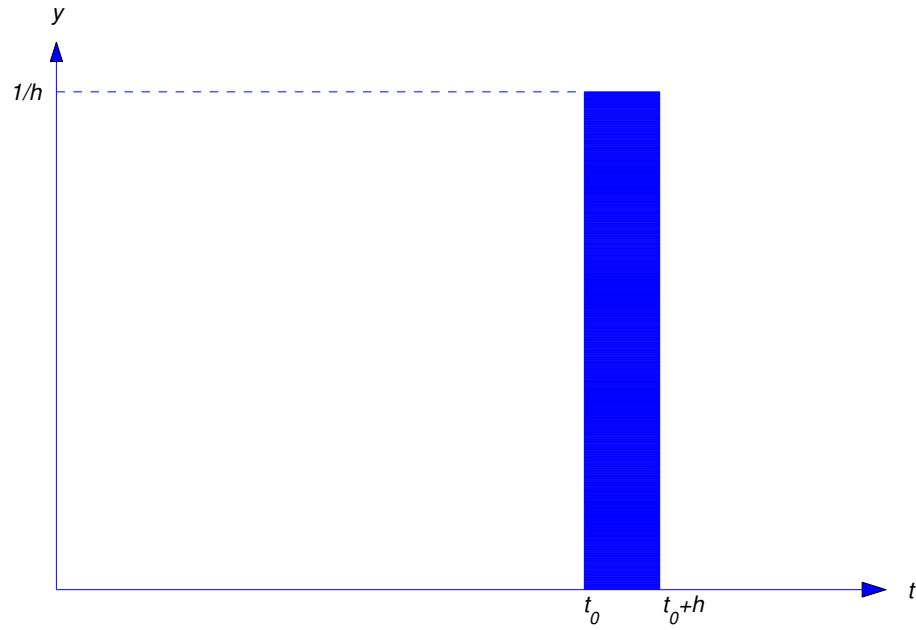
$$\lim_{h \rightarrow 0^+} y_h(t) = u(t - t_0)w(t - t_0), \quad (11.99)$$

where

$$w = \mathcal{L}^{-1} \left(\frac{1}{as^2 + bs + c} \right).$$

Proof. Taking Laplace transforms in (11.97) yields

$$(as^2 + bs + c)Y_h(s) = F_h(s),$$

Figure 11.14: $y = f_h(t)$

so

$$Y_h(s) = \frac{F_h(s)}{as^2 + bs + c}.$$

The convolution theorem implies that

$$y_h(t) = \int_0^t w(t - \tau) f_h(\tau) d\tau.$$

Therefore, (11.98) implies that

$$y_h(t) = \begin{cases} 0, & 0 \leq t < t_0, \\ \frac{1}{h} \int_{t_0}^t w(t - \tau) d\tau, & t_0 \leq t \leq t_0 + h, \\ \frac{1}{h} \int_{t_0}^{t_0+h} w(t - \tau) d\tau, & t > t_0 + h. \end{cases} \quad (11.100)$$

Since $y_h(t) = 0$ for all h if $0 \leq t \leq t_0$, it follows that

$$\lim_{h \rightarrow 0+} y_h(t) = 0 \quad \text{if} \quad 0 \leq t \leq t_0. \quad (11.101)$$

We'll now show that

$$\lim_{h \rightarrow 0+} y_h(t) = w(t - t_0) \quad \text{if} \quad t > t_0. \quad (11.102)$$

Suppose t is fixed and $t > t_0$. From (11.100),

$$y_h(t) = \frac{1}{h} \int_{t_0}^{t_0+h} w(t - \tau) d\tau \quad \text{if} \quad h < t - t_0. \quad (11.103)$$

Since

$$\frac{1}{h} \int_{t_0}^{t_0+h} d\tau = 1, \quad (11.104)$$

we can write

$$w(t - t_0) = \frac{1}{h} w(t - t_0) \int_{t_0}^{t_0+h} d\tau = \frac{1}{h} \int_{t_0}^{t_0+h} w(t - t_0) d\tau.$$

From this and (11.103),

$$y_h(t) - w(t - t_0) = \frac{1}{h} \int_{t_0}^{t_0+h} (w(t - \tau) - w(t - t_0)) d\tau.$$

Therefore

$$|y_h(t) - w(t - t_0)| \leq \frac{1}{h} \int_{t_0}^{t_0+h} |w(t - \tau) - w(t - t_0)| d\tau. \quad (11.105)$$

Now let M_h be the maximum value of $|w(t - \tau) - w(t - t_0)|$ as τ varies over the interval $[t_0, t_0 + h]$. (Remember that t and t_0 are fixed.) Then (11.104) and (11.105) imply that

$$|y_h(t) - w(t - t_0)| \leq \frac{1}{h} M_h \int_{t_0}^{t_0+h} d\tau = M_h. \quad (11.106)$$

But $\lim_{h \rightarrow 0^+} M_h = 0$, since w is continuous. Therefore (11.106) implies (11.102). This and (11.101) imply (11.99). ■

Theorem 11.7.1 motivates the next definition.

Definition 11.7.2 If $t_0 > 0$, then the solution of the initial value problem

$$ay'' + by' + cy = \delta(t - t_0), \quad y(0) = 0, \quad y'(0) = 0, \quad (11.107)$$

is defined to be

$$y = u(t - t_0)w(t - t_0),$$

where

$$w = \mathcal{L}^{-1} \left(\frac{1}{as^2 + bs + c} \right).$$

In physical applications where the input f and the output y of a device are related by the differential equation

$$ay'' + by' + cy = f(t),$$

w is called the *impulse response* of the device. Note that w is the solution of the initial value problem

$$aw'' + bw' + cw = 0, \quad w(0) = 0, \quad w'(0) = 1/a, \quad (11.108)$$

as can be seen by using the Laplace transform to solve this problem. (Verify.) On the other hand, we can solve (11.108) by the methods of Section 5.2 and show that w is defined on $(-\infty, \infty)$ by

$$w = \frac{e^{r_2 t} - e^{r_1 t}}{a(r_2 - r_1)}, \quad w = \frac{1}{a} t e^{r_1 t}, \quad \text{or} \quad w = \frac{1}{a\omega} e^{\lambda t} \sin \omega t, \quad (11.109)$$

depending upon whether the polynomial $p(r) = ar^2 + br + c$ has distinct real zeros r_1 and r_2 , a repeated zero r_1 , or complex conjugate zeros $\lambda \pm i\omega$. (In most physical applications, the zeros of the characteristic polynomial have negative real parts, so $\lim_{t \rightarrow \infty} w(t) = 0$.) This means that $y = u(t - t_0)w(t - t_0)$ is defined on $(-\infty, \infty)$ and has the following properties:

$$y(t) = 0, \quad t < t_0,$$

$$ay'' + by' + cy = 0 \quad \text{on} \quad (-\infty, t_0) \quad \text{and} \quad (t_0, \infty),$$

and

$$y'_-(t_0) = 0, \quad y'_+(t_0) = 1/a \quad (11.110)$$

(remember that $y'_-(t_0)$ and $y'_+(t_0)$ are derivatives from the right and left, respectively) and $y'(t_0)$ does not exist. Thus, even though we defined $y = u(t - t_0)w(t - t_0)$ to be the solution of (11.107), this function *doesn't satisfy* the differential equation in (11.107) *at* t_0 , since it isn't differentiable there; in fact (11.110) indicates that an impulse causes a jump discontinuity in velocity. (To see that this is reasonable, think of what happens when you hit a ball with a bat.) This means that the initial value problem (11.107) doesn't make sense if $t_0 = 0$, since $y'(0)$ doesn't exist in this case. However $y = u(t)w(t)$ can be defined to be the solution of the modified initial value problem

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'_-(0) = 0,$$

where the condition on the derivative at $t = 0$ has been replaced by a condition on the derivative from the left.

Figure 11.15 illustrates Theorem 11.7.1 for the case where the impulse response w is the first expression in (11.109) and r_1 and r_2 are distinct and both negative. The solid curve in the figure is the graph of w . The dashed curves are solutions of (11.97) for various values of h . As h decreases the graph of y_h moves to the left toward the graph of w .

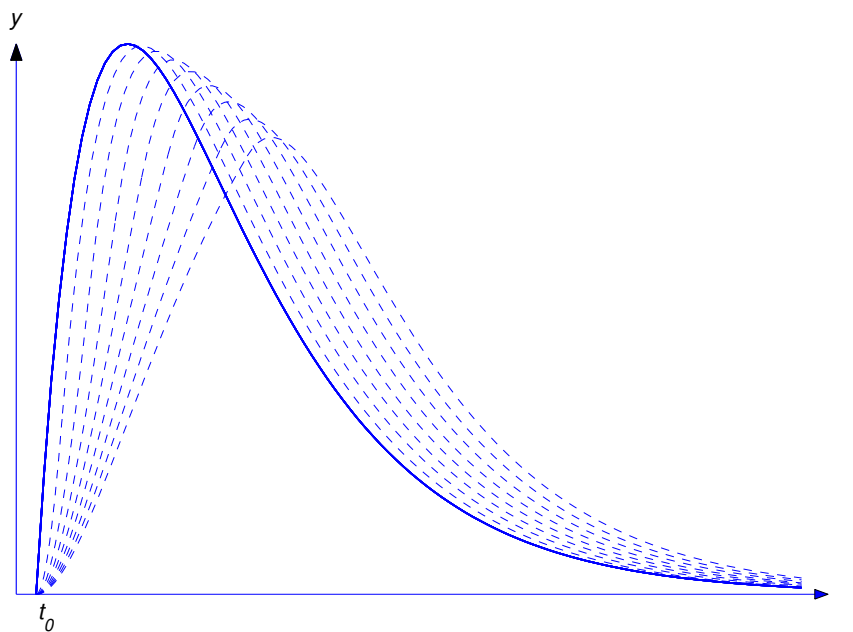


Figure 11.15: An illustration of Theorem 11.7.1

Example 11.7.1: Find the solution of the initial value problem

$$y'' - 2y' + y = \delta(t - t_0), \quad y(0) = 0, \quad y'(0) = 0, \quad (11.111)$$

where $t_0 > 0$. Then interpret the solution for the case where $t_0 = 0$.

Solution: Here

$$w = \mathcal{L}^{-1} \left(\frac{1}{s^2 - 2s + 1} \right) = \mathcal{L}^{-1} \left(\frac{1}{(s - 1)^2} \right) = te^{-t},$$

so Definition 11.7.2 yields

$$y = u(t - t_0)(t - t_0)e^{-(t-t_0)}$$

as the solution of (11.111) if $t_0 > 0$. If $t_0 = 0$, then (11.111) doesn't have a solution; however, $y = u(t)te^{-t}$ (which we would usually write simply as $y = te^{-t}$) is the solution of the modified initial value problem

$$y'' - 2y' + y = \delta(t), \quad y(0) = 0, \quad y'_-(0) = 0.$$

The graph of $y = u(t - t_0)(t - t_0)e^{-(t-t_0)}$ is shown in Figure 11.16

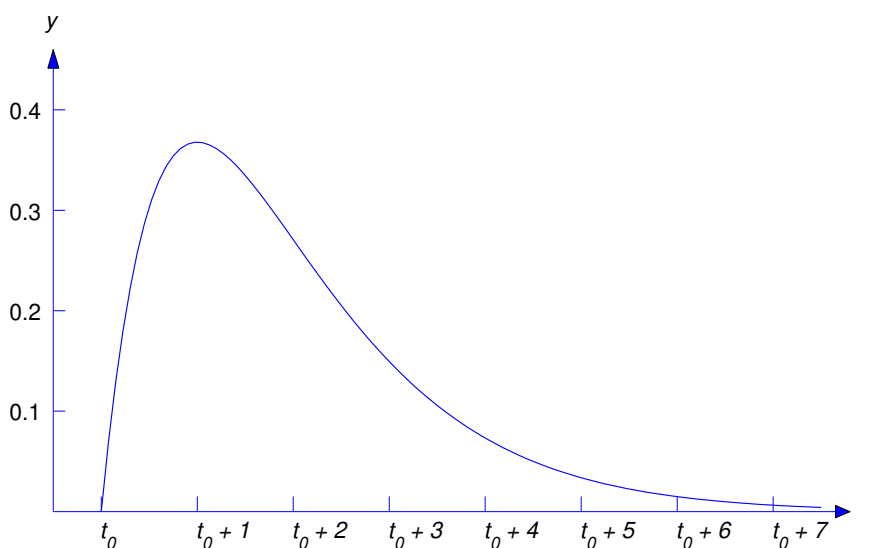


Figure 11.16: $y = u(t - t_0)(t - t_0)e^{-(t-t_0)}$

Definition 11.7.2 and the principle of superposition motivate the next definition.

Definition 11.7.3 Suppose α is a nonzero constant and f is piecewise continuous on $[0, \infty)$. If $t_0 > 0$, then the solution of the initial value problem

$$ay'' + by' + cy = f(t) + \alpha\delta(t - t_0), \quad y(0) = k_0, \quad y'(0) = k_1$$

is defined to be

$$y(t) = \hat{y}(t) + \alpha u(t - t_0)w(t - t_0),$$

where $\hat{y}$ is the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1.$$

This definition also applies if $t_0 = 0$, provided that the initial condition $y'(0) = k_1$ is replaced by $y'_-(0) = k_1$.

Example 11.7.2: Solve the initial value problem

$$y'' + 6y' + 5y = 3e^{-2t} + 2\delta(t - 1), \quad y(0) = -3, \quad y'(0) = 2. \quad (11.112)$$

Solution: We leave it to you to show that the solution of

$$y'' + 6y' + 5y = 3e^{-2t}, \quad y(0) = -3, \quad y'(0) = 2$$

is

$$\hat{y} = -e^{-2t} + \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t}.$$

Since

$$\begin{aligned} w(t) &= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 6s + 5}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s+5)}\right) \\ &= \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{s+5}\right) = \frac{e^{-t} - e^{-5t}}{4}, \end{aligned}$$

the solution of (11.112) is

$$y = -e^{-2t} + \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t} + u(t-1)\frac{e^{-(t-1)} - e^{-5(t-1)}}{2} \quad (11.113)$$

(Figure 11.17)

■.

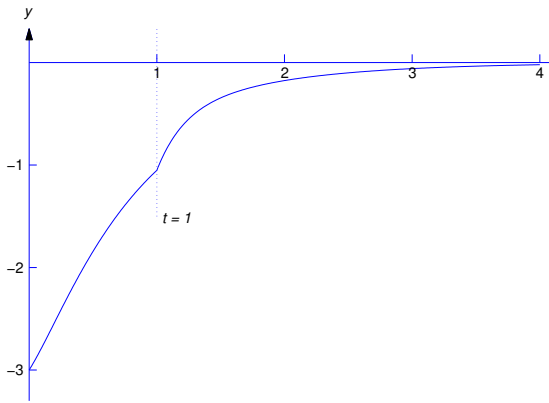


Figure 11.17: Graph of (11.113)

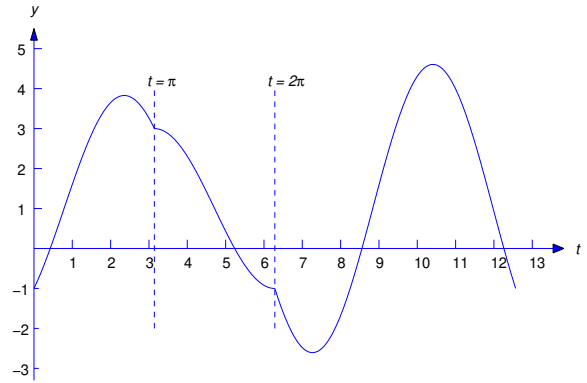


Figure 11.18: Graph of (11.115)

Definition 11.7.3 can be extended in the obvious way to cover the case where the forcing function contains more than one impulse.

Example 11.7.3: Solve the initial value problem

$$y'' + y = 1 + 2\delta(t - \pi) - 3\delta(t - 2\pi), \quad y(0) = -1, \quad y'(0) = 2. \quad (11.114)$$

Solution: We leave it to you to show that

$$\hat{y} = 1 - 2\cos t + 2\sin t$$

is the solution of

$$y'' + y = 1, \quad y(0) = -1, \quad y'(0) = 2.$$

Since

$$w = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) = \sin t,$$

the solution of (11.114) is

$$\begin{aligned} y &= 1 - 2\cos t + 2\sin t + 2u(t - \pi)\sin(t - \pi) - 3u(t - 2\pi)\sin(t - 2\pi) \\ &= 1 - 2\cos t + 2\sin t - 2u(t - \pi)\sin t - 3u(t - 2\pi)\sin t, \end{aligned}$$

or

$$y = \begin{cases} 1 - 2 \cos t + 2 \sin t, & 0 \leq t < \pi, \\ 1 - 2 \cos t, & \pi \leq t < 2\pi, \\ 1 - 2 \cos t - 3 \sin t, & t \geq 2\pi \end{cases} \quad (11.115)$$

(Figure 11.18).

11.7.1 Exercises

In Exercises 11.7.1–11.7.20 solve the initial value problem. Where indicated by C/G, graph the solution.

1. $y'' + 3y' + 2y = 6e^{2t} + 2\delta(t - 1), \quad y(0) = 2, \quad y'(0) = -6$
2. C/G $y'' + y' - 2y = -10e^{-t} + 5\delta(t - 1), \quad y(0) = 7, \quad y'(0) = -9$
3. $y'' - 4y = 2e^{-t} + 5\delta(t - 1), \quad y(0) = -1, \quad y'(0) = 2$
4. C/G $y'' + y = \sin 3t + 2\delta(t - \pi/2), \quad y(0) = 1, \quad y'(0) = -1$
5. $y'' + 4y = 4 + \delta(t - 3\pi), \quad y(0) = 0, \quad y'(0) = 1$
6. $y'' - y = 8 + 2\delta(t - 2), \quad y(0) = -1, \quad y'(0) = 1$
7. $y'' + y' = e^t + 3\delta(t - 6), \quad y(0) = -1, \quad y'(0) = 4$
8. $y'' + 4y = 8e^{2t} + \delta(t - \pi/2), \quad y(0) = 8, \quad y'(0) = 0$
9. C/G $y'' + 3y' + 2y = 1 + \delta(t - 1), \quad y(0) = 1, \quad y'(0) = -1$
10. $y'' + 2y' + y = e^t + 2\delta(t - 2), \quad y(0) = -1, \quad y'(0) = 2$
11. C/G $y'' + 4y = \sin t + \delta(t - \pi/2), \quad y(0) = 0, \quad y'(0) = 2$
12. $y'' + 2y' + 2y = \delta(t - \pi) - 3\delta(t - 2\pi), \quad y(0) = -1, \quad y'(0) = 2$
13. $y'' + 4y' + 13y = \delta(t - \pi/6) + 2\delta(t - \pi/3), \quad y(0) = 1, \quad y'(0) = 2$
14. $2y'' - 3y' - 2y = 1 + \delta(t - 2), \quad y(0) = -1, \quad y'(0) = 2$
15. $4y'' - 4y' + 5y = 4\sin t - 4\cos t + \delta(t - \pi/2) - \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 1$
16. $y'' + y = \cos 2t + 2\delta(t - \pi/2) - 3\delta(t - \pi), \quad y(0) = 0, \quad y'(0) = -1$
17. C/G $y'' - y = 4e^{-t} - 5\delta(t - 1) + 3\delta(t - 2), \quad y(0) = 0, \quad y'(0) = 0$
18. $y'' + 2y' + y = e^t - \delta(t - 1) + 2\delta(t - 2), \quad y(0) = 0, \quad y'(0) = -1$
19. $y'' + y = f(t) + \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1$, and

$$f(t) = \begin{cases} \sin 2t, & 0 \leq t < \pi, \\ 0, & t \geq \pi. \end{cases}$$
20. $y'' + 4y = f(t) + \delta(t - \pi) - 3\delta(t - 3\pi/2), \quad y(0) = 1, \quad y'(0) = -1$, and

$$f(t) = \begin{cases} 1, & 0 \leq t < \pi/2, \\ 2, & t \geq \pi/2 \end{cases}$$
21. $y'' + y = \delta(t), \quad y(0) = 1, \quad y'_-(0) = -2$
22. $y'' - 4y = 3\delta(t), \quad y(0) = -1, \quad y'_-(0) = 7$
23. $y'' + 3y' + 2y = -5\delta(t), \quad y(0) = 0, \quad y'_-(0) = 0$
24. $y'' + 4y' + 4y = -\delta(t), \quad y(0) = 1, \quad y'_-(0) = 5$
25. $4y'' + 4y' + y = 3\delta(t), \quad y(0) = 1, \quad y'_-(0) = -6$

In Exercises 11.7.26–11.7.28, solve the initial value problem

$$ay''_h + by'_h + cy_h = \begin{cases} 0, & 0 \leq t < t_0, \\ 1/h, & t_0 \leq t < t_0 + h, \\ 0, & t \geq t_0 + h, \end{cases} \quad y_h(0) = 0, \quad y'_h(0) = 0,$$

where $t_0 > 0$ and $h > 0$. Then find

$$w = \mathcal{L}^{-1} \left(\frac{1}{as^2 + bs + c} \right)$$

and verify Theorem 11.7.1 by graphing w and y_h on the same axes, for small positive values of h .

26. $\boxed{\text{L}}$ $y'' + 2y' + 2y = f_h(t), \quad y(0) = 0, \quad y'(0) = 0$

27. $\boxed{\text{L}}$ $y'' + 2y' + y = f_h(t), \quad y(0) = 0, \quad y'(0) = 0$

28. $\boxed{\text{L}}$ $y'' + 3y' + 2y = f_h(t), \quad y(0) = 0, \quad y'(0) = 0$

29. Recall from Section 6.2 that the displacement of an object of mass m in a spring–mass system in free damped oscillation is

$$my'' + cy' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

and that y can be written as

$$y = Re^{-ct/2m} \cos(\omega_1 t - \phi)$$

if the motion is underdamped. Suppose $y(\tau) = 0$. Find the impulse that would have to be applied to the object at $t = \tau$ to put it in equilibrium.

30. Solve the initial value problem. Find a formula that does not involve step functions and represents y on each subinterval of $[0, \infty)$ on which the forcing function is zero.

(a) $y'' - y = \sum_{k=1}^{\infty} \delta(t - k), \quad y(0) = 0, \quad y'(0) = 1$

(b) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi), \quad y(0) = 0, \quad y'(0) = 1$

(c) $y'' - 3y' + 2y = \sum_{k=1}^{\infty} \delta(t - k), \quad y(0) = 0, \quad y'(0) = 1$

(d) $y'' + y = \sum_{k=1}^{\infty} \delta(t - k\pi), \quad y(0) = 0, \quad y'(0) = 0$

11.8 Table of Laplace Transforms

On the next page is a brief table of common Laplace transforms.

Table of Laplace Transforms

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	(1) $\delta(t)$	1 (19)
$f'(t)$	$sF(s) - f(0)$	(2) $\delta(t - t_0)$	e^{-st_0} (20)
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	(3) $\frac{e^{at} - e^{bt}}{a - b}$	$\frac{1}{(s - a)(s - b)}$ (21)
$f^{(n)}(t)$	$s^n F(s) - s^{(n-1)}f(0) - \dots - f^{(n-1)}(0)$	$\frac{ae^{at} - be^{bt}}{a - b}$	$\frac{s}{(s - a)(s - b)}$ (22)
$e^{at}f(t)$	$F(s - a)$	(5) $\sinh(kt)$	$\frac{k}{s^2 - k^2}$ (23)
$f(t - a)\mathcal{U}(t - a)$	$e^{-as}F(s)$	(6) $\cosh(kt)$	$\frac{s}{s^2 - k^2}$ (24)
$f(t)\mathcal{U}(t - a)$	$e^{-as}F(a + s)$	(7) te^{at}	$\frac{1}{(s - a)^2}$ (25)
$tf(t)$	$-\frac{d}{ds}F(s)$	(8) $t^n e^{at}$	$\frac{n!}{(s - a)^{n+1}}$ (26)
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	(9) $e^{at} \sin(kt)$	$\frac{k}{(s - a)^2 + k^2}$ (27)
$\int_0^t f(\tau)g(t - \tau)d\tau$	$F(s)G(s)$	(10) $e^{at} \cos(kt)$	$\frac{s - a}{(s - a)^2 + k^2}$ (28)
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	(11) $e^{at} \sinh(kt)$	$\frac{k}{(s - a)^2 - k^2}$ (29)
1	$\frac{1}{s}$	(12) $e^{at} \cosh(kt)$	$\frac{s - a}{(s - a)^2 - k^2}$ (30)
$t^n (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$	(13) $t \sin(kt)$	$\frac{2ks}{(s^2 + k^2)^2}$ (31)
$t^a (a > -1)$	$\frac{\Gamma(a + 1)}{s^{a+1}}$	(14) $t \cos(kt)$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ (32)
$\sin(kt)$	$\frac{k}{s^2 + k^2}$	(15) $t \sinh(kt)$	$\frac{2ks}{(s^2 - k^2)^2}$ (33)
$\cos(kt)$	$\frac{s}{s^2 + k^2}$	(16) $t \cosh(kt)$	$\frac{s^2 - k^2}{(s^2 - k^2)^2}$ (34)
e^{at}	$\frac{1}{s - a}$	(17) $\frac{\sin(kt)}{t}$	$\arctan\left(\frac{k}{s}\right)$ (35)
$\mathcal{U}(t - a)$	$\frac{e^{-as}}{s}$	(18)	

FACTS: • $\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt$; $\Gamma(a + 1) = a\Gamma(a)$; $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ • $\cosh(t) = \frac{e^t + e^{-t}}{2}$; $\sinh(t) = \frac{e^t - e^{-t}}{2}$

Chapter 12

More about the Laplace Transform

12.1 The Laplace Transform

12.1.1 The Transform

In this chapter we will discuss the Laplace transform[†]. The Laplace transform is a very efficient method to solve certain ODE or PDE problems. The transform takes a differential equation and turns it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution. The Laplace transform also has applications in the analysis of electrical circuits, NMR spectroscopy, signal processing, and elsewhere. Finally, understanding the Laplace transform will also help with understanding the related Fourier transform, which, however, requires more understanding of complex numbers. We will not cover the Fourier transform.

The Laplace transform also gives a lot of insight into the nature of the equations we are dealing with. It can be seen as converting between the time and the frequency domain. For example, take the standard equation

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

We can think of t as time and $f(t)$ as incoming signal. The Laplace transform will convert the equation from a differential equation in time to an algebraic (no derivatives) equation, where the new independent variable s is the frequency.

We can think of the *Laplace transform* as a black box. It eats functions and spits out functions in a new variable. We write $\mathcal{L}\{f(t)\} = F(s)$ for the Laplace transform of $f(t)$. It is common to write lower case letters for functions in the time domain and upper case letters for functions in the frequency domain. We use the same letter to denote that one function is the Laplace transform of the other. For example, $F(s)$ is the Laplace transform of $f(t)$. Let us define the transform.

$$\mathcal{L}\{f(t)\} = F(s) \stackrel{\text{def}}{=} \int_0^{\infty} e^{-st} f(t) dt.$$

We note that we are only considering $t \geq 0$ in the transform. Of course, if we think of t as time there is no problem, we are generally interested in finding out what will happen in the future (Laplace transform is one place where it is safe to ignore the past). Let us compute some simple transforms.

Example 12.1.1: Suppose $f(t) = 1$, then

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_{t=0}^{\infty} = \lim_{h \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_{t=0}^h = \lim_{h \rightarrow \infty} \left(\frac{e^{-sh}}{-s} - \frac{1}{-s} \right) = \frac{1}{s}.$$

[†]Just like the Laplace equation and the Laplacian, the Laplace transform is also named after [Pierre-Simon, marquis de Laplace](#) (1749–1827).

The limit (the improper integral) only exists if $s > 0$. So $\mathcal{L}\{1\}$ is only defined for $s > 0$.

Example 12.1.2: Suppose $f(t) = e^{-at}$, then

$$\mathcal{L}\{e^{-at}\} = \int_0^{\infty} e^{-st} e^{-at} dt = \int_0^{\infty} e^{-(s+a)t} dt = \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_{t=0}^{\infty} = \frac{1}{s+a}.$$

The limit only exists if $s + a > 0$. So $\mathcal{L}\{e^{-at}\}$ is only defined for $s + a > 0$.

Example 12.1.3: Suppose $f(t) = t$, then using integration by parts

$$\begin{aligned} \mathcal{L}\{t\} &= \int_0^{\infty} e^{-st} t dt \\ &= \left[\frac{-te^{-st}}{s} \right]_{t=0}^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt \\ &= 0 + \frac{1}{s} \left[\frac{e^{-st}}{-s} \right]_{t=0}^{\infty} \\ &= \frac{1}{s^2}. \end{aligned}$$

Again, the limit only exists if $s > 0$.

Example 12.1.4: A common function is the *unit step function*, which is sometimes called the *Heaviside function**. This function is generally given as

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Let us find the Laplace transform of $u(t - a)$, where $a \geq 0$ is some constant. That is, the function that is 0 for $t < a$ and 1 for $t \geq a$.

$$\mathcal{L}\{u(t - a)\} = \int_0^{\infty} e^{-st} u(t - a) dt = \int_a^{\infty} e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_{t=a}^{\infty} = \frac{e^{-as}}{s},$$

where of course $s > 0$ (and $a \geq 0$ as we said before).

By applying similar procedures we can compute the transforms of many elementary functions. Many basic transforms are listed in [Table 12.1](#) on the next page.

Exercise 12.1.1: Verify [Table 12.1](#) on the facing page.

Since the transform is defined by an integral. We can use the linearity properties of the integral. For example, suppose C is a constant, then

$$\mathcal{L}\{Cf(t)\} = \int_0^{\infty} e^{-st} Cf(t) dt = C \int_0^{\infty} e^{-st} f(t) dt = C\mathcal{L}\{f(t)\}.$$

So we can “pull out” a constant out of the transform. Similarly we have linearity. Since linearity is very important we state it as a theorem.

Theorem 12.1.1 (Linearity of the Laplace transform). *Suppose that A , B , and C are constants, then*

$$\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\},$$

and in particular,

$$\mathcal{L}\{Cf(t)\} = C\mathcal{L}\{f(t)\}.$$

*The function is named after the English mathematician, engineer, and physicist [Oliver Heaviside](#) (1850–1925). Only by coincidence is the function “heavy” on “one side.”

$f(t)$	$\mathcal{L}\{f(t)\}$	$f(t)$	$\mathcal{L}\{f(t)\}$
C	$\frac{C}{s}$	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
t	$\frac{1}{s^2}$	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
t^2	$\frac{2}{s^3}$	$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
t^3	$\frac{6}{s^4}$	$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$u(t - a)$	$\frac{e^{-as}}{s}$
e^{-at}	$\frac{1}{s+a}$		

Table 12.1: Some Laplace transforms (C , ω , and a are constants).

Exercise 12.1.2: Verify the theorem. That is, show that $\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$.

These rules together with Table 12.1 make it easy to find the Laplace transform of a whole lot of functions already. But be careful. It is a common mistake to think that the Laplace transform of a product is the product of the transforms. In general

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

It must also be noted that not all functions have a Laplace transform. For example, the function $\frac{1}{t}$ does not have a Laplace transform as the integral diverges for all s . Similarly, $\tan t$ or e^{t^2} do not have Laplace transforms.

12.1.2 Existence and Uniqueness

When does the Laplace transform exist? A function $f(t)$ is of *exponential order* as t goes to infinity if

$$|f(t)| \leq Me^{ct},$$

for some constants M and c , for sufficiently large t (say for all $t > t_0$ for some t_0). The simplest way to check this condition is to try and compute

$$\lim_{t \rightarrow \infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

Exercise 12.1.3: Use L'Hopital's rule from calculus to show that a polynomial is of exponential order. Hint: Note that a sum of two exponential order functions is also of exponential order. Then show that t^n is of exponential order for any n .

For an exponential order function we have existence and uniqueness of the Laplace transform.

Theorem 12.1.2 (Existence). Let $f(t)$ be continuous on the interval $[0, \infty)$ and of exponential order for a certain constant c . Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

The existence is not difficult to see. Let $f(t)$ be of exponential order, that is $|f(t)| \leq Me^{ct}$ for all $t > 0$ (for simplicity $t_0 = 0$). Let $s > c$, or in other words $(s - c) > 0$. By the comparison theorem from calculus, the improper integral defining $\mathcal{L}\{f(t)\}$ exists if the following integral exists

$$\int_0^\infty e^{-st}(Me^{ct}) dt = M \int_0^\infty e^{-(s-c)t} dt = M \left[\frac{e^{-(s-c)t}}{-(s-c)} \right]_{t=0}^\infty = \frac{M}{s-c}.$$

The transform also exists for some other functions that are not of exponential order, but that will not be relevant to us. Before dealing with uniqueness, let us note that for exponential order functions, we obtain that their Laplace transform decays at infinity:

$$\lim_{s \rightarrow \infty} F(s) = 0.$$

Theorem 12.1.3 (Uniqueness). *Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C , such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.*

Both theorems hold for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points, where it has jump discontinuities like the Heaviside function. Uniqueness, however, does not “see” values at the discontinuities. So we can only conclude that $f(t) = g(t)$ outside of discontinuities. For example, the unit step function is sometimes defined using $u(0) = 1/2$. This new step function, however, has the exact same Laplace transform as the one we defined earlier where $u(0) = 1$.

12.1.3 The Inverse Transform

As we said, the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the frequency domain we will want to get back to the time domain, as that is what we are interested in. Given a function $F(s)$, we wish to find a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$. [Theorem 12.1.3](#) says that the solution $f(t)$ is unique. So we can without fear make the following definition.

Suppose $F(s) = \mathcal{L}\{f(t)\}$ for some function $f(t)$. Define the *inverse Laplace transform* as

$$\mathcal{L}^{-1}\{F(s)\} \stackrel{\text{def}}{=} f(t).$$

There is an integral formula for the inverse, but it is not as simple as the transform itself—it requires complex numbers and path integrals. For us it will suffice to compute the inverse using [Table 12.1](#) on the preceding page.

Example 12.1.5: Take $F(s) = \frac{1}{s+1}$. Find the inverse Laplace transform.

We look at the table to find

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

As the Laplace transform is linear, the inverse Laplace transform is also linear. That is,

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, we also have $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$. Let us demonstrate how linearity can be used.

Example 12.1.6: Take $F(s) = \frac{s^2+s+1}{s^3+s}$. Find the inverse Laplace transform.

First we use the *method of partial fractions* to write F in a form where we can use [Table 12.1](#) on the previous page. We factor the denominator as $s(s^2 + 1)$ and write

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

Putting the right-hand side over a common denominator and equating the numerators we get $A(s^2 + 1) + s(Bs + C) = s^2 + s + 1$. Expanding and equating coefficients, we obtain $A + B = 1$, $C = 1$, $A = 1$, and thus $B = 0$. In other words,

$$F(s) = \frac{s^2 + s + 1}{s^3 + s} = \frac{1}{s} + \frac{1}{s^2 + 1}.$$

By linearity of the inverse Laplace transform, we get

$$\mathcal{L}^{-1} \left\{ \frac{s^2 + s + 1}{s^3 + s} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} = 1 + \sin t.$$

Another useful property is the so-called *shifting property* or the *first shifting property*

$$\mathcal{L}\{e^{-at}f(t)\} = F(s + a),$$

where $F(s)$ is the Laplace transform of $f(t)$.

Exercise 12.1.4: Derive the first shifting property from the definition of the Laplace transform.

The shifting property can be used, for example, when the denominator is a more complicated quadratic that may come up in the method of partial fractions. We complete the square and write such quadratics as $(s + a)^2 + b$ and then use the shifting property.

Example 12.1.7: Find $\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4s + 8} \right\}$.

First, we complete the square to make the denominator $(s + 2)^2 + 4$. We find

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4} \right\} = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 2^2} \right\} = \frac{1}{2} \sin(2t).$$

Finally, we put it all together with the shifting property to find

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4s + 8} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s + 2)^2 + 4} \right\} = \frac{1}{2} e^{-2t} \sin(2t).$$

Often, we want to be able to apply the inverse Laplace transform to rational functions, that is, functions of the form

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials. If $\frac{F(s)}{G(s)}$ is the Laplace transform of an exponential order function, it goes to zero as $s \rightarrow \infty$, and so the degree of $F(s)$ must be smaller than that of $G(s)$. Such rational functions are called *proper rational functions* and we can always apply the method of partial fractions without polynomial division. Of course, we still need to be able to factor the denominator into linear and quadratic terms, which involves finding the roots of the denominator.

12.1.4 Exercises

Exercise 12.1.5: Find the Laplace transform of $3 + t^5 + \sin(\pi t)$.

Exercise 12.1.6: Find the Laplace transform of $a + bt + ct^2$ for some constants a , b , and c .

Exercise 12.1.7: Find the Laplace transform of $A \cos(\omega t) + B \sin(\omega t)$.

Exercise 12.1.8: Find the Laplace transform of $\cos^2(\omega t)$.

Exercise 12.1.9: Find the inverse Laplace transform of $\frac{4}{s^2 - 9}$.

Exercise 12.1.10: Find the inverse Laplace transform of $\frac{2s}{s^2 - 1}$.

Exercise 12.1.11: Find the inverse Laplace transform of $\frac{1}{(s-1)^2(s+1)}$.

Exercise 12.1.12: Find the Laplace transform of $f(t) = \begin{cases} t & \text{if } t \geq 1, \\ 0 & \text{if } t < 1. \end{cases}$

Exercise 12.1.13: Find the inverse Laplace transform of $\frac{s}{(s^2+s+2)(s+4)}$.

Exercise 12.1.14: Find the Laplace transform of $\sin(\omega(t-a))$.

Exercise 12.1.15: Find the Laplace transform of $t \sin(\omega t)$. Hint: Several integrations by parts.

Exercise 12.1.101: Find the Laplace transform of $4(t+1)^2$.

Exercise 12.1.102: Find the inverse Laplace transform of $\frac{8}{s^3(s+2)}$.

Exercise 12.1.103: Find the Laplace transform of te^{-t} . Hint: Integrate by parts.

Exercise 12.1.104: Find the Laplace transform of $\sin(t)e^{-t}$. Hint: Integrate by parts.

12.2 Transforms of Derivatives and ODEs

12.2.1 Transforms of Derivatives

Let us see how the Laplace transform is used for differential equations. First we find the Laplace transform of the derivative of a function. Suppose $g(t)$ is a differentiable function of exponential order, that is, $|g(t)| \leq Me^{ct}$ for some M and c . So $\mathcal{L}\{g(t)\}$ exists, and what is more, $\lim_{t \rightarrow \infty} e^{-st}g(t) = 0$ when $s > c$. Then

$$\mathcal{L}\{g'(t)\} = \int_0^{\infty} e^{-st} g'(t) dt = \left[e^{-st} g(t) \right]_{t=0}^{\infty} - \int_0^{\infty} (-s) e^{-st} g(t) dt = -g(0) + s \mathcal{L}\{g(t)\}.$$

We repeat this procedure for higher derivatives. The results are listed in [Table 12.2](#). The procedure also works for continuous piecewise smooth functions, that is, functions that are continuous with a piecewise continuous derivative.

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$

Table 12.2: Laplace transforms of derivatives ($G(s) = \mathcal{L}\{g(t)\}$ as usual).

Exercise 12.2.1: Verify [Table 12.2](#).

12.2.2 Solving ODEs with the Laplace Transform

Notice that the Laplace transform turns differentiation into multiplication by s . Let us see how to apply this fact to differential equations.

Example 12.2.1: Take the equation

$$x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

We will take the Laplace transform of both sides. By $X(s)$ we will, as usual, denote the Laplace transform of $x(t)$.

$$\begin{aligned} \mathcal{L}\{x''(t) + x(t)\} &= \mathcal{L}\{\cos(2t)\}, \\ s^2X(s) - sx(0) - x'(0) + X(s) &= \frac{s}{s^2 + 4}. \end{aligned}$$

We plug in the initial conditions now—this makes the computations more streamlined—to obtain

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2 + 4}.$$

We solve for $X(s)$,

$$X(s) = \frac{s}{(s^2 + 1)(s^2 + 4)} + \frac{1}{s^2 + 1}.$$

We use partial fractions (exercise) to write

$$X(s) = \frac{1}{3} \frac{s}{s^2 + 1} - \frac{1}{3} \frac{s}{s^2 + 4} + \frac{1}{s^2 + 1}.$$

Now take the inverse Laplace transform to obtain

$$x(t) = \frac{1}{3} \cos(t) - \frac{1}{3} \cos(2t) + \sin(t).$$

The procedure for linear constant coefficient equations is as follows: Take an ordinary differential equation in the time variable t . Apply the Laplace transform to transform the equation into an algebraic (non differential) equation in the frequency domain. All the $x(t)$, $x'(t)$, $x''(t)$, and so on, will be converted to $X(s)$, $sX(s) - x(0)$, $s^2X(s) - sx(0) - x'(0)$, and so on. Solve the equation for $X(s)$. Then taking the inverse transform, if possible, find $x(t)$.

It should be noted that since not every function has a Laplace transform, not every equation can be solved in this manner. Also if the equation is not a linear constant coefficient ODE, then by applying the Laplace transform we may not obtain an algebraic equation.

12.2.3 Using the Heaviside Function

Before we move on to more general equations than those we could solve before, we want to consider the Heaviside function. See Figure 12.1 for the graph.

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

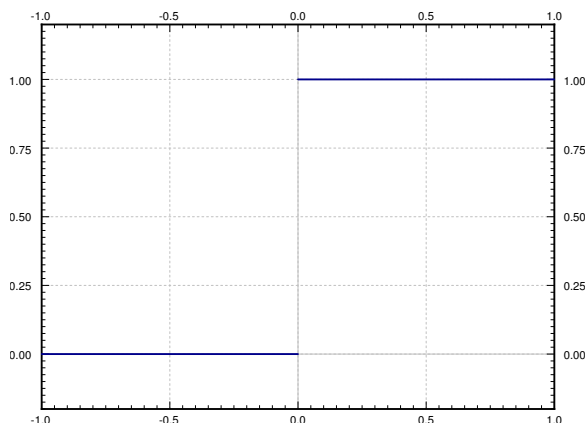


Figure 12.1: Plot of the Heaviside (unit step) function $u(t)$.

This function is useful for putting together functions, or cutting functions off. Most commonly it is used as $u(t - a)$ for some constant a . This just shifts the graph to the right by a . That is, it is a function that is 0 when $t < a$ and 1 when $t \geq a$. Suppose for example that $f(t)$ is a “signal” and you started receiving the signal $\sin t$ at time $t = \pi$. The function $f(t)$ should then be defined as

$$f(t) = \begin{cases} 0 & \text{if } t < \pi, \\ \sin t & \text{if } t \geq \pi. \end{cases}$$

Using the Heaviside function, $f(t)$ can be written as

$$f(t) = u(t - \pi) \sin t.$$

Similarly the step function that is 1 on the interval $[1, 2)$ and zero everywhere else can be written as

$$u(t - 1) - u(t - 2).$$

The Heaviside function is useful to define functions defined piecewise. If you want to define $f(t)$ such that $f(t) = t$ when t is in $[0, 1]$, $f(t) = -t + 2$ when t is in $[1, 2]$, and $f(t) = 0$ otherwise, then you can use the expression

$$f(t) = t(u(t) - u(t-1)) + (-t+2)(u(t-1) - u(t-2)).$$

Hence it is useful to know how the Heaviside function interacts with the Laplace transform. We have already seen that

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}.$$

This can be generalized into a *shifting property* or *second shifting property*.

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}. \quad (12.1)$$

Example 12.2.2: The forcing function in our setup need not be periodic. Consider the mass-spring system

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t) = 1$ if $1 \leq t < 5$ and zero otherwise. Imagine a rocket attached to the mass is fired for 4 seconds starting at $t = 1$. Or perhaps imagine an RLC circuit, where the voltage is raised at a constant rate for 4 seconds starting at $t = 1$, and then held steady again starting at $t = 5$.

We can write $f(t) = u(t-1) - u(t-5)$. We transform the equation and we plug in the initial conditions as before to obtain

$$s^2X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

We solve for $X(s)$ to obtain

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

We leave it as an exercise to the reader to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

In other words $\mathcal{L}\{1 - \cos t\} = \frac{1}{s(s^2+1)}$. So using (12.1) we find

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-s}\mathcal{L}\{1 - \cos t\}\} = (1 - \cos(t-1))u(t-1).$$

Similarly

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-5s}\mathcal{L}\{1 - \cos t\}\} = (1 - \cos(t-5))u(t-5).$$

Hence, the solution is

$$x(t) = (1 - \cos(t-1))u(t-1) - (1 - \cos(t-5))u(t-5).$$

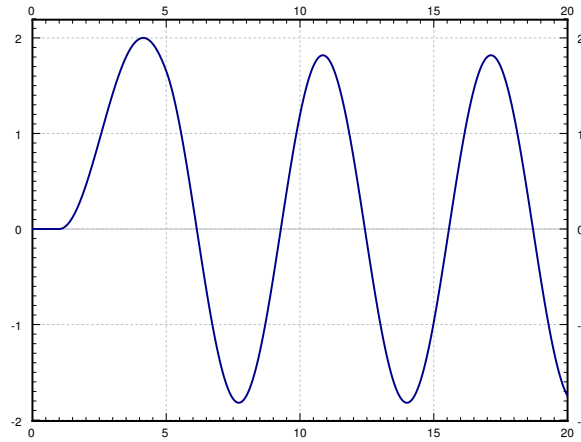
The plot of this solution is given in [Figure 12.2](#) on the next page.

12.2.4 Transfer Functions

The Laplace transform leads to the following useful concept for studying the steady state behavior of a linear system. Consider an equation of the form

$$Lx = f(t),$$

where L is a linear constant coefficient differential operator. Then $f(t)$ is usually thought of as input of the system and $x(t)$ is thought of as the output of the system. For example, for a mass-spring system the input

Figure 12.2: Plot of $x(t)$.

is the forcing function and the output is the behavior of the mass. We would like to have a convenient way to study the behavior of the system for different inputs.

Let us suppose that all the initial conditions are zero and take the Laplace transform of the equation, we obtain the equation

$$A(s)X(s) = F(s).$$

Solving for the ratio $X(s)/F(s)$ we obtain the so-called *transfer function* $H(s) = 1/A(s)$, that is,

$$H(s) = \frac{X(s)}{F(s)}.$$

In other words, $X(s) = H(s)F(s)$. We obtain an algebraic dependence of the output of the system based on the input. We can now easily study the steady state behavior of the system given different inputs by simply multiplying by the transfer function. Moreover, it is possible to compute the $H(s)$ without knowing exactly what the equation is by observing the output $X(s)$ for a given input $F(s)$. Once $H(s)$ is known, you can find the output for any input.

Example 12.2.3: Given $x'' + \omega_0^2 x = f(t)$, let us find the transfer function (assuming the initial conditions are zero).

First, we take the Laplace transform of the equation.

$$s^2 X(s) + \omega_0^2 X(s) = F(s).$$

Now we solve for the transfer function $X(s)/F(s)$.

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + \omega_0^2}.$$

Let us see how to use the transfer function. Suppose we have the constant input $f(t) = 1$. Hence $F(s) = 1/s$, and

$$X(s) = H(s)F(s) = \frac{1}{s^2 + \omega_0^2} \frac{1}{s}.$$

Taking the inverse Laplace transform of $X(s)$ we obtain

$$x(t) = \frac{1 - \cos(\omega_0 t)}{\omega_0^2}.$$

12.2.5 Transforms of Integrals

A feature of Laplace transforms is that it is also able to easily deal with integral equations. That is, equations in which integrals rather than derivatives of functions appear. The basic property, which can be proved by applying the definition and doing integration by parts, is

$$\mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} F(s).$$

It is sometimes useful (e.g. for computing the inverse transform) to write this as

$$\int_0^t f(\tau) d\tau = \mathcal{L}^{-1} \left\{ \frac{1}{s} F(s) \right\}.$$

Example 12.2.4: To compute $\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+1)} \right\}$ we could proceed by applying this integration rule.

$$\mathcal{L}^{-1} \left\{ \frac{1}{s} \frac{1}{s^2+1} \right\} = \int_0^t \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} d\tau = \int_0^t \sin \tau d\tau = 1 - \cos t.$$

Example 12.2.5: An equation containing an integral of the unknown function is called an *integral equation*. Consider

$$x(t) - t = \int_0^t x(\tau) d\tau,$$

where we wish to solve for $x(t)$. We apply the Laplace transform to get

$$X(s) - \frac{1}{s^2} = \frac{1}{s} X(s),$$

where $X(s) = \mathcal{L}\{x(t)\}$. Thus

$$X(s) = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s}.$$

Taking the inverse Laplace transform, we obtain

$$x(t) = e^t - 1.$$

12.2.6 Periodic Functions

The reader might ask: What about periodic functions as our input $f(t)$? That is, a function $f(t)$ where $f(t) = f(t+P)$ for some constant P (the period). Well, let us compute $F(s)$:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t) dt = \int_0^P e^{-st} f(t) dt + \int_P^\infty e^{-st} f(t) dt \\ &= \int_0^P e^{-st} f(t) dt + \int_0^\infty e^{-s(t+P)} f(t+P) dt = \int_0^P e^{-st} f(t) dt + e^{-Ps} \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^P e^{-st} f(t) dt + e^{-Ps} F(s). \end{aligned}$$

Solving for $F(s)$, we get

$$F(s) = \frac{1}{1 - e^{-Ps}} \int_0^P e^{-st} f(t) dt.$$

As before, computing the inverse would be more complex and possibly involve consulting a table. Let us not worry about computing the inverse here.

Example 12.2.6: Suppose our function $f(t)$ is a version of the *sawtooth*, that is, let $f(t) = t$ for $0 \leq t < 1$ and use $f(t) = f(t + 1)$ to extend it periodically. So $f(t) = t - 1$ for $1 \leq t < 2$, $f(t) = t - 2$ for $2 \leq t < 3$, etc. Then $P = 1$ and a short computation with integration by parts gets

$$F(s) = \frac{1}{1 - e^{-s}} \int_0^1 e^{-st} t \, dt = \frac{1}{1 - e^{-s}} \left(\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right) = \frac{-e^{-s}}{(1 - e^{-s})s} + \frac{1}{s^2}.$$

12.2.7 Exercises

Exercise 12.2.2: Using the Heaviside function write down the piecewise function that is 0 for $t < 0$, t^2 for t in $[0, 1]$ and t for $t > 1$.

Exercise 12.2.3: Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where $m > 0$, $c > 0$, $k > 0$, and $c^2 - 4km > 0$ (system is overdamped).

Exercise 12.2.4: Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where $m > 0$, $c > 0$, $k > 0$, and $c^2 - 4km < 0$ (system is underdamped).

Exercise 12.2.5: Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where $m > 0$, $c > 0$, $k > 0$, and $c^2 = 4km$ (system is critically damped).

Exercise 12.2.6: Solve $x'' + x = u(t - 1)$ for initial conditions $x(0) = 0$ and $x'(0) = 0$.

Exercise 12.2.7: Show the differentiation of the transform property. Suppose $\mathcal{L}\{f(t)\} = F(s)$, then show

$$\mathcal{L}\{-tf(t)\} = F'(s).$$

Hint: Differentiate under the integral sign.

Exercise 12.2.8: Solve $x''' + x = t^3 u(t - 1)$ for initial conditions $x(0) = 1$ and $x'(0) = 0$, $x''(0) = 0$.

Exercise 12.2.9: Show the second shifting property: $\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

Exercise 12.2.10: Let us think of the mass-spring system with a rocket from [Example 12.2.2](#). We noticed that the solution kept oscillating after the rocket stopped running. The amplitude of the oscillation depends on the time that the rocket was fired (for 4 seconds in the example).

- Find a formula for the amplitude of the resulting oscillation in terms of the amount of time the rocket is fired.
- Is there a nonzero time (if so what is it?) for which the rocket fires and the resulting oscillation has amplitude 0 (the mass is not moving)?

Exercise 12.2.11: Define

$$f(t) = \begin{cases} (t-1)^2 & \text{if } 1 \leq t < 2, \\ 3-t & \text{if } 2 \leq t < 3, \\ 0 & \text{otherwise.} \end{cases}$$

- Sketch the graph of $f(t)$.
- Write down $f(t)$ using the Heaviside function.
- Solve $x'' + x = f(t)$, $x(0) = 0$, $x'(0) = 0$ using Laplace transform.

Exercise 12.2.12: Find the transfer function for $mx'' + cx' + kx = f(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise 12.2.13: Suppose $Lx = f(t)$, $x(0) = 0$, $x'(0) = 0$ for the input function $f(t) = t$ has the output $x(t) = 2e^{-t} + te^{-t} + (t - 2)$.

a) Find $F(s)$, $X(s)$, and the transfer function $H(s)$.

b) If the input is instead $f(t) = \sin(t)$ instead, find the new output $x(t)$.

Exercise 12.2.14: Suppose $f(t) = 1$ if $0 \leq t < 1$ and $f(t) = 0$ if $1 \leq t < 2$, and then extend periodically for all $t \geq 0$ so that $f(t) = f(t + 2)$. Compute the Laplace transform $F(s)$.

Exercise 12.2.101: Using the Heaviside function $u(t)$, write down the function

$$f(t) = \begin{cases} 0 & \text{if } t < 1, \\ t - 1 & \text{if } 1 \leq t < 2, \\ 1 & \text{if } 2 \leq t. \end{cases}$$

Exercise 12.2.102: Solve $x'' - x = (t^2 - 1)u(t - 1)$ for initial conditions $x(0) = 1$, $x'(0) = 2$ using the Laplace transform.

Exercise 12.2.103: Find the transfer function for $x' + x = f(t)$, $x(0) = 0$.

Exercise 12.2.104: Suppose $Lx = f(t)$, $x(0) = 0$, $x'(0) = 0$ for the input function $f(t) = 4$ has the output $x(t) = 1 - \cos(2t)$. Find $F(s)$, $X(s)$, and the transfer function $H(s)$.

12.3 Convolution

12.3.1 The Convolution

The Laplace transformation of a product is not the product of the transforms. All hope is not lost however. We simply have to use a different type of a “product.” Take two functions $f(t)$ and $g(t)$ defined for $t \geq 0$, and define the *convolution** of $f(t)$ and $g(t)$ as

$$(f * g)(t) \stackrel{\text{def}}{=} \int_0^t f(\tau)g(t - \tau) d\tau. \quad (12.2)$$

As you can see, the convolution of two functions of t is another function of t .

Example 12.3.1: Take $f(t) = e^t$ and $g(t) = t$ for $t \geq 0$. Then

$$(f * g)(t) = \int_0^t e^\tau(t - \tau) d\tau = e^t - t - 1.$$

To solve the integral we did one integration by parts.

Example 12.3.2: Take $f(t) = \sin(\omega t)$ and $g(t) = \cos(\omega t)$ for $t \geq 0$. Then

$$(f * g)(t) = \int_0^t \sin(\omega \tau) \cos(\omega(t - \tau)) d\tau.$$

Apply the identity

$$\cos(\theta) \sin(\psi) = \frac{1}{2} (\sin(\theta + \psi) - \sin(\theta - \psi)),$$

to get

$$\begin{aligned} (f * g)(t) &= \int_0^t \frac{1}{2} (\sin(\omega t) - \sin(\omega t - 2\omega \tau)) d\tau \\ &= \left[\frac{1}{2} \tau \sin(\omega t) + \frac{1}{4\omega} \cos(2\omega \tau - \omega t) \right]_{\tau=0}^t \\ &= \frac{1}{2} t \sin(\omega t). \end{aligned}$$

The formula holds only for $t \geq 0$. The functions f , g , and $f * g$ are undefined for $t < 0$.

Convolution has many properties that make it behave like a product. Let c be a constant and f , g , and h be functions. Then it is a calculus exercise to show

$$\begin{aligned} f * g &= g * f, \\ (cf) * g &= f * (cg) = c(f * g), \\ (f + g) * h &= f * h + g * h, \\ (f * g) * h &= f * (g * h). \end{aligned}$$

The most interesting property for us is the following theorem.

Theorem 12.3.1. Let $f(t)$ and $g(t)$ be of exponential order, then

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t - \tau) d\tau\right\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

*For those that have seen convolution before, you may have seen it defined as $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau$. This definition agrees with (12.2) if you define $f(t)$ and $g(t)$ to be zero for $t < 0$. When discussing the Laplace transform the definition we gave is sufficient. Convolution does occur in many other applications, however, where you may have to use the more general definition with infinities.

In other words, the Laplace transform of a convolution is the product of the Laplace transforms. The simplest way to use this result is in reverse.

Example 12.3.3: Suppose we have the function of s defined by

$$\frac{1}{(s+1)s^2} = \frac{1}{s+1} \frac{1}{s^2}.$$

We recognize the two entries of [Table 12.2](#). That is,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1} \frac{1}{s^2}\right\} = \int_0^t \tau e^{-(t-\tau)} d\tau = e^{-t} + t - 1.$$

The calculation of the integral involved an integration by parts.

12.3.2 Solving ODEs

The next example demonstrates the full power of the convolution and the Laplace transform. We can give the solution to the forced oscillation problem for any forcing function as a definite integral.

Example 12.3.4: Find the solution to

$$x'' + \omega_0^2 x = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$.

We first apply the Laplace transform to the equation. Denote the transform of $x(t)$ by $X(s)$ and the transform of $f(t)$ by $F(s)$ as usual. We get

$$s^2 X(s) + \omega_0^2 X(s) = F(s),$$

or in other words we find the

$$X(s) = \frac{1}{s^2 + \omega_0^2} F(s).$$

Recall that $H(s) = \frac{1}{s^2 + \omega_0^2}$ is the transfer function. We know

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega_0^2}\right\} = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Therefore,

$$x(t) = \int_0^t \frac{\sin(\omega_0 \tau)}{\omega_0} f(t - \tau) d\tau,$$

or if we reverse the order

$$x(t) = \int_0^t f(\tau) \frac{\sin(\omega_0(t - \tau))}{\omega_0} d\tau.$$

Notice one more feature of the example above. We can now see how Laplace transform handles resonance. Suppose that $f(t) = \cos(\omega_0 t)$. Then

$$x(t) = \int_0^t \frac{\sin(\omega_0 \tau)}{\omega_0} \cos(\omega_0(t - \tau)) d\tau = \frac{1}{\omega_0} \int_0^t \sin(\omega_0 \tau) \cos(\omega_0(t - \tau)) d\tau.$$

We have computed the convolution of sine and cosine in [Example 12.3.2](#). Hence

$$x(t) = \left(\frac{1}{\omega_0}\right) \left(\frac{1}{2} t \sin(\omega_0 t)\right) = \frac{1}{2\omega_0} t \sin(\omega_0 t).$$

Note the t in front of the sine. The solution, therefore, grows without bound as t gets large, meaning we get resonance.

The general idea here is that if $H(s)$ is the transfer function, then $X(s) = H(s)F(s)$. If we find the $h(t) = \mathcal{L}^{-1}\{H(s)\}$, then

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{F(s)H(s)\} = (f * h)(t) = \int_0^t f(\tau)h(t - \tau) d\tau.$$

Hence, we can solve any constant coefficient equation with an arbitrary forcing function $f(t)$ as a definite integral using convolution. A definite integral, rather than a closed form solution, is usually enough for most practical purposes. It is not hard to numerically evaluate a definite integral.

12.3.3 Volterra Integral Equation

A common integral equation is the *Volterra integral equation**

$$x(t) = f(t) + \int_0^t g(t - \tau)x(\tau) d\tau,$$

where $f(t)$ and $g(t)$ are known functions and $x(t)$ is an unknown we wish to solve for. To find $x(t)$, we apply the Laplace transform to the equation to obtain

$$X(s) = F(s) + G(s)X(s),$$

where $X(s)$, $F(s)$, and $G(s)$ are the Laplace transforms of $x(t)$, $f(t)$, and $g(t)$ respectively. We find

$$X(s) = \frac{F(s)}{1 - G(s)}.$$

To find $x(t)$ we now need to find the inverse Laplace transform of $X(s)$.

Example 12.3.5: Solve

$$x(t) = e^{-t} + \int_0^t \sinh(t - \tau)x(\tau) d\tau.$$

We apply Laplace transform to obtain

$$X(s) = \frac{1}{s + 1} + \frac{1}{s^2 - 1}X(s),$$

or

$$X(s) = \frac{\frac{1}{s+1}}{1 - \frac{1}{s^2-1}} = \frac{s-1}{s^2-2} = \frac{s}{s^2-2} - \frac{1}{s^2-2}.$$

It is not hard to apply [Table 12.1](#) on page 549 to find

$$x(t) = \cosh(\sqrt{2}t) - \frac{1}{\sqrt{2}} \sinh(\sqrt{2}t).$$

12.3.4 Exercises

Exercise 12.3.1: Let $f(t) = t^2$ for $t \geq 0$, and $g(t) = u(t - 1)$. Compute $f * g$.

Exercise 12.3.2: Let $f(t) = t$ for $t \geq 0$, and $g(t) = \sin t$ for $t \geq 0$. Compute $f * g$.

*Named for the Italian mathematician [Vito Volterra](#) (1860–1940).

Exercise 12.3.3: Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0$, $c > 0$, $k > 0$, and $c^2 - 4km > 0$ (the system is overdamped). Write the solution as a definite integral.

Exercise 12.3.4: Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0$, $c > 0$, $k > 0$, and $c^2 - 4km < 0$ (the system is underdamped). Write the solution as a definite integral.

Exercise 12.3.5: Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0$, $c > 0$, $k > 0$, and $c^2 = 4km$ (the system is critically damped). Write the solution as a definite integral.

Exercise 12.3.6: Solve

$$x(t) = e^{-t} + \int_0^t \cos(t - \tau)x(\tau) d\tau.$$

Exercise 12.3.7: Solve

$$x(t) = \cos t + \int_0^t \cos(t - \tau)x(\tau) d\tau.$$

Exercise 12.3.8: Compute $\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+4)^2} \right\}$ using convolution.

Exercise 12.3.9: Write down the solution to $x'' - 2x = e^{-t^2}$, $x(0) = 0$, $x'(0) = 0$ as a definite integral. Hint: Do not try to compute the Laplace transform of e^{-t^2} .

Exercise 12.3.101: Let $f(t) = \cos t$ for $t \geq 0$, and $g(t) = e^{-t}$. Compute $f * g$.

Exercise 12.3.102: Compute $\mathcal{L}^{-1} \left\{ \frac{5}{s^4 + s^2} \right\}$ using convolution.

Exercise 12.3.103: Solve $x'' + x = \sin t$, $x(0) = 0$, $x'(0) = 0$ using convolution.

Exercise 12.3.104: Solve $x''' + x' = f(t)$, $x(0) = 0$, $x'(0) = 0$, $x''(0) = 0$ using convolution. Write the result as a definite integral.

12.4 Dirac Delta and Impulse Response

12.4.1 Rectangular Pulse

Often we study a physical system by putting in a short pulse and then seeing what the system does. The resulting behavior is often called *impulse response*, and understanding it tells us how the system responds to any input. Let us see what we mean by a pulse. The simplest kind of a pulse is a simple *rectangular pulse* defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } t < a, \\ M & \text{if } a \leq t < b, \\ 0 & \text{if } b \leq t. \end{cases}$$

See [Figure 12.3](#) on page 569 for a graph.

Notice that

$$\varphi(t) = M(u(t-a) - u(t-b)),$$

where $u(t)$ is the unit step function. Let us take the Laplace transform of a rectangular pulse,

$$\begin{aligned} \mathcal{L}\{\varphi(t)\} &= \mathcal{L}\{M(u(t-a) - u(t-b))\} \\ &= M \frac{e^{-as} - e^{-bs}}{s}. \end{aligned}$$

For simplicity, let $a = 0$. It is also convenient to set $M = 1/b$ so that

$$\int_0^\infty \varphi(t) dt = 1.$$

That is, to have the pulse have “unit mass.” For such a pulse,

$$\mathcal{L}\{\varphi(t)\} = \mathcal{L}\left\{\frac{u(t) - u(t-b)}{b}\right\} = \frac{1 - e^{-bs}}{bs}.$$

We want b to be very small; we wish to have the pulse be very short and very tall. By letting b go to zero, we arrive at the concept of the Dirac delta function. The limit of $\frac{1 - e^{-bs}}{bs}$ as $b \rightarrow \infty$ is 1, so we are looking for a “function” whose Laplace transform is 1.

12.4.2 The Dirac Delta Function

The *Dirac delta function*^{*} is not exactly a function; it is sometimes called a *generalized function*. We avoid unnecessary details and simply say that it is an object that does not really make sense unless we integrate it. The motivation is that we would like a “function” $\delta(t)$ such that for any continuous function $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0).$$

The formula should hold if we integrate over any interval that contains 0, not just $(-\infty, \infty)$. So $\delta(t)$ is a “function” with all its “mass” at the single point $t = 0$. For any interval[†] $[c, d]$,

$$\int_c^d \delta(t) dt = \begin{cases} 1 & \text{if the interval } [c, d] \text{ contains } 0, \text{ i.e. } c \leq 0 \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

^{*}Named after the English physicist and mathematician [Paul Adrien Maurice Dirac](#) (1902–1984).

[†]It is important that we consider c and d as part of the interval. One could write this integral as $\int_{c-}^{d+}$.

Unfortunately there is no such function in the classical sense. You could informally think that $\delta(t)$ is zero for $t \neq 0$ and somehow infinite at $t = 0$.

A good way to think about $\delta(t)$ is as a limit of short pulses whose integral is 1. For example, consider a rectangular pulse $\varphi(t)$ as above with $a = 0$ and $M = 1/b$, that is, $\varphi(t) = \frac{u(t) - u(t-b)}{b}$. Compute

$$\int_{-\infty}^{\infty} \varphi(t) f(t) dt = \int_{-\infty}^{\infty} \frac{u(t) - u(t-b)}{b} f(t) dt = \frac{1}{b} \int_0^b f(t) dt.$$

If $f(t)$ is continuous at $t = 0$, then for very small b , the function $f(t)$ is approximately equal to $f(0)$ on the interval $[0, b]$. We approximate the integral

$$\frac{1}{b} \int_0^b f(t) dt \approx \frac{1}{b} \int_0^b f(0) dt = f(0).$$

Hence,

$$\lim_{b \rightarrow 0} \int_{-\infty}^{\infty} \varphi(t) f(t) dt = \lim_{b \rightarrow 0} \frac{1}{b} \int_0^b f(t) dt = f(0).$$

Let us therefore accept $\delta(t)$ as an object that is possible to integrate. We often want to shift δ to another point, for example $\delta(t - a)$. In that case,

$$\int_{-\infty}^{\infty} \delta(t - a) f(t) dt = f(a).$$

Note that $\delta(a - t)$ is the same object as $\delta(t - a)$. With that, the convolution of $\delta(t)$ with $f(t)$ is again $f(t)$,

$$(f * \delta)(t) = \int_0^t f(\tau) \delta(t - \tau) d\tau = f(t).$$

As we can integrate $\delta(t)$, we compute its Laplace transform:

$$\mathcal{L}\{\delta(t - a)\} = \int_0^{\infty} e^{-st} \delta(t - a) dt = e^{-as}.$$

In particular,

$$\mathcal{L}\{\delta(t)\} = 1.$$

Remark 12.4.1: The Laplace transform of $\delta(t - a)$ would be the Laplace transform of the derivative of the Heaviside function $u(t - a)$, if the Heaviside function had a derivative. First,

$$\mathcal{L}\{u(t - a)\} = \frac{e^{-as}}{s}.$$

To obtain what the Laplace transform of the derivative would be, we multiply by s , to obtain e^{-as} , which is the Laplace transform of $\delta(t - a)$. We see the same thing using integration,

$$\int_0^t \delta(s - a) ds = u(t - a).$$

So in a certain sense

$$“ \quad \frac{d}{dt} [u(t - a)] = \delta(t - a). \quad ”$$

This line of reasoning allows us to talk about derivatives of functions with jump discontinuities. We can think of the derivative of the Heaviside function $u(t - a)$ as being somehow infinite at a , which is precisely our intuitive understanding of the delta function.

Example 12.4.1: Let us compute $\mathcal{L}^{-1} \left\{ \frac{s+1}{s} \right\}$. So far we only computed the inverse transform of proper rational functions in the s variable. That is, the numerator was of lower degree than the denominator. Not so with $\frac{s+1}{s}$. We can use the delta function to compute,

$$\mathcal{L}^{-1} \left\{ \frac{s+1}{s} \right\} = \mathcal{L}^{-1} \left\{ 1 + \frac{1}{s} \right\} = \mathcal{L}^{-1}\{1\} + \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} = \delta(t) + 1.$$

The resulting object is a generalized function and only makes sense when put underneath an integral.

12.4.3 Impulse Response

As we said before, in the differential equation $Lx = f(t)$, we think of $f(t)$ as input, and $x(t)$ as the output. We think of the delta function as an impulse, and so to find the response to an impulse, we use the delta function in place of $f(t)$. The solution to

$$Lx = \delta(t)$$

is called the *impulse response*.

Example 12.4.2: Solve (find the impulse response)

$$x'' + \omega_0^2 x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0. \quad (12.3)$$

Apply the Laplace transform to the equation, and denote the transform of $x(t)$ by $X(s)$:

$$s^2 X(s) + \omega_0^2 X(s) = 1, \quad \text{and so} \quad X(s) = \frac{1}{s^2 + \omega_0^2}.$$

The inverse Laplace transform produces (for $t > 0$)

$$x(t) = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Remark 12.4.2: Perhaps an astute reader will notice that it does not seem like $x'(0) = 0$. However, we really want to think of as $x(t) = 0$ for $t \leq 0$, so $x(t)$ has a “corner” at $t = 0$, so x' has a jump discontinuity there, which is what produces the $\delta(t)$ when we take x'' , see [Remark 12.4.1](#). The initial condition really is $x'(0-) = \lim_{t \uparrow 0} x'(t) = 0$.

Let us notice something about the example above. In [Example 12.3.4](#), we found that when the input is $f(t)$, the solution to $Lx = f(t)$ is given by

$$x(t) = \int_0^t f(\tau) \frac{\sin(\omega_0(t - \tau))}{\omega_0} d\tau.$$

That is, *the solution for an arbitrary input is given as convolution with the impulse response*. Let us see why. The key is to notice that for functions $f(t)$ and $h(t)$,

$$(f * h)''(t) = \frac{d^2}{dt^2} \left[\int_0^t f(\tau) h(t - \tau) d\tau \right] = \int_0^t f(\tau) h''(t - \tau) d\tau = (f * h'')(t).$$

We simply differentiate twice under the integral*. Suppose that $h(t)$ is the impulse response (solution to $h'' + \omega_0^2 h = \delta(t)$). If we convolve the entire equation (12.3), the left-hand side becomes

$$f * (h'' + \omega_0^2 h) = (f * h'') + \omega_0^2 (f * h) = (f * h)'' + \omega_0^2 (f * h).$$

The right-hand side becomes

$$(f * \delta)(t) = f(t).$$

*You should really think of the integral going over $(-\infty, \infty)$ rather than over $[0, t]$ and simply assume that $f(t)$ and $h(t)$ are zero for negative t .

Therefore, if h is the impulse response, then $x(t) = (f * h)(t)$ is the solution to

$$x'' + \omega_0^2 x = f(t).$$

The procedure works for any constant coefficient linear equation $Lx = f(t)$. If you find the impulse response h (solution to $Lh = \delta(t)$), you also know how to obtain the output $x(t)$ for any input $f(t)$. We simply convolve, $x(t) = (f * h)(t)$. As you may have noticed in the example, the impulse response is in fact just the inverse Laplace transform of the transfer function, that is, $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

12.4.4 Three-Point Beam Bending

Let us give another quite different example where the delta function turns up: Representing point loads on a steel beam. Consider a beam of length L , resting on two simple supports at the ends. Let x denote the position on the beam, and let $y(x)$ denote the deflection of the beam in the vertical direction. The deflection $y(x)$ satisfies the *Euler–Bernoulli equation*^{*},

$$EI \frac{d^4 y}{dx^4} = F(x),$$

where E and I are constants[†] and $F(x)$ is the force applied per unit length at position x . The situation we are interested in is when the force is applied at a single point as in [Figure 12.4](#).

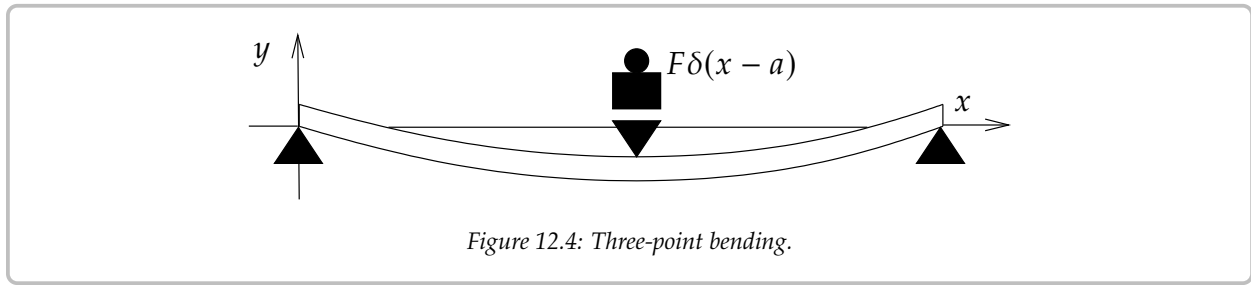


Figure 12.4: Three-point bending.

The equation becomes

$$EI \frac{d^4 y}{dx^4} = -F\delta(x - a),$$

where $x = a$ is the point where the mass is applied. The constant F is the force applied and the minus sign indicates that the force is downward, that is, in the negative y direction. The end points of the beam satisfy the conditions,

$$\begin{aligned} y(0) &= 0, & y''(0) &= 0, \\ y(L) &= 0, & y''(L) &= 0. \end{aligned}$$

See § ?? for further information about endpoint conditions applied to beams.

Example 12.4.3: Suppose that length of the beam is 2, and $EI = 1$ for simplicity. Further suppose that the force $F = 1$ is applied at $x = 1$. That is, we have the equation

$$\frac{d^4 y}{dx^4} = -\delta(x - 1),$$

and the endpoint conditions are

$$y(0) = 0, \quad y''(0) = 0, \quad y(2) = 0, \quad y''(2) = 0.$$

^{*}Named for the Swiss mathematicians [Jacob Bernoulli](#) (1654–1705), [Daniel Bernoulli](#) (1700–1782), the nephew of Jacob, and [Leonhard Paul Euler](#) (1707–1783).

[†] E is the elastic modulus and I is the second moment of area. Let us not worry about the details and simply think of these as some given constants.

We could integrate, but using the Laplace transform is even easier. We apply the transform in the x variable rather than the t variable. Let us again denote the transform of $y(x)$ as $Y(s)$.

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = -e^{-s}.$$

We notice that $y(0) = 0$ and $y''(0) = 0$. Let us call $C_1 = y'(0)$ and $C_2 = y'''(0)$. We solve for $Y(s)$,

$$Y(s) = \frac{-e^{-s}}{s^4} + \frac{C_1}{s^2} + \frac{C_2}{s^4}.$$

We take the inverse Laplace transform utilizing the second shifting property (12.1) to take the inverse of the first term.

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) + C_1 x + \frac{C_2}{6} x^3.$$

We still need to apply two of the endpoint conditions. As the conditions are at $x = 2$ we can simply replace $u(x-1) = 1$ when taking the derivatives. Therefore,

$$0 = y(2) = \frac{-(2-1)^3}{6} + C_1(2) + \frac{C_2}{6} 2^3 = \frac{-1}{6} + 2C_1 + \frac{4}{3} C_2,$$

and

$$0 = y''(2) = \frac{-3 \cdot 2 \cdot (2-1)}{6} + \frac{C_2}{6} 3 \cdot 2 \cdot 2 = -1 + 2C_2.$$

Hence $C_2 = \frac{1}{2}$ and solving for C_1 using the first equation we obtain $C_1 = \frac{-1}{4}$. Our solution for the beam deflection is

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) - \frac{x}{4} + \frac{x^3}{12}.$$

12.4.5 Exercises

Exercise 12.4.1: Solve (find the impulse response) $x'' + x' + x = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise 12.4.2: Solve (find the impulse response) $x'' + 2x' + x = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise 12.4.3: A pulse can come later and can be bigger. Solve $x'' + 4x = 4\delta(t-1)$, $x(0) = 0$, $x'(0) = 0$.

Exercise 12.4.4: Suppose that $f(t)$ and $g(t)$ are differentiable functions (and the derivatives are continuous) and suppose that $f(t) = g(t) = 0$ for all $t \leq 0$. Show that

$$(f * g)'(t) = (f' * g)(t) = (f * g')(t).$$

Exercise 12.4.5: Suppose that $Lx = \delta(t)$, $x(0) = 0$, $x'(0) = 0$, has the solution $x(t) = te^{-t}$ for $t > 0$. Find the solution to $Lx = t^2$, $x(0) = 0$, $x'(0) = 0$ for $t > 0$.

Exercise 12.4.6: Compute $\mathcal{L}^{-1} \left\{ \frac{s^2 + s + 1}{s^2} \right\}$.

Exercise 12.4.7 (challenging): Solve Example 12.4.3 via integrating 4 times in the x variable.

Exercise 12.4.8: Suppose we have a beam of length 1 simply supported at the ends and suppose that force $F = 1$ is applied at $x = 3/4$ in the downward direction. Suppose that $EI = 1$ for simplicity. Find the beam deflection $y(x)$.

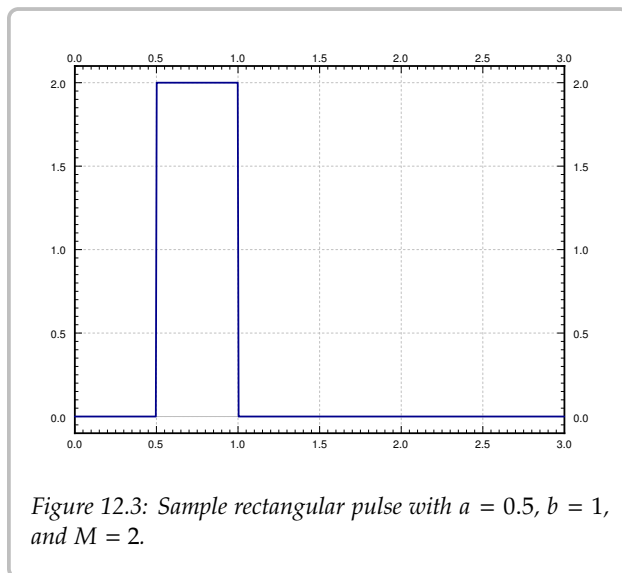
Exercise 12.4.101: Solve (find the impulse response) $x'' = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise 12.4.102: Solve (find the impulse response) $x' + ax = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise 12.4.103: Suppose that $Lx = \delta(t)$, $x(0) = 0$, $x'(0) = 0$, has the solution $x(t) = e^t \sin(t)$ for $t > 0$. Find (in closed form) the solution to $Lx = e^t$, $x(0) = 0$, $x'(0) = 0$ for $t > 0$.

Exercise 12.4.104: Compute $\mathcal{L}^{-1} \left\{ \frac{s^2}{s^2+1} \right\}$.

Exercise 12.4.105: Compute $\mathcal{L}^{-1} \left\{ \frac{3s^2 e^{-s} + 2}{s^2} \right\}$.



Chapter 13

Linear Systems of Differential Equations

13.1 Introduction to Systems of Differential Equations

Many physical situations are modelled by systems of n differential equations in n unknown functions, where $n \geq 2$. The next three examples illustrate physical problems that lead to systems of differential equations. In these examples and throughout this chapter we'll denote the independent variable by t .

Example 13.1.1: Tanks T_1 and T_2 contain 100 gallons and 300 gallons of salt solutions, respectively. Salt solutions are simultaneously added to both tanks from external sources, pumped from each tank to the other, and drained from both tanks (Figure 13.1). A solution with 1 pound of salt per gallon is pumped into T_1 from an external source at 5 gal/min, and a solution with 2 pounds of salt per gallon is pumped into T_2 from an external source at 4 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 3 gal/min. T_1 is drained at 6 gal/min and T_2 is drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2 , respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2 . Assume that both mixtures are well stirred.

Solution: As in Section 4.2, let *rate in* and *rate out* denote the rates (lb/min) at which salt enters and leaves a tank; thus,

$$Q_1' = (\text{rate in})_1 - (\text{rate out})_1,$$

$$Q_2' = (\text{rate in})_2 - (\text{rate out})_2.$$

Note that the volumes of the solutions in T_1 and T_2 remain constant at 100 gallons and 300 gallons, respectively.

T_1 receives salt from the external source at the rate of

$$(1 \text{ lb/gal}) \times (5 \text{ gal/min}) = 5 \text{ lb/min},$$

and from T_2 at the rate of

$$(\text{lb/gal in } T_2) \times (3 \text{ gal/min}) = \frac{1}{300}Q_2 \times 3 = \frac{1}{100}Q_2 \text{ lb/min}.$$

Therefore

$$(\text{rate in})_1 = 5 + \frac{1}{100}Q_2. \quad (13.1)$$

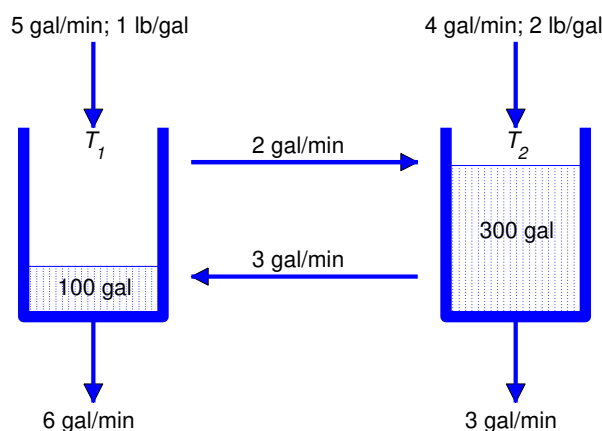


Figure 13.1

Solution leaves T_1 at the rate of 8 gal/min, since 6 gal/min are drained and 2 gal/min are pumped to T_2 ; hence,

$$(\text{rate out})_1 = (\text{lb/gal in } T_1) \times (8 \text{ gal/min}) = \frac{1}{100} Q_1 \times 8 = \frac{2}{25} Q_1. \quad (13.2)$$

Eqns. (13.1) and (13.2) imply that

$$Q'_1 = 5 + \frac{1}{100} Q_2 - \frac{2}{25} Q_1. \quad (13.3)$$

T_2 receives salt from the external source at the rate of

$$(2 \text{ lb/gal}) \times (4 \text{ gal/min}) = 8 \text{ lb/min},$$

and from T_1 at the rate of

$$(\text{lb/gal in } T_1) \times (2 \text{ gal/min}) = \frac{1}{100} Q_1 \times 2 = \frac{1}{50} Q_1 \text{ lb/min}.$$

Therefore

$$(\text{rate in})_2 = 8 + \frac{1}{50} Q_1. \quad (13.4)$$

Solution leaves T_2 at the rate of 6 gal/min, since 3 gal/min are drained and 3 gal/min are pumped to T_1 ; hence,

$$(\text{rate out})_2 = (\text{lb/gal in } T_2) \times (6 \text{ gal/min}) = \frac{1}{300} Q_2 \times 6 = \frac{1}{50} Q_2. \quad (13.5)$$

Eqns. (13.4) and (13.5) imply that

$$Q'_2 = 8 + \frac{1}{50} Q_1 - \frac{1}{50} Q_2. \quad (13.6)$$

We say that (13.3) and (13.6) form a *system of two first order equations in two unknowns*, and write them together as

$$\begin{aligned} Q_1' &= 5 - \frac{2}{25}Q_1 + \frac{1}{100}Q_2 \\ Q_2' &= 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \quad \blacksquare \end{aligned}$$

Example 13.1.2: A mass m_1 is suspended from a rigid support on a spring S_1 and a second mass m_2 is suspended from the first on a spring S_2 (Figure 13.2). The springs obey Hooke's law, with spring constants k_1 and k_2 . Internal friction causes the springs to exert damping forces proportional to the rates of change of their lengths, with damping constants c_1 and c_2 . Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be the displacements of the two masses from their equilibrium positions at time t , measured positive upward. Derive a system of differential equations for y_1 and y_2 , assuming that the masses of the springs are negligible and that vertical external forces F_1 and F_2 also act on the objects.

Solution: In equilibrium, S_1 supports both m_1 and m_2 and S_2 supports only m_2 . Therefore, if $\Delta\ell_1$ and $\Delta\ell_2$ are the elongations of the springs in equilibrium then

$$(m_1 + m_2)g = k_1\Delta\ell_1 \quad \text{and} \quad m_2g = k_2\Delta\ell_2. \quad (13.7)$$

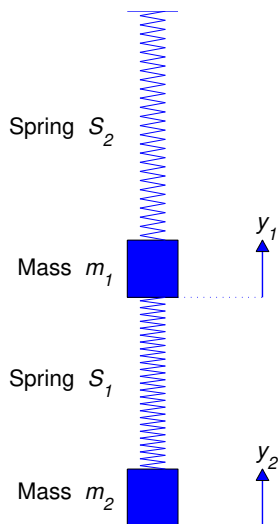


Figure 13.2

Let H_1 be the Hooke's law force acting on m_1 , and let D_1 be the damping force on m_1 . Similarly, let H_2 and D_2 be the Hooke's law and damping forces acting on m_2 . According to Newton's second law of motion,

$$\begin{aligned} m_1 y_1'' &= -m_1 g + H_1 + D_1 + F_1, \\ m_2 y_2'' &= -m_2 g + H_2 + D_2 + F_2. \end{aligned} \quad (13.8)$$

When the displacements are y_1 and y_2 , the change in length of S_1 is $-y_1 + \Delta\ell_1$ and the change in length of S_2 is $-y_2 + y_1 + \Delta\ell_2$. Both springs exert Hooke's law forces on m_1 , while only S_2 exerts a Hooke's law force

on m_2 . These forces are in directions that tend to restore the springs to their natural lengths. Therefore

$$H_1 = k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \quad \text{and} \quad H_2 = k_2(-y_2 + y_1 + \Delta\ell_2). \quad (13.9)$$

When the velocities are y'_1 and y'_2 , S_1 and S_2 are changing length at the rates $-y'_1$ and $-y'_2 + y'_1$, respectively. Both springs exert damping forces on m_1 , while only S_2 exerts a damping force on m_2 . Since the force due to damping exerted by a spring is proportional to the rate of change of length of the spring and in a direction that opposes the change, it follows that

$$D_1 = -c_1y'_1 + c_2(y'_2 - y'_1) \quad \text{and} \quad D_2 = -c_2(y'_2 - y'_1). \quad (13.10)$$

From (13.8), (13.9), and (13.10),

$$\begin{aligned} m_1y''_1 &= -m_1g + k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \\ &\quad -c_1y'_1 + c_2(y'_2 - y'_1) + F_1 \\ &= -(m_1g - k_1\Delta\ell_1 + k_2\Delta\ell_2) - k_1y_1 + k_2(y_2 - y_1) \\ &\quad -c_1y'_1 + c_2(y'_2 - y'_1) + F_1 \end{aligned} \quad (13.11)$$

and

$$\begin{aligned} m_2y''_2 &= -m_2g + k_2(-y_2 + y_1 + \Delta\ell_2) - c_2(y'_2 - y'_1) + F_2 \\ &= -(m_2g - k_2\Delta\ell_2) - k_2(y_2 - y_1) - c_2(y'_2 - y'_1) + F_2. \end{aligned} \quad (13.12)$$

From (13.7),

$$m_1g - k_1\Delta\ell_1 + k_2\Delta\ell_2 = -m_2g + k_2\Delta\ell_2 = 0.$$

Therefore we can rewrite (13.11) and (13.12) as

$$\begin{aligned} m_1y''_1 &= -(c_1 + c_2)y'_1 + c_2y'_2 - (k_1 + k_2)y_1 + k_2y_2 + F_1 \\ m_2y''_2 &= c_2y'_1 - c_2y'_2 + k_2y_1 - k_2y_2 + F_2. \quad \blacksquare \end{aligned}$$

Example 13.1.3: Let $\mathbf{X} = \mathbf{X}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be the position vector at time t of an object with mass m , relative to a rectangular coordinate system with origin at Earth's center (Figure 13.3). According to Newton's law of gravitation, Earth's gravitational force $\mathbf{F} = \mathbf{F}(x, y, z)$ on the object is inversely proportional to the square of the distance of the object from Earth's center, and directed toward the center; thus,

$$\mathbf{F} = \frac{K}{\|\mathbf{X}\|^2} \left(-\frac{\mathbf{X}}{\|\mathbf{X}\|} \right) = -K \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}, \quad (13.13)$$

where K is a constant. To determine K , we observe that the magnitude of $\mathbf{F}$ is

$$\|\mathbf{F}\| = K \frac{\|\mathbf{X}\|}{\|\mathbf{X}\|^3} = \frac{K}{\|\mathbf{X}\|^2} = \frac{K}{(x^2 + y^2 + z^2)}.$$

Let R be Earth's radius. Since $\|\mathbf{F}\| = mg$ when the object is at Earth's surface,

$$mg = \frac{K}{R^2}, \quad \text{so} \quad K = mgR^2.$$

Therefore we can rewrite (13.13) as

$$\mathbf{F} = -mgR^2 \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

Now suppose $\mathbf{F}$ is the only force acting on the object. According to Newton's second law of motion, $\mathbf{F} = m\mathbf{X}''$; that is,

$$m(x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k}) = -mgR^2 \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

Cancelling the common factor m and equating components on the two sides of this equation yields the system

$$\begin{aligned} x'' &= -\frac{gR^2x}{(x^2 + y^2 + z^2)^{3/2}} \\ y'' &= -\frac{gR^2y}{(x^2 + y^2 + z^2)^{3/2}} \\ z'' &= -\frac{gR^2z}{(x^2 + y^2 + z^2)^{3/2}}. \end{aligned} \tag{13.14}$$

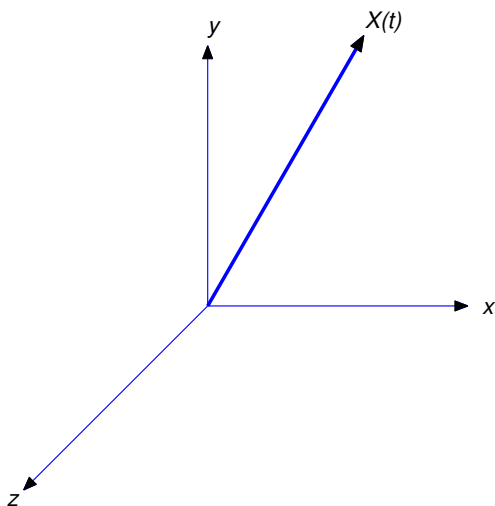


Figure 13.3

13.1.1 Rewriting Higher Order Systems as First Order Systems

A system of the form

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2, \dots, y_n) \\ y_2' &= g_2(t, y_1, y_2, \dots, y_n) \\ &\vdots \\ y_n' &= g_n(t, y_1, y_2, \dots, y_n) \end{aligned} \tag{13.15}$$

is called a *first order system*, since the only derivatives occurring in it are first derivatives. The derivative of each of the unknowns may depend upon the independent variable and all the unknowns, but not on the derivatives of other unknowns. When we wish to emphasize the number of unknown functions in (13.15) we will say that (13.15) is an $n \times n$ system.

Systems involving higher order derivatives can often be reformulated as first order systems by introducing additional unknowns. The next two examples illustrate this.

Example 13.1.4: Rewrite the system

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2)y_1' + c_2 y_2' - (k_1 + k_2)y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2. \end{aligned} \tag{13.16}$$

derived in Example 13.1.2 as a system of first order equations.

Solution: If we define $v_1 = y_1'$ and $v_2 = y_2'$, then $v_1' = y_1''$ and $v_2' = y_2''$, so (13.16) becomes

$$\begin{aligned} m_1 v_1' &= -(c_1 + c_2)v_1 + c_2 v_2 - (k_1 + k_2)y_1 + k_2 y_2 + F_1 \\ m_2 v_2' &= c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2. \end{aligned}$$

Therefore $\{y_1, y_2, v_1, v_2\}$ satisfies the 4×4 first order system

$$\begin{aligned} y_1' &= v_1 \\ y_2' &= v_2 \\ v_1' &= \frac{1}{m_1} [-(c_1 + c_2)v_1 + c_2 v_2 - (k_1 + k_2)y_1 + k_2 y_2 + F_1] \\ v_2' &= \frac{1}{m_2} [c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2]. \end{aligned} \tag{13.17}$$

Remark 13.1.1: The difference in form between (13.15) and (13.17), due to the way in which the unknowns are *denoted* in the two systems, isn't important; (13.17) is a first order system, in that each equation in (13.17) expresses the first derivative of one of the unknown functions in a way that does not involve derivatives of any of the other unknowns.

Example 13.1.5: Rewrite the system

$$\begin{aligned} x'' &= f(t, x, x', y, y', y'') \\ y''' &= g(t, x, x', y, y', y'') \end{aligned}$$

as a first order system.

Solution: We regard x, x', y, y' , and y'' as unknown functions, and rename them

$$x = x_1, \quad x' = x_2, \quad y = y_1, \quad y' = y_2, \quad y'' = y_3.$$

These unknowns satisfy the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2, y_1, y_2, y_3) \\ y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= g(t, x_1, x_2, y_1, y_2, y_3). \end{aligned}$$

13.1.2 Rewriting Scalar Differential Equations as Systems

In this chapter we'll refer to differential equations involving only one unknown function as *scalar* differential equations. Scalar differential equations can be rewritten as systems of first order equations by the method illustrated in the next two examples.

Example 13.1.6: Rewrite the equation

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0 \tag{13.18}$$

as a 4×4 first order system.

Solution: We regard $y, y', y'',$ and y''' as unknowns and rename them

$$y = y_1, \quad y' = y_2, \quad y'' = y_3, \quad \text{and} \quad y''' = y_4.$$

Then $y^{(4)} = y'_4$, so (13.18) can be written as

$$y'_4 + 4y_4 + 6y_3 + 4y_2 + y_1 = 0.$$

Therefore $\{y_1, y_2, y_3, y_4\}$ satisfies the system

$$\begin{aligned} y'_1 &= y_2 \\ y'_2 &= y_3 \\ y'_3 &= y_4 \\ y'_4 &= -4y_4 - 6y_3 - 4y_2 - y_1. \blacksquare \end{aligned}$$

Example 13.1.7: Rewrite

$$x''' = f(t, x, x', x'')$$

as a system of first order equations.

Solution: We regard x , x' , and x'' as unknowns and rename them

$$x = y_1, \quad x' = y_2, \quad \text{and} \quad x'' = y_3.$$

Then

$$y'_1 = x' = y_2, \quad y'_2 = x'' = y_3, \quad \text{and} \quad y'_3 = x'''. \quad \blacksquare$$

Therefore $\{y_1, y_2, y_3\}$ satisfies the first order system

$$\begin{aligned} y'_1 &= y_2 \\ y'_2 &= y_3 \\ y'_3 &= f(t, y_1, y_2, y_3). \end{aligned}$$

Since systems of differential equations involving higher derivatives can be rewritten as first order systems by the method used in Examples 13.1.5–13.1.7, we'll consider only first order systems.

13.1.3 Numerical Solution of Systems

The numerical methods that we studied in Chapter 3 can be extended to systems, and most differential equation software packages include programs to solve systems of equations. We won't go into detail on numerical methods for systems; however, for illustrative purposes we'll describe the Runge-Kutta method for the numerical solution of the initial value problem

$$\begin{aligned} y'_1 &= g_1(t, y_1, y_2), & y_1(t_0) &= y_{10}, \\ y'_2 &= g_2(t, y_1, y_2), & y_2(t_0) &= y_{20} \end{aligned}$$

at equally spaced points $t_0, t_1, \dots, t_n = b$ in an interval $[t_0, b]$. Thus,

$$t_i = t_0 + ih, \quad i = 0, 1, \dots, n,$$

where

$$h = \frac{b - t_0}{n}.$$

We'll denote the approximate values of y_1 and y_2 at these points by $y_{10}, y_{11}, \dots, y_{1n}$ and $y_{20}, y_{21}, \dots, y_{2n}$. The Runge-Kutta method computes these approximate values as follows: given y_{1i} and y_{2i} , compute

$$\begin{aligned} I_{1i} &= g_1(t_i, y_{1i}, y_{2i}), \\ J_{1i} &= g_2(t_i, y_{1i}, y_{2i}), \\ I_{2i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2}I_{1i}, y_{2i} + \frac{h}{2}J_{1i}\right), \\ J_{2i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2}I_{1i}, y_{2i} + \frac{h}{2}J_{1i}\right), \\ I_{3i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2}I_{2i}, y_{2i} + \frac{h}{2}J_{2i}\right), \\ J_{3i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2}I_{2i}, y_{2i} + \frac{h}{2}J_{2i}\right), \\ I_{4i} &= g_1(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \\ J_{4i} &= g_2(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \end{aligned}$$

and

$$\begin{aligned} y_{1,i+1} &= y_{1i} + \frac{h}{6}(I_{1i} + 2I_{2i} + 2I_{3i} + I_{4i}), \\ y_{2,i+1} &= y_{2i} + \frac{h}{6}(J_{1i} + 2J_{2i} + 2J_{3i} + J_{4i}) \end{aligned}$$

for $i = 0, \dots, n-1$. Under appropriate conditions on g_1 and g_2 , it can be shown that the global truncation error for the Runge-Kutta method is $O(h^4)$, as in the scalar case considered in Section 3.3.

13.1.4 Exercises

1. Tanks T_1 and T_2 contain 50 gallons and 100 gallons of salt solutions, respectively. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 1 gal/min, and a solution with 3 pounds of salt per gallon is pumped into T_2 from an external source at 2 gal/min. The solution from T_1 is pumped into T_2 at 3 gal/min, and the solution from T_2 is pumped into T_1 at 4 gal/min. T_1 is drained at 2 gal/min and T_2 is drained at 1 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2 , respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2 . Assume that both mixtures are well stirred.
2. Two 500 gallon tanks T_1 and T_2 initially contain 100 gallons each of salt solution. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 6 gal/min, and a solution with 1 pound of salt per gallon is pumped into T_2 from an external source at 5 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 1 gal/min. Both tanks are drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2 , respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2 that's valid until a tank is about to overflow. Assume that both mixtures are well stirred.
3. A mass m_1 is suspended from a rigid support on a spring S_1 with spring constant k_1 and damping constant c_1 . A second mass m_2 is suspended from the first on a spring S_2 with spring constant k_2 and damping constant c_2 , and a third mass m_3 is suspended from the second on a spring S_3 with spring constant k_3 and damping constant c_3 . Let $y_1 = y_1(t)$, $y_2 = y_2(t)$, and $y_3 = y_3(t)$ be the displacements of the three masses from their equilibrium positions at time t , measured positive upward. Derive a system of differential equations for y_1 , y_2 and y_3 , assuming that the masses of the springs are negligible and that vertical external forces F_1 , F_2 , and F_3 also act on the masses.
4. Let $\mathbf{X} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of an object with mass m , expressed in terms of a rectangular coordinate system with origin at Earth's center (Figure 13.3). Derive a system of differential equations

for x , y , and z , assuming that the object moves under Earth's gravitational force (given by Newton's law of gravitation, as in Example 13.1.3) and a resistive force proportional to the speed of the object. Let α be the constant of proportionality.

5. Rewrite the given system as a first order system.

$$\begin{array}{ll} \text{(a)} & \begin{array}{l} x''' = f(t, x, y, y') \\ y'' = g(t, y, y') \end{array} & \begin{array}{l} u' = f(t, u, v, v', w') \\ v'' = g(t, u, v, v', w) \\ w'' = h(t, u, v, v', w, w') \end{array} \\ \text{(b)} & \end{array}$$

$$\begin{array}{ll} \text{(c)} & y''' = f(t, y, y', y'') \\ \text{(d)} & y^{(4)} = f(t, y) \end{array}$$

$$\text{(e)} \quad \begin{array}{l} x'' = f(t, x, y) \\ y'' = g(t, x, y) \end{array}$$

6. Rewrite the system (13.14) of differential equations derived in Example 13.1.3 as a first order system.
7. Formulate a version of Euler's method (Section 3.1) for the numerical solution of the initial value problem

$$\begin{array}{ll} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{array}$$

on an interval $[t_0, b]$.

8. Formulate a version of the improved Euler method (Section 3.2) for the numerical solution of the initial value problem

$$\begin{array}{ll} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{array}$$

on an interval $[t_0, b]$.

13.2 Linear Systems of Differential Equations

A first order system of differential equations that can be written in the form

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t) \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t) \end{aligned} \quad (13.19)$$

is called a *linear system*.

The linear system (13.19) can be written in matrix form as

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{f}(t),$$

or more briefly as

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{f}(t), \quad (13.20)$$

where

$$\mathbf{y} = y_n, \quad \mathbf{A}(t) = a_{nn}, \quad \text{and} \quad \mathbf{f}(t) = f_n.$$

We call $\mathbf{A}$ the *coefficient matrix* of (13.20) and $\mathbf{f}$ the *forcing function*. We'll say that $\mathbf{A}$ and $\mathbf{f}$ are *continuous* if their entries are continuous. If $\mathbf{f} = \mathbf{0}$, then (13.20) is *homogeneous*; otherwise, (13.20) is *nonhomogeneous*.

An initial value problem for (13.20) consists of finding a solution of (13.20) that equals a given constant vector

$$\mathbf{k} = k_n.$$

at some initial point t_0 . We write this initial value problem as

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}.$$

The next theorem gives sufficient conditions for the existence of solutions of initial value problems for (13.20). We omit the proof.

Theorem 13.2.1. *Suppose the coefficient matrix $\mathbf{A}$ and the forcing function $\mathbf{f}$ are continuous on (a, b) , let t_0 be in (a, b) , and let $\mathbf{k}$ be an arbitrary constant n -vector. Then the initial value problem*

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}$$

has a unique solution on (a, b) .

Example 13.2.1:

(a) Write the system

$$\begin{aligned} y_1' &= y_1 + 2y_2 + 2e^{4t} \\ y_2' &= 2y_1 + y_2 + e^{4t} \end{aligned} \quad (13.21)$$

in matrix form and conclude from Theorem 13.2.1 that every initial value problem for (13.21) has a unique solution on $(-\infty, \infty)$.

(b) Verify that

$$\mathbf{y} = \frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \quad (13.22)$$

is a solution of (13.21) for all values of the constants c_1 and c_2 .

(c) Find the solution of the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{y}(0) = \frac{1}{5} \begin{bmatrix} 3 \\ 22 \end{bmatrix}. \quad (13.23)$$

Solution (a): The system (13.21) can be written in matrix form as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}.$$

An initial value problem for (13.21) can be written as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{y}(t_0) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

Since the coefficient matrix and the forcing function are both continuous on $(-\infty, \infty)$, Theorem 13.2.1 implies that this problem has a unique solution on $(-\infty, \infty)$.

Solution (b): If $\mathbf{y}$ is given by (13.22), then

$$\begin{aligned} A\mathbf{y} + \mathbf{f} &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \\ &\quad + c_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5} \begin{bmatrix} 22 \\ 23 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5} \begin{bmatrix} 32 \\ 28 \end{bmatrix} e^{4t} + 3c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} = \mathbf{y}'. \end{aligned}$$

Solution (c): We must choose c_1 and c_2 in (13.22) so that

$$\frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 \\ 22 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Solving this system yields $c_1 = 1$, $c_2 = -2$, so

$$\mathbf{y} = \frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

is the solution of (13.23).

Remark 13.2.1: The theory of $n \times n$ linear systems of differential equations is analogous to the theory of the scalar n -th order equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = F(t), \quad (13.24)$$

as developed in Sections 9.1. For example, by rewriting (13.24) as an equivalent linear system it can be shown that Theorem 13.2.1 implies Theorem ?? (Exercise 13.2.12).

13.2.1 Exercises

1. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2 .

$$(a) \quad \begin{aligned} y_1' &= 2y_1 + 4y_2 \\ y_2' &= 4y_1 + 2y_2; \end{aligned} \quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

$$(b) \quad \begin{aligned} y_1' &= -2y_1 - 2y_2 \\ y_2' &= -5y_1 + y_2; \end{aligned} \quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{3t}$$

$$(c) \quad \begin{aligned} y_1' &= -4y_1 - 10y_2 \\ y_2' &= 3y_1 + 7y_2; \end{aligned} \quad \mathbf{y} = c_1 \begin{bmatrix} -5 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t$$

$$(d) \quad \begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= y_1 + 2y_2; \end{aligned} \quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

2. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 , c_2 , and c_3 .

$$(a) \quad \begin{aligned} y_1' &= -y_1 + 2y_2 + 3y_3 \\ y_2' &= y_2 + 6y_3 \\ y_3' &= -2y_3; \end{aligned} \quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-2t}$$

$$(b) \quad \begin{aligned} y_1' &= 2y_2 + 2y_3 \\ y_2' &= 2y_1 + 2y_3 \\ y_3' &= 2y_1 + 2y_2; \end{aligned} \quad \mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$$

$$(c) \quad \begin{aligned} y_1' &= -y_1 + 2y_2 + 2y_3 \\ y_2' &= 2y_1 - y_2 + 2y_3 \\ y_3' &= 2y_1 + 2y_2 - y_3; \end{aligned} \quad \mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t}$$

$$(d) \quad \begin{aligned} y_1' &= 3y_1 - y_2 - y_3 \\ y_2' &= -2y_1 + 3y_2 + 2y_3 \\ y_3' &= 4y_1 - y_2 - 2y_3; \end{aligned} \quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$$

3. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

$$(a) \quad \begin{aligned} y_1' &= y_1 + y_2 & y_1(0) &= 1 \\ y_2' &= -2y_1 + 4y_2, & y_2(0) &= 0; \end{aligned} \quad \mathbf{y} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

$$(b) \quad \begin{aligned} y_1' &= 5y_1 + 3y_2 & y_1(0) &= 12 \\ y_2' &= -y_1 + y_2, & y_2(0) &= -6; \end{aligned} \quad \mathbf{y} = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + 3 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{4t}$$

4. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

$$(a) \quad \begin{aligned} y_1' &= 6y_1 + 4y_2 + 4y_3 & y_1(0) &= 3 \\ y_2' &= -7y_1 - 2y_2 - y_3, & y_2(0) &= -6 \\ y_3' &= 7y_1 + 4y_2 + 3y_3 & y_3(0) &= 4 \end{aligned}$$

$$\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{6t} + 2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-t}$$

$$\begin{aligned} \text{(b)} \quad y_1' &= 8y_1 + 7y_2 + 7y_3 & y_1(0) &= 2 \\ y_2' &= -5y_1 - 6y_2 - 9y_3, & y_2(0) &= -4 \\ y_3' &= 5y_1 + 7y_2 + 10y_3, & y_3(0) &= 3 \end{aligned}$$

$$\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{8t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^t$$

5. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2 .

$$\begin{aligned} \text{(a)} \quad y_1' &= -3y_1 + 2y_2 + 3 - 2t \\ y_2' &= -5y_1 + 3y_2 + 6 - 3t \end{aligned}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \cos t \\ 3 \cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} 2 \sin t \\ 3 \sin t + \cos t \end{bmatrix} + \begin{bmatrix} 1 \\ t \end{bmatrix}$$

$$\begin{aligned} \text{(b)} \quad y_1' &= 3y_1 + y_2 - 5e^t \\ y_2' &= -y_1 + y_2 + e^t \end{aligned}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1+t \\ -t \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^t$$

$$\begin{aligned} \text{(c)} \quad y_1' &= -y_1 - 4y_2 + 4e^t + 8te^t \\ y_2' &= -y_1 - y_2 + e^{3t} + (4t+2)e^t \end{aligned}$$

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} e^{3t} \\ 2te^t \end{bmatrix}$$

$$\begin{aligned} \text{(d)} \quad y_1' &= -6y_1 - 3y_2 + 14e^{2t} + 12e^t \\ y_2' &= y_1 - 2y_2 + 7e^{2t} - 12e^t \end{aligned}$$

$$\mathbf{y} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} e^{2t} + 3e^t \\ 2e^{2t} - 3e^t \end{bmatrix}$$

6. Convert the linear scalar equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y(t) = F(t) \quad (\text{A})$$

into an equivalent $n \times n$ system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

and show that A and $\mathbf{f}$ are continuous on an interval (a, b) if and only if (A) is normal on (a, b) .

7. A matrix function

$$Q(t) = qrs$$

is said to be *differentiable* if its entries $\{q_{ij}\}$ are differentiable. Then the *derivative* Q' is defined by

$$Q'(t) = q'rs.$$

- (a) Prove: If P and Q are differentiable matrices such that $P + Q$ is defined and if c_1 and c_2 are constants, then

$$(c_1P + c_2Q)' = c_1P' + c_2Q'.$$

- (b) Prove: If P and Q are differentiable matrices such that PQ is defined, then

$$(PQ)' = P'Q + PQ'.$$

8. Verify that $Y' = AY$.

$$(a) \quad Y = \begin{bmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

$$(b) \quad Y = \begin{bmatrix} e^{-4t} & -2e^{3t} \\ e^{-4t} & 5e^{3t} \end{bmatrix}, \quad A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}$$

$$(c) \quad Y = \begin{bmatrix} -5e^{2t} & 2e^t \\ 3e^{2t} & -e^t \end{bmatrix}, \quad A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}$$

$$(d) \quad Y = \begin{bmatrix} e^{3t} & e^t \\ e^{3t} & -e^t \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(e) \quad Y = \begin{bmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & 0 & -2e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(f) \quad Y = -e^{-2t} - e^{-2t} e^{4t} 0 \quad e^{-2t} e^{4t} e^{-2t} 0 e^{4t}, \quad A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

$$(g) \quad Y = e^{3t} e^{-3t} 0 e^{3t} 0 - e^{-3t} e^{3t} e^{-3t} \quad e^{-3t}, \quad A = \begin{bmatrix} -9 & 6 & 6 \\ -6 & 3 & 6 \\ -6 & 6 & 3 \end{bmatrix}$$

$$(h) \quad Y = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix}$$

9. Suppose

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$$

are solutions of the homogeneous system

$$\mathbf{y}' = A(t)\mathbf{y}, \tag{A}$$

and define

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}.$$

(a) Show that $Y' = AY$.

(b) Show that if $\mathbf{c}$ is a constant vector then $\mathbf{y} = Y\mathbf{c}$ is a solution of (A).

(c) State generalizations of (a) and (b) for $n \times n$ systems.

10. Suppose Y is a differentiable square matrix.
- (a) Find a formula for the derivative of Y^2 .
 - (b) Find a formula for the derivative of Y^n , where n is any positive integer.
 - (c) State how the results obtained in (a) and (b) are analogous to results from calculus concerning scalar functions.
11. It can be shown that if Y is a differentiable and invertible square matrix function, then Y^{-1} is differentiable.
- (a) Show that $(Y^{-1})' = -Y^{-1}Y'Y^{-1}$. (Hint: Differentiate the identity $Y^{-1}Y = I$.)
 - (b) Find the derivative of $Y^{-n} = (Y^{-1})^n$, where n is a positive integer.
 - (c) State how the results obtained in (a) and (b) are analogous to results from calculus concerning scalar functions.
12. Show that Theorem 13.2.1 implies Theorem ?? HINT: Write the scalar equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x)$$

as an $n \times n$ system of linear equations.

13. Suppose $\mathbf{y}$ is a solution of the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) , and that the $n \times n$ matrix P is invertible and differentiable on (a, b) . Find a matrix B such that the function $\mathbf{x} = P\mathbf{y}$ is a solution of $\mathbf{x}' = B\mathbf{x}$ on (a, b) .

13.3 Basic Theory of Homogeneous Linear Systems

In this section we consider homogeneous linear systems $\mathbf{y}' = A(t)\mathbf{y}$, where $A = A(t)$ is a continuous $n \times n$ matrix function on an interval (a, b) . The theory of linear homogeneous systems has much in common with the theory of linear homogeneous scalar equations, which we considered in Sections 2.1, 5.1, and 9.1.

Whenever we refer to solutions of $\mathbf{y}' = A(t)\mathbf{y}$ we'll mean solutions on (a, b) . Since $\mathbf{y} \equiv \mathbf{0}$ is obviously a solution of $\mathbf{y}' = A(t)\mathbf{y}$, we call it the *trivial* solution. Any other solution is *nontrivial*.

If $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are vector functions defined on an interval (a, b) and $c_1, c_2, \dots, c_n$ are constants, then

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \dots + c_n\mathbf{y}_n \quad (13.25)$$

is a *linear combination* of $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$. It's easy to show that if $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) , then so is any linear combination of $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ (Exercise 13.3.1). We say that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a *fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b)* on if every solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) can be written as a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$, as in (13.25). In this case we say that (13.25) is the *general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b)* .

It can be shown that if A is continuous on (a, b) then $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b) (Exercises 13.3.15 and 13.3.16). The next definition will help to characterize fundamental sets of solutions of $\mathbf{y}' = A(t)\mathbf{y}$.

We say that a set $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of n -vector functions is *linearly independent* on (a, b) if the only constants $c_1, c_2, \dots, c_n$ such that

$$c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \dots + c_n\mathbf{y}_n(t) = \mathbf{0}, \quad a < t < b, \quad (13.26)$$

are $c_1 = c_2 = \dots = c_n = 0$. If (13.26) holds for some set of constants $c_1, c_2, \dots, c_n$ that are not all zero, then $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is *linearly dependent* on (a, b) .

The next theorem is analogous to Theorems 6.1.3 and ??.

Theorem 13.3.1. *Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) . Then a set $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of n solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is a fundamental set if and only if it's linearly independent on (a, b) .*

Example 13.3.1: Show that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}$$

are linearly independent on every interval (a, b) .

Solution: Suppose

$$c_1 \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

We must show that $c_1 = c_2 = c_3 = 0$. Rewriting this equation in matrix form yields

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

Expanding the determinant of this system in cofactors of the entries of the first row yields

$$\begin{aligned} \begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} &= e^t \begin{vmatrix} e^{3t} & e^{3t} \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 0 \end{vmatrix} + e^{2t} \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 1 \end{vmatrix} \\ &= e^t(-e^{3t}) + e^{2t}(-e^{2t}) = -2e^{4t}. \end{aligned}$$

Since this determinant is never zero, $c_1 = c_2 = c_3 = 0$. ■

We can use the method in Example 13.3.1 to test n solutions $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of any $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ for linear independence on an interval (a, b) on which A is continuous. To explain this (and for other purposes later), it's useful to write a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ in a different way. We first write the vector functions in terms of their components as

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix}, \quad \dots, \quad \mathbf{y}_n = \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}.$$

If

$$\mathbf{y} = c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 + \dots + c_n \mathbf{y}_n$$

then

$$\begin{aligned} \mathbf{y} &= c_1 \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix} + c_2 \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix} + \dots + c_n \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix} \\ &= \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \mathbf{c}. \end{aligned}$$

This shows that

$$c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 + \dots + c_n \mathbf{y}_n = Y \mathbf{c}, \quad (13.27)$$

where

$$\mathbf{c} = c n$$

and

$$Y = [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n] = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}; \quad (13.28)$$

that is, the columns of Y are the vector functions $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$.

For reference below, note that

$$\begin{aligned} Y' &= [\mathbf{y}'_1 \ \mathbf{y}'_2 \ \cdots \ \mathbf{y}'_n] \\ &= [A\mathbf{y}_1 \ A\mathbf{y}_2 \ \cdots \ A\mathbf{y}_n] \\ &= A[\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n] = AY; \end{aligned}$$

that is, Y satisfies the matrix differential equation

$$Y' = AY.$$

The determinant of Y ,

$$W = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} \quad (13.29)$$

is called the *Wronskian* of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$. It can be shown (Exercises 13.3.2 and 13.3.3) that this definition is analogous to definitions of the Wronskian of scalar functions given in Sections 5.1 and 9.1. The next theorem is analogous to Theorems 6.1.4 and 9.1. The proof is sketched in Exercise 13.3.4 for $n = 2$ and in Exercise 13.3.5 for general n .

Theorem 13.3.2. [Abel's Formula] Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) , let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) , and let t_0 be in (a, b) . Then the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is given by

$$W(t) = W(t_0) \exp \left(\int_{t_0}^t [a_{11}(s) + a_{22}(s) + \dots + a_{nn}(s)] ds \right), \quad a < t < b. \quad (13.30)$$

Therefore, either W has no zeros in (a, b) or $W \equiv 0$ on (a, b) .

Remark 13.3.1: The sum of the diagonal entries of a square matrix A is called the *trace* of A , denoted by $\text{tr}(A)$. Thus, for an $n \times n$ matrix A ,

$$\text{tr}(A) = a_{11} + a_{22} + \dots + a_{nn},$$

and (13.30) can be written as

$$W(t) = W(t_0) \exp \left(\int_{t_0}^t \text{tr}(A(s)) ds \right), \quad a < t < b.$$

The next theorem is analogous to Theorems 6.1.6 and ??.

Theorem 13.3.3. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) . Then the following statements are equivalent; that is, they are either all true or all false:

- (a) The general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \dots + c_n\mathbf{y}_n$, where $c_1, c_2, \dots, c_n$ are arbitrary constants.
- (b) $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) .
- (c) $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is linearly independent on (a, b) .
- (d) The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is nonzero at some point in (a, b) .
- (e) The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is nonzero at all points in (a, b) .

We say that Y in (13.28) is a *fundamental matrix* for $\mathbf{y}' = A(t)\mathbf{y}$ if any (and therefore all) of the statements (a)-(e) of Theorem 13.3.2 are true for the columns of Y . In this case, (13.27) implies that the general solution of $\mathbf{y}' = A(t)\mathbf{y}$ can be written as $\mathbf{y} = Y\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant n -vector.

Example 13.3.2: The vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

are solutions of the constant coefficient system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y} \quad (13.31)$$

on $(-\infty, \infty)$. (Verify.)

- (a) Compute the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$ directly from the definition (13.29)
- (b) Verify Abel's formula (13.30) for the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.
- (c) Find the general solution of (13.31).
- (d) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}. \quad (13.32)$$

Solution (a): From (13.29)

$$W(t) = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = e^{2t} e^{-t} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = e^t. \quad (13.33)$$

Solution (b): Here

$$A = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix},$$

so $\text{tr}(A) = -4 + 5 = 1$. If t_0 is an arbitrary real number then (13.30) implies that

$$W(t) = W(t_0) \exp \left(\int_{t_0}^t 1 \, ds \right) = \begin{vmatrix} -e^{2t_0} & -e^{-t_0} \\ 2e^{2t_0} & e^{-t_0} \end{vmatrix} e^{(t-t_0)} = e^{t_0} e^{t-t_0} = e^t,$$

which is consistent with (13.33).

Solution (c): Since $W(t) \neq 0$, Theorem 13.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions of (13.31) and

$$Y = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix for (13.31). Therefore the general solution of (13.31) is

$$\mathbf{y} = c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 = c_1 \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \quad (13.34)$$

Solution (d): Setting $t = 0$ in (13.34) and imposing the initial condition in (13.32) yields

$$c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}.$$

Thus,

$$\begin{aligned} -c_1 - c_2 &= 4 \\ 2c_1 + c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = -1$, $c_2 = -3$. Substituting these values into (13.34) yields

$$\mathbf{y} = - \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} - 3 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{2t} + 3e^{-t} \\ -2e^{2t} - 3e^{-t} \end{bmatrix}$$

as the solution of (13.32).

13.3.1 Exercises

1. Prove: If $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) , then any linear combination of $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ is also a solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) .
2. In Section 5.1 the Wronskian of two solutions y_1 and y_2 of the scalar second order equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \quad (\text{A})$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

- (a) Rewrite (A) as a system of first order equations and show that W is the Wronskian (as defined in this section) of two solutions of this system.
- (b) Apply Eqn. (13.30) to the system derived in (a), and show that

$$W(x) = W(x_0) \exp \left\{ - \int_{x_0}^x \frac{P_1(s)}{P_0(s)} \, ds \right\},$$

3. In Section 9.1 the Wronskian of n solutions $y_1, y_2, \dots, y_n$ of the n -th order equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \dots + P_n(x)y = 0 \quad (\text{A})$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

- (a) Rewrite (A) as a system of first order equations and show that W is the Wronskian (as defined in this section) of n solutions of this system.
 (b) Apply Eqn. (13.30) to the system derived in (a), and show that

$$W(x) = W(x_0) \exp \left\{ - \int_{x_0}^x \frac{P_1(s)}{P_0(s)} ds \right\},$$

which is the form of Abel's formula given in Theorem 9.1.3.

4. Suppose

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$$

are solutions of the 2×2 system $\mathbf{y}' = A\mathbf{y}$ on (a, b) , and let

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \quad \text{and} \quad W = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix};$$

thus, W is the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.

- (a) Deduce from the definition of determinant that

$$W' = \begin{vmatrix} y'_{11} & y'_{12} \\ y_{21} & y_{22} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{12} \\ y'_{21} & y'_{22} \end{vmatrix}.$$

- (b) Use the equation $Y' = A(t)Y$ and the definition of matrix multiplication to show that

$$[y'_{11} \quad y'_{12}] = a_{11}[y_{11} \quad y_{12}] + a_{12}[y_{21} \quad y_{22}]$$

and

$$[y'_{21} \quad y'_{22}] = a_{21}[y_{11} \quad y_{12}] + a_{22}[y_{21} \quad y_{22}].$$

- (c) Use properties of determinants to deduce from (a) and (b) that

$$\begin{vmatrix} y'_{11} & y'_{12} \\ y_{21} & y_{22} \end{vmatrix} = a_{11}W \quad \text{and} \quad \begin{vmatrix} y_{11} & y_{12} \\ y'_{21} & y'_{22} \end{vmatrix} = a_{22}W.$$

- (d) Conclude from (c) that

$$W' = (a_{11} + a_{22})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0) \exp \left(\int_{t_0}^t [a_{11}(s) + a_{22}(s)] ds \right) \quad a < t < b.$$

5. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) . Let

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix},$$

where the columns of Y are solutions of $\mathbf{y}' = A(t)\mathbf{y}$. Let

$$r_i = [y_{i1} \ y_{i2} \ \cdots \ y_{in}]$$

be the i th row of Y , and let W be the determinant of Y .

- (a) Deduce from the definition of determinant that

$$W' = W_1 + W_2 + \cdots + W_n,$$

where, for $1 \leq m \leq n$, the i th row of W_m is r_i if $i \neq m$, and r'_m if $i = m$.

- (b) Use the equation $Y' = AY$ and the definition of matrix multiplication to show that

$$r'_m = a_{m1}r_1 + a_{m2}r_2 + \cdots + a_{mn}r_n.$$

- (c) Use properties of determinants to deduce from (b) that

$$\det(W_m) = a_{mm}W.$$

- (d) Conclude from (a) and (c) that

$$W' = (a_{11} + a_{22} + \cdots + a_{nn})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0) \exp \left(\int_{t_0}^t [a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)] ds \right), \quad a < t < b.$$

6. Suppose the $n \times n$ matrix A is continuous on (a, b) and t_0 is a point in (a, b) . Let Y be a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) .

- (a) Show that $Y(t_0)$ is invertible.

- (b) Show that if $\mathbf{k}$ is an arbitrary n -vector then the solution of the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{k}$$

is

$$\mathbf{y} = Y(t)Y^{-1}(t_0)\mathbf{k}.$$

7. Let

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}.$$

- (a) Verify that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.

- (b) Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \quad (\text{A})$$

- (c) Use the result of Exercise 13.3.6(b) to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

8. Repeat Exercise 13.3.7 with

$$A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -2e^{3t} \\ 5e^{3t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}.$$

9. Repeat Exercise 13.3.7 with

$$A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} -5e^{2t} \\ 3e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 2e^t \\ -e^t \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -19 \\ 11 \end{bmatrix}.$$

10. Repeat Exercise 13.3.7 with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$

11. Let

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-t} \\ -3e^{-t} \\ 7e^{-t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ -7 \\ 20 \end{bmatrix}.$$

(a) Verify that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.

(b) Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \quad (\text{A})$$

(c) Use the result of Exercise 13.3.6(b) to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

12. Repeat Exercise 13.3.11 with

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} -e^{-2t} \\ 0 \\ e^{-2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ -9 \\ 12 \end{bmatrix}.$$

13. Repeat Exercise 13.3.11 with

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix}.$$

14. Suppose Y and Z are fundamental matrices for the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$. Then some of the four matrices YZ^{-1} , $Y^{-1}Z$, $Z^{-1}Y$, ZY^{-1} are necessarily constant. Identify them and prove that they are constant.
15. Suppose the columns of an $n \times n$ matrix Y are solutions of the $n \times n$ system $\mathbf{y}' = A\mathbf{y}$ and C is an $n \times n$ constant matrix.

- (a) Show that the matrix $Z = YC$ satisfies the differential equation $Z' = AZ$.
- (b) Show that Z is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ if and only if C is invertible and Y is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$.
16. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and t_0 is in (a, b) . For $i = 1, 2, \dots, n$, let $\mathbf{y}_i$ be the solution of the initial value problem $\mathbf{y}'_i = A(t)\mathbf{y}_i$, $\mathbf{y}_i(t_0) = \mathbf{e}_i$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

that is, the j th component of $\mathbf{e}_i$ is 1 if $j = i$, or 0 if $j \neq i$.

- (a) Show that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) .
- (b) Conclude from (a) and Exercise 13.3.15 that $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b) .
17. Show that Y is a fundamental matrix for the system $\mathbf{y}' = A(t)\mathbf{y}$ if and only if Y^{-1} is a fundamental matrix for $\mathbf{y}' = -A^T(t)\mathbf{y}$, where A^T denotes the transpose of A . HINT: See Exercise 13.2.11.
18. Let Z be the fundamental matrix for the constant coefficient system $\mathbf{y}' = A\mathbf{y}$ such that $Z(0) = I$.
- (a) Show that $Z(t)Z(s) = Z(t+s)$ for all s and t . HINT: For fixed s let $\Gamma_1(t) = Z(t)Z(s)$ and $\Gamma_2(t) = Z(t+s)$. Show that Γ_1 and Γ_2 are both solutions of the matrix initial value problem $\Gamma' = A\Gamma$, $\Gamma(0) = Z(s)$. Then conclude from Theorem 13.2.1 that $\Gamma_1 = \Gamma_2$.
- (b) Show that $(Z(t))^{-1} = Z(-t)$.
- (c) The matrix Z defined above is sometimes denoted by e^{tA} . Discuss the motivation for this notation.

13.4 Constant Coefficient Homogeneous Systems

I

We'll now begin our study of the homogeneous system

$$\mathbf{y}' = A\mathbf{y}, \quad (13.35)$$

where A is an $n \times n$ constant matrix. Since A is continuous on $(-\infty, \infty)$, Theorem 13.2.1 implies that all solutions of (13.35) are defined on $(-\infty, \infty)$. Therefore, when we speak of solutions of $\mathbf{y}' = A\mathbf{y}$, we'll mean solutions on $(-\infty, \infty)$.

In this section we assume that all the eigenvalues of A are real and that A has a set of n linearly independent eigenvectors. In the next two sections we consider the cases where some of the eigenvalues of A are complex, or where A does not have n linearly independent eigenvectors.

In Example 13.3.2 we showed that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

form a fundamental set of solutions of the system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \quad (13.36)$$

but we did not show how we obtained $\mathbf{y}_1$ and $\mathbf{y}_2$ in the first place. To see how these solutions can be obtained we write (13.36) as

$$\begin{aligned} y_1' &= -4y_1 - 3y_2 \\ y_2' &= 6y_1 + 5y_2 \end{aligned} \quad (13.37)$$

and look for solutions of the form

$$y_1 = x_1 e^{\lambda t} \quad \text{and} \quad y_2 = x_2 e^{\lambda t}, \quad (13.38)$$

where x_1 , x_2 , and λ are constants to be determined. Differentiating (13.38) yields

$$y_1' = \lambda x_1 e^{\lambda t} \quad \text{and} \quad y_2' = \lambda x_2 e^{\lambda t}.$$

Substituting this and (13.38) into (13.37) and canceling the common factor $e^{\lambda t}$ yields

$$\begin{aligned} -4x_1 - 3x_2 &= \lambda x_1 \\ 6x_1 + 5x_2 &= \lambda x_2. \end{aligned}$$

For a given λ , this is a homogeneous algebraic system, since it can be rewritten as

$$\begin{aligned} (-4 - \lambda)x_1 - 3x_2 &= 0 \\ 6x_1 + (5 - \lambda)x_2 &= 0. \end{aligned} \quad (13.39)$$

The trivial solution $x_1 = x_2 = 0$ of this system isn't useful, since it corresponds to the trivial solution $y_1 \equiv y_2 \equiv 0$ of (13.37), which can't be part of a fundamental set of solutions of (13.36). Therefore we consider only those values of λ for which (13.39) has nontrivial solutions. These are the values of λ for which the determinant of (13.39) is zero; that is,

$$\begin{aligned} \begin{vmatrix} -4 - \lambda & -3 \\ 6 & 5 - \lambda \end{vmatrix} &= (-4 - \lambda)(5 - \lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1) = 0, \end{aligned}$$

which has the solutions $\lambda_1 = 2$ and $\lambda_2 = -1$.

Taking $\lambda = 2$ in (13.39) yields

$$\begin{aligned} -6x_1 - 3x_2 &= 0 \\ 6x_1 + 3x_2 &= 0, \end{aligned}$$

which implies that $x_1 = -x_2/2$, where x_2 can be chosen arbitrarily. Choosing $x_2 = 2$ yields the solution $y_1 = -e^{2t}$, $y_2 = 2e^{2t}$ of (13.37). We can write this solution in vector form as

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{2t}. \quad (13.40)$$

Taking $\lambda = -1$ in (13.39) yields the system

$$\begin{aligned} -3x_1 - 3x_2 &= 0 \\ 6x_1 + 6x_2 &= 0, \end{aligned}$$

so $x_1 = -x_2$. Taking $x_2 = 1$ here yields the solution $y_1 = -e^{-t}$, $y_2 = e^{-t}$ of (13.37). We can write this solution in vector form as

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}. \quad (13.41)$$

In (13.40) and (13.41) the constant coefficients in the arguments of the exponential functions are the eigenvalues of the coefficient matrix in (13.36), and the vector coefficients of the exponential functions are associated eigenvectors. This illustrates the next theorem.

Theorem 13.4.1. *Suppose the $n \times n$ constant matrix A has n real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Then the functions*

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t}, \mathbf{y}_2 = \mathbf{x}_2 e^{\lambda_2 t}, \dots, \mathbf{y}_n = \mathbf{x}_n e^{\lambda_n t}$$

form a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$; that is, the general solution of this system is

$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t} + \dots + c_n \mathbf{x}_n e^{\lambda_n t}.$$

Proof. Differentiating $\mathbf{y}_i = \mathbf{x}_i e^{\lambda_i t}$ and recalling that $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$ yields

$$\mathbf{y}'_i = \lambda_i \mathbf{x}_i e^{\lambda_i t} = A\mathbf{x}_i e^{\lambda_i t} = A\mathbf{y}_i.$$

This shows that $\mathbf{y}_i$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is

$$\begin{vmatrix} x_{11}e^{\lambda_1 t} & x_{12}e^{\lambda_2 t} & \dots & x_{1n}e^{\lambda_n t} \\ x_{21}e^{\lambda_1 t} & x_{22}e^{\lambda_2 t} & \dots & x_{2n}e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}e^{\lambda_1 t} & x_{n2}e^{\lambda_2 t} & \dots & x_{nn}e^{\lambda_n t} \end{vmatrix} = e^{\lambda_1 t} e^{\lambda_2 t} \dots e^{\lambda_n t} \begin{vmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix}.$$

Since the columns of the determinant on the right are $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, which are assumed to be linearly independent, the determinant is nonzero. Therefore Theorem 13.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$.

Example 13.4.1:

(a) Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}. \quad (13.42)$$

(b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}. \quad (13.43)$$

Solution (a): The characteristic polynomial of the coefficient matrix A in (13.42) is

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} &= (\lambda-2)^2 - 16 \\ &= (\lambda-2-4)(\lambda-2+4) \\ &= (\lambda-6)(\lambda+2). \end{aligned}$$

Hence, $\lambda_1 = 6$ and $\lambda_2 = -2$ are eigenvalues of A . To obtain the eigenvectors, we must solve the system

$$\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (13.44)$$

with $\lambda = 6$ and $\lambda = -2$. Setting $\lambda = 6$ in (13.44) yields

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

is a solution of (13.42). Setting $\lambda = -2$ in (13.44) yields

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = -x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

is a solution of (13.42). From Theorem 13.4.1, the general solution of (13.42) is

$$\mathbf{y} = c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}. \quad (13.45)$$

Solution (b): To satisfy the initial condition in (13.43), we must choose c_1 and c_2 in (13.45) so that

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

This is equivalent to the system

$$\begin{aligned}c_1 - c_2 &= 5 \\c_1 + c_2 &= -1,\end{aligned}$$

so $c_1 = 2$, $c_2 = -3$. Therefore the solution of (13.43) is

$$\mathbf{y} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} - 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

or, in terms of components,

$$y_1 = 2e^{6t} + 3e^{-2t}, \quad y_2 = 2e^{6t} - 3e^{-2t}.$$

Example 13.4.2:

(a) Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}. \quad (13.46)$$

(b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}. \quad (13.47)$$

Solution (a): The characteristic polynomial of the coefficient matrix A in (13.46) is

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = -(\lambda-2)(\lambda-3)(\lambda+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$. To find the eigenvectors, we must solve the system

$$\begin{bmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (13.48)$$

with $\lambda = 2, 3, -1$. With $\lambda = 2$, the augmented matrix of (13.48) is

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ 4 & -1 & -4 & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

as a solution of (13.46). With $\lambda = 3$, the augmented matrix of (13.48) is

$$\begin{bmatrix} 0 & -1 & -1 & \vdots & 0 \\ -2 & 0 & 2 & \vdots & 0 \\ 4 & -1 & -5 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}$$

as a solution of (13.46). With $\lambda = -1$, the augmented matrix of (13.48) is

$$\begin{bmatrix} 4 & -1 & -1 & \vdots & 0 \\ -2 & 4 & 2 & \vdots & 0 \\ 4 & -1 & -1 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -\frac{1}{7} & \vdots & 0 \\ 0 & 1 & \frac{3}{7} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3/7$ and $x_2 = -3x_3/7$. Taking $x_3 = 7$ yields

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$$

as a solution of (13.46). By Theorem 13.4.1, the general solution of (13.46) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t},$$

which can also be written as

$$\mathbf{y} = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \quad (13.49)$$

Solution (b): To satisfy the initial condition in (13.47) we must choose c_1, c_2, c_3 in (13.49) so that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}.$$

Solving this system yields $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Hence, the solution of (13.47) is

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} - 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}. \end{aligned}$$

Example 13.4.3: Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \mathbf{y}. \quad (13.50)$$

Solution: The characteristic polynomial of the coefficient matrix A in (13.50) is

$$\begin{vmatrix} -3-\lambda & 2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+5)^2.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 1, while $\lambda_2 = -5$ is an eigenvalue of multiplicity 2. Eigenvectors associated with $\lambda_1 = 1$ are solutions of the system with augmented matrix

$$\begin{bmatrix} -4 & 2 & 2 & \vdots & 0 \\ 2 & -4 & 2 & \vdots & 0 \\ 2 & 2 & -4 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_2 = x_3$, and we choose $x_3 = 1$ to obtain the solution

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t \quad (13.51)$$

of (13.50). Eigenvectors associated with $\lambda_2 = -5$ are solutions of the system with augmented matrix

$$\begin{bmatrix} 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \end{bmatrix}.$$

Hence, the components of these eigenvectors need only satisfy the single condition

$$x_1 + x_2 + x_3 = 0.$$

Since there's only one equation here, we can choose x_2 and x_3 arbitrarily. We obtain one eigenvector by choosing $x_2 = 0$ and $x_3 = 1$, and another by choosing $x_2 = 1$ and $x_3 = 0$. In both cases $x_1 = -1$. Therefore

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent eigenvectors associated with $\lambda_2 = -5$, and the corresponding solutions of (13.50) are

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Because of this and (13.51), Theorem 13.4.1 implies that the general solution of (13.50) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

13.4.1 Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \quad (13.52)$$

It is convenient to think of a “ y_1 - y_2 plane,” where a point is identified by rectangular coordinates (y_1, y_2) . If $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is a non-constant solution of (13.52), then the point $(y_1(t), y_2(t))$ moves along a curve C in the y_1 - y_2 plane as t varies from $-\infty$ to ∞ . We call C the *trajectory* of $\mathbf{y}$. (We also say that C is a trajectory of the system (13.52).) It's important to note that C is the trajectory of infinitely many solutions of (13.52), since if τ is any real number, then $\mathbf{y}(t - \tau)$ is a solution of (13.52) (Exercise 13.4.28(b)), and $(y_1(t - \tau), y_2(t - \tau))$ also moves along C as t varies from $-\infty$ to ∞ . Moreover, Exercise 13.4.28(c) implies that distinct trajectories of (13.52) can't intersect, and that two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of (13.52) have the same trajectory if and only if $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$ for some τ .

From Exercise 13.4.28(a), a trajectory of a nontrivial solution of (13.52) can't contain $(0, 0)$, which we define to be the trajectory of the trivial solution $\mathbf{y} \equiv \mathbf{0}$. More generally, if $\mathbf{y} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq \mathbf{0}$ is a constant solution of (13.52) (which could occur if zero is an eigenvalue of the matrix of (13.52)), we define the trajectory of $\mathbf{y}$ to be the single point (k_1, k_2) .

To be specific, this is the question: What do the trajectories look like, and how are they traversed? In this section we'll answer this question, assuming that the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

of (13.52) has real eigenvalues λ_1 and λ_2 with associated linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then the general solution of (13.52) is

$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t}. \quad (13.53)$$

We'll consider other situations in the next two sections.

We leave it to you (Exercise 13.4.35) to classify the trajectories of (13.52) if zero is an eigenvalue of A . We'll confine our attention here to the case where both eigenvalues are nonzero. In this case the simplest situation is where $\lambda_1 = \lambda_2 \neq 0$, so (13.53) becomes

$$\mathbf{y} = (c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2) e^{\lambda_1 t}.$$

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, an arbitrary vector $\mathbf{x}$ can be written as $\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2$. Therefore the general solution of (13.52) can be written as $\mathbf{y} = \mathbf{x} e^{\lambda_1 t}$ where $\mathbf{x}$ is an arbitrary 2-vector, and the trajectories of nontrivial solutions of (13.52) are half-lines through (but not including) the origin. The direction of motion is away from the origin if $\lambda_1 > 0$ (Figure 13.4), toward it if $\lambda_1 < 0$ (Figure 13.5). (In these and the next figures an arrow through a point indicates the direction of motion along the trajectory through the point.)

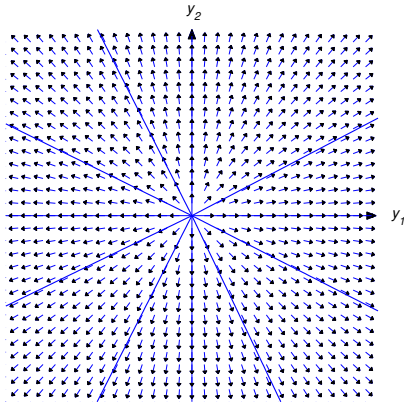


Figure 13.4: Trajectories of a 2×2 system with a repeated positive eigenvalue

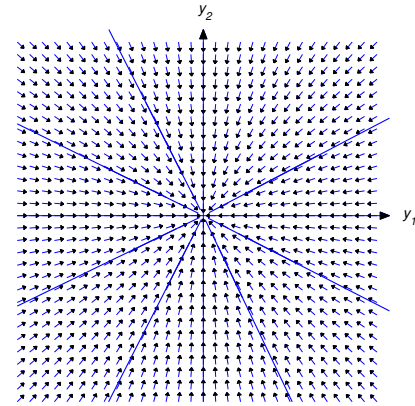


Figure 13.5: Trajectories of a 2×2 system with a repeated negative eigenvalue

Now suppose $\lambda_2 > \lambda_1$, and let L_1 and L_2 denote lines through the origin parallel to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. By a half-line of L_1 (or L_2), we mean either of the rays obtained by removing the origin from L_1 (or L_2).

Letting $c_2 = 0$ in (13.53) yields $\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t}$. If $c_1 \neq 0$, the trajectory defined by this solution is a half-line of L_1 . The direction of motion is away from the origin if $\lambda_1 > 0$, toward the origin if $\lambda_1 < 0$. Similarly, the trajectory of $\mathbf{y} = c_2 \mathbf{x}_2 e^{\lambda_2 t}$ with $c_2 \neq 0$ is a half-line of L_2 .

Henceforth, we assume that c_1 and c_2 in (13.53) are both nonzero. In this case, the trajectory of (13.53) can't intersect L_1 or L_2 , since every point on these lines is on the trajectory of a solution for which either $c_1 = 0$ or $c_2 = 0$. (Remember: distinct trajectories can't intersect!). Therefore the trajectory of (13.53) must lie entirely in one of the four open sectors bounded by L_1 and L_2 , but do not any point on L_1 or L_2 . Since the initial point $(y_1(0), y_2(0))$ defined by

$$\mathbf{y}(0) = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2$$

is on the trajectory, we can determine which sector contains the trajectory from the signs of c_1 and c_2 , as shown in Figure 13.6.

The direction of $\mathbf{y}(t)$ in (13.53) is the same as that of

$$e^{-\lambda_2 t} \mathbf{y}(t) = c_1 \mathbf{x}_1 e^{-(\lambda_2 - \lambda_1)t} + c_2 \mathbf{x}_2 \quad (13.54)$$

and of

$$e^{-\lambda_1 t} \mathbf{y}(t) = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 e^{(\lambda_2 - \lambda_1)t}. \quad (13.55)$$

Since the right side of (13.54) approaches $c_2 \mathbf{x}_2$ as $t \rightarrow \infty$, the trajectory is asymptotically parallel to L_2 as $t \rightarrow \infty$. Since the right side of (13.55) approaches $c_1 \mathbf{x}_1$ as $t \rightarrow -\infty$, the trajectory is asymptotically parallel to L_1 as $t \rightarrow -\infty$.

The shape and direction of traversal of the trajectory of (13.53) depend upon whether λ_1 and λ_2 are both positive, both negative, or of opposite signs. We'll now analyze these three cases.

Henceforth $\|\mathbf{u}\|$ denote the length of the vector $\mathbf{u}$.

13.4.2 Case 1: $\lambda_2 > \lambda_1 > 0$

Figure 13.7 shows some typical trajectories. In this case, $\lim_{t \rightarrow -\infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is not only asymptotically parallel to L_1 as $t \rightarrow -\infty$, but is actually asymptotically tangent to L_1 at the origin. On the other hand, $\lim_{t \rightarrow \infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t \rightarrow \infty} \|\mathbf{y}(t) - c_2 \mathbf{x}_2 e^{\lambda_2 t}\| = \lim_{t \rightarrow \infty} \|c_1 \mathbf{x}_1 e^{\lambda_1 t}\| = \infty,$$

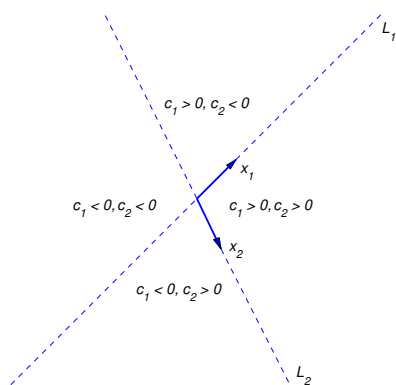
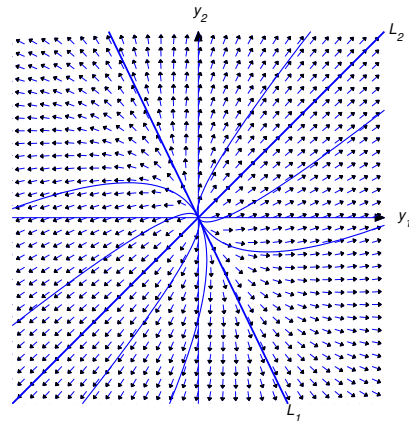
Figure 13.6: Four open sectors bounded by L_1 and L_2 

Figure 13.7: Two positive eigenvalues; motion away from origin

so, although the trajectory is asymptotically parallel to L_2 as $t \rightarrow \infty$, it's not asymptotically tangent to L_2 . The direction of motion along each trajectory is away from the origin.

13.4.3 Case 2: $0 > \lambda_2 > \lambda_1$

Figure 13.8 shows some typical trajectories. In this case, $\lim_{t \rightarrow \infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is asymptotically tangent to L_2 at the origin as $t \rightarrow \infty$. On the other hand, $\lim_{t \rightarrow -\infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t \rightarrow -\infty} \|\mathbf{y}(t) - c_1 \mathbf{x}_1 e^{\lambda_1 t}\| = \lim_{t \rightarrow -\infty} \|c_2 \mathbf{x}_2 e^{\lambda_2 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to L_1 as $t \rightarrow -\infty$, it's not asymptotically tangent to it. The direction of motion along each trajectory is toward the origin.

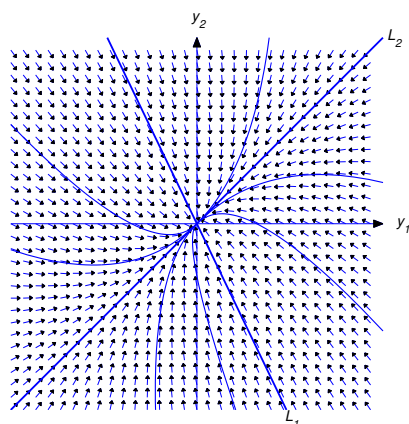


Figure 13.8: Two negative eigenvalues; motion toward the origin

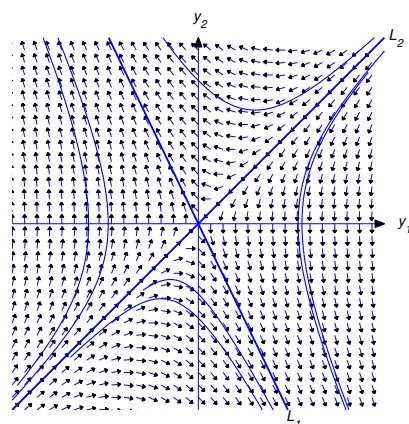


Figure 13.9: Eigenvalues of different signs

13.4.4 Case 3: $\lambda_2 > 0 > \lambda_1$

Figure 13.9 shows some typical trajectories. In this case,

$$\lim_{t \rightarrow \infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\mathbf{y}(t) - c_2 \mathbf{x}_2 e^{\lambda_2 t}\| = \lim_{t \rightarrow \infty} \|c_1 \mathbf{x}_1 e^{\lambda_1 t}\| = 0,$$

so the trajectory is asymptotically tangent to L_2 as $t \rightarrow \infty$. Similarly,

$$\lim_{t \rightarrow -\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t \rightarrow -\infty} \|\mathbf{y}(t) - c_1 \mathbf{x}_1 e^{\lambda_1 t}\| = \lim_{t \rightarrow -\infty} \|c_2 \mathbf{x}_2 e^{\lambda_2 t}\| = 0,$$

so the trajectory is asymptotically tangent to L_1 as $t \rightarrow -\infty$. The direction of motion is toward the origin on L_1 and away from the origin on L_2 . The direction of motion along any other trajectory is away from L_1 , toward L_2 .

13.4.5 Exercises

In Exercises 13.4.1–13.4.15 find the general solution.

$$1. \quad \mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} \qquad 2. \quad \mathbf{y}' = \frac{1}{4} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \mathbf{y}$$

$$3. \quad \mathbf{y}' = \frac{1}{5} \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y} \qquad 4. \quad \mathbf{y}' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$$

$$5. \quad \mathbf{y}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y} \qquad 6. \quad \mathbf{y}' = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \mathbf{y}$$

$$7. \quad \mathbf{y}' = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{y} \qquad 8. \quad \mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} \mathbf{y}$$

$$9. \quad \mathbf{y}' = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \mathbf{y} \qquad 10. \quad \mathbf{y}' = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}$$

$$11. \quad \mathbf{y}' = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \mathbf{y} \qquad 12. \quad \mathbf{y}' = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{y}$$

$$13. \quad \mathbf{y}' = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y} \qquad 14. \quad \mathbf{y}' = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \mathbf{y}$$

$$15. \quad \mathbf{y}' = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \mathbf{y}$$

In Exercises 13.4.16–13.4.27 solve the initial value problem.

$$16. \quad \mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$17. \quad \mathbf{y}' = \frac{1}{6} \begin{bmatrix} 7 & 2 \\ -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$18. \quad \mathbf{y}' = \begin{bmatrix} 21 & -12 \\ 24 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$19. \quad \mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

$$20. \quad \mathbf{y}' = \frac{1}{6} \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

$$21. \quad \mathbf{y}' = \frac{1}{3} \begin{bmatrix} 2 & -2 & 3 \\ -4 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$22. \quad \mathbf{y}' = \begin{bmatrix} 6 & -3 & -8 \\ 2 & 1 & -2 \\ 3 & -3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$23. \quad \mathbf{y}' = \frac{1}{3} \begin{bmatrix} 2 & 4 & -7 \\ 1 & 5 & -5 \\ -4 & 4 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$24. \quad \mathbf{y}' = \begin{bmatrix} 3 & 0 & 1 \\ 11 & -2 & 7 \\ 1 & 0 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$$

$$25. \quad \mathbf{y}' = \begin{bmatrix} -2 & -5 & -1 \\ -4 & -1 & 1 \\ 4 & 5 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ -10 \\ -4 \end{bmatrix}$$

$$26. \quad \mathbf{y}' = \begin{bmatrix} 3 & -1 & 0 \\ 4 & -2 & 0 \\ 4 & -4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 7 \\ 10 \\ 2 \end{bmatrix}$$

$$27. \quad \mathbf{y}' = \begin{bmatrix} -2 & 2 & 6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ -10 \\ 7 \end{bmatrix}$$

28. Let A be an $n \times n$ constant matrix. Then Theorem 13.2.1 implies that the solutions of

$$\mathbf{y}' = A\mathbf{y} \tag{A}$$

are all defined on $(-\infty, \infty)$.

- (a) Use Theorem 13.2.1 to show that the only solution of (A) that can ever equal the zero vector is $\mathbf{y} \equiv \mathbf{0}$.
- (b) Suppose $\mathbf{y}_1$ is a solution of (A) and $\mathbf{y}_2$ is defined by $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$, where τ is an arbitrary real number. Show that $\mathbf{y}_2$ is also a solution of (A).
- (c) Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of (A) and there are real numbers t_1 and t_2 such that $\mathbf{y}_1(t_1) = \mathbf{y}_2(t_2)$. Show that $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$ for all t , where $\tau = t_2 - t_1$. *HINT: Show that $\mathbf{y}_1(t - \tau)$ and $\mathbf{y}_2(t)$ are solutions of the same initial value problem for (A), and apply the uniqueness assertion of Theorem 13.2.1.*

In Exercises 13.4.29–13.4.34 describe and graph trajectories of the given system.

$$29. \quad \boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y} \qquad 30. \quad \boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y}$$

$$31. \quad \boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} \mathbf{y} \qquad 32. \quad \boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -1 & -10 \\ -5 & 4 \end{bmatrix} \mathbf{y}$$

$$33. \quad \boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 1 & 10 \end{bmatrix} \mathbf{y} \qquad 34. \quad \boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -7 & 1 \\ 3 & -5 \end{bmatrix} \mathbf{y}$$

35. Suppose the eigenvalues of the 2×2 matrix A are $\lambda = 0$ and $\mu \neq 0$, with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Let L_1 be the line through the origin parallel to $\mathbf{x}_1$.
- (a) Show that every point on L_1 is the trajectory of a constant solution of $\mathbf{y}' = A\mathbf{y}$.
- (b) Show that the trajectories of nonconstant solutions of $\mathbf{y}' = A\mathbf{y}$ are half-lines parallel to $\mathbf{x}_2$ and on either side of L_1 , and that the direction of motion along these trajectories is away from L_1 if $\mu > 0$, or toward L_1 if $\mu < 0$.

The matrices of the systems in Exercises 13.4.36–13.4.41 are singular. Describe and graph the trajectories of nonconstant solutions of the given systems.

36. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y}$

37. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix} \mathbf{y}$

38. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \mathbf{y}$

39. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \mathbf{y}$

40. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} -4 & -4 \\ 1 & 1 \end{bmatrix} \mathbf{y}$

41. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \mathbf{y}$

42. $\boxed{\text{L}}$ Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time t , and assume that each population would grow exponentially if the other didn't exist; that is, in the absence of competition,

$$P' = aP \quad \text{and} \quad Q' = bQ, \quad (\text{A})$$

where a and b are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so (A) is replaced by

$$\begin{aligned} P' &= aP - \alpha Q \\ Q' &= -\beta P + bQ, \end{aligned}$$

where α and β are positive constants. (Since negative population doesn't make sense, this system holds only while P and Q are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$.

- (a) For several choices of a , b , α , and β , verify experimentally (by graphing trajectories of (A) in the P - Q plane) that there's a constant $\rho > 0$ (depending upon a , b , α , and β) with the following properties:
- (i) If $Q_0 > \rho P_0$, then P decreases monotonically to zero in finite time, during which Q remains positive.
 - (ii) If $Q_0 < \rho P_0$, then Q decreases monotonically to zero in finite time, during which P remains positive.
- (b) Conclude from (a) that exactly one of the species becomes extinct in finite time if $Q_0 \neq \rho P_0$. Determine experimentally what happens if $Q_0 = \rho P_0$.
- (c) Confirm your experimental results and determine γ by expressing the eigenvalues and associated eigenvectors of

$$A = \begin{bmatrix} a & -\alpha \\ -\beta & b \end{bmatrix}$$

in terms of a , b , α , and β , and applying the geometric arguments developed at the end of this section.

13.5 Constant Coefficient Homogeneous Systems II

We saw in Section 10.4 that if an $n \times n$ constant matrix A has n real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, then the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t} + \dots + c_n \mathbf{x}_n e^{\lambda_n t}.$$

In this section we consider the case where A has n real eigenvalues, but does not have n linearly independent eigenvectors. It is shown in linear algebra that this occurs if and only if A has at least one eigenvalue of multiplicity $r > 1$ such that the associated eigenspace has dimension less than r . In this case A is said to be *defective*. Since it's beyond the scope of this book to give a complete analysis of systems with defective coefficient matrices, we will restrict our attention to some commonly occurring special cases.

Example 13.5.1: Show that the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \quad (13.56)$$

does not have a fundamental set of solutions of the form $\{\mathbf{x}_1 e^{\lambda_1 t}, \mathbf{x}_2 e^{\lambda_2 t}\}$, where λ_1 and λ_2 are eigenvalues of the coefficient matrix A of (13.56) and $\mathbf{x}_1$, and $\mathbf{x}_2$ are associated linearly independent eigenvectors.

Solution: The characteristic polynomial of A is

$$\begin{aligned} \begin{vmatrix} 11 - \lambda & -25 \\ 4 & -9 - \lambda \end{vmatrix} &= (\lambda - 11)(\lambda + 9) + 100 \\ &= \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2. \end{aligned}$$

Hence, $\lambda = 1$ is the only eigenvalue of A . The augmented matrix of the system $(A - I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 10 & -25 & \vdots & 0 \\ 4 & -10 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & -\frac{5}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = 5x_2/2$ where x_2 is arbitrary. Therefore all eigenvectors of A are scalar multiples of $\mathbf{x}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, so A does not have a set of two linearly independent eigenvectors. ■

From Example 13.5.1, we know that all scalar multiples of $\mathbf{y}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t$ are solutions of (13.56); however, to find the general solution we must find a second solution $\mathbf{y}_2$ such that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is linearly independent. Based on your recollection of the procedure for solving a constant coefficient scalar equation

$$ay'' + by' + cy = 0$$

in the case where the characteristic polynomial has a repeated root, you might expect to obtain a second solution of (13.56) by multiplying the first solution by t . However, this yields $\mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t$, which doesn't

work, since

$$\mathbf{y}'_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} (te^t + e^t), \quad \text{while} \quad \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

The next theorem shows what to do in this situation.

Theorem 13.5.1. *Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 2 and the associated eigenspace has dimension 1; that is, all λ_1 -eigenvectors of A are scalar multiples of an eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that*

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \quad (13.57)$$

Moreover, if $\mathbf{u}$ is any such vector then

$$\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t} \quad \text{and} \quad \mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \quad (13.58)$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

A complete proof of this theorem is beyond the scope of this book. The difficulty is in proving that there's a vector $\mathbf{u}$ satisfying (13.57), since $\det(A - \lambda_1 I) = 0$. We'll take this without proof and verify the other assertions of the theorem.

We already know that $\mathbf{y}_1$ in (13.58) is a solution of $\mathbf{y}' = A\mathbf{y}$. To see that $\mathbf{y}_2$ is also a solution, we compute

$$\begin{aligned} \mathbf{y}'_2 - A\mathbf{y}_2 &= \lambda_1 \mathbf{u}e^{\lambda_1 t} + \mathbf{x}e^{\lambda_1 t} + \lambda_1 \mathbf{x}te^{\lambda_1 t} - A\mathbf{u}e^{\lambda_1 t} - A\mathbf{x}te^{\lambda_1 t} \\ &= (\lambda_1 \mathbf{u} + \mathbf{x} - A\mathbf{u})e^{\lambda_1 t} + (\lambda_1 \mathbf{x} - A\mathbf{x})te^{\lambda_1 t}. \end{aligned}$$

Since $A\mathbf{x} = \lambda_1 \mathbf{x}$, this can be written as

$$\mathbf{y}'_2 - A\mathbf{y}_2 = -((A - \lambda_1 I)\mathbf{u} - \mathbf{x})e^{\lambda_1 t},$$

and now (13.57) implies that $\mathbf{y}'_2 = A\mathbf{y}_2$.

To see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent, suppose c_1 and c_2 are constants such that

$$c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 = c_1 \mathbf{x}e^{\lambda_1 t} + c_2 (\mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}) = \mathbf{0}. \quad (13.59)$$

We must show that $c_1 = c_2 = 0$. Multiplying (13.59) by $e^{-\lambda_1 t}$ shows that

$$c_1 \mathbf{x} + c_2 (\mathbf{u} + \mathbf{x}t) = \mathbf{0}. \quad (13.60)$$

By differentiating this with respect to t , we see that $c_2 \mathbf{x} = \mathbf{0}$, which implies $c_2 = 0$, because $\mathbf{x} \neq \mathbf{0}$. Substituting $c_2 = 0$ into (13.60) yields $c_1 \mathbf{x} = \mathbf{0}$, which implies that $c_1 = 0$, again because $\mathbf{x} \neq \mathbf{0}$.

Example 13.5.2: Use Theorem 13.5.1 to find the general solution of the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \quad (13.61)$$

considered in Example 13.5.1.

Solution: In Example 13.5.1 we saw that $\lambda_1 = 1$ is an eigenvalue of multiplicity 2 of the coefficient matrix A in (13.61), and that all of the eigenvectors of A are multiples of

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t$$

is a solution of (13.61). From Theorem 13.5.1, a second solution is given by $\mathbf{y}_2 = \mathbf{u}e^t + \mathbf{x}te^t$, where $(A - I)\mathbf{u} = \mathbf{x}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 10 & -25 & \vdots & 5 \\ 4 & -10 & \vdots & 2 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & -\frac{5}{2} & \vdots & \frac{1}{2} \\ 0 & 0 & \vdots & 0 \end{array} \right].$$

Therefore the components of $\mathbf{u}$ must satisfy

$$u_1 - \frac{5}{2}u_2 = \frac{1}{2},$$

where u_2 is arbitrary. We choose $u_2 = 0$, so that $u_1 = 1/2$ and

$$\mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

Thus,

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent by Theorem 13.5.1, they form a fundamental set of solutions of (13.61). Therefore the general solution of (13.61) is

$$\mathbf{y} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t \right). \blacksquare$$

Note that choosing the arbitrary constant u_2 to be nonzero is equivalent to adding a scalar multiple of $\mathbf{y}_1$ to the second solution $\mathbf{y}_2$ (Exercise 13.5.33).

Example 13.5.3: Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & 4 & -10 \\ 2 & 1 & -2 \\ 2 & 2 & -5 \end{bmatrix} \mathbf{y}. \quad (13.62)$$

Solution: The characteristic polynomial of the coefficient matrix A in (13.62) is

$$\begin{vmatrix} 3 - \lambda & 4 & -10 \\ 2 & 1 - \lambda & -2 \\ 2 & 2 & -5 - \lambda \end{vmatrix} = -(\lambda - 1)(\lambda + 1)^2.$$

Hence, the eigenvalues are $\lambda_1 = 1$ with multiplicity 1 and $\lambda_2 = -1$ with multiplicity 2.

Eigenvectors associated with $\lambda_1 = 1$ must satisfy $(A - I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & \vdots & 0 \\ 2 & 0 & -2 & \vdots & 0 \\ 2 & 2 & -6 & \vdots & 0 \end{array} \right],$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3$ and $x_2 = 2x_3$, where x_3 is arbitrary. Choosing $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t$$

is a solution of (13.62).

Eigenvectors associated with $\lambda_2 = -1$ satisfy $(A + I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\begin{bmatrix} 4 & 4 & -10 & \vdots & 0 \\ 2 & 2 & -2 & \vdots & 0 \\ 2 & 2 & -4 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_3 = 0$ and $x_1 = -x_2$, where x_2 is arbitrary. Choosing $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

is a solution of (13.62).

Since all the eigenvectors of A associated with $\lambda_2 = -1$ are multiples of $\mathbf{x}_2$, we must now use Theorem 13.5.1 to find a third solution of (13.62) in the form

$$\mathbf{y}_3 = \mathbf{u}e^{-t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t}, \quad (13.63)$$

where $\mathbf{u}$ is a solution of $(A + I)\mathbf{u} = \mathbf{x}_2$. The augmented matrix of this system is

$$\begin{bmatrix} 4 & 4 & -10 & \vdots & -1 \\ 2 & 2 & -2 & \vdots & 1 \\ 2 & 2 & -4 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $u_3 = 1/2$ and $u_1 = 1 - u_2$, where u_2 is arbitrary. Choosing $u_2 = 0$ yields

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix},$$

and substituting this into (13.63) yields the solution

$$\mathbf{y}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} t e^{-t}$$

of (13.62).

Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2},$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (13.62). Therefore the general solution of (13.62) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} t e^{-t} \right).$$

Theorem 13.5.2. Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is one-dimensional; that is, all eigenvectors associated with λ_1 are scalar multiples of the eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}, \quad (13.64)$$

and, if $\mathbf{u}$ is any such vector, there are infinitely many vectors $\mathbf{v}$ such that

$$(A - \lambda_1 I)\mathbf{v} = \mathbf{u}. \quad (13.65)$$

If $\mathbf{u}$ satisfies (13.64) and $\mathbf{v}$ satisfies (13.65), then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x} e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{u} e^{\lambda_1 t} + \mathbf{x} t e^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{v} e^{\lambda_1 t} + \mathbf{u} t e^{\lambda_1 t} + \mathbf{x} \frac{t^2 e^{\lambda_1 t}}{2} \end{aligned}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Again, it's beyond the scope of this book to prove that there are vectors $\mathbf{u}$ and $\mathbf{v}$ that satisfy (13.64) and (13.65). Theorem 13.5.1 implies that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave the rest of the proof to you (Exercise 13.5.34).

Example 13.5.4: Use Theorem 13.5.2 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \mathbf{y}. \quad (13.66)$$

Solution: The characteristic polynomial of the coefficient matrix A in (13.66) is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 2 & 2-\lambda \end{vmatrix} = -(\lambda - 2)^3.$$

Hence, $\lambda_1 = 2$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A - 2I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\begin{bmatrix} -1 & 1 & 1 & \vdots & 0 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3$ and $x_2 = 0$, so the eigenvectors are all scalar multiples of

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of (13.66).

We now find a second solution of (13.66) in the form

$$\mathbf{y}_2 = \mathbf{u}e^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t},$$

where $\mathbf{u}$ satisfies $(A - 2I)\mathbf{u} = \mathbf{x}_1$. The augmented matrix of this system is

$$\begin{bmatrix} -1 & 1 & 1 & \vdots & 1 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 1 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & -\frac{1}{2} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Letting $u_3 = 0$ yields $u_1 = -1/2$ and $u_2 = 1/2$; hence,

$$\mathbf{u} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t e^{2t}$$

is a solution of (13.66).

We now find a third solution of (13.66) in the form

$$\mathbf{y}_3 = \mathbf{v} e^{2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{t e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2}$$

where $\mathbf{v}$ satisfies $(A - 2I)\mathbf{v} = \mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & \vdots & -\frac{1}{2} \\ 1 & 1 & -1 & \vdots & \frac{1}{2} \\ 0 & 2 & 0 & \vdots & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & \vdots & \frac{1}{2} \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array} \right].$$

Letting $v_3 = 0$ yields $v_1 = 1/2$ and $v_2 = 0$; hence,

$$\mathbf{v} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{t e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2}$$

is a solution of (13.66). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 13.5.2, they form a fundamental set of solutions of (13.66). Therefore the general solution of (13.66) is

$$\begin{aligned} \mathbf{y} = & c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t e^{2t} \right) \\ & + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{t e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2} \right). \end{aligned}$$

Theorem 13.5.3. Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is two-dimensional; that is, all eigenvectors of A associated with λ_1 are linear combinations of two linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then there are constants α and β (not both zero) such that if

$$\mathbf{x}_3 = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2, \quad (13.67)$$

then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}_3. \quad (13.68)$$

If $\mathbf{u}$ satisfies (13.68), then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_1 e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{x}_2 e^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{u} e^{\lambda_1 t} + \mathbf{x}_3 t e^{\lambda_1 t}, \end{aligned} \quad (13.69)$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Example 13.5.5: Use Theorem 13.5.3 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \mathbf{y}. \quad (13.70)$$

Solution: The characteristic polynomial of the coefficient matrix A in (13.70) is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} = -(\lambda-1)^3.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A - I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & \vdots & 0 & 0 \\ -1 & 0 & 1 & \vdots & 0 & 0 \\ -1 & 0 & 1 & \vdots & 0 & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \end{array} \right].$$

Hence, $x_1 = x_3$ and x_2 is arbitrary, so the eigenvectors are of the form

$$\mathbf{x}_1 = \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (13.71)$$

form a basis for the eigenspace, and

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$$

are linearly independent solutions of (13.70).

To find a third linearly independent solution of (13.70), we must find constants α and β (not both zero) such that the system

$$(A - I)\mathbf{u} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 \quad (13.72)$$

has a solution $\mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & \vdots & \alpha & 0 \\ -1 & 0 & 1 & \vdots & 0 & \beta \\ -1 & 0 & 1 & \vdots & \alpha & 0 \end{array} \right],$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & -\alpha \\ 0 & 0 & 0 & \vdots & \beta - \alpha \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}. \quad (13.73)$$

Therefore (13.72) has a solution if and only if $\beta = \alpha$, where α is arbitrary. If $\alpha = \beta = 1$ then (13.67) and (13.71) yield

$$\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and the augmented matrix (13.73) becomes

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

This implies that $u_1 = -1 + u_3$, while u_2 and u_3 are arbitrary. Choosing $u_2 = u_3 = 0$ yields

$$\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore (13.69) implies that

$$\mathbf{y}_3 = \mathbf{u}e^t + \mathbf{x}_3te^t = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}e^t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}te^t$$

is a solution of (13.70). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 13.5.3, they form a fundamental set of solutions for (13.70). Therefore the general solution of (13.70) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^t \right). \blacksquare$$

13.5.1 Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (13.74)$$

under the assumptions of this section; that is, when the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a repeated eigenvalue λ_1 and the associated eigenspace is one-dimensional. In this case we know from Theorem 13.5.1 that the general solution of (13.74) is

$$\mathbf{y} = c_1 \mathbf{x}e^{\lambda_1 t} + c_2 (\mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}), \quad (13.75)$$

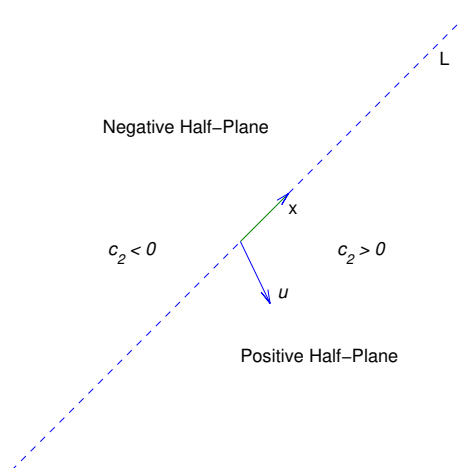


Figure 13.10: Positive and negative half-planes

where $\mathbf{x}$ is an eigenvector of A and $\mathbf{u}$ is any one of the infinitely many solutions of

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \quad (13.76)$$

We assume that $\lambda_1 \neq 0$.

Let L denote the line through the origin parallel to $\mathbf{x}$. By a *half-line* of L we mean either of the rays obtained by removing the origin from L . Eqn. (13.75) is a parametric equation of the half-line of L in the direction of $\mathbf{x}$ if $c_1 > 0$, or of the half-line of L in the direction of $-\mathbf{x}$ if $c_1 < 0$. The origin is the trajectory of the trivial solution $\mathbf{y} \equiv \mathbf{0}$.

Henceforth, we assume that $c_2 \neq 0$. In this case, the trajectory of (13.75) can't intersect L , since every point of L is on a trajectory obtained by setting $c_2 = 0$. Therefore the trajectory of (13.75) must lie entirely in one of the open half-planes bounded by L , but does not contain any point on L . Since the initial point $(y_1(0), y_2(0))$ defined by $\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{u}$ is on the trajectory, we can determine which half-plane contains the trajectory from the sign of c_2 , as shown in Figure 615. For convenience we'll call the half-plane where $c_2 > 0$ the *positive half-plane*. Similarly, the half-plane where $c_2 < 0$ is the *negative half-plane*. You should convince yourself (Exercise 13.5.35) that even though there are infinitely many vectors $\mathbf{u}$ that satisfy (13.76), they all define the same positive and negative half-planes. In the figures simply regard $\mathbf{u}$ as an arrow pointing to the positive half-plane, since we've attempted to give $\mathbf{u}$ its proper length or direction in comparison with $\mathbf{x}$. For our purposes here, only the relative orientation of $\mathbf{x}$ and $\mathbf{u}$ is important; that is, whether the positive half-plane is to the right of an observer facing the direction of $\mathbf{x}$ (as in Figures 13.11 and 13.14), or to the left of the observer (as in Figures 13.12 and 13.13).

Multiplying (13.75) by $e^{-\lambda_1 t}$ yields

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x} + c_2\mathbf{u} + c_2t\mathbf{x}.$$

Since the last term on the right is dominant when $|t|$ is large, this provides the following information on the direction of $\mathbf{y}(t)$:

- (a) Along trajectories in the positive half-plane ($c_2 > 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $\mathbf{x}$ as $t \rightarrow \infty$ and the direction of $-\mathbf{x}$ as $t \rightarrow -\infty$.
- (b) Along trajectories in the negative half-plane ($c_2 < 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $-\mathbf{x}$ as $t \rightarrow \infty$ and the direction of $\mathbf{x}$ as $t \rightarrow -\infty$.

Since

$$\lim_{t \rightarrow \infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t \rightarrow -\infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 > 0,$$

or

$$\lim_{t \rightarrow \infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 < 0,$$

there are four possible patterns for the trajectories of (13.74), depending upon the signs of c_2 and λ_1 . Figures 13.11-13.14 illustrate these patterns, and reveal the following principle:

If λ_1 and c_2 have the same sign then the direction of the trajectory approaches the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \rightarrow 0$ and the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \rightarrow \infty$. If λ_1 and c_2 have opposite signs then the direction of the trajectory approaches the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \rightarrow 0$ and the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \rightarrow \infty$.

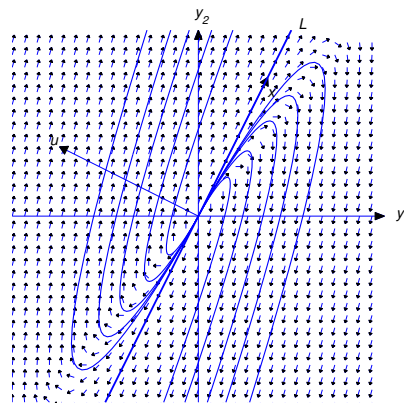
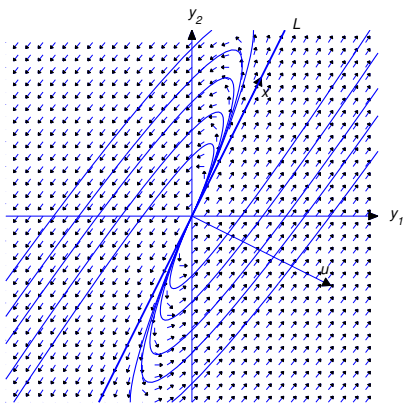


Figure 13.11: Positive eigenvalue; motion away from the origin Figure 13.12: Positive eigenvalue; motion away from the origin

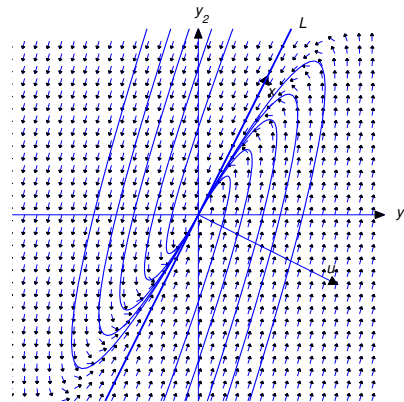
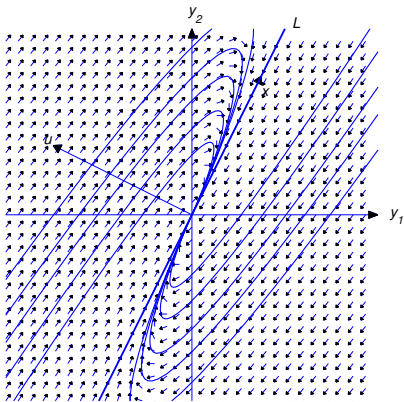


Figure 13.13: Negative eigenvalue; motion toward the origin Figure 13.14: Negative eigenvalue; motion toward the origin

13.5.2 Exercises

In Exercises 13.5.1–13.5.12 find the general solution.

$$1. \quad \mathbf{y}' = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \mathbf{y} \qquad 2. \quad \mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{y}$$

$$3. \quad \mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} \mathbf{y} \qquad 4. \quad \mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{y}$$

$$5. \quad \mathbf{y}' = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} \mathbf{y} \qquad 6. \quad \mathbf{y}' = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix} \mathbf{y}$$

$$7. \quad \mathbf{y}' = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix} \mathbf{y} \qquad 8. \quad \mathbf{y}' = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix} \mathbf{y}$$

$$9. \quad \mathbf{y}' = \frac{1}{3} \begin{bmatrix} 1 & 1 & -3 \\ -4 & -4 & 3 \\ -2 & 1 & 0 \end{bmatrix} \mathbf{y} \qquad 10. \quad \mathbf{y}' = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix} \mathbf{y}$$

$$11. \quad \mathbf{y}' = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \mathbf{y} \qquad 12. \quad \mathbf{y}' = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{y}$$

In Exercises 13.5.13–13.5.23 solve the initial value problem.

$$13. \quad \mathbf{y}' = \begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$14. \quad \mathbf{y}' = \begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$15. \quad \mathbf{y}' = \begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$16. \quad \mathbf{y}' = \begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$17. \quad \mathbf{y}' = \begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$18. \quad \mathbf{y}' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix}$$

$$19. \quad \mathbf{y}' = \begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -6 \\ -2 \\ 0 \end{bmatrix}$$

$$20. \quad \mathbf{y}' = \begin{bmatrix} -7 & -4 & 4 \\ -1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -6 \\ 9 \\ -1 \end{bmatrix}$$

$$21. \quad \mathbf{y}' = \begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$$22. \quad \mathbf{y}' = \begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -3 \\ 1 & -1 & 9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$$

$$23. \quad \mathbf{y}' = \begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

The coefficient matrices in Exercises 13.5.24–13.5.32 have eigenvalues of multiplicity 3. Find the general solution.

$$24. \quad \mathbf{y}' = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix} \mathbf{y} \qquad 25. \quad \mathbf{y}' = \begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix} \mathbf{y}$$

$$26. \quad \mathbf{y}' = \begin{bmatrix} -6 & -4 & -4 \\ 2 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \mathbf{y} \qquad 27. \quad \mathbf{y}' = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{y}$$

$$28. \quad \mathbf{y}' = \begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} \mathbf{y} \qquad 29. \quad \mathbf{y}' = \begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} \mathbf{y}$$

$$30. \quad \mathbf{y}' = \begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} \mathbf{y} \qquad 31. \quad \mathbf{y}' = \begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} \mathbf{y}$$

$$32. \quad \mathbf{y}' = \begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} \mathbf{y}$$

33. Under the assumptions of Theorem 13.5.1, suppose $\mathbf{u}$ and $\hat{\mathbf{u}}$ are vectors such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x} \quad \text{and} \quad (A - \lambda_1 I)\hat{\mathbf{u}} = \mathbf{x},$$

and let

$$\mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \quad \text{and} \quad \hat{\mathbf{y}}_2 = \hat{\mathbf{u}}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}.$$

Show that $\mathbf{y}_2 - \hat{\mathbf{y}}_2$ is a scalar multiple of $\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t}$.

34. Under the assumptions of Theorem 13.5.2, let

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t} + \mathbf{u}te^{\lambda_1 t} + \mathbf{x}\frac{t^2 e^{\lambda_1 t}}{2}. \end{aligned}$$

Complete the proof of Theorem 13.5.2 by showing that $\mathbf{y}_3$ is a solution of $\mathbf{y}' = A\mathbf{y}$ and that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is linearly independent.

35. Suppose the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a repeated eigenvalue λ_1 and the associated eigenspace is one-dimensional. Let $\mathbf{x}$ be a λ_1 -eigenvector of A . Show that if $(A - \lambda_1 I)\mathbf{u}_1 = \mathbf{x}$ and $(A - \lambda_1 I)\mathbf{u}_2 = \mathbf{x}$, then $\mathbf{u}_2 - \mathbf{u}_1$ is parallel to $\mathbf{x}$. Conclude from this that all vectors $\mathbf{u}$ such that $(A - \lambda_1 I)\mathbf{u} = \mathbf{x}$ define the same positive and negative half-planes with respect to the line L through the origin parallel to $\mathbf{x}$.

In Exercises 13.5.36- 13.5.45 plot trajectories of the given system.

36. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} \mathbf{y}$ 37. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{y}$
38. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ 3 & 5 \end{bmatrix} \mathbf{y}$ 39. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} -5 & 3 \\ -3 & 1 \end{bmatrix} \mathbf{y}$
40. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \mathbf{y}$ 41. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix} \mathbf{y}$
42. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{y}$ 43. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}$
44. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y}$ 45. $\boxed{\text{C/G}}$ $\mathbf{y}' = \begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix} \mathbf{y}$

13.6 Constant Coefficient Homogeneous Systems III

We now consider the system $\mathbf{y}' = A\mathbf{y}$, where A has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$. We continue to assume that A has real entries, so the characteristic polynomial of A has real coefficients. This implies that $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue of A .

An eigenvector $\mathbf{x}$ of A associated with $\lambda = \alpha + i\beta$ will have complex entries, so we'll write

$$\mathbf{x} = \mathbf{u} + i\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ have real entries; that is, $\mathbf{u}$ and $\mathbf{v}$ are the real and imaginary parts of $\mathbf{x}$. Since $A\mathbf{x} = \lambda\mathbf{x}$,

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v}). \quad (13.77)$$

Taking complex conjugates here and recalling that A has real entries yields

$$A(\mathbf{u} - i\mathbf{v}) = (\alpha - i\beta)(\mathbf{u} - i\mathbf{v}),$$

which shows that $\mathbf{x} = \mathbf{u} - i\mathbf{v}$ is an eigenvector associated with $\bar{\lambda} = \alpha - i\beta$. The complex conjugate eigenvalues λ and $\bar{\lambda}$ can be separately associated with linearly independent solutions $\mathbf{y}' = A\mathbf{y}$; however, we won't pursue this approach, since solutions obtained in this way turn out to be complex-valued. Instead, we'll obtain solutions of $\mathbf{y}' = A\mathbf{y}$ in the form

$$\mathbf{y} = f_1\mathbf{u} + f_2\mathbf{v} \quad (13.78)$$

where f_1 and f_2 are real-valued scalar functions. The next theorem shows how to do this.

Theorem 13.6.1. *Let A be an $n \times n$ matrix with real entries. Let $\lambda = \alpha + i\beta$ ($\beta \neq 0$) be a complex eigenvalue of A and let $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ be an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Then $\mathbf{u}$ and $\mathbf{v}$ are both nonzero and*

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t),$$

which are the real and imaginary parts of

$$e^{\alpha t}(\cos \beta t + i \sin \beta t)(\mathbf{u} + i\mathbf{v}), \quad (13.79)$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Proof. A function of the form (13.78) is a solution of $\mathbf{y}' = A\mathbf{y}$ if and only if

$$f_1'\mathbf{u} + f_2'\mathbf{v} = f_1A\mathbf{u} + f_2A\mathbf{v}. \quad (13.80)$$

Carrying out the multiplication indicated on the right side of (13.77) and collecting the real and imaginary parts of the result yields

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha\mathbf{u} - \beta\mathbf{v}) + i(\alpha\mathbf{v} + \beta\mathbf{u}).$$

Equating real and imaginary parts on the two sides of this equation yields

$$\begin{aligned} A\mathbf{u} &= \alpha\mathbf{u} - \beta\mathbf{v} \\ A\mathbf{v} &= \alpha\mathbf{v} + \beta\mathbf{u}. \end{aligned}$$

We leave it to you (Exercise 13.6.25) to show from this that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero. Substituting from these equations into (13.80) yields

$$\begin{aligned} f_1'\mathbf{u} + f_2'\mathbf{v} &= f_1(\alpha\mathbf{u} - \beta\mathbf{v}) + f_2(\alpha\mathbf{v} + \beta\mathbf{u}) \\ &= (\alpha f_1 + \beta f_2)\mathbf{u} + (-\beta f_1 + \alpha f_2)\mathbf{v}. \end{aligned}$$

This is true if

$$\begin{aligned} f_1' &= \alpha f_1 + \beta f_2 \\ f_2' &= -\beta f_1 + \alpha f_2, \end{aligned} \quad \text{or, equivalently,} \quad \begin{aligned} f_1' - \alpha f_1 &= \beta f_2 \\ f_2' - \alpha f_2 &= -\beta f_1. \end{aligned}$$

If we let $f_1 = g_1 e^{\alpha t}$ and $f_2 = g_2 e^{\alpha t}$, where g_1 and g_2 are to be determined, then the last two equations become

$$\begin{aligned} g_1' &= \beta g_2 \\ g_2' &= -\beta g_1, \end{aligned}$$

which implies that

$$g_1'' = \beta g_2' = -\beta^2 g_1,$$

so

$$g_1'' + \beta^2 g_1 = 0.$$

The general solution of this equation is

$$g_1 = c_1 \cos \beta t + c_2 \sin \beta t.$$

Moreover, since $g_2 = g_1'/\beta$,

$$g_2 = -c_1 \sin \beta t + c_2 \cos \beta t.$$

Multiplying g_1 and g_2 by $e^{\alpha t}$ shows that

$$\begin{aligned} f_1 &= e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t), \\ f_2 &= e^{\alpha t} (-c_1 \sin \beta t + c_2 \cos \beta t). \end{aligned}$$

Substituting these into (13.78) shows that

$$\begin{aligned} \mathbf{y} &= e^{\alpha t} [(c_1 \cos \beta t + c_2 \sin \beta t)\mathbf{u} + (-c_1 \sin \beta t + c_2 \cos \beta t)\mathbf{v}] \\ &= c_1 e^{\alpha t} (\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) + c_2 e^{\alpha t} (\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t) \end{aligned} \quad (13.81)$$

is a solution of $\mathbf{y}' = A\mathbf{y}$ for any choice of the constants c_1 and c_2 . In particular, by first taking $c_1 = 1$ and $c_2 = 0$ and then taking $c_1 = 0$ and $c_2 = 1$, we see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave it to you to verify that they are, respectively, the real and imaginary parts of (13.79) (Exercise 13.6.26), and that they are linearly independent (Exercise 13.6.27).

Example 13.6.1: Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix} \mathbf{y}. \quad (13.82)$$

Solution: The characteristic polynomial of the coefficient matrix A in (13.82) is

$$\begin{vmatrix} 4 - \lambda & -5 \\ 5 & -2 - \lambda \end{vmatrix} = (\lambda - 1)^2 + 16.$$

Hence, $\lambda = 1 + 4i$ is an eigenvalue of A . The associated eigenvectors satisfy $(A - (1 + 4i)I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 3 - 4i & -5 & \vdots & 0 \\ 5 & -3 - 4i & \vdots & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & -\frac{3+4i}{5} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array} \right].$$

Therefore $x_1 = (3 + 4i)x_2/5$. Taking $x_2 = 5$ yields $x_1 = 3 + 4i$, so

$$\mathbf{x} = \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 4t + i \sin 4t) \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t \begin{bmatrix} 3 \cos 4t - 4 \sin 4t \\ 5 \cos 4t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t \begin{bmatrix} 3 \sin 4t + 4 \cos 4t \\ 5 \sin 4t \end{bmatrix},$$

which are linearly independent solutions of (13.82). The general solution of (13.82) is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} 3 \cos 4t - 4 \sin 4t \\ 5 \cos 4t \end{bmatrix} + c_2 e^t \begin{bmatrix} 3 \sin 4t + 4 \cos 4t \\ 5 \sin 4t \end{bmatrix}.$$

Example 13.6.2: Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} \mathbf{y}. \quad (13.83)$$

Solution: The characteristic polynomial of the coefficient matrix A in (13.83) is

$$\begin{vmatrix} -14 - \lambda & 39 \\ -6 & 16 - \lambda \end{vmatrix} = (\lambda - 1)^2 + 9.$$

Hence, $\lambda = 1 + 3i$ is an eigenvalue of A . The associated eigenvectors satisfy $(A - (1 + 3i)I)\mathbf{x} = \mathbf{0}$. The augmented augmented matrix of this system is

$$\left[\begin{array}{cc|cc} -15 - 3i & 39 & \vdots & 0 \\ -6 & 15 - 3i & \vdots & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{cc|cc} 1 & \frac{-5+i}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array} \right].$$

Therefore $x_1 = (5 - i)/2$. Taking $x_2 = 2$ yields $x_1 = 5 - i$, so

$$\mathbf{x} = \begin{bmatrix} 5 - i \\ 2 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 3t + i \sin 3t) \begin{bmatrix} 5 - i \\ 2 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t \begin{bmatrix} \sin 3t + 5 \cos 3t \\ 2 \cos 3t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t \begin{bmatrix} -\cos 3t + 5 \sin 3t \\ 2 \sin 3t \end{bmatrix},$$

which are linearly independent solutions of (13.83). The general solution of (13.83) is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} \sin 3t + 5 \cos 3t \\ 2 \cos 3t \end{bmatrix} + c_2 e^t \begin{bmatrix} -\cos 3t + 5 \sin 3t \\ 2 \sin 3t \end{bmatrix}.$$

Example 13.6.3: Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}. \quad (13.84)$$

Solution: The characteristic polynomial of the coefficient matrix A in (13.84) is

$$\begin{vmatrix} -5-\lambda & 5 & 4 \\ -8 & 7-\lambda & 6 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda-2)(\lambda^2+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = i$, and $\lambda_3 = -i$. The augmented matrix of $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -7 & 5 & 4 & \vdots & 0 \\ -8 & 5 & 6 & \vdots & 0 \\ 1 & 0 & -2 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -2 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = x_2 = 2x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of (13.84).

The augmented matrix of $(A - iI)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -5-i & 5 & 4 & \vdots & 0 \\ -8 & 7-i & 6 & \vdots & 0 \\ 1 & 0 & -i & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -i & \vdots & 0 \\ 0 & 1 & 1-i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = ix_3$ and $x_2 = -(1-i)x_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}.$$

The real and imaginary parts of

$$(\cos t + i \sin t) \begin{bmatrix} i \\ -1 + i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2 = \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix},$$

which are solutions of (13.84). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (13.84). The general solution of (13.84) is

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix}.$$

Example 13.6.4: Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} \mathbf{y}. \quad (13.85)$$

Solution: The characteristic polynomial of the coefficient matrix A in (13.85) is

$$\begin{vmatrix} 1 - \lambda & -1 & -2 \\ 1 & 3 - \lambda & 2 \\ 1 & -1 & 2 - \lambda \end{vmatrix} = -(\lambda - 2)((\lambda - 2)^2 + 4).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2 + 2i$, and $\lambda_3 = 2 - 2i$. The augmented matrix of $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -1 & -1 & -2 & \vdots & 0 \\ 1 & 1 & 2 & \vdots & 0 \\ 1 & -1 & 0 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of (13.85).

The augmented matrix of $(A - (2 + 2i)I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -1-2i & -1 & -2 & \vdots & 0 \\ 1 & 1-2i & 2 & \vdots & 0 \\ 1 & -1 & -2i & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -i & \vdots & 0 \\ 0 & 1 & i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = ix_3$ and $x_2 = -ix_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

The real and imaginary parts of

$$e^{2t}(\cos 2t + i \sin 2t) \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2 = e^{2t} \begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_3 = e^{2t} \begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix},$$

which are solutions of (13.85). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} -1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -2,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (13.85). The general solution of (13.85) is

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t} + c_2 e^{2t} \begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix}.$$

13.6.1 Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (13.86)$$

under the assumptions of this section; that is, when the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a complex eigenvalue $\lambda = \alpha + i\beta$ ($\beta \neq 0$) and $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ is an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. To describe the trajectories accurately it's necessary to introduce a new rectangular

coordinate system in the y_1 - y_2 plane. This raises a point that hasn't come up before: It is always possible to choose $\mathbf{x}$ so that $(\mathbf{u}, \mathbf{v}) = 0$. A special effort is required to do this, since not every eigenvector has this property. However, if we know an eigenvector that doesn't, we can multiply it by a suitable complex constant to obtain one that does. To see this, note that if $\mathbf{x}$ is a λ -eigenvector of A and k is an arbitrary real number, then

$$\mathbf{x}_1 = (1 + ik)\mathbf{x} = (1 + ik)(\mathbf{u} + i\mathbf{v}) = (\mathbf{u} - k\mathbf{v}) + i(\mathbf{v} + k\mathbf{u})$$

is also a λ -eigenvector of A , since

$$A\mathbf{x}_1 = A((1 + ik)\mathbf{x}) = (1 + ik)A\mathbf{x} = (1 + ik)\lambda\mathbf{x} = \lambda((1 + ik)\mathbf{x}) = \lambda\mathbf{x}_1.$$

The real and imaginary parts of $\mathbf{x}_1$ are

$$\mathbf{u}_1 = \mathbf{u} - k\mathbf{v} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} + k\mathbf{u}, \quad (13.87)$$

so

$$(\mathbf{u}_1, \mathbf{v}_1) = (\mathbf{u} - k\mathbf{v}, \mathbf{v} + k\mathbf{u}) = -[(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v})].$$

Therefore $(\mathbf{u}_1, \mathbf{v}_1) = 0$ if

$$(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v}) = 0. \quad (13.88)$$

If $(\mathbf{u}, \mathbf{v}) \neq 0$ we can use the quadratic formula to find two real values of k such that $(\mathbf{u}_1, \mathbf{v}_1) = 0$ (Exercise 13.6.28).

Example 13.6.5: In Example 13.6.1 we found the eigenvector

$$\mathbf{x} = \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + i \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

for the matrix of the system (13.82). Here $\mathbf{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ are not orthogonal, since $(\mathbf{u}, \mathbf{v}) = 12$. Since $\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 = -18$, (13.88) is equivalent to

$$2k^2 - 3k - 2 = 0.$$

The zeros of this equation are $k_1 = 2$ and $k_2 = -1/2$. Letting $k = 2$ in (13.87) yields

$$\mathbf{u}_1 = \mathbf{u} - 2\mathbf{v} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} + 2\mathbf{u} = \begin{bmatrix} 10 \\ 10 \end{bmatrix},$$

and $(\mathbf{u}_1, \mathbf{v}_1) = 0$. Letting $k = -1/2$ in (13.87) yields

$$\mathbf{u}_1 = \mathbf{u} + \frac{\mathbf{v}}{2} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} - \frac{\mathbf{u}}{2} = \frac{1}{2} \begin{bmatrix} -5 \\ 5 \end{bmatrix},$$

and again $(\mathbf{u}_1, \mathbf{v}_1) = 0$. ■

(The numbers don't always work out as nicely as in this example. You'll need a calculator or computer to do Exercises 13.6.29-13.6.40.)

Henceforth, we'll assume that $(\mathbf{u}, \mathbf{v}) = 0$. Let $\mathbf{U}$ and $\mathbf{V}$ be unit vectors in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively; that is, $\mathbf{U} = \mathbf{u}/\|\mathbf{u}\|$ and $\mathbf{V} = \mathbf{v}/\|\mathbf{v}\|$. The new rectangular coordinate system will have the same origin as the y_1 - y_2 system. The coordinates of a point in this system will be denoted by (z_1, z_2) , where z_1 and z_2 are the displacements in the directions of $\mathbf{U}$ and $\mathbf{V}$, respectively.

From (13.81), the solutions of (13.86) are given by

$$\mathbf{y} = e^{at} [(c_1 \cos \beta t + c_2 \sin \beta t)\mathbf{u} + (-c_1 \sin \beta t + c_2 \cos \beta t)\mathbf{v}]. \quad (13.89)$$

For convenience, let's call the curve traversed by $e^{-\alpha t}\mathbf{y}(t)$ a *shadow trajectory* of (13.86). Multiplying (13.89) by $e^{-\alpha t}$ yields

$$e^{-\alpha t}\mathbf{y}(t) = z_1(t)\mathbf{U} + z_2(t)\mathbf{V},$$

where

$$\begin{aligned} z_1(t) &= \|\mathbf{u}\|(c_1 \cos \beta t + c_2 \sin \beta t) \\ z_2(t) &= \|\mathbf{v}\|(-c_1 \sin \beta t + c_2 \cos \beta t). \end{aligned}$$

Therefore

$$\frac{(z_1(t))^2}{\|\mathbf{u}\|^2} + \frac{(z_2(t))^2}{\|\mathbf{v}\|^2} = c_1^2 + c_2^2$$

(verify!), which means that the shadow trajectories of (13.86) are ellipses centered at the origin, with axes of symmetry parallel to $\mathbf{U}$ and $\mathbf{V}$. Since

$$z_1' = \frac{\beta\|\mathbf{u}\|}{\|\mathbf{v}\|}z_2 \quad \text{and} \quad z_2' = -\frac{\beta\|\mathbf{v}\|}{\|\mathbf{u}\|}z_1,$$

the vector from the origin to a point on the shadow ellipse rotates in the same direction that $\mathbf{V}$ would have to be rotated by $\pi/2$ radians to bring it into coincidence with $\mathbf{U}$ (Figures 13.15 and 13.16).

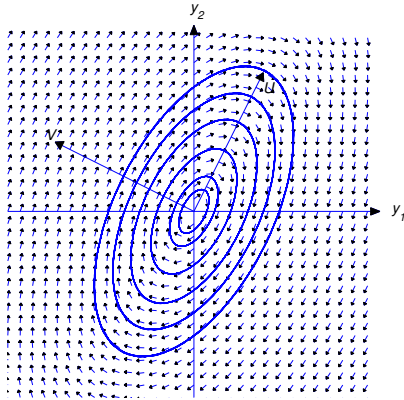


Figure 13.15: Shadow trajectories traversed clockwise

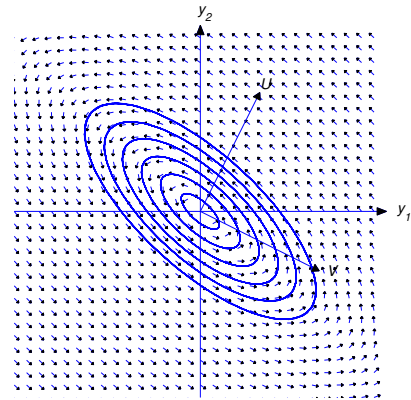


Figure 13.16: Shadow trajectories traversed counterclockwise

If $\alpha = 0$, then any trajectory of (13.86) is a shadow trajectory of (13.86); therefore, if λ is purely imaginary, then the trajectories of (13.86) are ellipses traversed periodically as indicated in Figures 13.15 and 13.16.

If $\alpha > 0$, then

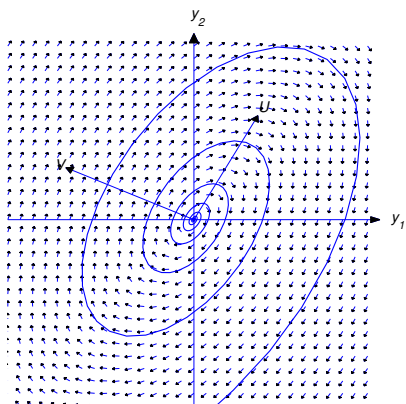
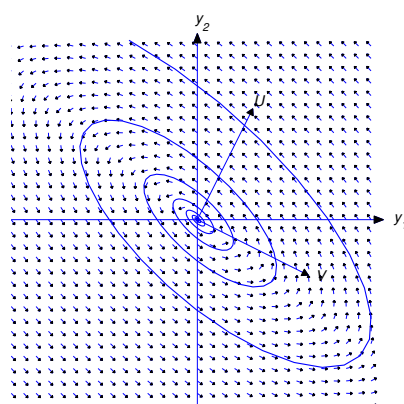
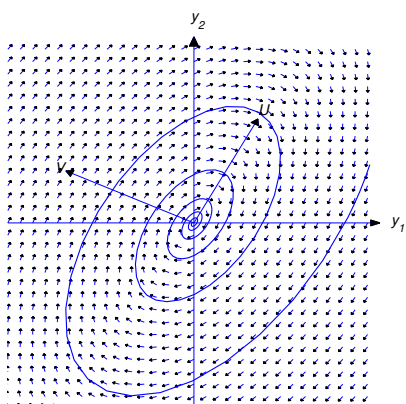
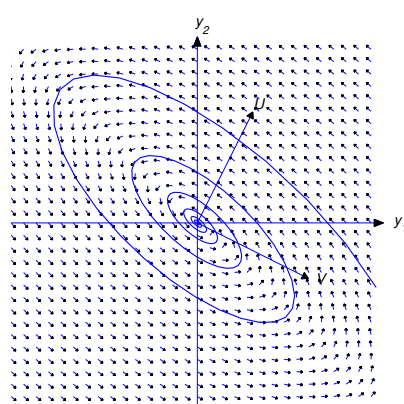
$$\lim_{t \rightarrow \infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t \rightarrow -\infty} \mathbf{y}(t) = 0,$$

so the trajectory spirals away from the origin as t varies from $-\infty$ to ∞ . The direction of the spiral depends upon the relative orientation of $\mathbf{U}$ and $\mathbf{V}$, as shown in Figures 13.17 and 13.18.

If $\alpha < 0$, then

$$\lim_{t \rightarrow -\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \mathbf{y}(t) = 0,$$

so the trajectory spirals toward the origin as t varies from $-\infty$ to ∞ . Again, the direction of the spiral depends upon the relative orientation of $\mathbf{U}$ and $\mathbf{V}$, as shown in Figures 13.19 and 13.20.

Figure 13.17: $\alpha > 0$; shadow trajectory spiraling outwardFigure 13.18: $\alpha > 0$; shadow trajectory spiraling outwardFigure 13.19: $\alpha < 0$; shadow trajectory spiraling inwardFigure 13.20: $\alpha < 0$; shadow trajectory spiraling inward

13.6.2 Exercises

In Exercises 13.6.1–13.6.16 find the general solution.

1. $\mathbf{y}' = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} \mathbf{y}$

2. $\mathbf{y}' = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \mathbf{y}$

3. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \mathbf{y}$

4. $\mathbf{y}' = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \mathbf{y}$

5. $\mathbf{y}' = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \mathbf{y}$

6. $\mathbf{y}' = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \mathbf{y}$

7. $\mathbf{y}' = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{y}$

8. $\mathbf{y}' = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \mathbf{y}$

9. $\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 10 & 1 \end{bmatrix} \mathbf{y}$

10. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 7 & -5 \\ 2 & 5 \end{bmatrix} \mathbf{y}$

11. $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \mathbf{y}$

12. $\mathbf{y}' = \begin{bmatrix} 34 & 52 \\ -20 & -30 \end{bmatrix} \mathbf{y}$

13. $\mathbf{y}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -2 & -1 \end{bmatrix} \mathbf{y}$

14. $\mathbf{y}' = \begin{bmatrix} 3 & -4 & -2 \\ -5 & 7 & -8 \\ -10 & 13 & -8 \end{bmatrix} \mathbf{y}$

15. $\mathbf{y}' = \begin{bmatrix} 6 & 0 & -3 \\ -3 & 3 & 3 \\ 1 & -2 & 6 \end{bmatrix} \mathbf{y}'$

16. $\mathbf{y}' = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}'$

In Exercises 13.6.17–13.6.24 solve the initial value problem.

17. $\mathbf{y}' = \begin{bmatrix} 4 & -6 \\ 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

18. $\mathbf{y}' = \begin{bmatrix} 7 & 15 \\ -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

19. $\mathbf{y}' = \begin{bmatrix} 7 & -15 \\ 3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$

20. $\mathbf{y}' = \frac{1}{6} \begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

21. $\mathbf{y}' = \begin{bmatrix} 5 & 2 & -1 \\ -3 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} 4 & 4 & 0 \\ 8 & 10 & -20 \\ 2 & 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

23. $\mathbf{y}' = \begin{bmatrix} 1 & 15 & -15 \\ -6 & 18 & -22 \\ -3 & 11 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 15 \\ 17 \\ 10 \end{bmatrix}$

24. $\mathbf{y}' = \begin{bmatrix} 4 & -4 & 4 \\ -10 & 3 & 15 \\ 2 & -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 16 \\ 14 \\ 6 \end{bmatrix}$

25. Suppose an $n \times n$ matrix A with real entries has a complex eigenvalue $\lambda = \alpha + i\beta$ ($\beta \neq 0$) with associated eigenvector $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Show that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero.

26. Verify that

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t),$$

are the real and imaginary parts of

$$e^{\alpha t}(\cos \beta t + i \sin \beta t)(\mathbf{u} + i\mathbf{v}).$$

27. Show that if the vectors $\mathbf{u}$ and $\mathbf{v}$ are not both $\mathbf{0}$ and $\beta \neq 0$ then the vector functions

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t)$$

are linearly independent on every interval. **HINT:** There are two cases to consider: (i) $\{\mathbf{u}, \mathbf{v}\}$ linearly independent, and (ii) $\{\mathbf{u}, \mathbf{v}\}$ linearly dependent. In either case, exploit the linear independence of $\{\cos \beta t, \sin \beta t\}$ on every interval.

28. Suppose $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are not orthogonal; that is, $(\mathbf{u}, \mathbf{v}) \neq 0$.

(a) Show that the quadratic equation

$$(\mathbf{u}, \mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u}, \mathbf{v}) = 0$$

has a positive root k_1 and a negative root $k_2 = -1/k_1$.

- (b) Let $\mathbf{u}_1^{(1)} = \mathbf{u} - k_1 \mathbf{v}$, $\mathbf{v}_1^{(1)} = \mathbf{v} + k_1 \mathbf{u}$, $\mathbf{u}_1^{(2)} = \mathbf{u} - k_2 \mathbf{v}$, and $\mathbf{v}_1^{(2)} = \mathbf{v} + k_2 \mathbf{u}$, so that $(\mathbf{u}_1^{(1)}, \mathbf{v}_1^{(1)}) = (\mathbf{u}_1^{(2)}, \mathbf{v}_1^{(2)}) = 0$, from the discussion given above. Show that

$$\mathbf{u}_1^{(2)} = \frac{\mathbf{v}_1^{(1)}}{k_1} \quad \text{and} \quad \mathbf{v}_1^{(2)} = -\frac{\mathbf{u}_1^{(1)}}{k_1}.$$

- (c) Let $\mathbf{U}_1$, $\mathbf{V}_1$, $\mathbf{U}_2$, and $\mathbf{V}_2$ be unit vectors in the directions of $\mathbf{u}_1^{(1)}$, $\mathbf{v}_1^{(1)}$, $\mathbf{u}_1^{(2)}$, and $\mathbf{v}_1^{(2)}$, respectively. Conclude from (a) that $\mathbf{U}_2 = \mathbf{V}_1$ and $\mathbf{V}_2 = -\mathbf{U}_1$, and that therefore the counterclockwise angles from $\mathbf{U}_1$ to $\mathbf{V}_1$ and from $\mathbf{U}_2$ to $\mathbf{V}_2$ are both $\pi/2$ or both $-\pi/2$.

In Exercises 13.6.29–13.6.32 find vectors $\mathbf{U}$ and $\mathbf{V}$ parallel to the axes of symmetry of the trajectories, and plot some typical trajectories.

29. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} 3 & -5 \\ 5 & -3 \end{bmatrix} \mathbf{y}$ 30. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -15 & 10 \\ -25 & 15 \end{bmatrix} \mathbf{y}$
31. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -4 & 8 \\ -4 & 4 \end{bmatrix} \mathbf{y}$ 32. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -3 & -15 \\ 3 & 3 \end{bmatrix} \mathbf{y}$

In Exercises 13.6.33–13.6.40 find vectors $\mathbf{U}$ and $\mathbf{V}$ parallel to the axes of symmetry of the shadow trajectories, and plot a typical trajectory.

33. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -5 & 6 \\ -12 & 7 \end{bmatrix} \mathbf{y}$ 34. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} 5 & -12 \\ 6 & -7 \end{bmatrix} \mathbf{y}$
35. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 9 & -2 \end{bmatrix} \mathbf{y}$ 36. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix} \mathbf{y}$
37. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -1 & 10 \\ -10 & -1 \end{bmatrix} \mathbf{y}$ 38. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -1 & -5 \\ 20 & -1 \end{bmatrix} \mathbf{y}$
39. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix} \mathbf{y}$ 40. $\boxed{\text{C/G}} \quad \mathbf{y}' = \begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} \mathbf{y}$

Chapter 14

More about Systems of ODEs

14.1 Introduction to Systems of ODEs

14.1.1 Systems

Often we do not have just one dependent variable and one equation. And as we will see, we may end up with systems of several equations and several dependent variables even if we start with a single equation.

If we have several dependent variables, suppose $y_1, y_2, \dots, y_n$, then we can have a differential equation involving all of them and their derivatives with respect to one independent variable x . For example, $y_1'' = f(y_1', y_2', y_1, y_2, x)$. Usually, when we have two dependent variables we have two equations such as

$$\begin{aligned}y_1'' &= f_1(y_1', y_2', y_1, y_2, x), \\y_2'' &= f_2(y_1', y_2', y_1, y_2, x),\end{aligned}$$

for some functions f_1 and f_2 . We call the above a *system of differential equations*. More precisely, the above is a *second order system* of ODEs as second order derivatives appear. The system

$$\begin{aligned}x_1' &= g_1(x_1, x_2, x_3, t), \\x_2' &= g_2(x_1, x_2, x_3, t), \\x_3' &= g_3(x_1, x_2, x_3, t),\end{aligned}$$

is a *first order system*, where x_1, x_2, x_3 are the dependent variables, and t is the independent variable.

The terminology for systems is essentially the same as for single equations. For the system above, a *solution* is a set of three functions $x_1(t), x_2(t), x_3(t)$, such that

$$\begin{aligned}x_1'(t) &= g_1(x_1(t), x_2(t), x_3(t), t), \\x_2'(t) &= g_2(x_1(t), x_2(t), x_3(t), t), \\x_3'(t) &= g_3(x_1(t), x_2(t), x_3(t), t).\end{aligned}$$

We usually also have an *initial condition*. Just like for single equations, we specify x_1, x_2 , and x_3 for some fixed t . For example, $x_1(0) = a_1, x_2(0) = a_2, x_3(0) = a_3$, where a_1, a_2 , and a_3 are some constants. For the second order system, we would also specify the first derivatives at a point. If we find a solution with some arbitrary constants in it, where by solving for the constants we find a solution for any initial condition, we call this solution the *general solution*. Best to look at a simple example.

Example 14.1.1: Sometimes a system is easy to solve by solving for one variable and then for the second variable. Take the first order system

$$\begin{aligned}y_1' &= y_1, \\y_2' &= y_1 - y_2,\end{aligned}$$

with y_1, y_2 as the dependent variables and x as the independent variable. And consider initial conditions $y_1(0) = 1, y_2(0) = 2$.

We note that $y_1 = C_1 e^x$ is the general solution of the first equation. We then plug this y_1 into the second equation and get the equation $y_2' = C_1 e^x - y_2$, which is a linear first order equation that is easily solved for y_2 . By the method of integrating factor we get

$$e^x y_2 = \frac{C_1}{2} e^{2x} + C_2,$$

or $y_2 = \frac{C_1}{2} e^x + C_2 e^{-x}$. The general solution to the system is, therefore,

$$y_1 = C_1 e^x, \quad y_2 = \frac{C_1}{2} e^x + C_2 e^{-x}.$$

We solve for C_1 and C_2 given the initial conditions. We substitute $x = 0$ and find that $C_1 = 1$ and $C_2 = 3/2$. Thus the solution is $y_1 = e^x$, and $y_2 = (1/2)e^x + (3/2)e^{-x}$.

Generally, we will not be so lucky to be able to solve for each variable separately as in the example above, and we will have to solve for all variables at once. While we won't generally be able to solve for one variable and then the next, we will try to salvage as much as possible from this technique. It will turn out that in a certain sense we will still (try to) solve a bunch of single equations and put their solutions together. Let us not worry right now about how to solve systems yet.

We will mostly consider the *linear systems*. The example above is a so-called *linear first order system*. It is linear as none of the dependent variables or their derivatives appear in nonlinear functions or with powers higher than one (x, y, x' and y' , constants, and functions of t can appear, but not xy or $(y')^2$ or x^3). Another, more complicated, example of a linear system is

$$\begin{aligned} y_1'' &= e^t y_1' + t^2 y_1 + 5y_2 + \sin(t), \\ y_2'' &= t y_1' - y_2' + 2y_1 + \cos(t). \end{aligned}$$

14.1.2 Applications

Let us consider some simple applications of systems and how to set up the equations.

Example 14.1.2: First, we consider salt and brine tanks, but this time water flows from one to the other and back. We again consider that the tanks are evenly mixed.

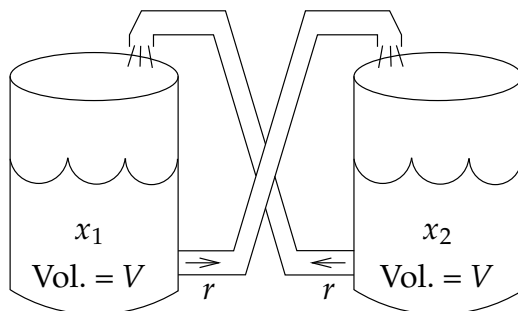


Figure 14.1: A closed system of two brine tanks.

Suppose we have two tanks, each containing volume V liters of salt brine. The amount of salt in the first tank is x_1 grams, and the amount of salt in the second tank is x_2 grams. The liquid is perfectly mixed and flows at the rate r liters per second out of each tank into the other. See [Figure 14.1](#).

The rate of change of x_1 , that is x_1' , is the rate of salt coming in minus the rate going out. The rate coming in is the density of the salt in tank 2, that is $\frac{x_2}{V}$, times the rate r . The rate coming out is the density of the salt in tank 1, that is $\frac{x_1}{V}$, times the rate r . In other words it is

$$x_1' = \frac{x_2}{V}r - \frac{x_1}{V}r = \frac{r}{V}x_2 - \frac{r}{V}x_1 = \frac{r}{V}(x_2 - x_1).$$

Similarly we find the rate x_2' , where the roles of x_1 and x_2 are reversed. All in all, the system of ODEs for this problem is

$$\begin{aligned}x_1' &= \frac{r}{V}(x_2 - x_1), \\x_2' &= \frac{r}{V}(x_1 - x_2).\end{aligned}$$

In this system we cannot solve for x_1 or x_2 separately. We must solve for both x_1 and x_2 at once, which is intuitively clear since the amount of salt in one tank affects the amount in the other. We can't know x_1 before we know x_2 , and vice versa.

We do not yet know how to find all the solutions, but intuitively we can at least find some solutions. Suppose we know that initially the tanks have the same amount of salt. That is, we have an initial condition such as $x_1(0) = x_2(0) = C$. Then clearly the amount of salt coming and out of each tank is the same, so the amounts are not changing. In other words, $x_1 = C$ and $x_2 = C$ (the constant functions) is a solution: $x_1' = x_2' = 0$, and $x_2 - x_1 = x_1 - x_2 = 0$, so the equations are satisfied.

Let us think about the setup a little bit more without solving it. Suppose the initial conditions are $x_1(0) = A$ and $x_2(0) = B$, for two different constants A and B . Since no salt is coming in or out of this closed system, the total amount of salt is constant. That is, $x_1 + x_2$ is constant, and so it equals $A + B$. Intuitively if A is bigger than B , then more salt will flow out of tank one than into it. Eventually, after a long time we would then expect the amount of salt in each tank to equalize. In other words, the solutions of both x_1 and x_2 should tend towards $\frac{A+B}{2}$. Once you know how to solve systems you will find out that this really is so.

Example 14.1.3: Let us look at a second order example. We return to the mass and spring setup, but this time we consider two masses.

Consider one spring with constant k and two masses m_1 and m_2 . Think of the masses as carts that ride along a straight track with no friction. Let x_1 be the displacement of the first cart and x_2 be the displacement of the second cart. That is, we put the two carts somewhere with no tension on the spring, and we mark the position of the first and second cart and call those the zero positions. Then x_1 measures how far the first cart is from its zero position, and x_2 measures how far the second cart is from its zero position. The force exerted by the spring on the first cart is $k(x_2 - x_1)$, since $x_2 - x_1$ is how far the string is stretched (or compressed) from the rest position. The force exerted on the second cart is the opposite, thus the same thing with a negative sign. Newton's second law states that force equals mass times acceleration. So the system of equations is

$$\begin{aligned}m_1x_1'' &= k(x_2 - x_1), \\m_2x_2'' &= -k(x_2 - x_1).\end{aligned}$$

Again, we cannot solve for the x_1 or x_2 variable separately. That we must solve for both x_1 and x_2 at once is intuitively clear, since where the first cart goes depends on exactly where the second cart goes and vice versa.

14.1.3 Changing to First Order

Before we talk about how to handle systems, let us note that in some sense we need only consider first order systems. Let us take an n^{th} order differential equation

$$y^{(n)} = F(y^{(n-1)}, \dots, y', y, x).$$

We define new variables $u_1, u_2, \dots, u_n$ and write the system

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= u_3, \\ &\vdots \\ u_{n-1}' &= u_n, \\ u_n' &= F(u_n, u_{n-1}, \dots, u_2, u_1, x). \end{aligned}$$

We solve this system for $u_1, u_2, \dots, u_n$. Once we have solved for the u , we can discard u_2 through u_n and let $y = u_1$. This y solves the original equation.

Example 14.1.4: Take $x''' = 2x'' + 8x' + x + t$. Letting $u_1 = x$, $u_2 = x'$, $u_3 = x''$, we find the system:

$$u_1' = u_2, \quad u_2' = u_3, \quad u_3' = 2u_3 + 8u_2 + u_1 + t.$$

A similar process can be followed for a system of higher order differential equations. For example, a system of k differential equations in k unknowns, all of order n , can be transformed into a first order system of $n \times k$ equations and $n \times k$ unknowns.

Example 14.1.5: Consider the system from the carts example,

$$m_1 x_1'' = k(x_2 - x_1), \quad m_2 x_2'' = -k(x_2 - x_1).$$

Let $u_1 = x_1$, $u_2 = x_1'$, $u_3 = x_2$, $u_4 = x_2'$. The second order system becomes the first order system

$$u_1' = u_2, \quad m_1 u_2' = k(u_3 - u_1), \quad u_3' = u_4, \quad m_2 u_4' = -k(u_3 - u_1).$$

Example 14.1.6: The idea works in reverse as well. Consider the system

$$x' = 2y - x, \quad y' = x,$$

where the independent variable is t . We wish to solve for the initial conditions $x(0) = 1$, $y(0) = 0$.

If we differentiate the second equation, we get $y'' = x'$. We know what x' is in terms of x and y , and we know that $x = y'$. So,

$$y'' = x' = 2y - x = 2y - y'.$$

We now have the equation $y'' + y' - 2y = 0$. We know how to solve this equation and we find that $y = C_1 e^{-2t} + C_2 e^t$. Once we have y , we use the equation $y' = x$ to get x .

$$x = y' = -2C_1 e^{-2t} + C_2 e^t.$$

We solve for the initial conditions $1 = x(0) = -2C_1 + C_2$ and $0 = y(0) = C_1 + C_2$. Hence, $C_1 = -C_2$ and $1 = 3C_2$. So $C_1 = -1/3$ and $C_2 = 1/3$. Our solution is

$$x = \frac{2e^{-2t} + e^t}{3}, \quad y = \frac{-e^{-2t} + e^t}{3}.$$

Exercise 14.1.1: Plug in and check that this really is the solution.

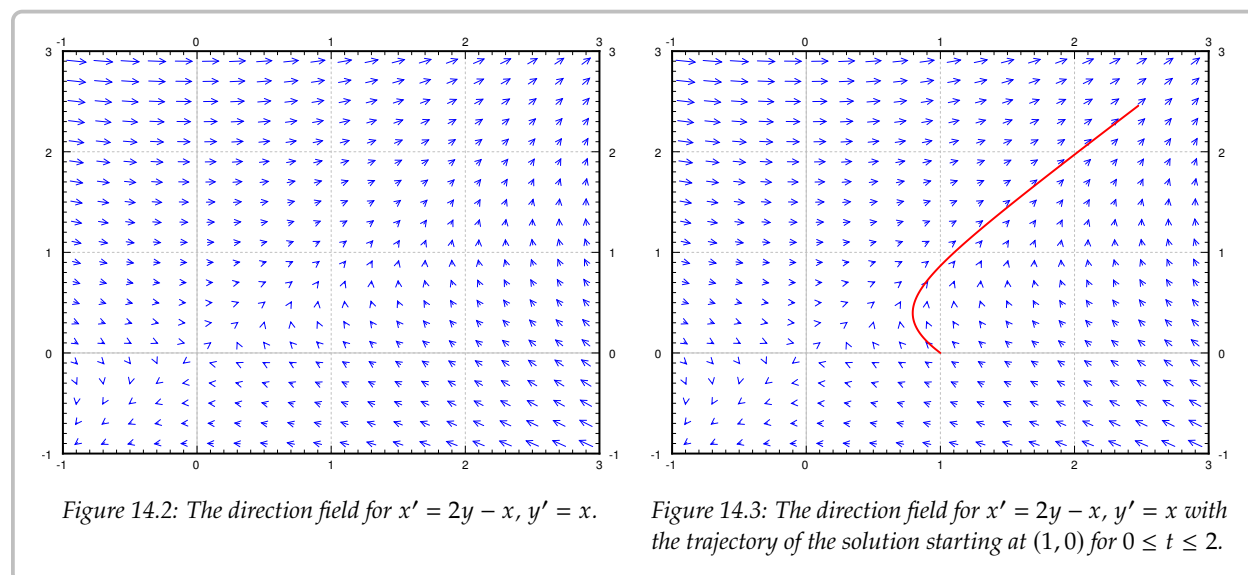
It is useful to go back and forth between systems and higher order equations for other reasons. For example, software for solving ODE numerically (approximation) is generally for first order systems. To use it, you take whatever ODE you want to solve and convert it to a first order system. It is not very hard to adapt computer code for the Euler or Runge-Kutta method for first order equations to handle first order systems. We simply treat the dependent variable not as a number but as a vector. In many mathematical computer languages there is almost no distinction in syntax.

14.1.4 Autonomous Systems and Vector Fields

A system where the equations do not depend on the independent variable is called an *autonomous system*. For example the system $y' = 2y - x$, $y' = x$ is autonomous as t is the independent variable but does not appear in the equations.

For autonomous systems we can draw the so-called *direction field* or *vector field*, a plot similar to a slope field, but instead of giving a slope at each point, we give a direction (and a magnitude). The previous example, $x' = 2y - x$, $y' = x$, says that at the point (x, y) the direction in which we should travel to satisfy the equations should be the direction of the vector $(2y - x, x)$ with the speed equal to the magnitude of this vector. So we draw the vector $(2y - x, x)$ at the point (x, y) and we do this for many points on the xy -plane. For example, at the point $(1, 2)$ we draw the vector $(2(2) - 1, 1) = (3, 1)$, a vector pointing to the right and a little bit up, while at the point $(2, 1)$ we draw the vector $(2(1) - 2, 2) = (0, 2)$ a vector that points straight up. When drawing the vectors, we will scale down their size to fit many of them on the same direction field. If we drew the arrows at the actual size, the diagram would be a jumbled mess once you would draw more than a couple of arrows. So we scale them all so that not even the longest one interferes with the others. We are mostly interested in their direction and relative size. See Figure 14.2.

We can draw a path of the solution in the plane. Suppose the solution is given by $x = f(t)$, $y = g(t)$. We pick an interval of t (say $0 \leq t \leq 2$ for our example) and plot all the points $(f(t), g(t))$ for t in the selected range. The resulting picture is called the *phase portrait* (or phase plane portrait). The particular curve obtained is called the *trajectory* or *solution curve*. See an example plot in Figure 14.3. In the figure the solution starts at $(1, 0)$ and travels along the vector field for a distance of 2 units of t . We solved this system precisely, so we compute $x(2)$ and $y(2)$ to find $x(2) \approx 2.475$ and $y(2) \approx 2.457$. This point corresponds to the top right end of the plotted solution curve in the figure.



Notice the similarity to the diagrams we drew for autonomous systems in one dimension. But note how much more complicated things become when we allow just one extra dimension.

We can draw phase portraits and trajectories in the xy -plane even if the system is not autonomous. In this case, however, we cannot draw the direction field, since the field changes as t changes. For each t we would get a different direction field.

14.1.5 Picard's Theorem

Perhaps before going further, let us mention that Picard's theorem on existence and uniqueness still holds for systems of ODE. Let us restate this theorem in the setting of systems. A general first order system is of

the form

$$\begin{aligned}x'_1 &= F_1(x_1, x_2, \dots, x_n, t), \\x'_2 &= F_2(x_1, x_2, \dots, x_n, t), \\&\vdots \\x'_n &= F_n(x_1, x_2, \dots, x_n, t).\end{aligned}\tag{14.1}$$

Theorem 14.1.1 (Picard's theorem on existence and uniqueness for systems). *If for every $j = 1, 2, \dots, n$ and every $k = 1, 2, \dots, n$ each F_j is continuous and the derivative $\frac{\partial F_j}{\partial x_k}$ exists and is continuous near some $(x_1^0, x_2^0, \dots, x_n^0, t^0)$, then a solution to (14.1) subject to the initial condition $x_1(t^0) = x_1^0, x_2(t^0) = x_2^0, \dots, x_n(t^0) = x_n^0$ exists (at least for t in some small interval) and is unique.*

That is, a unique solution exists for any initial condition given that the system is reasonable (F_j and its partial derivatives in the x variables are continuous). As for single equations we may not have a solution for all time t , but at least for some short period of time.

As we can change any n th order ODE into a first order system, then we notice that this theorem provides also the existence and uniqueness of solutions for higher order equations that we have until now not stated explicitly.

14.1.6 Exercises

Exercise 14.1.2: Find the general solution of $x'_1 = x_2 - x_1 + t, x'_2 = x_2$.

Exercise 14.1.3: Find the general solution of $x'_1 = 3x_1 - x_2 + e^t, x'_2 = x_1$.

Exercise 14.1.4: Write $ay'' + by' + cy = f(x)$ as a first order system of ODEs.

Exercise 14.1.5: Write $x'' + y^2y' - x^3 = \sin(t), y'' + (x' + y')^2 - x = 0$ as a first order system of ODEs.

Exercise 14.1.6: Suppose two masses on carts on frictionless surface are at displacements x_1 and x_2 as in [Example 14.1.3](#) on page 633. Suppose that a rocket applies force F in the positive direction on cart x_1 . Set up the system of equations.

Exercise 14.1.7: Suppose the tanks are as in [Example 14.1.2](#) on page 632, starting both at volume V , but now the rate of flow from tank 1 to tank 2 is r_1 , and rate of flow from tank 2 to tank one is r_2 . Notice that the volumes are now not constant. Set up the system of equations.

Exercise 14.1.101: Find the general solution to $y'_1 = 3y_1, y'_2 = y_1 + y_2, y'_3 = y_1 + y_3$.

Exercise 14.1.102: Solve $y' = 2x, x' = x + y, x(0) = 1, y(0) = 3$.

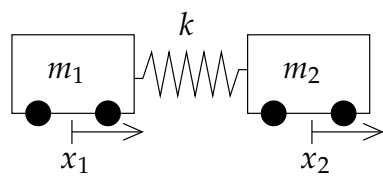
Exercise 14.1.103: Write $x''' = x + t$ as a first order system.

Exercise 14.1.104: Write $y''_1 + y_1 + y_2 = t, y''_2 + y_1 - y_2 = t^2$ as a first order system.

Exercise 14.1.105: Suppose two masses on carts on frictionless surface are at displacements x_1 and x_2 as in [Example 14.1.3](#) on page 633. Suppose initial displacement is $x_1(0) = x_2(0) = 0$, and initial velocity is $x'_1(0) = x'_2(0) = a$ for some number a . Use your intuition to solve the system, explain your reasoning.

Exercise 14.1.106: Suppose the tanks are as in [Example 14.1.2](#) on page 632 except that clean water flows in at the rate s liters per second into tank 1, and brine flows out of tank 2 and into the sewer also at the rate of s liters per second. The rate of flow from tank 1 into tank 2 is still r , but the rate of flow from tank 2 back into tank 1 is $r - s$ (assume $r > s$).

- Draw the picture.
- Set up the system of equations.
- Intuitively, what happens as t goes to infinity, explain.



14.2 Matrices and Linear Systems

14.2.1 Matrices and Vectors

Before we start talking about linear systems of ODEs, we need to talk about matrices, so let us review these briefly. A *matrix* is an $m \times n$ array of numbers (m rows and n columns). For example, we denote a 3×5 matrix as follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}.$$

The numbers a_{ij} are called *elements* or *entries*.

By a *vector* we usually mean a *column vector*, that is an $m \times 1$ matrix. If we mean a *row vector*, we will explicitly say so (a row vector is a $1 \times n$ matrix). We usually denote matrices by upper case letters and vectors by lower case letters with an arrow such as $\vec{x}$ or $\vec{b}$. By $\vec{0}$ we mean the vector of all zeros.

We define some operations on matrices. We want 1×1 matrices to really act like numbers, so our operations have to be compatible with this viewpoint.

First, we can multiply a matrix by a *scalar* (a number). We simply multiply each entry in the matrix by the scalar. For example,

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Matrix addition is also easy. We add matrices element by element. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 7 & 10 \end{bmatrix}.$$

If the sizes do not match, then addition is not defined.

If we denote by 0 the matrix with all zero entries, by c, d scalars, and by A, B, C matrices, we have the following familiar rules:

$$\begin{aligned} A + 0 &= A = 0 + A, \\ A + B &= B + A, \\ (A + B) + C &= A + (B + C), \\ c(A + B) &= cA + cB, \\ (c + d)A &= cA + dA. \end{aligned}$$

Another useful operation for matrices is the so-called *transpose*. This operation just swaps rows and columns of a matrix. The transpose of A is denoted by A^T . Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

14.2.2 Matrix Multiplication

Let us now define matrix multiplication. First we define the so-called *dot product* (or *inner product*) of two vectors. Usually this will be a row vector multiplied with a column vector of the same size. For the dot product we multiply each pair of entries from the first and the second vector and we sum these products. The result is a single number. For example,

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3.$$

Similarly for larger (or smaller) vectors.

Armed with the dot product we define the *product of matrices*. First let us denote by $\text{row}_i(A)$ the i^{th} row of A and by $\text{column}_j(A)$ the j^{th} column of A . For an $m \times n$ matrix A and an $n \times p$ matrix B , we can define the product AB . We let AB be an $m \times p$ matrix whose ij^{th} entry is the dot product

$$\text{row}_i(A) \cdot \text{column}_j(B).$$

Do note how the sizes match up: $m \times n$ multiplied by $n \times p$ is $m \times p$. Example:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} &= \\ &= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 & 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 0 & 1 \cdot (-1) + 2 \cdot 1 + 3 \cdot 0 \\ 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 1 & 4 \cdot 0 + 5 \cdot 1 + 6 \cdot 0 & 4 \cdot (-1) + 5 \cdot 1 + 6 \cdot 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 1 \\ 15 & 5 & 1 \end{bmatrix} \end{aligned}$$

For multiplication we want an analogue of a 1. This analogue is the so-called *identity matrix*. The identity matrix is a square matrix with 1s on the diagonal and zeros everywhere else. It is usually denoted by I . For each size we have a different identity matrix and so sometimes we may denote the size as a subscript. For example, the I_3 would be the 3×3 identity matrix

$$I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have the following rules for matrix multiplication. Suppose that A, B, C are matrices of the correct sizes so that the following make sense. Let α denote a scalar (number).

$$\begin{aligned} A(BC) &= (AB)C, \\ A(B + C) &= AB + AC, \\ (B + C)A &= BA + CA, \\ \alpha(AB) &= (\alpha A)B = A(\alpha B), \\ IA &= A = AI. \end{aligned}$$

A few warnings are in order.

- (i) $AB \neq BA$ in general (it may be true by fluke sometimes). That is, matrices do not commute. For example, take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.
- (ii) $AB = AC$ does not necessarily imply $B = C$, even if A is not 0.
- (iii) $AB = 0$ does not necessarily mean that $A = 0$ or $B = 0$. Try, for example, $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

For the last two items to hold we would need to “divide” by a matrix. This is where the *matrix inverse* comes in. Suppose that A and B are $n \times n$ matrices such that

$$AB = I = BA.$$

Then we call B the inverse of A and we denote B by A^{-1} . If the inverse of A exists, then we call A *invertible*. If A is not invertible, we sometimes say A is *singular*.

If A is invertible, then $AB = AC$ does imply that $B = C$ (in particular, the inverse of A is unique). We just multiply both sides by A^{-1} (on the left) to get $A^{-1}AB = A^{-1}AC$ or $IB = IC$ or $B = C$. It is also not hard to see that $(A^{-1})^{-1} = A$.

14.2.3 The Determinant

For square matrices we define a useful quantity called the *determinant*. We define the determinant of a 1×1 matrix as the value of its only entry. For a 2×2 matrix we define

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{\text{def}}{=} ad - bc.$$

Before trying to define the determinant for larger matrices, let us note the meaning of the determinant. Consider an $n \times n$ matrix as a mapping of the n -dimensional euclidean space $\mathbb{R}^n$ to itself, where $\vec{x}$ gets sent to $A\vec{x}$. In particular, a 2×2 matrix A is a mapping of the plane to itself. The determinant of A is the factor by which the area of objects changes. If we take the unit square (square of side 1) in the plane, then A takes the square to a parallelogram of area $|\det(A)|$. The sign of $\det(A)$ denotes changing of orientation (negative if the axes get flipped). For example, let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then $\det(A) = 1 + 1 = 2$. Let us see where the (unit) square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$ gets sent. Clearly $(0, 0)$ gets sent to $(0, 0)$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The image of the square is another square with vertices $(0, 0)$, $(1, -1)$, $(1, 1)$, and $(2, 0)$. The image square has a side of length $\sqrt{2}$ and is therefore of area 2.

If you think back to high school geometry, you may have seen a formula for computing the area of a parallelogram with vertices $(0, 0)$, (a, c) , (b, d) and $(a + b, c + d)$. And it is precisely

$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|.$$

The vertical lines above mean absolute value. The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ carries the unit square to the given parallelogram.

Let us look at the determinant for larger matrices. We define A_{ij} as the matrix A with the i^{th} row and the j^{th} column deleted. To compute the determinant of a matrix, pick one row, say the i^{th} row and compute:

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

For the first row we get

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) - \cdots \begin{cases} +a_{1n} \det(A_{1n}) & \text{if } n \text{ is odd,} \\ -a_{1n} \det(A_{1n}) & \text{if } n \text{ even.} \end{cases}$$

We alternately add and subtract the determinants of the submatrices A_{ij} multiplied by a_{ij} for a fixed i and all j . For a 3×3 matrix, picking the first row, we get $\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13})$. For example,

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= 1 \cdot \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0. \end{aligned}$$

The numbers $(-1)^{i+j} \det(A_{ij})$ are called *cofactors* of the matrix and this way of computing the determinant is called the *cofactor expansion*. No matter which row you pick, you always get the same number. It is also possible to compute the determinant by expanding along columns (picking a column instead of a row above). It is true that $\det(A) = \det(A^T)$.

A common notation for the determinant is a pair of vertical lines:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).$$

I personally find this notation confusing as vertical lines usually mean a positive quantity, while determinants can be negative. Also think about how to write the absolute value of a determinant. I will not use this notation in this book.

Think of the determinants telling you the scaling of a mapping. If B doubles the sizes of geometric objects and A triples them, then AB (which applies B to an object and then A) should make size go up by a factor of 6. This is true in general:

$$\det(AB) = \det(A) \det(B).$$

This property is one of the most useful, and it is employed often to actually compute determinants. A particularly interesting consequence is to note what it means for existence of inverses. Take A and B to be inverses of each other, that is $AB = I$. Then

$$\det(A) \det(B) = \det(AB) = \det(I) = 1.$$

Neither $\det(A)$ nor $\det(B)$ can be zero. Let us state this as a theorem as it will be very important in the context of this course.

Theorem 14.2.1. *An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.*

In fact, $\det(A^{-1}) \det(A) = 1$ says that $\det(A^{-1}) = \frac{1}{\det(A)}$. So we even know what the determinant of A^{-1} is before we know how to compute A^{-1} .

There is a simple formula for the inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Notice the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the denominator of the fraction. The formula only works if the determinant is nonzero, otherwise we are dividing by zero.

14.2.4 Solving Linear Systems

One application of matrices we will need is to solve systems of linear equations. This is best shown by example. Suppose that we have the following system of linear equations

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + x_2 + 3x_3 &= 5, \\ x_1 + 4x_2 + x_3 &= 10. \end{aligned}$$

Without changing the solution, we could swap equations in this system, we could multiply any of the equations by a nonzero number, and we could add a multiple of one equation to another equation. It turns out these operations always suffice to find a solution.

It is easier to write the system as a matrix equation. The system above can be written as

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}.$$

To solve the system we put the coefficient matrix (the matrix on the left-hand side of the equation) together with the vector on the right and side and get the so-called *augmented matrix*

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right].$$

We apply the following three elementary operations.

- (i) Swap two rows.
- (ii) Multiply a row by a nonzero number.
- (iii) Add a multiple of one row to another row.

We keep doing these operations until we get into a state where it is easy to read off the answer, or until we get into a contradiction indicating no solution, for example if we come up with an equation such as $0 = 1$.

Let us work through the example. First multiply the first row by $1/2$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right].$$

Now subtract the first row from the second and third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 9 \end{array} \right]$$

Multiply the last row by $1/3$ and the second row by $1/2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{array} \right]$$

Swap rows 2 and 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Subtract the last row from the first, then subtract the second row from the first.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

If we think about what equations this augmented matrix represents, we see that $x_1 = -4$, $x_2 = 3$, and $x_3 = 2$. We try this solution in the original system and, voilà, it works!

Exercise 14.2.1: Check that the solution above really solves the given equations.

We write this equation in matrix notation as

$$A\vec{x} = \vec{b},$$

where A is the matrix $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix}$ and $\vec{b}$ is the vector $\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$. The solution can also be computed via the inverse,

$$\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}.$$

It is possible that the solution is not unique, or that no solution exists. It is easy to tell if a solution does not exist. If during the row reduction you come up with a row where all the entries except the last one are zero (the last entry in a row corresponds to the right-hand side of the equation), then the system is *inconsistent* and has no solution. For example, for a system of 3 equations and 3 unknowns, if you find a row such as $[0 \ 0 \ 0 \mid 1]$ in the augmented matrix, you know the system is inconsistent. That row corresponds to $0 = 1$.

You generally try to use row operations until the following conditions are satisfied. The first (from the left) nonzero entry in each row is called the *leading entry*.

- (i) The leading entry in any row is strictly to the right of the leading entry of the row above.
- (ii) Any zero rows are below all the nonzero rows.
- (iii) All leading entries are 1.
- (iv) All the entries above and below a leading entry are zero.

Such a matrix is said to be in *reduced row echelon form*. The variables corresponding to columns with no leading entries are said to be *free variables*. Free variables mean that we can pick those variables to be anything we want and then solve for the rest of the unknowns.

Example 14.2.1: The following augmented matrix is in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Suppose the variables are x_1 , x_2 , and x_3 . Then x_2 is the free variable, $x_1 = 3 - 2x_2$, and $x_3 = 1$.

On the other hand if during the row reduction process you come up with the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 13 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array} \right],$$

there is no need to go further. The last row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 3$, which is preposterous. Hence, no solution exists.

14.2.5 Computing the Inverse

If the matrix A is square and there exists a unique solution $\vec{x}$ to $A\vec{x} = \vec{b}$ for any $\vec{b}$ (there are no free variables), then A is invertible. Multiplying both sides by A^{-1} , you can see that $\vec{x} = A^{-1}\vec{b}$. So it is useful to compute the inverse if you want to solve the equation for many different right-hand sides $\vec{b}$.

We have a formula for the 2×2 inverse, but it is also not hard to compute inverses of larger matrices. While we will not have too much occasion to compute inverses for larger matrices than 2×2 by hand, let us touch on how to do it. Finding the inverse of A is actually just solving a bunch of linear equations. If we can solve $A\vec{x}_k = \vec{e}_k$ where $\vec{e}_k$ is the vector with all zeros except a 1 at the k^{th} position, then the inverse is the matrix with the columns $\vec{x}_k$ for $k = 1, 2, \dots, n$ (exercise: why?). Therefore, to find the inverse we write a larger $n \times 2n$ augmented matrix $[A \mid I]$, where I is the identity matrix. We then perform row reduction. The reduced row echelon form of $[A \mid I]$ will be of the form $[I \mid A^{-1}]$ if and only if A is invertible. We then just read off the inverse A^{-1} .

14.2.6 Exercises

Exercise 14.2.2: Solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ by using matrix inverse.

Exercise 14.2.3: Compute determinant of $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}$.

Exercise 14.2.4: Compute determinant of $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 0 & 5 & 0 \\ 6 & 0 & 7 & 0 \\ 8 & 0 & 10 & 1 \end{bmatrix}$. Hint: Expand along the proper row or column to make the calculations simpler.

Exercise 14.2.5: Compute inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Exercise 14.2.6: For which h is $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & h \end{bmatrix}$ not invertible? Is there only one such h ? Are there several? Infinitely many?

Exercise 14.2.7: For which h is $\begin{bmatrix} h & 1 & 1 \\ 0 & h & 0 \\ 1 & 1 & h \end{bmatrix}$ not invertible? Find all such h .

Exercise 14.2.8: Solve $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Exercise 14.2.9: Solve $\begin{bmatrix} 5 & 3 & 7 \\ 8 & 4 & 4 \\ 6 & 3 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

Exercise 14.2.10: Solve $\begin{bmatrix} 3 & 2 & 3 & 0 \\ 3 & 3 & 3 & 3 \\ 0 & 2 & 4 & 2 \\ 2 & 3 & 4 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$.

Exercise 14.2.11: Find 3 nonzero 2×2 matrices A , B , and C such that $AB = AC$ but $B \neq C$.

Exercise 14.2.101: Compute determinant of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -5 \\ 1 & -1 & 0 \end{bmatrix}$

Exercise 14.2.102: Find t such that $\begin{bmatrix} 1 & t \\ -1 & 2 \end{bmatrix}$ is not invertible.

Exercise 14.2.103: Solve $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$.

Exercise 14.2.104: Suppose a, b, c are nonzero numbers. Let $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $N = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

a) Compute M^{-1} .

b) Compute N^{-1} .

14.3 Linear Systems of ODEs

First let us talk about matrix- or vector-valued functions. Such a function is just a matrix or vector whose entries depend on some variable. If t is the independent variable, we write a *vector-valued function* $\vec{x}(t)$ as

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Similarly a *matrix-valued function* $A(t)$ is

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}.$$

The derivative $A'(t)$ or $\frac{dA}{dt}$ is just the matrix-valued function whose ij^{th} entry is $a'_{ij}(t)$.

Rules of differentiation of matrix-valued functions are similar to rules for normal functions. Let $A(t)$ and $B(t)$ be matrix-valued functions. Let c a scalar and let C be a constant matrix. Then

$$\begin{aligned} (A(t) + B(t))' &= A'(t) + B'(t), \\ (A(t)B(t))' &= A'(t)B(t) + A(t)B'(t), \\ (cA(t))' &= cA'(t), \\ (CA(t))' &= CA'(t), \\ (A(t)C)' &= A'(t)C. \end{aligned}$$

Note the order of the multiplication in the last two expressions.

A *first order linear system of ODEs* is a system that can be written as the vector equation

$$\vec{x}'(t) = P(t)\vec{x}(t) + \vec{f}(t),$$

where $P(t)$ is a matrix-valued function, and $\vec{x}(t)$ and $\vec{f}(t)$ are vector-valued functions. We will often suppress the dependence on t and only write $\vec{x}' = P\vec{x} + \vec{f}$. A solution of the system is a vector-valued function $\vec{x}$ satisfying the vector equation.

For example, the equations

$$\begin{aligned} x_1' &= 2tx_1 + e^t x_2 + t^2, \\ x_2' &= \frac{x_1}{t} - x_2 + e^t, \end{aligned}$$

can be written as

$$\vec{x}' = \begin{bmatrix} 2t & e^t \\ 1/t & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} t^2 \\ e^t \end{bmatrix}.$$

We will mostly concentrate on equations that are not just linear, but are in fact *constant coefficient* equations. That is, the matrix P will be constant; it will not depend on t .

When $\vec{f} = \vec{0}$ (the zero vector), then we say the system is *homogeneous*. For homogeneous linear systems we have the principle of superposition, just like for single homogeneous equations.

Theorem 14.3.1 (Superposition). *Let $\vec{x}' = P\vec{x}$ be a linear homogeneous system of ODEs. Suppose that $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are n solutions of the equation and $c_1, c_2, \dots, c_n$ are any constants, then*

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n, \quad (14.2)$$

is also a solution. Furthermore, if this is a system of n equations (P is $n \times n$), and $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly independent, then every solution $\vec{x}$ can be written as (14.2).

Linear independence for vector-valued functions is the same idea as for normal functions. The vector-valued functions $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly independent when

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = \vec{0}$$

has only the solution $c_1 = c_2 = \dots = c_n = 0$, where the equation must hold for all t .

Example 14.3.1: $\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 1+t \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$ are linearly dependent because $\vec{x}_1 + \vec{x}_3 = \vec{x}_2$, and this holds for all t . So $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$ above will work.

On the other hand if we change the example just slightly $\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$, then the functions are linearly independent. First write $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \vec{0}$ and note that it has to hold for all t . We get that

$$c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \begin{bmatrix} c_1t^2 - c_3t^2 \\ c_1t + c_2t + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words $c_1t^2 - c_3t^2 = 0$ and $c_1t + c_2t + c_3 = 0$. If we set $t = 0$, then the second equation becomes $c_3 = 0$. But then the first equation becomes $c_1t^2 = 0$ for all t and so $c_1 = 0$. Thus the second equation is just $c_2t = 0$, which means $c_2 = 0$. So $c_1 = c_2 = c_3 = 0$ is the only solution and $\vec{x}_1, \vec{x}_2$, and $\vec{x}_3$ are linearly independent.

The linear combination $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n$ could always be written as

$$X(t)\vec{c},$$

where $X(t)$ is the matrix with columns $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$, and $\vec{c}$ is the column vector with entries $c_1, c_2, \dots, c_n$. Assuming that $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly independent, the matrix-valued function $X(t)$ is called a *fundamental matrix*, or a *fundamental matrix solution*.

To solve nonhomogeneous first order linear systems, we use the same technique as we applied to solve single linear nonhomogeneous equations.

Theorem 14.3.2. Let $\vec{x}' = P\vec{x} + \vec{f}$ be a linear system of ODEs. Suppose $\vec{x}_p$ is one particular solution. Then every solution can be written as

$$\vec{x} = \vec{x}_c + \vec{x}_p,$$

where $\vec{x}_c$ is a solution to the associated homogeneous equation ($\vec{x}' = P\vec{x}$).

The procedure for systems is the same as for single equations. We find a particular solution to the nonhomogeneous equation, then we find the general solution to the associated homogeneous equation, and finally we add the two together.

Alright, suppose you have found the general solution of $\vec{x}' = P\vec{x} + \vec{f}$. Next suppose you are given an initial condition of the form

$$\vec{x}(t_0) = \vec{b}$$

for some fixed t_0 and a constant vector $\vec{b}$. Let $X(t)$ be a fundamental matrix solution of the associated homogeneous equation (i.e. columns of $X(t)$ are solutions). The general solution can be written as

$$\vec{x}(t) = X(t)\vec{c} + \vec{x}_p(t).$$

We are seeking a vector $\vec{c}$ such that

$$\vec{b} = \vec{x}(t_0) = X(t_0)\vec{c} + \vec{x}_p(t_0).$$

In other words, we are solving for $\vec{c}$ the nonhomogeneous system of linear equations

$$X(t_0)\vec{c} = \vec{b} - \vec{x}_p(t_0).$$

Example 14.3.2: In § 14.1 we solved the system

$$\begin{aligned}x_1' &= x_1, \\x_2' &= x_1 - x_2,\end{aligned}$$

with initial conditions $x_1(0) = 1$, $x_2(0) = 2$. Let us consider this problem in the language of this section.

The system is homogeneous, so $\vec{f}(t) = \vec{0}$. We write the system and the initial conditions as

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We found the general solution is $x_1 = c_1 e^t$ and $x_2 = \frac{c_1}{2} e^t + c_2 e^{-t}$. Letting $c_1 = 1$ and $c_2 = 0$, we obtain the solution $\begin{bmatrix} e^t \\ (1/2)e^t \end{bmatrix}$. Letting $c_1 = 0$ and $c_2 = 1$, we obtain $\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$. These two solutions are linearly independent, as can be seen by setting $t = 0$, and noting that the resulting constant vectors are linearly independent. In matrix notation, a fundamental matrix solution is, therefore,

$$X(t) = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}e^t & e^{-t} \end{bmatrix}.$$

To solve the initial value problem we solve for $\vec{c}$ in the equation

$$X(0) \vec{c} = \vec{b},$$

or in other words,

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

A single elementary row operation shows $\vec{c} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$. Our solution is

$$\vec{x}(t) = X(t) \vec{c} = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}e^t & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} e^t \\ \frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{bmatrix}.$$

This new solution agrees with our previous solution from § 14.1.

14.3.1 Exercises

Exercise 14.3.1: Write the system $x_1' = 2x_1 - 3tx_2 + \sin t$, $x_2' = e^t x_1 + 3x_2 + \cos t$ in the form $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$.

Exercise 14.3.2:

- Verify that the system $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{x}$ has the two solutions $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$.
- Write down the general solution.
- Write down the general solution in the form $x_1 = ?$, $x_2 = ?$ (i.e. write down a formula for each element of the solution).

Exercise 14.3.3: Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$ are linearly independent. Hint: Just plug in $t = 0$.

Exercise 14.3.4: Verify that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^t$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^t$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$ are linearly independent. Hint: You must be a bit more tricky than in the previous exercise.

Exercise 14.3.5: Verify that $\begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $\begin{bmatrix} t^3 \\ t^4 \end{bmatrix}$ are linearly independent.

Exercise 14.3.6: Take the system $x_1' + x_2' = x_1$, $x_1' - x_2' = x_2$.

- a) Write it in the form $A\vec{x}' = B\vec{x}$ for matrices A and B .
- b) Compute A^{-1} and use that to write the system in the form $\vec{x}' = P\vec{x}$.

Exercise 14.3.101: Are $\begin{bmatrix} e^{2t} \\ e^t \end{bmatrix}$ and $\begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}$ linearly independent? Justify.

Exercise 14.3.102: Are $\begin{bmatrix} \cosh(t) \\ 1 \end{bmatrix}$, $\begin{bmatrix} e^t \\ 1 \end{bmatrix}$, and $\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}$ linearly independent? Justify.

Exercise 14.3.103: Write $x' = 3x - y + e^t$, $y' = tx$ in matrix notation.

Exercise 14.3.104:

- a) Write $x'_1 = 2tx_2$, $x'_2 = 2tx_2$ in matrix notation.
- b) Solve and write the solution in matrix notation.

14.4 Eigenvalue Method

In this section we will learn how to solve linear homogeneous constant coefficient systems of ODEs by the eigenvalue method. Suppose we have such a system

$$\vec{x}' = P\vec{x},$$

where P is a constant square matrix. We wish to adapt the method for the single constant coefficient equation by trying the function $e^{\lambda t}$. However, $\vec{x}$ is a vector. So we try $\vec{x} = \vec{v}e^{\lambda t}$, where $\vec{v}$ is an arbitrary constant vector. We plug this $\vec{x}$ into the equation to get

$$\underbrace{\lambda \vec{v} e^{\lambda t}}_{\vec{x}'} = \underbrace{P \vec{v} e^{\lambda t}}_{P\vec{x}}.$$

We divide by $e^{\lambda t}$ and notice that we are looking for a scalar λ and a vector $\vec{v}$ that satisfy the equation

$$\lambda \vec{v} = P\vec{v}.$$

To solve this equation we need a little bit more linear algebra, which we now review.

14.4.1 Eigenvalues and Eigenvectors of a Matrix

Let A be a constant square matrix. Suppose there is a scalar λ and a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

We call λ an *eigenvalue* of A and we call $\vec{v}$ a corresponding *eigenvector*.

Example 14.4.1: The matrix $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ has an eigenvalue $\lambda = 2$ with a corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let us see how to compute eigenvalues for any matrix. Rewrite the equation for an eigenvalue as

$$(A - \lambda I)\vec{v} = \vec{0}.$$

This equation has a nonzero solution $\vec{v}$ only if $A - \lambda I$ is not invertible. Were it invertible, we could write $(A - \lambda I)^{-1}(A - \lambda I)\vec{v} = (A - \lambda I)^{-1}\vec{0}$, which implies $\vec{v} = \vec{0}$. Therefore, A has the eigenvalue λ if and only if λ solves the equation

$$\det(A - \lambda I) = 0.$$

Consequently, we can find an eigenvalue of A without finding a corresponding eigenvector at the same time. An eigenvector will have to be found later, once λ is known.

Example 14.4.2: Find all eigenvalues of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

We write

$$\begin{aligned} \det\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = \\ &= (2-\lambda)((2-\lambda)^2 - 1) = -(\lambda-1)(\lambda-2)(\lambda-3). \end{aligned}$$

So the eigenvalues are $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$.

For an $n \times n$ matrix, the polynomial we get by computing $\det(A - \lambda I)$ is of degree n , and hence in general, we have n eigenvalues. Some may be repeated, some may be complex.

To find an eigenvector corresponding to an eigenvalue λ , we write

$$(A - \lambda I)\vec{v} = \vec{0},$$

and solve for a nontrivial (nonzero) vector $\vec{v}$. If λ is an eigenvalue, there will be at least one free variable, and so for each distinct eigenvalue λ , we can always find an eigenvector.

Example 14.4.3: Find an eigenvector of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$.

We write

$$(A - \lambda I)\vec{v} = \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0}.$$

It is easy to solve this system of linear equations. We write down the augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array} \right],$$

and perform row operations (exercise: which ones?) until we get:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

The entries of $\vec{v}$ have to satisfy the equations $v_1 - v_2 = 0$, $v_3 = 0$, and v_2 is a free variable. We can pick v_2 to be arbitrary (but nonzero), let $v_1 = v_2$, and of course $v_3 = 0$. For example, if we pick $v_2 = 1$, then $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Let us verify that $\vec{v}$ really is an eigenvector corresponding to $\lambda = 3$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Yay! It worked.

Exercise 14.4.1 (easy): Are eigenvectors unique? Can you find a different eigenvector for $\lambda = 3$ in the example above? How are the two eigenvectors related?

Exercise 14.4.2: When the matrix is 2×2 you do not need to do row operations when computing an eigenvector, you can read it off from $A - \lambda I$ (if you have computed the eigenvalues correctly). Can you see why? Explain. Try it for the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

14.4.2 The Eigenvalue Method with Distinct Real Eigenvalues

OK. We have the system of equations

$$\vec{x}' = P\vec{x}.$$

We find the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the matrix P , and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Now we notice that the functions $\vec{v}_1 e^{\lambda_1 t}, \vec{v}_2 e^{\lambda_2 t}, \dots, \vec{v}_n e^{\lambda_n t}$ are solutions of the system of equations and hence $\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}$ is a solution.

Theorem 14.4.1. Take $\vec{x}' = P\vec{x}$. If P is an $n \times n$ constant matrix that has n distinct real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then there exist n linearly independent corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, and the general solution to $\vec{x}' = P\vec{x}$ can be written as

$$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}.$$

The corresponding fundamental matrix solution is

$$X(t) = \begin{bmatrix} \vec{v}_1 e^{\lambda_1 t} & \vec{v}_2 e^{\lambda_2 t} & \cdots & \vec{v}_n e^{\lambda_n t} \end{bmatrix}.$$

That is, $X(t)$ is the matrix whose j^{th} column is $\vec{v}_j e^{\lambda_j t}$.

Example 14.4.4: Consider the system

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}.$$

Find the general solution.

Earlier, we found the eigenvalues are 1, 2, 3. We found the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ for the eigenvalue 3. Similarly we find the eigenvector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for the eigenvalue 1, and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ for the eigenvalue 2 (exercise: check). Hence our general solution is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t} = \begin{bmatrix} c_1 e^t + c_3 e^{3t} \\ -c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ -c_2 e^{2t} \end{bmatrix}.$$

In terms of a fundamental matrix solution,

$$\vec{x} = X(t) \vec{c} = \begin{bmatrix} e^t & 0 & e^{3t} \\ -e^t & e^{2t} & e^{3t} \\ 0 & -e^{2t} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Exercise 14.4.3: Check that this $\vec{x}$ really solves the system.

Note: If we write a single homogeneous linear constant coefficient n^{th} order equation as a first order system (as we did in § 14.1), then the eigenvalue equation

$$\det(P - \lambda I) = 0$$

is essentially the same as the characteristic equation we got in § 7.2 and § 7.3.

14.4.3 Complex Eigenvalues

A matrix may very well have complex eigenvalues even if all the entries are real. Take, for example,

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}.$$

Let us compute the eigenvalues of the matrix $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

$$\det(P - \lambda I) = \det \left(\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} \right) = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0.$$

Thus $\lambda = 1 \pm i$. Corresponding eigenvectors are also complex. Start with $\lambda = 1 - i$.

$$(P - (1 - i)I) \vec{v} = \vec{0},$$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \vec{v} = \vec{0}.$$

The equations $iv_1 + v_2 = 0$ and $-v_1 + iv_2 = 0$ are multiples of each other. So we only need to consider one of them. After picking $v_2 = 1$, for example, we have an eigenvector $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. In similar fashion we find that $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $1 + i$.

We could write the solution as

$$\vec{x} = c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t} + c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} c_1 i e^{(1-i)t} - c_2 i e^{(1+i)t} \\ c_1 e^{(1-i)t} + c_2 e^{(1+i)t} \end{bmatrix}.$$

We would then need to look for complex values c_1 and c_2 to solve any initial conditions. It is perhaps not completely clear that we get a real solution. After solving for c_1 and c_2 , we could use [Euler's formula](#) and do the whole song and dance we did before, but we will not. We will apply the formula in a smarter way first to find independent real solutions.

We claim that we did not have to look for a second eigenvector (nor for the second eigenvalue). All complex eigenvalues come in pairs (because the matrix P is real).

First a small detour. The real part of a complex number z can be computed as $\frac{z+\bar{z}}{2}$, where the bar above z means $\overline{a+ib} = a-ib$. This operation is called the *complex conjugate*. If a is a real number, then $\bar{a} = a$. Similarly we bar whole vectors or matrices by taking the complex conjugate of every entry. Suppose a matrix P is real. Then $\bar{P} = P$, and so $\overline{P\vec{x}} = \bar{P}\bar{\vec{x}} = P\bar{\vec{x}}$. Also the complex conjugate of 0 is still 0, therefore,

$$\vec{0} = \bar{\vec{0}} = \overline{(P - \lambda I)\vec{v}} = (P - \bar{\lambda}I)\bar{\vec{v}}.$$

In other words, if $\lambda = a+ib$ is an eigenvalue, then so is $\bar{\lambda} = a-ib$. And if $\vec{v}$ is an eigenvector corresponding to the eigenvalue λ , then $\bar{\vec{v}}$ is an eigenvector corresponding to the eigenvalue $\bar{\lambda}$.

Suppose $a+ib$ is a complex eigenvalue of P , and $\vec{v}$ is a corresponding eigenvector. Then

$$\vec{x}_1 = \vec{v}e^{(a+ib)t}$$

is a solution (complex-valued) of $\vec{x}' = P\vec{x}$. [Euler's formula](#) shows that $\overline{e^{a+ib}} = e^{a-ib}$, and so

$$\vec{x}_2 = \bar{\vec{x}}_1 = \bar{\vec{v}}e^{(a-ib)t}$$

is also a solution. As $\vec{x}_1$ and $\vec{x}_2$ are solutions, the function

$$\vec{x}_3 = \operatorname{Re} \vec{x}_1 = \operatorname{Re} \vec{v}e^{(a+ib)t} = \frac{\vec{x}_1 + \bar{\vec{x}}_1}{2} = \frac{\vec{x}_1 + \vec{x}_2}{2} = \frac{1}{2}\vec{x}_1 + \frac{1}{2}\vec{x}_2$$

is also a solution. And $\vec{x}_3$ is real-valued! Similarly as $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$ is the imaginary part, we find that

$$\vec{x}_4 = \operatorname{Im} \vec{x}_1 = \frac{\vec{x}_1 - \bar{\vec{x}}_1}{2i} = \frac{\vec{x}_1 - \vec{x}_2}{2i}.$$

is also a real-valued solution. It turns out that $\vec{x}_3$ and $\vec{x}_4$ are linearly independent. We will use [Euler's formula](#) to separate out the real and imaginary part.

Returning to our problem,

$$\vec{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t} = \begin{bmatrix} i \\ 1 \end{bmatrix} (e^t \cos t - ie^t \sin t) = \begin{bmatrix} ie^t \cos t + e^t \sin t \\ e^t \cos t - ie^t \sin t \end{bmatrix} = \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + i \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix}.$$

Then

$$\operatorname{Re} \vec{x}_1 = \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix}, \quad \text{and} \quad \operatorname{Im} \vec{x}_1 = \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix},$$

are the two real-valued linearly independent solutions we seek.

Exercise 14.4.4: Check that these really are solutions.

The general solution is

$$\vec{x} = c_1 \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + c_2 \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix} = \begin{bmatrix} c_1 e^t \sin t + c_2 e^t \cos t \\ c_1 e^t \cos t - c_2 e^t \sin t \end{bmatrix}.$$

This solution is real-valued for real c_1 and c_2 . At this point, we would solve for any initial conditions we may have to find c_1 and c_2 .

Let us summarize the discussion as a theorem.

Theorem 14.4.2. *Let P be a real-valued constant matrix. If P has a complex eigenvalue $a + ib$ and a corresponding eigenvector $\vec{v}$, then P also has a complex eigenvalue $a - ib$ with a corresponding eigenvector $\bar{\vec{v}}$. Furthermore, $\vec{x}' = P\vec{x}$ has two linearly independent real-valued solutions*

$$\vec{x}_1 = \operatorname{Re} \vec{v} e^{(a+ib)t}, \quad \text{and} \quad \vec{x}_2 = \operatorname{Im} \vec{v} e^{(a+ib)t}.$$

For each pair of complex eigenvalues $a + ib$ and $a - ib$, we get two real-valued linearly independent solutions. We then go on to the next eigenvalue, which is either a real eigenvalue or another complex eigenvalue pair. If we have n distinct eigenvalues (real or complex), then we end up with n linearly independent solutions. If we had only two equations ($n = 2$) as in the example above, then once we found two solutions we are finished, and our general solution is

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 = c_1 (\operatorname{Re} \vec{v} e^{(a+ib)t}) + c_2 (\operatorname{Im} \vec{v} e^{(a+ib)t}).$$

We can now find a real-valued general solution to any homogeneous system where the matrix has distinct eigenvalues. When we have repeated eigenvalues, matters get a bit more complicated and we will look at that situation in § 14.7.

14.4.4 Exercises

Exercise 14.4.5 (easy): Let A be a 3×3 matrix with an eigenvalue of 3 and a corresponding eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. Find $A\vec{v}$.

Exercise 14.4.6:

- Find the general solution of $x'_1 = 2x_1$, $x'_2 = 3x_2$ using the eigenvalue method (first write the system in the form $\vec{x}' = A\vec{x}$).
- Solve the system by solving each equation separately and verify you get the same general solution.

Exercise 14.4.7: Find the general solution of $x'_1 = 3x_1 + x_2$, $x'_2 = 2x_1 + 4x_2$ using the eigenvalue method.

Exercise 14.4.8: Find the general solution of $x'_1 = x_1 - 2x_2$, $x'_2 = 2x_1 + x_2$ using the eigenvalue method. Do not use complex exponentials in your solution.

Exercise 14.4.9:

- Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}$.
- Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 14.4.10: Compute eigenvalues and eigenvectors of $\begin{bmatrix} -2 & -1 & -1 \\ 3 & 2 & 1 \\ -3 & -1 & 0 \end{bmatrix}$.

Exercise 14.4.11: Let a, b, c, d, e, f be numbers. Find the eigenvalues of $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

Exercise 14.4.101:

a) Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}$.

b) Solve the system $\vec{x}' = A\vec{x}$.

Exercise 14.4.102:

a) Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$.

b) Solve the system $\vec{x}' = A\vec{x}$.

Exercise 14.4.103: Solve $x_1' = x_2$, $x_2' = x_1$ using the eigenvalue method.

Exercise 14.4.104: Solve $x_1' = x_2$, $x_2' = -x_1$ using the eigenvalue method.

14.5 Two-dimensional Systems and their Vector Fields

Let us take a moment to talk about constant coefficient linear homogeneous systems in the plane. Much intuition can be obtained by studying this simple case. We use coordinates (x, y) for the plane as usual, and suppose $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix. Consider the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = P \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \quad (14.3)$$

The system is autonomous (compare this section to § 4.6) and so we can draw a vector field (see the end of § 14.1). We will be able to visually tell what the vector field looks like and how the solutions behave, once we find the eigenvalues and eigenvectors of the matrix P . For this section, we assume that P has two eigenvalues and two corresponding eigenvectors.

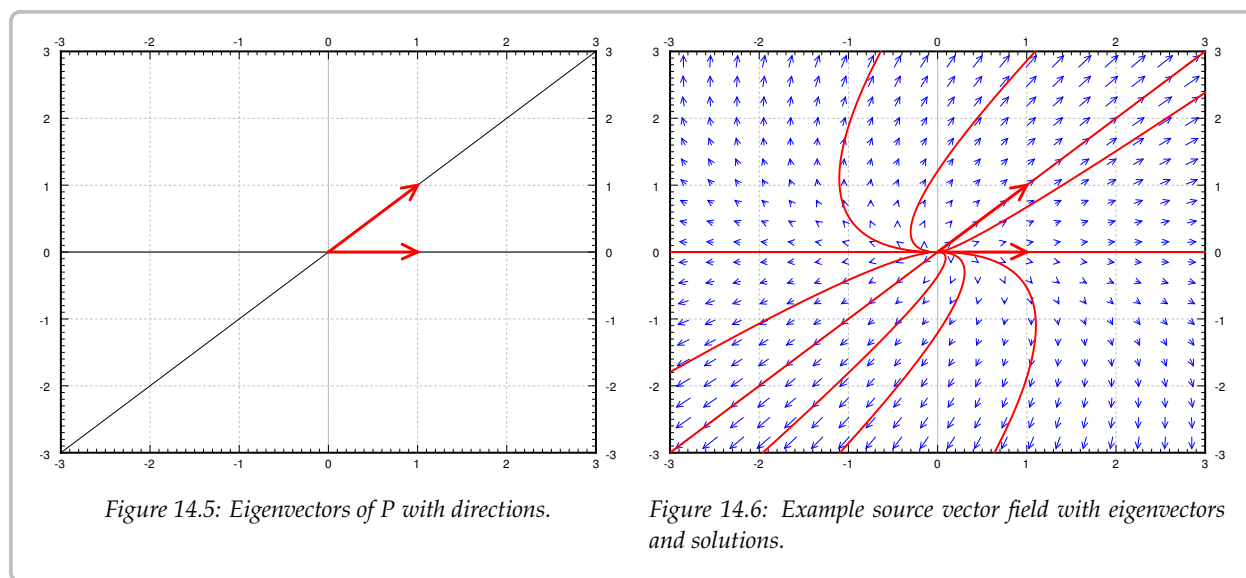
Case 1. Suppose that the eigenvalues of P are real and positive. We find two corresponding eigenvectors and plot them in the plane. For example, take the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. The eigenvalues are 1 and 2 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. See Figure 14.4 on page 659.

Let (x, y) be a point on the line determined by an eigenvector $\vec{v}$ for an eigenvalue λ . That is, $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \vec{v}$ for some scalar α . Then

$$\begin{bmatrix} x \\ y \end{bmatrix}' = P \begin{bmatrix} x \\ y \end{bmatrix} = P(\alpha \vec{v}) = \alpha(P\vec{v}) = \alpha\lambda\vec{v}.$$

The derivative is a multiple of $\vec{v}$ and hence points along the line determined by $\vec{v}$. As $\lambda > 0$, the derivative points in the direction of $\vec{v}$ when α is positive and in the opposite direction when α is negative. We draw the lines determined by the eigenvectors, and we draw arrows on the lines to indicate the directions. See Figure 14.5.

We fill in the rest of the arrows for the vector field and we also draw a few solutions. See Figure 14.6. The picture looks like a source with arrows coming out from the origin. Hence we call this type of picture a *source* or sometimes an *unstable node*.



Case 2. Suppose both eigenvalues are negative. For example, take the negation of the matrix in case 1, $\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$. The eigenvalues are -1 and -2 and corresponding eigenvectors are the same, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The calculation and the picture are almost the same. The only difference is that the eigenvalues are negative and

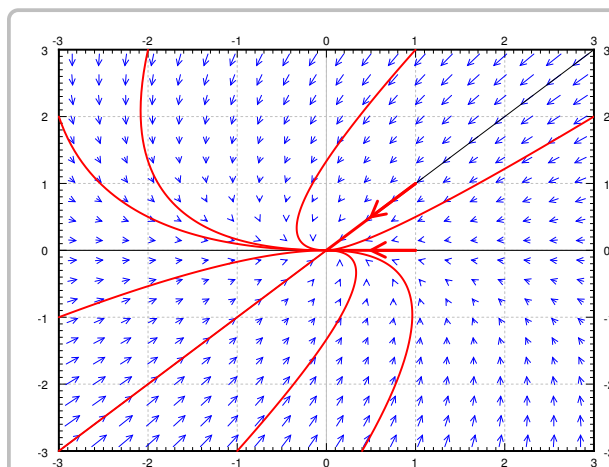


Figure 14.7: Example sink vector field with eigenvectors and solutions.

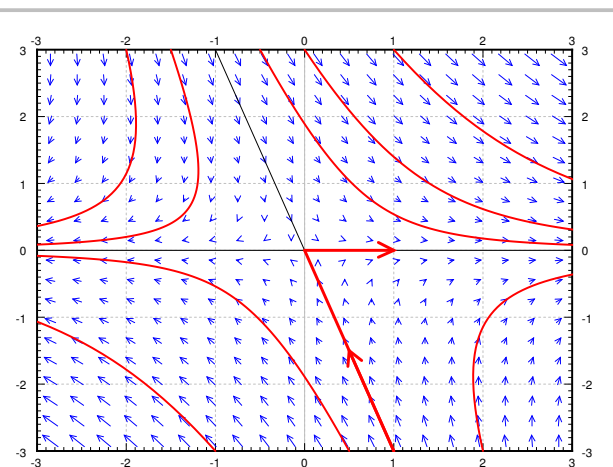


Figure 14.8: Example saddle vector field with eigenvectors and solutions.

hence all arrows are reversed. We get the picture in Figure 14.7. We call this kind of picture a *sink* or a *stable node*.

Case 3. Suppose one eigenvalue is positive and one is negative. For example the matrix $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$. The eigenvalues are 1 and -2 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. We reverse the arrows on one line (corresponding to the negative eigenvalue) and we obtain the picture in Figure 14.8. We call this picture a *saddle point*.

For the next three cases we will assume the eigenvalues are complex. In this case the eigenvectors are also complex and we cannot just plot them in the plane.

Case 4. Suppose the eigenvalues are purely imaginary, that is, $\pm ib$. For example, let $P = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$. The eigenvalues are $\pm 2i$ and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$. Consider the eigenvalue $2i$ and its eigenvector $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$. The real and imaginary parts of $\vec{v}e^{2it}$ are

$$\operatorname{Re} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} = \begin{bmatrix} \cos(2t) \\ -2 \sin(2t) \end{bmatrix}, \quad \operatorname{Im} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} = \begin{bmatrix} \sin(2t) \\ 2 \cos(2t) \end{bmatrix}.$$

We can take any linear combination of them to get other solutions, which one we take depends on the initial conditions. Now note that the real part is a parametric equation for an ellipse. Same with the imaginary part and in fact any linear combination of the two. This is what happens in general when the eigenvalues are purely imaginary. So when the eigenvalues are purely imaginary, we get *ellipses* for the solutions. This type of picture is sometimes called a *center*. See Figure 14.9 on the next page.

Case 5. Now suppose the complex eigenvalues have a positive real part. That is, suppose the eigenvalues are $a \pm ib$ for some $a > 0$. For example, let $P = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$. The eigenvalues turn out to be $1 \pm 2i$ and eigenvectors are $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$. We take $1 + 2i$ and its eigenvector $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ and find the real and imaginary parts of $\vec{v}e^{(1+2i)t}$ are

$$\operatorname{Re} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} = e^t \begin{bmatrix} \cos(2t) \\ -2 \sin(2t) \end{bmatrix}, \quad \operatorname{Im} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} = e^t \begin{bmatrix} \sin(2t) \\ 2 \cos(2t) \end{bmatrix}.$$

Note the e^t in front of the solutions. The solutions grow in magnitude while spinning around the origin. Hence we get a *spiral source*. See Figure 14.10 on the facing page.

Case 6. Finally suppose the complex eigenvalues have a negative real part. That is, suppose the eigenvalues are $-a \pm ib$ for some $a > 0$. For example, let $P = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}$. The eigenvalues turn out to be $-1 \pm 2i$ and

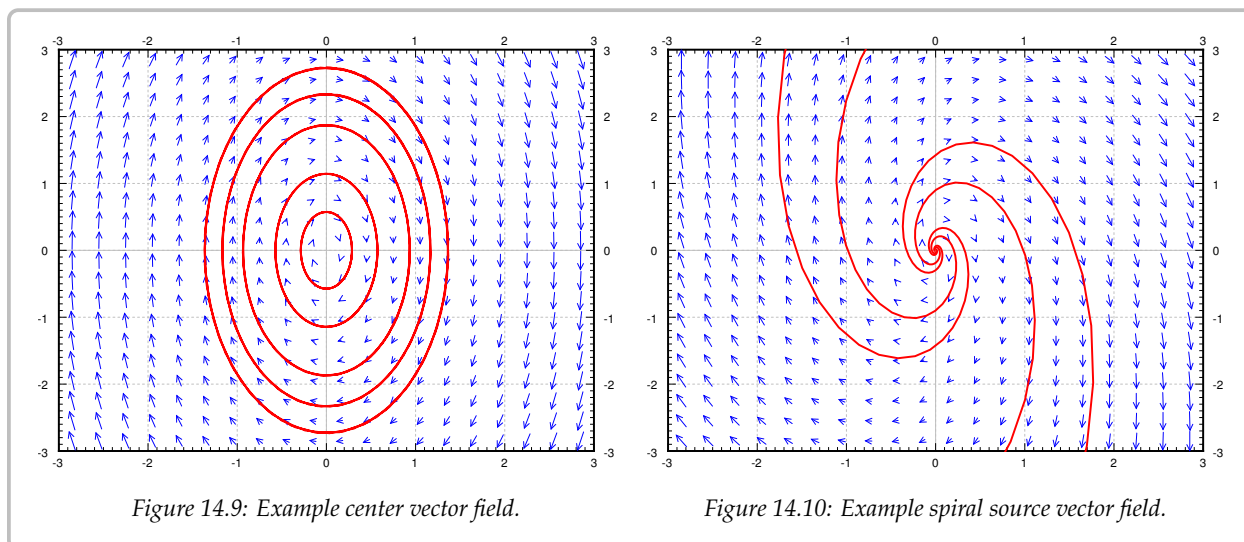
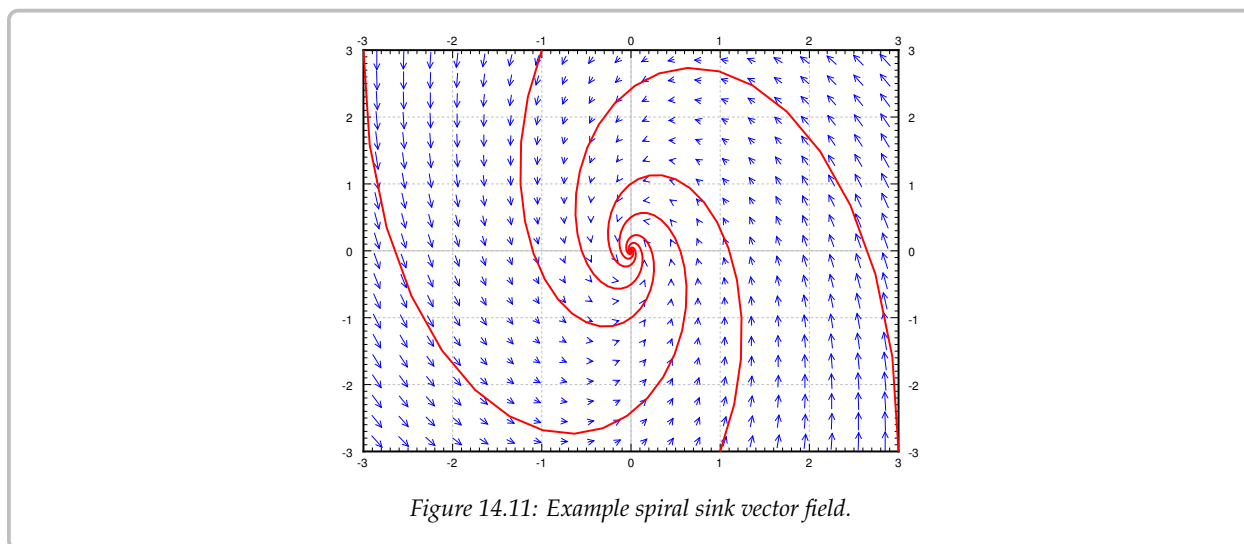


Figure 14.10: Example spiral source vector field.

eigenvectors are $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$. We take $-1 - 2i$ and its eigenvector $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ and find the real and imaginary parts of $\vec{v}e^{(-1-2i)t}$ are

$$\operatorname{Re} \left[\begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1-2i)t} \right] = e^{-t} \begin{bmatrix} \cos(2t) \\ 2 \sin(2t) \end{bmatrix}, \quad \operatorname{Im} \left[\begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1-2i)t} \right] = e^{-t} \begin{bmatrix} -\sin(2t) \\ 2 \cos(2t) \end{bmatrix}.$$

Note the e^{-t} in front of the solutions. The solutions shrink in magnitude while spinning around the origin. Hence we get a *spiral sink*. See Figure 14.11.



We summarize the behavior of linear homogeneous two-dimensional systems given by a nonsingular matrix in Table 14.1. Systems where one of the eigenvalues is zero (the matrix is singular) come up in practice from time to time, see Example 14.1.2 on page 632, and the pictures are somewhat different (simpler in a way). See the exercises.

14.5.1 Exercises

Exercise 14.5.1: Take the equation $mx'' + cx' + kx = 0$, with $m > 0$, $c \geq 0$, $k > 0$ for the mass-spring system.

Eigenvalues	Behavior
real and both positive	source / unstable node
real and both negative	sink / stable node
real and opposite signs	saddle
purely imaginary	center point / ellipses
complex with positive real part	spiral source
complex with negative real part	spiral sink

Table 14.1: Summary of behavior of linear homogeneous two-dimensional systems.

- Convert this to a system of first order equations.
- Classify for what m, c, k do you get which behavior.
- Explain from physical intuition why you do not get all the different kinds of behavior here?

Exercise 14.5.2: What happens in the case when $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$? In this case the eigenvalue is repeated and there is only one independent eigenvector. What picture does this look like?

Exercise 14.5.3: What happens in the case when $P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$? Does this look like any of the pictures we have drawn?

Exercise 14.5.4: Which behaviors are possible if P is diagonal, that is $P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$? You can assume that a and b are not zero.

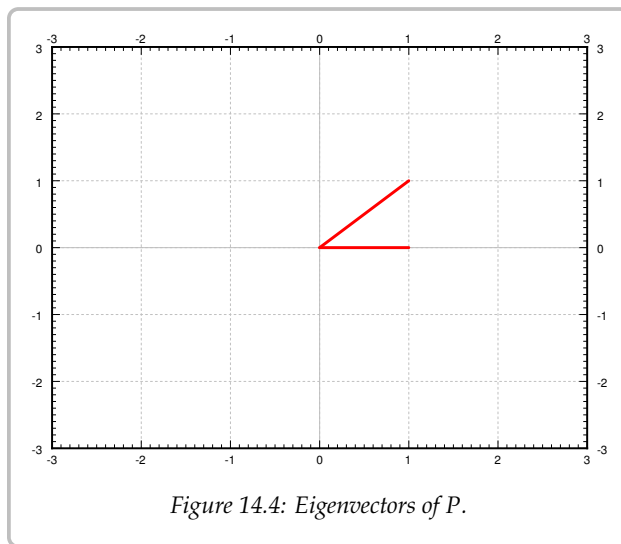
Exercise 14.5.5: Take the system from [Example 14.1.2](#) on page 632, $x'_1 = \frac{r}{V}(x_2 - x_1)$, $x'_2 = \frac{r}{V}(x_1 - x_2)$. As we said, one of the eigenvalues is zero. What is the other eigenvalue, how does the picture look like and what happens when t goes to infinity.

Exercise 14.5.101: Describe the behavior of the following systems without solving:

- $x' = x + y, \quad y' = x - y.$
- $x'_1 = x_1 + x_2, \quad x'_2 = 2x_2.$
- $x'_1 = -2x_2, \quad x'_2 = 2x_1.$
- $x' = x + 3y, \quad y' = -2x - 4y.$
- $x' = x - 4y, \quad y' = -4x + y.$

Exercise 14.5.102: Suppose that $\vec{x}' = A\vec{x}$ where A is a 2 by 2 matrix with eigenvalues $2 \pm i$. Describe the behavior.

Exercise 14.5.103: Take $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Draw the vector field and describe the behavior. Is it one of the behaviors that we have seen before?



14.6 Second Order Systems and Applications

14.6.1 Undamped Mass-Spring Systems

While we did say that we will usually only look at first order systems, it is sometimes more convenient to study the system in the way it arises naturally. For example, suppose we have 3 masses connected by springs between two walls. We could pick any higher number, and the math would be essentially the same, but for simplicity we pick 3 right now. Let us also assume no friction, that is, the system is undamped. The masses are m_1 , m_2 , and m_3 and the spring constants are k_1 , k_2 , k_3 , and k_4 . Let x_1 be the displacement from rest position of the first mass, and x_2 and x_3 the displacement of the second and third mass. We make, as usual, positive values go right (as x_1 grows, the first mass is moving right). See Figure 14.12.

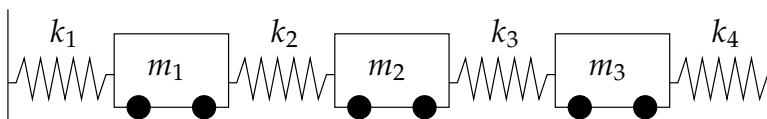


Figure 14.12: System of masses and springs.

This simple system turns up in unexpected places. For example, our world really consists of many small particles of matter interacting together. When we try the system above with many more masses, we obtain a good approximation to how an elastic material behaves. By somehow taking a limit of the number of masses going to infinity, we obtain the continuous one-dimensional wave equation (that we study in § ??). But we digress.

Let us set up the equations for the three mass system. By Hooke's law, the force acting on the mass equals the spring compression times the spring constant. By Newton's second law, force is mass times acceleration. So if we sum the forces acting on each mass, put the right sign in front of each term, depending on the direction in which it is acting, and set this equal to mass times the acceleration, we end up with the desired system of equations.

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) &= -(k_1 + k_2)x_1 + k_2 x_2, \\ m_2 x_2'' &= -k_2(x_2 - x_1) + k_3(x_3 - x_2) &= k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3, \\ m_3 x_3'' &= -k_3(x_3 - x_2) - k_4 x_3 &= k_3 x_2 - (k_3 + k_4)x_3. \end{aligned}$$

We define the matrices

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} -(k_1 + k_2) & k_2 & 0 \\ k_2 & -(k_2 + k_3) & k_3 \\ 0 & k_3 & -(k_3 + k_4) \end{bmatrix}.$$

We write the equation simply as

$$M\vec{x}'' = K\vec{x}.$$

At this point we could introduce 3 new variables and write out a system of 6 first order equations. We claim this simple setup is easier to handle as a second order system. We call $\vec{x}$ the *displacement vector*, M the *mass matrix*, and K the *stiffness matrix*.

Exercise 14.6.1: Repeat this setup for 4 masses (find the matrices M and K). Do it for 5 masses. Can you find a prescription to do it for n masses?

As with a single equation we want to “divide by M .” This means computing the inverse of M . The masses

are all nonzero and M is a diagonal matrix, so computing the inverse is easy:

$$M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}.$$

This fact follows readily by how we multiply diagonal matrices. As an exercise, you should verify that $MM^{-1} = M^{-1}M = I$.

Let $A = M^{-1}K$. We look at the system $\vec{x}'' = M^{-1}K\vec{x}$, or

$$\vec{x}'' = A\vec{x}.$$

Many real world systems can be modeled by this equation. For simplicity, we will only talk about the given masses-and-springs problem. We try a solution of the form

$$\vec{x} = \vec{v}e^{\alpha t}.$$

We compute that for this guess, $\vec{x}'' = \alpha^2\vec{v}e^{\alpha t}$. We plug our guess into the equation and get

$$\alpha^2\vec{v}e^{\alpha t} = A\vec{v}e^{\alpha t}.$$

We divide by $e^{\alpha t}$ to arrive at $\alpha^2\vec{v} = A\vec{v}$. Hence if α^2 is an eigenvalue of A and $\vec{v}$ is a corresponding eigenvector, we have found a solution.

In our example, and in other common applications, A has only real negative eigenvalues (and possibly a zero eigenvalue). So we study only this case. When an eigenvalue λ is negative, it means that $\alpha^2 = \lambda$ is negative. Hence there is some real number ω such that $-\omega^2 = \lambda$. Then $\alpha = \pm i\omega$. The solution we guessed was

$$\vec{x} = \vec{v}(\cos(\omega t) + i \sin(\omega t)).$$

By taking the real and imaginary parts (note that $\vec{v}$ is real), we find that $\vec{v} \cos(\omega t)$ and $\vec{v} \sin(\omega t)$ are linearly independent solutions.

If an eigenvalue is zero, it turns out that both $\vec{v}$ and $\vec{v}t$ are solutions, where $\vec{v}$ is an eigenvector corresponding to the eigenvalue 0.

Exercise 14.6.2: Show that if A has a zero eigenvalue and $\vec{v}$ is a corresponding eigenvector, then $\vec{x} = \vec{v}(a + bt)$ is a solution of $\vec{x}'' = A\vec{x}$ for arbitrary constants a and b .

Theorem 14.6.1. Let A be a real $n \times n$ matrix with n distinct real negative (or zero) eigenvalues we denote by $-\omega_1^2 > -\omega_2^2 > \dots > -\omega_n^2$, and corresponding eigenvectors by $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. If A is invertible (that is, if $\omega_1 > 0$), then

$$\vec{x}(t) = \sum_{i=1}^n \vec{v}_i (a_i \cos(\omega_i t) + b_i \sin(\omega_i t)),$$

is the general solution of

$$\vec{x}'' = A\vec{x},$$

for some arbitrary constants a_i and b_i . If A has a zero eigenvalue, that is $\omega_1 = 0$, and all other eigenvalues are distinct and negative, then the general solution can be written as

$$\vec{x}(t) = \vec{v}_1(a_1 + b_1 t) + \sum_{i=2}^n \vec{v}_i (a_i \cos(\omega_i t) + b_i \sin(\omega_i t)).$$

We use this solution and the setup from the introduction of this section even when some of the masses and springs are missing. For example, when there are only 2 masses and only 2 springs, simply take only the equations for the two masses and set all the spring constants for the springs that are missing to zero.

14.6.2 Examples

Example 14.6.1: Consider the setup in Figure 14.13, with $m_1 = 2$ kg, $m_2 = 1$ kg, $k_1 = 4$ N/m, and $k_2 = 2$ N/m.

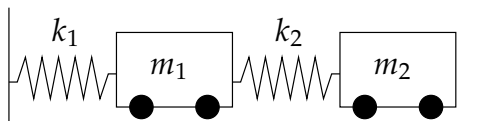


Figure 14.13: System of masses and springs.

The equations we write down are

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -(4+2) & 2 \\ 2 & -2 \end{bmatrix} \vec{x},$$

or

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

We find the eigenvalues of A to be $\lambda = -1, -4$ (exercise). We find corresponding eigenvectors to be $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively (exercise).

We check the theorem and note that $\omega_1 = 1$ and $\omega_2 = 2$. Hence the general solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (a_1 \cos(t) + b_1 \sin(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_2 \cos(2t) + b_2 \sin(2t)).$$

The two terms in the solution represent the two so-called *natural* or *normal modes of oscillation*. And the two (angular) frequencies are the *natural frequencies*. The first natural frequency is 1, and second natural frequency is 2. The two modes are plotted in Figure 14.14.

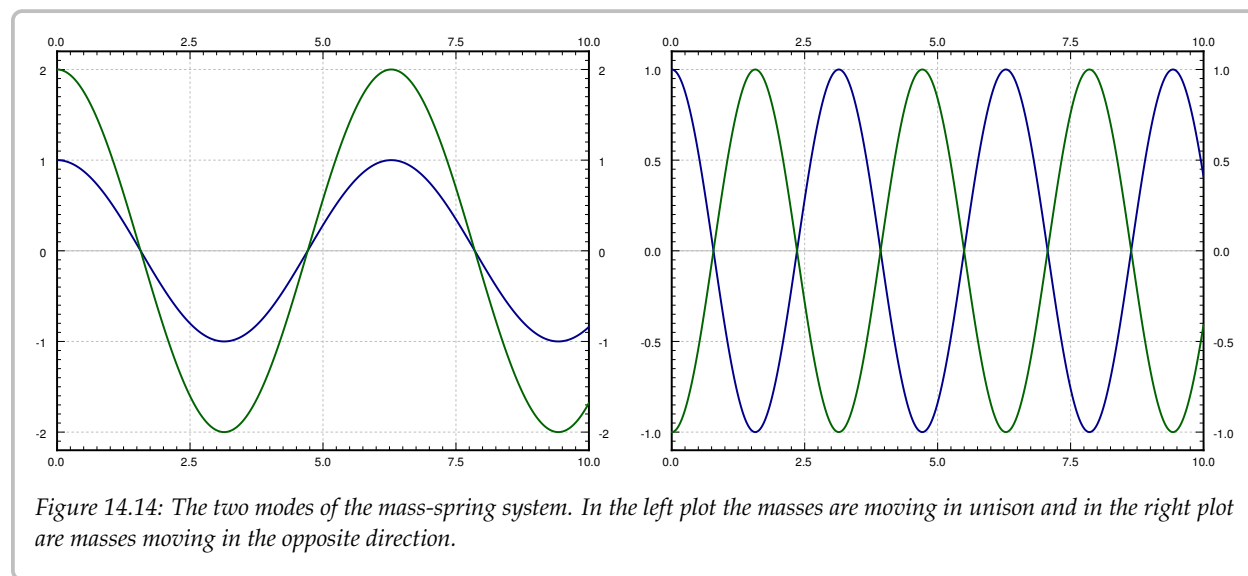


Figure 14.14: The two modes of the mass-spring system. In the left plot the masses are moving in unison and in the right plot are masses moving in the opposite direction.

Let us write the solution as

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1 \cos(t - \alpha_1) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} c_2 \cos(2t - \alpha_2).$$

The first term,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1 \cos(t - \alpha_1) = \begin{bmatrix} c_1 \cos(t - \alpha_1) \\ 2c_1 \cos(t - \alpha_1) \end{bmatrix},$$

corresponds to the mode where the masses move synchronously in the same direction.

The second term,

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} c_2 \cos(2t - \alpha_2) = \begin{bmatrix} c_2 \cos(2t - \alpha_2) \\ -c_2 \cos(2t - \alpha_2) \end{bmatrix},$$

corresponds to the mode where the masses move synchronously but in opposite directions.

The general solution is a combination of the two modes. That is, the initial conditions determine the amplitude and phase shift of each mode. As an example, suppose we have initial conditions

$$\vec{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \vec{x}'(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}.$$

We use the a_j, b_j constants to solve for initial conditions. First

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2 = \begin{bmatrix} a_1 + a_2 \\ 2a_1 - a_2 \end{bmatrix}.$$

We solve (exercise) to find $a_1 = 0, a_2 = 1$. To find the b_1 and b_2 , we differentiate first:

$$\vec{x}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (-a_1 \sin(t) + b_1 \cos(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-2a_2 \sin(2t) + 2b_2 \cos(2t)).$$

Now we solve:

$$\begin{bmatrix} 0 \\ 6 \end{bmatrix} = \vec{x}'(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} b_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} 2b_2 = \begin{bmatrix} b_1 + 2b_2 \\ 2b_1 - 2b_2 \end{bmatrix}.$$

Again solve (exercise) to find $b_1 = 2, b_2 = -1$. So our solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2 \sin(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (\cos(2t) - \sin(2t)) = \begin{bmatrix} 2 \sin(t) + \cos(2t) - \sin(2t) \\ 4 \sin(t) - \cos(2t) + \sin(2t) \end{bmatrix}.$$

The graphs of the two displacements, x_1 and x_2 of the two carts is in [Figure 14.15](#).

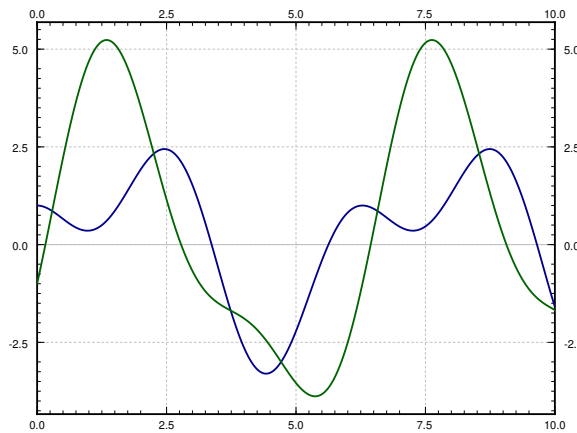


Figure 14.15: Superposition of the two modes given the initial conditions.

Example 14.6.2: We have two toy rail cars. Car 1 of mass 2 kg is traveling at 3 m/s towards the second rail car of mass 1 kg. There is a bumper on the second rail car that engages at the moment the cars hit (it connects

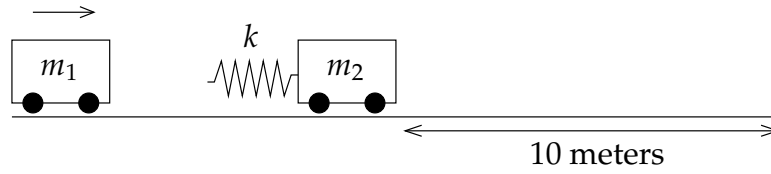


Figure 14.16: The crash of two rail cars.

to two cars) and does not let go. The bumper acts like a spring of spring constant $k = 2 \text{ N/m}$. The second car is 10 meters from a wall. See Figure 14.16.

We want to ask several questions. At what time after the cars link does impact with the wall happen? What is the speed of car 2 when it hits the wall?

OK, let us first set the system up. Let $t = 0$ be the time when the two cars link up. Let x_1 be the displacement of the first car from the position at $t = 0$, and let x_2 be the displacement of the second car from its original location. Then the time when $x_2(t) = 10$ is exactly the time when impact with wall occurs. For this t , $x'_2(t)$ is the speed at impact. This system acts just like the system of the previous example but without k_1 . Hence the equation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{x},$$

or

$$\vec{x}'' = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

We compute the eigenvalues of A . It is not hard to see that the eigenvalues are 0 and -3 (exercise). Furthermore, eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively (exercise). Then $\omega_1 = 0$, $\omega_2 = \sqrt{3}$, and by the second part of the theorem the general solution is

$$\begin{aligned} \vec{x} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} (a_1 + b_1 t) + \begin{bmatrix} 1 \\ -2 \end{bmatrix} (a_2 \cos(\sqrt{3} t) + b_2 \sin(\sqrt{3} t)) \\ &= \begin{bmatrix} a_1 + b_1 t + a_2 \cos(\sqrt{3} t) + b_2 \sin(\sqrt{3} t) \\ a_1 + b_1 t - 2a_2 \cos(\sqrt{3} t) - 2b_2 \sin(\sqrt{3} t) \end{bmatrix}. \end{aligned}$$

We now apply the initial conditions. First the cars start at position 0 so $x_1(0) = 0$ and $x_2(0) = 0$. The first car is traveling at 3 m/s , so $x'_1(0) = 3$ and the second car starts at rest, so $x'_2(0) = 0$. The first conditions says

$$\vec{0} = \vec{x}(0) = \begin{bmatrix} a_1 + a_2 \\ a_1 - 2a_2 \end{bmatrix}.$$

It is not hard to see that $a_1 = a_2 = 0$. We set $a_1 = 0$ and $a_2 = 0$ in $\vec{x}(t)$ and differentiate to get

$$\vec{x}'(t) = \begin{bmatrix} b_1 + \sqrt{3} b_2 \cos(\sqrt{3} t) \\ b_1 - 2\sqrt{3} b_2 \cos(\sqrt{3} t) \end{bmatrix}.$$

So

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \vec{x}'(0) = \begin{bmatrix} b_1 + \sqrt{3} b_2 \\ b_1 - 2\sqrt{3} b_2 \end{bmatrix}.$$

Solving these two equations we find $b_1 = 2$ and $b_2 = \frac{1}{\sqrt{3}}$. Hence the position of our cars is (until the impact with the wall)

$$\vec{x} = \begin{bmatrix} 2t + \frac{1}{\sqrt{3}} \sin(\sqrt{3} t) \\ 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3} t) \end{bmatrix}.$$

Note how the presence of the zero eigenvalue resulted in a term containing t . This means that the cars will be traveling in the positive direction as time grows, which is what we expect.

What we are really interested in is the second expression, the one for x_2 . We have $x_2(t) = 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3}t)$. See Figure 14.17 on page 667 for the plot of x_2 versus time.

Just from the graph we can see that time of impact will be a little more than 5 seconds from time zero. For this we have to solve the equation $10 = x_2(t) = 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3}t)$. Using a computer (or even a graphing calculator) we find that $t_{\text{impact}} \approx 5.22$ seconds.

The speed of the second car is $x_2' = 2 - 2 \cos(\sqrt{3}t)$. At the time of impact (5.22 seconds from $t = 0$) we get $x_2'(t_{\text{impact}}) \approx 3.85$. The maximum speed is the maximum of $2 - 2 \cos(\sqrt{3}t)$, which is 4. We are traveling at almost the maximum speed when we hit the wall.

Suppose that Bob is a tiny person sitting on car 2. Bob has a Martini in his hand and would like not to spill it. Let us suppose Bob would not spill his Martini when the first car links up with car 2, but if car 2 hits the wall at any speed greater than zero, Bob will spill his drink. Suppose Bob can move car 2 a few meters towards or away from the wall (he cannot go all the way to the wall, nor can he get out of the way of the first car). Is there a “safe” distance for him to be at? A distance such that the impact with the wall is at zero speed?

The answer is yes. On Figure 14.17 on page 667, note the “plateau” between $t = 3$ and $t = 4$. There is a point where the speed is zero. To find it we solve $x_2'(t) = 0$. This is when $\cos(\sqrt{3}t) = 1$ or in other words when $t = \frac{2\pi}{\sqrt{3}}, \frac{4\pi}{\sqrt{3}}, \dots$ and so on. We plug in the first value to obtain $x_2\left(\frac{2\pi}{\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}} \approx 7.26$. So a “safe” distance is about 7 and a quarter meters from the wall.

Alternatively Bob could move away from the wall towards the incoming car 2, where another safe distance is $x_2\left(\frac{4\pi}{\sqrt{3}}\right) = \frac{8\pi}{\sqrt{3}} \approx 14.51$ and so on. We can use all the different t such that $x_2'(t) = 0$. Of course $t = 0$ is also a solution, corresponding to $x_2 = 0$, but that means standing right at the wall.

14.6.3 Forced Oscillations

Finally we move to forced oscillations. Suppose that now our system is

$$\vec{x}'' = A\vec{x} + \vec{F} \cos(\omega t). \quad (14.4)$$

That is, we are adding periodic forcing to the system in the direction of the vector $\vec{F}$.

As before, this system just requires us to find one particular solution $\vec{x}_p$, add it to the general solution of the associated homogeneous system $\vec{x}_c$, and we will have the general solution to (14.4). Let us suppose that ω is not one of the natural frequencies of $\vec{x}'' = A\vec{x}$, then we can guess

$$\vec{x}_p = \vec{c} \cos(\omega t),$$

where $\vec{c}$ is an unknown constant vector. Note that we do not need to use sine since there are only second derivatives. We solve for $\vec{c}$ to find $\vec{x}_p$. This is really just the method of *undetermined coefficients* for systems. Let us differentiate $\vec{x}_p$ twice to get

$$\vec{x}_p'' = -\omega^2 \vec{c} \cos(\omega t).$$

Plug $\vec{x}_p$ and $\vec{x}_p''$ into equation (14.4):

$$\underbrace{\vec{x}_p''}_{-\omega^2 \vec{c} \cos(\omega t)} = \underbrace{A\vec{x}_p}_{A\vec{c} \cos(\omega t)} + \vec{F} \cos(\omega t).$$

We cancel out the cosine and rearrange the equation to obtain

$$(A + \omega^2 I)\vec{c} = -\vec{F}.$$

So

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}).$$

Of course this is possible only if $(A + \omega^2 I) = (A - (-\omega^2)I)$ is invertible. That matrix is invertible if and only if $-\omega^2$ is not an eigenvalue of A . That is true if and only if ω is not a natural frequency of the system.

We simplified things a little bit. If we wish to have the forcing term to be in the units of force, say Newtons, then we must write

$$M\vec{x}'' = K\vec{x} + \vec{G} \cos(\omega t).$$

If we then write things in terms of $A = M^{-1}K$, we have

$$\vec{x}'' = M^{-1}K\vec{x} + M^{-1}\vec{G} \cos(\omega t) \quad \text{or} \quad \vec{x}'' = A\vec{x} + \vec{F} \cos(\omega t),$$

where $\vec{F} = M^{-1}\vec{G}$.

Example 14.6.3: Let us take the example in Figure 14.13 on page 662 with the same parameters as before: $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, and $k_2 = 2$. Now suppose that there is a force $2 \cos(3t)$ acting on the second cart.

The equation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -(4+2) & 2 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(3t) \quad \text{or} \quad \vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(3t).$$

We solved the associated homogeneous equation before and found the complementary solution to be

$$\vec{x}_c = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (a_1 \cos(t) + b_1 \sin(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_2 \cos(2t) + b_2 \sin(2t)).$$

The natural frequencies are 1 and 2. As 3 is not a natural frequency, we try $\vec{c} \cos(3t)$. We invert $(A + 3^2 I)$:

$$\left(\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} + 3^2 I \right)^{-1} = \begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{bmatrix}.$$

Hence,

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}) = \begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{bmatrix}.$$

Combining with the general solution of the associated homogeneous problem, we get that the general solution to $\vec{x}'' = A\vec{x} + \vec{F} \cos(\omega t)$ is

$$\vec{x} = \vec{x}_c + \vec{x}_p = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (a_1 \cos(t) + b_1 \sin(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_2 \cos(2t) + b_2 \sin(2t)) + \begin{bmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{bmatrix} \cos(3t).$$

We would then solve for the constants a_1 , a_2 , b_1 , and b_2 using any given initial conditions.

Note that given force $\vec{f}$, we write the equation as $M\vec{x}'' = K\vec{x} + \vec{f}$ to get the units right. Then we write $\vec{x}'' = M^{-1}K\vec{x} + M^{-1}\vec{f}$. The term $\vec{g} = M^{-1}\vec{f}$ in $\vec{x}'' = A\vec{x} + \vec{g}$ is in units of force per unit mass.

If ω is a natural frequency of the system, *resonance* may occur, because we will have to try a particular solution of the form

$$\vec{x}_p = \vec{c} t \sin(\omega t) + \vec{d} \cos(\omega t).$$

That is assuming that the eigenvalues of the coefficient matrix are distinct. Next, note that the amplitude of this solution grows without bound as t grows.

14.6.4 Exercises

Exercise 14.6.3: Find a particular solution to

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(2t).$$

Exercise 14.6.4 (challenging): Let us take the example in [Figure 14.13](#) on page 662 with the same parameters as before: $m_1 = 2$, $k_1 = 4$, and $k_2 = 2$, except for m_2 , which is unknown. Suppose that there is a force $\cos(5t)$ acting on the first mass. Find an m_2 such that there exists a particular solution where the first mass does not move.

Note: This idea is called *dynamic damping*. In practice there will be a small amount of damping and so any transient solution will disappear and after long enough time, the first mass will always come to a stop.

Exercise 14.6.5: Let us take the [Example 14.6.2](#) on page 663, but that at time of impact, car 2 is moving to the left at the speed of 3 m/s.

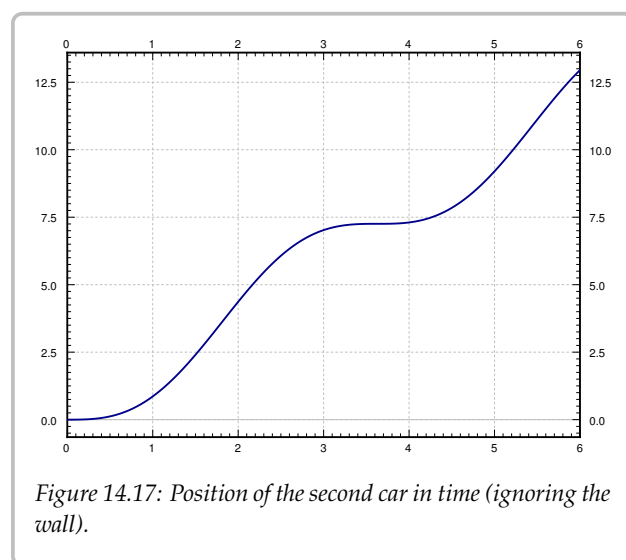
- Find the behavior of the system after linkup.
- Will the second car hit the wall, or will it be moving away from the wall as time goes on?
- At what speed would the first car have to be traveling for the system to essentially stay in place after linkup?

Exercise 14.6.6: Let us take the example in [Figure 14.13](#) on page 662 with parameters $m_1 = m_2 = 1$, $k_1 = k_2 = 1$. Does there exist a set of initial conditions for which the first cart moves but the second cart does not? If so, find those conditions. If not, argue why not.

Exercise 14.6.101: Find the general solution to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & 6 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} \cos(2t) \\ 0 \\ 0 \end{bmatrix}$.

Exercise 14.6.102: Suppose there are three carts of equal mass m and connected by two springs of constant k (and no connections to walls). Set up the system and find its general solution.

Exercise 14.6.103: Suppose a cart of mass 2 kg is attached by a spring of constant $k = 1$ to a cart of mass 3 kg, which is attached to the wall by a spring also of constant $k = 1$. Suppose that the initial position of the first cart is 1 meter in the positive direction from the rest position, and the second mass starts at the rest position. The masses are not moving and are let go. Find the position of the second mass as a function of time.



14.7 Multiple Eigenvalues

It may happen that a matrix A has some “repeated” eigenvalues. That is, the characteristic equation $\det(A - \lambda I) = 0$ may have repeated roots. This is actually unlikely to happen for a random matrix. If we take a small perturbation of A (we change the entries of A slightly), we get a matrix with distinct eigenvalues. As any system we want to solve in practice is an approximation to reality anyway, it is not absolutely indispensable to know how to solve these corner cases. On the other hand, these cases do come up in applications from time to time. Furthermore, if we have distinct but very close eigenvalues, the behavior is similar to that of repeated eigenvalues, and so understanding that case will give us insight into what is going on.

14.7.1 Geometric Multiplicity

Take the diagonal matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

A has an eigenvalue 3 of multiplicity 2. We call the multiplicity of the eigenvalue in the characteristic equation the *algebraic multiplicity*. In this case, there also exist 2 linearly independent eigenvectors, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue 3. This means that the so-called *geometric multiplicity* of this eigenvalue is also 2.

In all the theorems where we required a matrix to have n distinct eigenvalues, we only really needed to have n linearly independent eigenvectors. For example, $\vec{x}' = A\vec{x}$ has the general solution

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}.$$

Let us restate the theorem about real eigenvalues. In the following theorem we will repeat eigenvalues according to (algebraic) multiplicity. So for the matrix A above, we would say that it has eigenvalues 3 and 3.

Theorem 14.7.1. Suppose the $n \times n$ matrix P has n real eigenvalues (not necessarily distinct), $\lambda_1, \lambda_2, \dots, \lambda_n$, and there are n linearly independent corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Then the general solution to $\vec{x}' = P\vec{x}$ can be written as

$$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}.$$

The *geometric multiplicity* of an eigenvalue of algebraic multiplicity n is equal to the number of corresponding linearly independent eigenvectors. The geometric multiplicity is always less than or equal to the algebraic multiplicity. The theorem handles the case when these two multiplicities are equal for all eigenvalues. If for an eigenvalue the geometric multiplicity is equal to the algebraic multiplicity, then we say the eigenvalue is *complete*.

In other words, the hypothesis of the theorem could be stated as saying that if all the eigenvalues of P are complete, then there are n linearly independent eigenvectors and thus we have the given general solution.

If the geometric multiplicity of an eigenvalue is 2 or greater, then the set of linearly independent eigenvectors is not unique up to multiples as it was before. For example, for the diagonal matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ we could also pick eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, or in fact any pair of two linearly independent vectors. The number of linearly independent eigenvectors corresponding to λ is the number of free variables we obtain when solving $A\vec{v} = \lambda\vec{v}$. We pick specific values for those free variables to obtain eigenvectors. If you pick different values, you may get different eigenvectors.

14.7.2 Defective Eigenvalues

If an $n \times n$ matrix has less than n linearly independent eigenvectors, it is said to be *deficient*. Then there is at least one eigenvalue with an algebraic multiplicity that is higher than its geometric multiplicity. We call this eigenvalue *defective* and the difference between the two multiplicities we call the *defect*.

Example 14.7.1: The matrix

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

has an eigenvalue 3 of algebraic multiplicity 2. Let us try to compute eigenvectors.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}.$$

We must have that $v_2 = 0$. Hence any eigenvector is of the form $\begin{bmatrix} v_1 \\ 0 \end{bmatrix}$. Any two such vectors are linearly dependent, and hence the geometric multiplicity of the eigenvalue is 1. Therefore, the defect is 1, and we can no longer apply the eigenvalue method directly to a system of ODEs with such a coefficient matrix.

Roughly, the key observation is that if λ is an eigenvalue of A of algebraic multiplicity m , then we can find certain m linearly independent vectors solving $(A - \lambda I)^k \vec{v} = \vec{0}$ for various powers k . We will call these *generalized eigenvectors*.

Let us continue with the example $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and the equation $\vec{x}' = A\vec{x}$. We found an eigenvalue $\lambda = 3$ of (algebraic) multiplicity 2 and defect 1. We found one eigenvector $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We have one solution

$$\vec{x}_1 = \vec{v}e^{3t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}.$$

We are now stuck, we get no other solutions from standard eigenvectors. But we need two linearly independent solutions to find the general solution of the equation.

Let us try (in the spirit of repeated roots of the characteristic equation for a single equation) another solution of the form

$$\vec{x}_2 = (\vec{v}_2 + \vec{v}_1 t) e^{3t}.$$

We differentiate to get

$$\vec{x}_2' = \vec{v}_1 e^{3t} + 3(\vec{v}_2 + \vec{v}_1 t) e^{3t} = (3\vec{v}_2 + \vec{v}_1) e^{3t} + 3\vec{v}_1 t e^{3t}.$$

As we are assuming that $\vec{x}_2$ is a solution, $\vec{x}_2'$ must equal $A\vec{x}_2$. So we compute $A\vec{x}_2$:

$$A\vec{x}_2 = A(\vec{v}_2 + \vec{v}_1 t) e^{3t} = A\vec{v}_2 e^{3t} + A\vec{v}_1 t e^{3t}.$$

By looking at the coefficients of e^{3t} and te^{3t} , we see $3\vec{v}_2 + \vec{v}_1 = A\vec{v}_2$ and $3\vec{v}_1 = A\vec{v}_1$. This means that

$$(A - 3I)\vec{v}_2 = \vec{v}_1, \quad \text{and} \quad (A - 3I)\vec{v}_1 = \vec{0}.$$

Therefore, $\vec{x}_2$ is a solution if these two equations are satisfied. The second equation is satisfied if $\vec{v}_1$ is an eigenvector, and we found the eigenvector above, so let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, if we can find a $\vec{v}_2$ that solves $(A - 3I)\vec{v}_2 = \vec{v}_1$, then we are done. This is just a bunch of linear equations to solve and we are by now very good at that. Let us solve $(A - 3I)\vec{v}_2 = \vec{v}_1$. Write

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

By inspection we see that letting $a = 0$ (a could be anything in fact) and $b = 1$ does the job. Hence we can take $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Our general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \right) e^{3t} = \begin{bmatrix} c_1 e^{3t} + c_2 t e^{3t} \\ c_2 e^{3t} \end{bmatrix}.$$

Let us check that we really do have the solution. First $x_1' = c_1 3e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t} = 3x_1 + x_2$. Good. Now $x_2' = 3c_2 e^{3t} = 3x_2$. Good.

In the example, if we plug $(A - 3I)\vec{v}_2 = \vec{v}_1$ into $(A - 3I)\vec{v}_1 = \vec{0}$ we find

$$(A - 3I)(A - 3I)\vec{v}_2 = \vec{0}, \quad \text{or} \quad (A - 3I)^2 \vec{v}_2 = \vec{0}.$$

Furthermore, if $(A - 3I)\vec{w} \neq \vec{0}$, then $(A - 3I)\vec{w}$ is an eigenvector, a multiple of $\vec{v}_1$. In this 2×2 case $(A - 3I)^2$ is just the zero matrix (exercise). So any vector $\vec{w}$ solves $(A - 3I)^2\vec{w} = \vec{0}$ and we just need a $\vec{w}$ such that $(A - 3I)\vec{w} \neq \vec{0}$. Then we could use $\vec{w}$ for $\vec{v}_2$, and $(A - 3I)\vec{w}$ for $\vec{v}_1$.

Note that the system $\vec{x}' = A\vec{x}$ has a simpler solution since A is a so-called *upper triangular matrix*, that is every entry below the diagonal is zero. In particular, the equation for x_2 does not depend on x_1 . Mind you, not every defective matrix is triangular.

Exercise 14.7.1: Solve $\vec{x}' = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \vec{x}$ by first solving for x_2 and then for x_1 independently. Check that you got the same solution as we did above.

Let us describe the general algorithm. Suppose that λ is an eigenvalue of multiplicity 2, defect 1. First find an eigenvector $\vec{v}_1$ of λ . That is, $\vec{v}_1$ solves $(A - \lambda I)\vec{v}_1 = \vec{0}$. Then, find a vector $\vec{v}_2$ such that

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1.$$

This gives us two linearly independent solutions

$$\begin{aligned}\vec{x}_1 &= \vec{v}_1 e^{\lambda t}, \\ \vec{x}_2 &= (\vec{v}_2 + \vec{v}_1 t) e^{\lambda t}.\end{aligned}$$

Example 14.7.2: Consider the system

$$\vec{x}' = \begin{bmatrix} 2 & -5 & 0 \\ 0 & 2 & 0 \\ -1 & 4 & 1 \end{bmatrix} \vec{x}.$$

Compute the eigenvalues,

$$0 = \det(A - \lambda I) = \det \left(\begin{bmatrix} 2-\lambda & -5 & 0 \\ 0 & 2-\lambda & 0 \\ -1 & 4 & 1-\lambda \end{bmatrix} \right) = (2-\lambda)^2(1-\lambda).$$

The eigenvalues are 1 and 2, where 2 has multiplicity 2. We leave it to the reader to find that $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1$.

Let us focus on $\lambda = 2$. We compute eigenvectors:

$$\vec{0} = (A - 2I)\vec{v} = \begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The first equation says that $v_2 = 0$, so the last equation is $-v_1 - v_3 = 0$. Let v_3 be the free variable to find that $v_1 = -v_3$. Perhaps let $v_3 = -1$ to find an eigenvector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Problem is that setting v_3 to anything else just gets multiples of this vector and so we have a defect of 1. Let $\vec{v}_1$ be the eigenvector and let us look for a generalized eigenvector $\vec{v}_2$:

$$(A - 2I)\vec{v}_2 = \vec{v}_1,$$

or

$$\begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

where we used a, b, c as components of $\vec{v}_2$ for simplicity. The first equation says $-5b = 1$ so $b = -1/5$. The second equation says nothing. The last equation is $-a + 4b - c = -1$, or $a + 4/5 + c = 1$, or $a + c = 1/5$. We let c be the free variable and we choose $c = 0$. We find $\vec{v}_2 = \begin{bmatrix} 1/5 \\ -1/5 \\ 0 \end{bmatrix}$.

The general solution is therefore,

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 1/5 \\ -1/5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} t \right) e^{2t}.$$

This machinery can also be generalized to higher multiplicities and higher defects. We will not go over this method in detail, but let us just sketch the ideas. Suppose that A has an eigenvalue λ of multiplicity m . We find vectors such that

$$(A - \lambda I)^k \vec{v} = \vec{0}, \quad \text{but} \quad (A - \lambda I)^{k-1} \vec{v} \neq \vec{0}.$$

Such vectors are called *generalized eigenvectors* (then $\vec{v}_1 = (A - \lambda I)^{k-1} \vec{v}$ is an eigenvector). For the eigenvector $\vec{v}_1$ there is a chain of generalized eigenvectors $\vec{v}_2$ through $\vec{v}_k$ such that:

$$\begin{aligned} (A - \lambda I) \vec{v}_1 &= \vec{0}, \\ (A - \lambda I) \vec{v}_2 &= \vec{v}_1, \\ &\vdots \\ (A - \lambda I) \vec{v}_k &= \vec{v}_{k-1}. \end{aligned}$$

Really once you find the $\vec{v}_k$ such that $(A - \lambda I)^k \vec{v}_k = \vec{0}$ but $(A - \lambda I)^{k-1} \vec{v}_k \neq \vec{0}$, you find the entire chain since you can compute the rest, $\vec{v}_{k-1} = (A - \lambda I) \vec{v}_k$, $\vec{v}_{k-2} = (A - \lambda I) \vec{v}_{k-1}$, etc. We form the linearly independent solutions

$$\begin{aligned} \vec{x}_1 &= \vec{v}_1 e^{\lambda t}, \\ \vec{x}_2 &= (\vec{v}_2 + \vec{v}_1 t) e^{\lambda t}, \\ &\vdots \\ \vec{x}_k &= \left(\vec{v}_k + \vec{v}_{k-1} t + \vec{v}_{k-2} \frac{t^2}{2} + \cdots + \vec{v}_2 \frac{t^{k-2}}{(k-2)!} + \vec{v}_1 \frac{t^{k-1}}{(k-1)!} \right) e^{\lambda t}. \end{aligned}$$

Recall that $k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k$ is the factorial. If you have an eigenvalue of geometric multiplicity ℓ , you will have to find ℓ such chains (some of them might be short: just the single eigenvector equation). We go until we form m linearly independent solutions where m is the algebraic multiplicity. We don't quite know which specific eigenvectors go with which chain, so start by finding $\vec{v}_k$ first for the longest possible chain and go from there.

For example, if λ is an eigenvalue of A of algebraic multiplicity 3 and defect 2, then solve

$$(A - \lambda I) \vec{v}_1 = \vec{0}, \quad (A - \lambda I) \vec{v}_2 = \vec{v}_1, \quad (A - \lambda I) \vec{v}_3 = \vec{v}_2.$$

That is, find $\vec{v}_3$ such that $(A - \lambda I)^3 \vec{v}_3 = \vec{0}$, but $(A - \lambda I)^2 \vec{v}_3 \neq \vec{0}$. Then you are done as $\vec{v}_2 = (A - \lambda I) \vec{v}_3$ and $\vec{v}_1 = (A - \lambda I) \vec{v}_2$. The 3 linearly independent solutions are

$$\vec{x}_1 = \vec{v}_1 e^{\lambda t}, \quad \vec{x}_2 = (\vec{v}_2 + \vec{v}_1 t) e^{\lambda t}, \quad \vec{x}_3 = \left(\vec{v}_3 + \vec{v}_2 t + \vec{v}_1 \frac{t^2}{2} \right) e^{\lambda t}.$$

If, on the other hand, A has an eigenvalue λ of algebraic multiplicity 3 and defect 1, then solve

$$(A - \lambda I) \vec{v}_1 = \vec{0}, \quad (A - \lambda I) \vec{v}_2 = \vec{0}, \quad (A - \lambda I) \vec{v}_3 = \vec{v}_2.$$

Here $\vec{v}_1$ and $\vec{v}_2$ are actual honest eigenvectors, and $\vec{v}_3$ is a generalized eigenvector. So there are two chains. To solve, first find a $\vec{v}_3$ such that $(A - \lambda I)^2 \vec{v}_3 = \vec{0}$, but $(A - \lambda I) \vec{v}_3 \neq \vec{0}$. Then $\vec{v}_2 = (A - \lambda I) \vec{v}_3$ is going to be an eigenvector. Then solve for an eigenvector $\vec{v}_1$ that is linearly independent from $\vec{v}_2$. You get 3 linearly independent solutions

$$\vec{x}_1 = \vec{v}_1 e^{\lambda t}, \quad \vec{x}_2 = \vec{v}_2 e^{\lambda t}, \quad \vec{x}_3 = (\vec{v}_3 + \vec{v}_2 t) e^{\lambda t}.$$

14.7.3 Exercises

Exercise 14.7.2: Let $A = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$. Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 14.7.3: Let $A = \begin{bmatrix} 5 & -4 & 4 \\ 0 & 3 & 0 \\ -2 & 4 & -1 \end{bmatrix}$.

- What are the eigenvalues?
- What is/are the defect(s) of the eigenvalue(s)?
- Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 14.7.4: Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

- What are the eigenvalues?
- What is/are the defect(s) of the eigenvalue(s)?
- Find the general solution of $\vec{x}' = A\vec{x}$ in two different ways and verify you get the same answer.

Exercise 14.7.5: Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -2 & -2 \\ -4 & 4 & 7 \end{bmatrix}$.

- What are the eigenvalues?
- What is/are the defect(s) of the eigenvalue(s)?
- Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 14.7.6: Let $A = \begin{bmatrix} 0 & 4 & -2 \\ -1 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

- What are the eigenvalues?
- What is/are the defect(s) of the eigenvalue(s)?
- Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 14.7.7: Let $A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix}$.

- What are the eigenvalues?
- What is/are the defect(s) of the eigenvalue(s)?
- Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 14.7.8: Suppose that A is a 2×2 matrix with a repeated eigenvalue λ . Suppose that there are two linearly independent eigenvectors. Show that $A = \lambda I$.

Exercise 14.7.101: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

- What are the eigenvalues?
- What is/are the defect(s) of the eigenvalue(s)?
- Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 14.7.102: Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$.

- What are the eigenvalues?
- What is/are the defect(s) of the eigenvalue(s)?
- Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 14.7.103: Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 9 \\ 0 & -1 & 5 \end{bmatrix}$.

- a) What are the eigenvalues?
- b) What is/are the defect(s) of the eigenvalue(s)?
- c) Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise 14.7.104: Let $A = \begin{bmatrix} a & a \\ b & c \end{bmatrix}$, where a , b , and c are unknowns. Suppose that 5 is a doubled eigenvalue of defect 1, and suppose that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a corresponding eigenvector. Find A and show that there is only one such matrix A .

14.8 Matrix Exponentials

14.8.1 Definition

There is another way of finding a fundamental matrix solution of a system. Consider the constant coefficient equation

$$\vec{x}' = P\vec{x}.$$

If this would be just one equation (when P is a number or a 1×1 matrix), then the solution would be

$$\vec{x} = e^{Pt}.$$

That doesn't make sense if P is a larger matrix, but essentially the same computation that led to the above works for matrices when we define e^{Pt} properly. First let us write down the Taylor series for e^{at} for some number a :

$$e^{at} = 1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \frac{(at)^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}.$$

Recall $k! = 1 \cdot 2 \cdot 3 \cdots k$ is the factorial, and $0! = 1$. We differentiate this series term by term

$$\frac{d}{dt}(e^{at}) = 0 + a + a^2t + \frac{a^3t^2}{2} + \frac{a^4t^3}{6} + \cdots = a \left(1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \cdots \right) = ae^{at}.$$

Maybe we can try the same trick with matrices. For an $n \times n$ matrix A we define the *matrix exponential* as

$$e^A \stackrel{\text{def}}{=} I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{k!}A^k + \cdots$$

Let us not worry about convergence. The series really does always converge. We usually write Pt as tP by convention when P is a matrix. With this small change and by the exact same calculation as above we have that

$$\frac{d}{dt}(e^{tP}) = Pe^{tP}.$$

Now P and hence e^{tP} is an $n \times n$ matrix. What we are looking for is a vector. In the 1×1 case we would at this point multiply by an arbitrary constant to get the general solution. In the matrix case we multiply by a column vector $\vec{c}$.

Theorem 14.8.1. *Let P be an $n \times n$ matrix. Then the general solution to $\vec{x}' = P\vec{x}$ is*

$$\vec{x} = e^{tP}\vec{c},$$

where $\vec{c}$ is an arbitrary constant vector. In fact, $\vec{x}(0) = \vec{c}$.

Let us check:

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}(e^{tP}\vec{c}) = Pe^{tP}\vec{c} = P\vec{x}.$$

Hence e^{tP} is a fundamental matrix solution of the homogeneous system. So if we can compute the matrix exponential, we have another method of solving constant coefficient homogeneous systems. It also makes it easy to solve for initial conditions. To solve $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{b}$, we take the solution

$$\vec{x} = e^{tA}\vec{b}.$$

This equation follows because $e^{0A} = I$, so $\vec{x}(0) = e^{0A}\vec{b} = \vec{b}$.

We mention a drawback of matrix exponentials. In general $e^{A+B} \neq e^A e^B$. The trouble is that matrices do not commute, that is, in general $AB \neq BA$. If you try to prove $e^{A+B} \neq e^A e^B$ using the Taylor series, you will see why the lack of commutativity becomes a problem. However, it is still true that if $AB = BA$, that is, if A and B commute, then $e^{A+B} = e^A e^B$. We will find this fact useful. Let us restate this as a theorem to make a point.

Theorem 14.8.2. If $AB = BA$, then $e^{A+B} = e^A e^B$. Otherwise, $e^{A+B} \neq e^A e^B$ in general.

14.8.2 Simple Cases

In some instances it may work to just plug into the series definition. Suppose the matrix is diagonal. For example, $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then

$$D^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix},$$

and

$$\begin{aligned} e^D &= I + D + \frac{1}{2}D^2 + \frac{1}{6}D^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix} + \cdots = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}. \end{aligned}$$

So by this rationale

$$e^I = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} \quad \text{and} \quad e^{aI} = \begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix}.$$

This makes exponentials of certain other matrices easy to compute. For example, the matrix $A = \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$ can be written as $3I + B$ where $B = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$. Notice that $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So $B^k = 0$ for all $k \geq 2$. Therefore, $e^B = I + B$. Suppose we actually want to compute e^{tA} . The matrices $3tI$ and tB commute (exercise: check this) and $e^{tB} = I + tB$, since $(tB)^2 = t^2 B^2 = 0$. We write

$$\begin{aligned} e^{tA} &= e^{3tI+tB} = e^{3tI} e^{tB} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} (I + tB) = \\ &= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} = \begin{bmatrix} (1+2t)e^{3t} & 4te^{3t} \\ -te^{3t} & (1-2t)e^{3t} \end{bmatrix}. \end{aligned}$$

We found a fundamental matrix solution for the system $\vec{x}' = A\vec{x}$. Note that this matrix has a repeated eigenvalue with a defect; there is only one eigenvector for the eigenvalue 3. So we found a perhaps easier way to handle this case. In fact, if a matrix A is 2×2 and has an eigenvalue λ of multiplicity 2, then either $A = \lambda I$, or $A = \lambda I + B$ where $B^2 = 0$. This is a good exercise.

Exercise 14.8.1: Suppose that A is 2×2 and λ is the only eigenvalue. Show that $(A - \lambda I)^2 = 0$, and therefore that we can write $A = \lambda I + B$, where $B^2 = 0$ (and possibly $B = 0$). Hint: First write down what does it mean for the eigenvalue to be of multiplicity 2. You will get an equation for the entries. Now compute the square of B .

Matrices B such that $B^k = 0$ for some k are called *nilpotent*. Computation of the matrix exponential for nilpotent matrices is easy by just writing down the first k terms of the Taylor series.

14.8.3 General Matrices

In general, the exponential is not as easy to compute as above. We usually cannot write a matrix as a sum of commuting matrices where the exponential is simple for each one. But fear not, it is still not too difficult provided we can find enough eigenvectors. First we need the following interesting result about matrix exponentials. For two square matrices A and B , with B invertible, we have

$$e^{BAB^{-1}} = B e^A B^{-1}.$$

This can be seen by writing down the Taylor series. First

$$(BAB^{-1})^2 = BAB^{-1}BAB^{-1} = B A I A B^{-1} = B A^2 B^{-1}.$$

And by the same reasoning $(BAB^{-1})^k = BA^k B^{-1}$. Now write the Taylor series for $e^{BAB^{-1}}$:

$$\begin{aligned} e^{BAB^{-1}} &= I + BAB^{-1} + \frac{1}{2}(BAB^{-1})^2 + \frac{1}{6}(BAB^{-1})^3 + \cdots \\ &= BB^{-1} + BAB^{-1} + \frac{1}{2}BA^2B^{-1} + \frac{1}{6}BA^3B^{-1} + \cdots \\ &= B\left(I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots\right)B^{-1} \\ &= Be^A B^{-1}. \end{aligned}$$

Given a square matrix A , we can usually write $A = EDE^{-1}$, where D is diagonal and E invertible. This procedure is called *diagonalization*. If we can do that, the computation of the exponential becomes easy as e^D is just taking the exponential of the entries on the diagonal. Adding t into the mix, we can then compute the exponential

$$e^{tA} = Ee^{tD}E^{-1}.$$

To diagonalize A we need n linearly independent eigenvectors of A . Otherwise, this method of computing the exponential does not work and we need to be trickier, but we will not get into such details. Let E be the matrix with the eigenvectors as columns. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues and let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be the eigenvectors, then $E = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$. Make a diagonal matrix D with the eigenvalues on the diagonal:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

We compute

$$\begin{aligned} AE &= A[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] \\ &= [A\vec{v}_1 \ A\vec{v}_2 \ \cdots \ A\vec{v}_n] \\ &= [\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \ \cdots \ \lambda_n\vec{v}_n] \\ &= [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]D \\ &= ED. \end{aligned}$$

The columns of E are linearly independent as these are linearly independent eigenvectors of A . Hence E is invertible. Since $AE = ED$, we multiply on the right by E^{-1} and we get

$$A = EDE^{-1}.$$

This means that $e^A = Ee^DE^{-1}$. Multiplying the matrix by t we obtain

$$e^{tA} = Ee^{tD}E^{-1} = E \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} E^{-1}. \quad (14.5)$$

The formula (14.5), therefore, gives the formula for computing a fundamental matrix solution e^{tA} for the system $\vec{x}' = A\vec{x}$, in the case where we have n linearly independent eigenvectors.

This computation still works when the eigenvalues and eigenvectors are complex, though then you have to compute with complex numbers. It is clear from the definition that if A is real, then e^{tA} is real. So you will only need complex numbers in the computation and not for the result. You may need to apply [Euler's formula](#) to simplify the result. If simplified properly, the final matrix will not have any complex numbers in it.

Example 14.8.1: Compute a fundamental matrix solution using the matrix exponential for the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then compute the particular solution for the initial conditions $x(0) = 4$ and $y(0) = 2$.

Let A be the coefficient matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. We first compute (exercise) that the eigenvalues are 3 and -1 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence the diagonalization of A is

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_E \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}_{E^{-1}}.$$

We write

$$\begin{aligned} e^{tA} &= E e^{tD} E^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{-1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} -e^{3t} - e^{-t} & -e^{3t} + e^{-t} \\ -e^{3t} + e^{-t} & -e^{3t} - e^{-t} \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix}. \end{aligned}$$

The initial conditions are $x(0) = 4$ and $y(0) = 2$. Hence, by the property that $e^{0A} = I$ we find that the particular solution we are looking for is $e^{tA}\vec{b}$ where $\vec{b}$ is $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Then the particular solution we are looking for is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2e^{3t} + 2e^{-t} + e^{3t} - e^{-t} \\ 2e^{3t} - 2e^{-t} + e^{3t} + e^{-t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} + e^{-t} \\ 3e^{3t} - e^{-t} \end{bmatrix}.$$

14.8.4 Fundamental Matrix Solutions

We note that if you can compute a fundamental matrix solution in a different way, you can use this to find the matrix exponential e^{tA} . A fundamental matrix solution of a system of ODEs is not unique. The exponential is the fundamental matrix solution with the property that for $t = 0$ we get the identity matrix. So we must find the right fundamental matrix solution. Let X be any fundamental matrix solution to $\vec{x}' = A\vec{x}$. Then we claim

$$e^{tA} = X(t) [X(0)]^{-1}.$$

Clearly, if we plug $t = 0$ into $X(t) [X(0)]^{-1}$ we get the identity. We can multiply a fundamental matrix solution on the right by any constant invertible matrix and we still get a fundamental matrix solution. All we are doing is changing what are the arbitrary constants in the general solution $\vec{x}(t) = X(t) \vec{c}$.

14.8.5 Approximations

If you think about it, the computation of any fundamental matrix solution X using the eigenvalue method is just as difficult as the computation of e^{tA} . So perhaps we did not gain much by this new tool. However, the Taylor series expansion actually gives us a way to approximate solutions, which the eigenvalue method did not.

The simplest thing we can do is to just compute the series up to a certain number of terms. There are better ways to approximate the exponential*. In many cases, however, few terms of the Taylor series give a reasonable approximation for the exponential and may suffice for the application. For example, let us compute the first 4 terms of the series for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

$$\begin{aligned} e^{tA} &\approx I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 = I + t \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + t^2 \begin{bmatrix} \frac{5}{2} & 2 \\ 2 & \frac{5}{2} \end{bmatrix} + t^3 \begin{bmatrix} \frac{13}{6} & \frac{7}{3} \\ \frac{7}{3} & \frac{13}{6} \end{bmatrix} = \\ &= \begin{bmatrix} 1 + t + \frac{5}{2}t^2 + \frac{13}{6}t^3 & 2t + 2t^2 + \frac{7}{3}t^3 \\ 2t + 2t^2 + \frac{7}{3}t^3 & 1 + t + \frac{5}{2}t^2 + \frac{13}{6}t^3 \end{bmatrix}. \end{aligned}$$

Just like the scalar version of the Taylor series approximation, the approximation will be better for small t and worse for larger t . For larger t , we will generally have to compute more terms. Let us see how we stack up against the real solution with $t = 0.1$. The approximate solution is approximately (rounded to 8 decimal places)

$$e^{0.1A} \approx I + 0.1A + \frac{0.1^2}{2}A^2 + \frac{0.1^3}{6}A^3 = \begin{bmatrix} 1.12716667 & 0.22233333 \\ 0.22233333 & 1.12716667 \end{bmatrix}.$$

And plugging $t = 0.1$ into the real solution (rounded to 8 decimal places) we get

$$e^{0.1A} = \begin{bmatrix} 1.12734811 & 0.22251069 \\ 0.22251069 & 1.12734811 \end{bmatrix}.$$

Not bad at all! Although if we take the same approximation for $t = 1$ we get

$$I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 = \begin{bmatrix} 6.66666667 & 6.33333333 \\ 6.33333333 & 6.66666667 \end{bmatrix},$$

while the real value is (again rounded to 8 decimal places)

$$e^A = \begin{bmatrix} 10.22670818 & 9.85882874 \\ 9.85882874 & 10.22670818 \end{bmatrix}.$$

So the approximation is not very good once we get up to $t = 1$. To get a good approximation at $t = 1$ (say up to 2 decimal places) we would need to go up to the 11th power (exercise).

14.8.6 Exercises

Exercise 14.8.2: Using the matrix exponential, find a fundamental matrix solution for the system $x' = 3x + y$, $y' = x + 3y$.

Exercise 14.8.3: Find e^{tA} for the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

Exercise 14.8.4: Find a fundamental matrix solution for the system $x'_1 = 7x_1 + 4x_2 + 12x_3$, $x'_2 = x_1 + 2x_2 + x_3$, $x'_3 = -3x_1 - 2x_2 - 5x_3$. Then find the solution that satisfies $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

Exercise 14.8.5: Compute the matrix exponential e^A for $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Exercise 14.8.6 (challenging): Suppose $AB = BA$. Show that under this assumption, $e^{A+B} = e^A e^B$.

Exercise 14.8.7: Use [Exercise 14.8.6](#) to show that $(e^A)^{-1} = e^{-A}$. In particular, e^A is invertible even if A is not.

Exercise 14.8.8: Let A be a 2×2 matrix with eigenvalues $-1, 1$, and corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

*C. Moler and C.F. Van Loan, *Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*, SIAM Review 45 (1), 2003, 3–49

- a) Find matrix A with these properties.
- b) Find a fundamental matrix solution to $\vec{x}' = A\vec{x}$.
- c) Solve the system in with initial conditions $\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Exercise 14.8.9: Suppose that A is an $n \times n$ matrix with a repeated eigenvalue λ of multiplicity n with n linearly independent eigenvectors. Show that the matrix is diagonal, in fact, $A = \lambda I$. Hint: Use diagonalization and the fact that the identity matrix commutes with every other matrix.

Exercise 14.8.10: Let $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$.

- a) Find e^{tA} .
- b) Solve $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Exercise 14.8.11: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Approximate e^{tA} by expanding the power series up to the third order.

Exercise 14.8.12: For any positive integer n , find a formula (or a recipe) for A^n for the following matrices:

- a) $\begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}$
- b) $\begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$
- c) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
- d) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Exercise 14.8.101: Compute e^{tA} where $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

Exercise 14.8.102: Compute e^{tA} where $A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 1 & 2 \\ -1 & -3 & 4 \end{bmatrix}$.

Exercise 14.8.103:

- a) Compute e^{tA} where $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.
- b) Solve $\vec{x}' = A\vec{x}$ for $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Exercise 14.8.104: Compute the first 3 terms (up to the second degree) of the Taylor expansion of e^{tA} where $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$. Write it as a single matrix. Then use it to approximate $e^{0.1A}$.

Exercise 14.8.105: For any positive integer n , find a formula (or a recipe) for A^n for the following matrices:

- a) $\begin{bmatrix} 7 & 4 \\ -5 & -2 \end{bmatrix}$
- b) $\begin{bmatrix} -3 & 4 \\ -6 & 7 \end{bmatrix}$
- c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

14.9 Nonhomogeneous Systems

14.9.1 First Order Constant Coefficient

Integrating factor

Let us first focus on the nonhomogeneous first order equation

$$\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t),$$

where A is a constant matrix. The first method we look at is the *integrating factor method*. For simplicity we rewrite the equation as

$$\vec{x}'(t) + P\vec{x}(t) = \vec{f}(t),$$

where $P = -A$. We multiply both sides of the equation by e^{tP} (being mindful that we are dealing with matrices that may not commute) to obtain

$$e^{tP}\vec{x}'(t) + e^{tP}P\vec{x}(t) = e^{tP}\vec{f}(t).$$

We notice that $Pe^{tP} = e^{tP}P$. This fact follows by writing down the series definition of e^{tP} :

$$\begin{aligned} Pe^{tP} &= P \left(I + tP + \frac{1}{2}(tP)^2 + \cdots \right) = P + tP^2 + \frac{1}{2}t^2P^3 + \cdots = \\ &= \left(I + tP + \frac{1}{2}(tP)^2 + \cdots \right) P = e^{tP}P. \end{aligned}$$

So $\frac{d}{dt}(e^{tP}) = Pe^{tP} = e^{tP}P$. The product rule says

$$\frac{d}{dt}(e^{tP}\vec{x}(t)) = e^{tP}\vec{x}'(t) + e^{tP}P\vec{x}(t),$$

and so

$$\frac{d}{dt}(e^{tP}\vec{x}(t)) = e^{tP}\vec{f}(t).$$

We can now integrate. That is, we integrate each component of the vector separately

$$e^{tP}\vec{x}(t) = \int e^{tP}\vec{f}(t) dt + \vec{c}.$$

Recall from [Exercise 14.8.7](#) that $(e^{tP})^{-1} = e^{-tP}$. Therefore, we obtain

$$\vec{x}(t) = e^{-tP} \int e^{tP}\vec{f}(t) dt + e^{-tP}\vec{c}.$$

Perhaps it is better understood as a definite integral. In this case it will be easy to also solve for the initial conditions. Consider the equation with initial conditions

$$\vec{x}'(t) + P\vec{x}(t) = \vec{f}(t), \quad \vec{x}(0) = \vec{b}.$$

The solution can then be written as

$$\boxed{\vec{x}(t) = e^{-tP} \int_0^t e^{sP}\vec{f}(s) ds + e^{-tP}\vec{b}.} \quad (14.6)$$

Again, the integration means that each component of the vector $e^{sP}\vec{f}(s)$ is integrated separately. It is not hard to see that (14.6) really does satisfy the initial condition $\vec{x}(0) = \vec{b}$.

$$\vec{x}(0) = e^{-0P} \int_0^0 e^{sP} \vec{f}(s) ds + e^{-0P} \vec{b} = I\vec{b} = \vec{b}.$$

Example 14.9.1: Suppose that we have the system

$$\begin{aligned} x_1' + 5x_1 - 3x_2 &= e^t, \\ x_2' + 3x_1 - x_2 &= 0, \end{aligned}$$

with initial conditions $x_1(0) = 1, x_2(0) = 0$.

Let us write the system as

$$\vec{x}' + \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The matrix $P = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ has a doubled eigenvalue 2 with defect 1, and we leave it as an exercise to double check we computed e^{tP} correctly. Once we have e^{tP} , we find e^{-tP} , simply by negating t .

$$e^{tP} = \begin{bmatrix} (1+3t)e^{2t} & -3te^{2t} \\ 3te^{2t} & (1-3t)e^{2t} \end{bmatrix}, \quad e^{-tP} = \begin{bmatrix} (1-3t)e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)e^{-2t} \end{bmatrix}.$$

Instead of computing the whole formula at once, let us do it in stages. First

$$\begin{aligned} \int_0^t e^{sP} \vec{f}(s) ds &= \int_0^t \begin{bmatrix} (1+3s)e^{2s} & -3se^{2s} \\ 3se^{2s} & (1-3s)e^{2s} \end{bmatrix} \begin{bmatrix} e^s \\ 0 \end{bmatrix} ds \\ &= \int_0^t \begin{bmatrix} (1+3s)e^{3s} \\ 3se^{3s} \end{bmatrix} ds \\ &= \begin{bmatrix} \int_0^t (1+3s)e^{3s} ds \\ \int_0^t 3se^{3s} ds \end{bmatrix} \\ &= \begin{bmatrix} \frac{te^{3t}}{(3t-1)e^{3t}+1} \end{bmatrix} \quad (\text{used integration by parts}). \end{aligned}$$

Then

$$\begin{aligned} \vec{x}(t) &= e^{-tP} \int_0^t e^{sP} \vec{f}(s) ds + e^{-tP} \vec{b} \\ &= \begin{bmatrix} (1-3t)e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)e^{-2t} \end{bmatrix} \begin{bmatrix} \frac{te^{3t}}{(3t-1)e^{3t}+1} \end{bmatrix} + \begin{bmatrix} (1-3t)e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} te^{-2t} \\ -\frac{e^t}{3} + (\frac{1}{3} + t)e^{-2t} \end{bmatrix} + \begin{bmatrix} (1-3t)e^{-2t} \\ -3te^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} (1-2t)e^{-2t} \\ -\frac{e^t}{3} + (\frac{1}{3} - 2t)e^{-2t} \end{bmatrix}. \end{aligned}$$

Phew!

Let us check that this really works.

$$x_1' + 5x_1 - 3x_2 = (4te^{-2t} - 4e^{-2t}) + 5(1-2t)e^{-2t} + e^t - (1-6t)e^{-2t} = e^t.$$

Similarly (exercise) $x_2' + 3x_1 - x_2 = 0$. The initial conditions are also satisfied (exercise).

For systems, the integrating factor method only works if P does not depend on t , that is, P is constant. The problem is that in general

$$\frac{d}{dt} \left[e^{\int P(t) dt} \right] \neq P(t) e^{\int P(t) dt},$$

because matrix multiplication is not commutative.

Eigenvector decomposition

For the next method, note that eigenvectors of a matrix give the directions in which the matrix acts like a scalar. If we solve the system along these directions, the computations are simpler as we treat the matrix as a scalar. We then put those solutions together to get the general solution for the system.

Take the equation

$$\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t). \quad (14.7)$$

Assume A has n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Write

$$\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \dots + \vec{v}_n \xi_n(t). \quad (14.8)$$

That is, we wish to write our solution as a linear combination of eigenvectors of A . If we solve for the scalar functions ξ_1 through ξ_n , we have our solution $\vec{x}$. Let us decompose $\vec{f}$ in terms of the eigenvectors as well. We wish to write

$$\vec{f}(t) = \vec{v}_1 g_1(t) + \vec{v}_2 g_2(t) + \dots + \vec{v}_n g_n(t). \quad (14.9)$$

That is, we wish to find g_1 through g_n that satisfy (14.9). Since all the eigenvectors are independent, the matrix $E = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$ is invertible. Write the equation (14.9) as $\vec{f} = E\vec{g}$, where the components of $\vec{g}$ are the functions g_1 through g_n . Then $\vec{g} = E^{-1}\vec{f}$. Hence it is always possible to find $\vec{g}$ when there are n linearly independent eigenvectors.

We plug (14.8) into (14.7), and note that $A\vec{v}_k = \lambda_k \vec{v}_k$:

$$\begin{aligned} \overbrace{\vec{v}_1 \xi_1' + \vec{v}_2 \xi_2' + \dots + \vec{v}_n \xi_n'}^{\vec{x}'} &= \overbrace{A(\vec{v}_1 \xi_1 + \vec{v}_2 \xi_2 + \dots + \vec{v}_n \xi_n)}^{A\vec{x}} + \overbrace{\vec{v}_1 g_1 + \vec{v}_2 g_2 + \dots + \vec{v}_n g_n}^{\vec{f}} \\ &= A\vec{v}_1 \xi_1 + A\vec{v}_2 \xi_2 + \dots + A\vec{v}_n \xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \dots + \vec{v}_n g_n \\ &= \vec{v}_1 \lambda_1 \xi_1 + \vec{v}_2 \lambda_2 \xi_2 + \dots + \vec{v}_n \lambda_n \xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \dots + \vec{v}_n g_n \\ &= \vec{v}_1 (\lambda_1 \xi_1 + g_1) + \vec{v}_2 (\lambda_2 \xi_2 + g_2) + \dots + \vec{v}_n (\lambda_n \xi_n + g_n). \end{aligned}$$

If we identify the coefficients of the vectors $\vec{v}_1$ through $\vec{v}_n$, we get the equations

$$\begin{aligned} \xi_1' &= \lambda_1 \xi_1 + g_1, \\ \xi_2' &= \lambda_2 \xi_2 + g_2, \\ &\vdots \\ \xi_n' &= \lambda_n \xi_n + g_n. \end{aligned}$$

Each one of these equations is independent of the others. They are all linear first order equations and can easily be solved by the standard integrating factor method for single equations. That is, for the k^{th} equation we write

$$\xi_k'(t) - \lambda_k \xi_k(t) = g_k(t).$$

We use the integrating factor $e^{-\lambda_k t}$ to find that

$$\frac{d}{dt} [\xi_k(t) e^{-\lambda_k t}] = e^{-\lambda_k t} g_k(t).$$

We integrate and solve for ξ_k to get

$$\xi_k(t) = e^{\lambda_k t} \int e^{-\lambda_k t} g_k(t) dt + C_k e^{\lambda_k t}.$$

If we are looking for just any particular solution, we can set C_k to be zero. If we leave these constants in, we get the general solution. Write $\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \dots + \vec{v}_n \xi_n(t)$, and we are done.

As always, it is perhaps better to write these integrals as definite integrals. Suppose that we have an initial condition $\vec{x}(0) = \vec{b}$. Take $\vec{a} = E^{-1}\vec{b}$ to find $\vec{b} = \vec{v}_1 a_1 + \vec{v}_2 a_2 + \cdots + \vec{v}_n a_n$, just like before. Then if we write

$$\xi_k(t) = e^{\lambda_k t} \int_0^t e^{-\lambda_k s} g_k(s) ds + a_k e^{\lambda_k t},$$

we get the particular solution $\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \cdots + \vec{v}_n \xi_n(t)$ satisfying $\vec{x}(0) = \vec{b}$, because $\xi_k(0) = a_k$.

We remark that the technique we just outlined is the eigenvalue method applied to nonhomogeneous systems. If a system is homogeneous, that is, if $\vec{f} = \vec{0}$, then the equations we get are $\xi'_k = \lambda_k \xi_k$, and so $\xi_k = C_k e^{\lambda_k t}$ are the solutions and that is precisely what we got in § 14.4.

Example 14.9.2: Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. Solve $\vec{x}' = A\vec{x} + \vec{f}$ where $\vec{f}(t) = \begin{bmatrix} 2e^t \\ 2t \end{bmatrix}$ for $\vec{x}(0) = \begin{bmatrix} 3/16 \\ -5/16 \end{bmatrix}$.

The eigenvalues of A are -2 and 4 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively. This calculation is left as an exercise. We write down the matrix E of the eigenvectors and compute its inverse (using the inverse formula for 2×2 matrices)

$$E = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad E^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

We are looking for a solution of the form $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xi_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_2$. We first need to write $\vec{f}$ in terms of the eigenvectors. That is we wish to write $\vec{f} = \begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} g_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} g_2$. Thus

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1} \begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \begin{bmatrix} e^t - t \\ e^t + t \end{bmatrix}.$$

So $g_1 = e^t - t$ and $g_2 = e^t + t$.

We further need to write $\vec{x}(0)$ in terms of the eigenvectors. That is, we wish to write $\vec{x}(0) = \begin{bmatrix} 3/16 \\ -5/16 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_2$. Hence

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = E^{-1} \begin{bmatrix} 3/16 \\ -5/16 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -1/16 \end{bmatrix}.$$

So $a_1 = 1/4$ and $a_2 = -1/16$. We plug our $\vec{x}$ into the equation and get

$$\begin{aligned} \overbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \xi'_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi'_2}^{\vec{x}'} &= \overbrace{A \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xi_1 + A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_2}^{A\vec{x}} + \overbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix} g_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} g_2}^{\vec{f}} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-2\xi_1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 4\xi_2 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (e^t - t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (e^t + t). \end{aligned}$$

We get the two equations

$$\begin{aligned} \xi'_1 &= -2\xi_1 + e^t - t, & \text{where } \xi_1(0) &= a_1 = \frac{1}{4}, \\ \xi'_2 &= 4\xi_2 + e^t + t, & \text{where } \xi_2(0) &= a_2 = -\frac{1}{16}. \end{aligned}$$

We solve with integrating factor. Computation of the integral is left as an exercise to the student. You will need integration by parts.

$$\xi_1 = e^{-2t} \int e^{2t} (e^t - t) dt + C_1 e^{-2t} = \frac{e^t}{3} - \frac{t}{2} + \frac{1}{4} + C_1 e^{-2t}.$$

C_1 is the constant of integration. As $\xi_1(0) = 1/4$, then $1/4 = 1/3 + 1/4 + C_1$ and hence $C_1 = -1/3$. Similarly

$$\xi_2 = e^{4t} \int e^{-4t} (e^t + t) dt + C_2 e^{4t} = -\frac{e^t}{3} - \frac{t}{4} - \frac{1}{16} + C_2 e^{4t}.$$

As $\xi_2(0) = -1/16$ we have $-1/16 = -1/3 - 1/16 + C_2$ and hence $C_2 = 1/3$. The solution is

$$\vec{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \underbrace{\left(\frac{e^t - e^{-2t}}{3} + \frac{1 - 2t}{4} \right)}_{\xi_1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \underbrace{\left(\frac{e^{4t} - e^t}{3} - \frac{4t + 1}{16} \right)}_{\xi_2} = \begin{bmatrix} \frac{e^{4t} - e^{-2t}}{3} + \frac{3 - 12t}{16} \\ \frac{e^{-2t} + e^{4t} - 2e^t}{3} + \frac{4t - 5}{16} \end{bmatrix}.$$

That is, $x_1 = \frac{e^{4t} - e^{-2t}}{3} + \frac{3 - 12t}{16}$ and $x_2 = \frac{e^{-2t} + e^{4t} - 2e^t}{3} + \frac{4t - 5}{16}$.

Exercise 14.9.1: Check that x_1 and x_2 solve the problem. Check both that they satisfy the differential equation and that they satisfy the initial conditions.

Undetermined coefficients

The method of undetermined coefficients also works for systems. The only difference is that we use unknown vectors rather than just numbers. Same caveats apply to undetermined coefficients for systems as for single equations. This method does not always work. Furthermore, if the right-hand side is complicated, we have to solve for lots of variables. Each element of an unknown vector is an unknown number. In system of 3 equations with say 4 unknown vectors (this would not be uncommon), we already have 12 unknown numbers to solve for. The method can turn into a lot of tedious work if done by hand. As the method is essentially the same as for single equations, let us just do an example.

Example 14.9.3: Let $A = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$. Find a particular solution of $\vec{x}' = A\vec{x} + \vec{f}$ where $\vec{f}(t) = \begin{bmatrix} e^t \\ t \end{bmatrix}$.

Note that we can solve this system in an easier way (can you see how?), but for the purposes of the example, let us use the eigenvalue method plus undetermined coefficients. The eigenvalues of A are -1 and 1 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. Hence our complementary solution is

$$\vec{x}_c = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t,$$

for some arbitrary constants α_1 and α_2 .

We would want to guess a particular solution of

$$\vec{x} = \vec{a}e^t + \vec{b}t + \vec{c}.$$

However, something of the form $\vec{a}e^t$ appears in the complementary solution. Because we do not yet know if the vector $\vec{a}$ is a multiple of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we do not know if a conflict arises. It is possible that there is no conflict, but to be safe we should also try $\vec{b}te^t$. Here we find the crux of the difference between a single equation and systems. We try *both* terms $\vec{a}e^t$ and $\vec{b}te^t$ in the solution, not just the term $\vec{b}te^t$. Therefore, we try

$$\vec{x} = \vec{a}e^t + \vec{b}te^t + \vec{c}t + \vec{d}.$$

Thus we have 8 unknowns. We write $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. We plug $\vec{x}$ into the equation. First let us compute $\vec{x}'$.

$$\vec{x}' = (\vec{a} + \vec{b})e^t + \vec{b}te^t + \vec{c} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Now $\vec{x}'$ must equal $A\vec{x} + \vec{f}$, which is

$$\begin{aligned} A\vec{x} + \vec{f} &= A\vec{a}e^t + A\vec{b}te^t + A\vec{c}t + A\vec{d} + \vec{f} \\ &= \begin{bmatrix} -a_1 \\ -2a_1 + a_2 \end{bmatrix} e^t + \begin{bmatrix} -b_1 \\ -2b_1 + b_2 \end{bmatrix} te^t + \begin{bmatrix} -c_1 \\ -2c_1 + c_2 \end{bmatrix} t + \begin{bmatrix} -d_1 \\ -2d_1 + d_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t \\ &= \begin{bmatrix} -a_1 + 1 \\ -2a_1 + a_2 \end{bmatrix} e^t + \begin{bmatrix} -b_1 \\ -2b_1 + b_2 \end{bmatrix} te^t + \begin{bmatrix} -c_1 \\ -2c_1 + c_2 + 1 \end{bmatrix} t + \begin{bmatrix} -d_1 \\ -2d_1 + d_2 \end{bmatrix}. \end{aligned}$$

We identify the coefficients of e^t , te^t , t and any constant vectors in $\vec{x}'$ and in $A\vec{x} + \vec{f}$ to find the equations:

$$\begin{aligned} a_1 + b_1 &= -a_1 + 1, & 0 &= -c_1, \\ a_2 + b_2 &= -2a_1 + a_2, & 0 &= -2c_1 + c_2 + 1, \\ b_1 &= -b_1, & c_1 &= -d_1, \\ b_2 &= -2b_1 + b_2, & c_2 &= -2d_1 + d_2. \end{aligned}$$

We could write the 8×9 augmented matrix and start row reduction, but it is easier to just solve the equations in an ad hoc manner. Immediately we see that $b_1 = 0$, $c_1 = 0$, $d_1 = 0$. Plugging these back in, we get that $c_2 = -1$ and $d_2 = -1$. The remaining equations that tell us something are

$$\begin{aligned} a_1 &= -a_1 + 1, \\ a_2 + b_2 &= -2a_1 + a_2. \end{aligned}$$

So $a_1 = 1/2$ and $b_2 = -1$. Finally, a_2 can be arbitrary and still satisfy the equations. We are looking for just a single solution so presumably the simplest one is when $a_2 = 0$. Therefore,

$$\vec{x} = \vec{a}e^t + \vec{b}te^t + \vec{c}t + \vec{d} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^t \\ -te^t - t - 1 \end{bmatrix}.$$

That is, $x_1 = \frac{1}{2}e^t$, $x_2 = -te^t - t - 1$. We would add this to the complementary solution to get the general solution of the problem. Notice that both $\vec{a}e^t$ and $\vec{b}te^t$ were really needed.

Exercise 14.9.2: Check that x_1 and x_2 solve the problem. Try setting $a_2 = 1$ and check we get a solution as well. What is the difference between the two solutions we obtained (one with $a_2 = 0$ and one with $a_2 = 1$)?

As you can see, other than the handling of conflicts, undetermined coefficients works exactly the same as it did for single equations. However, the computations can get out of hand pretty quickly for systems. The equation we considered was pretty simple.

14.9.2 First Order Variable Coefficient

Variation of parameters

Just as for a single equation, there is the method of variation of parameters. For constant coefficient systems, it is essentially the same thing as the integrating factor method we discussed earlier. However, this method works for any linear system, even if it is not constant coefficient, provided we somehow solve the associated homogeneous problem.

Suppose we have the equation

$$\vec{x}' = A(t)\vec{x} + \vec{f}(t). \quad (14.10)$$

Further, suppose we solved the associated homogeneous equation $\vec{x}' = A(t)\vec{x}$ and found a fundamental matrix solution $X(t)$. The general solution to the associated homogeneous equation is $X(t)\vec{c}$ for a constant vector $\vec{c}$. Just like for variation of parameters for single equation we try the solution to the nonhomogeneous equation of the form

$$\vec{x}_p = X(t)\vec{u}(t),$$

where $\vec{u}(t)$ is a vector-valued function instead of a constant. We substitute $\vec{x}_p$ into (14.10) to obtain

$$\underbrace{X'(t)\vec{u}(t) + X(t)\vec{u}'(t)}_{\vec{x}_p'(t)} = \underbrace{A(t)X(t)\vec{u}(t)}_{A(t)\vec{x}_p(t)} + \vec{f}(t).$$

But $X(t)$ is a fundamental matrix solution to the homogeneous problem. So $X'(t) = A(t)X(t)$, and

$$\cancel{X'(t)\vec{u}(t)} + X(t)\vec{u}'(t) = \cancel{X'(t)\vec{u}(t)} + \vec{f}(t).$$

Hence $X(t)\vec{u}'(t) = \vec{f}(t)$. If we compute $[X(t)]^{-1}$, then $\vec{u}'(t) = [X(t)]^{-1}\vec{f}(t)$. We integrate to obtain $\vec{u}$ and we have the particular solution $\vec{x}_p = X(t)\vec{u}(t)$. Let us write this as a formula

$$\vec{x}_p = X(t) \int [X(t)]^{-1} \vec{f}(t) dt.$$

If A is a constant matrix and $X(t) = e^{tA}$, then $[X(t)]^{-1} = e^{-tA}$. We obtain a solution $\vec{x}_p = e^{tA} \int e^{-tA} \vec{f}(t) dt$, which is precisely what we got using the integrating factor method.

Example 14.9.4: Find a particular solution to

$$\vec{x}' = \frac{1}{t^2 + 1} \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix} \vec{x} + \begin{bmatrix} t \\ 1 \end{bmatrix} (t^2 + 1). \quad (14.11)$$

Here $A = \frac{1}{t^2+1} \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}$ is most definitely not constant. Perhaps by a lucky guess, we find that $X = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$ solves $X'(t) = A(t)X(t)$. Once we know the complementary solution we can easily find a solution to (14.11). First we find

$$[X(t)]^{-1} = \frac{1}{t^2 + 1} \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}.$$

Next we know a particular solution to (14.11) is

$$\begin{aligned} \vec{x}_p &= X(t) \int [X(t)]^{-1} \vec{f}(t) dt \\ &= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \int \frac{1}{t^2 + 1} \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} (t^2 + 1) dt \\ &= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \int \begin{bmatrix} 2t \\ -t^2 + 1 \end{bmatrix} dt \\ &= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \begin{bmatrix} t^2 \\ -\frac{1}{3}t^3 + t \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3}t^4 \\ \frac{2}{3}t^3 + t \end{bmatrix}. \end{aligned}$$

Adding the complementary solution we find the general solution to (14.11):

$$\vec{x} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}t^4 \\ \frac{2}{3}t^3 + t \end{bmatrix} = \begin{bmatrix} c_1 - c_2t + \frac{1}{3}t^4 \\ c_2 + (c_1 + 1)t + \frac{2}{3}t^3 \end{bmatrix}.$$

Exercise 14.9.3: Check that $x_1 = \frac{1}{3}t^4$ and $x_2 = \frac{2}{3}t^3 + t$ really solve (14.11).

In the variation of parameters, just like in the integrating factor method we can obtain the general solution by adding in constants of integration. That is, we will add $X(t)\vec{c}$ for a vector of arbitrary constants. But that is precisely the complementary solution.

14.9.3 Second Order Constant Coefficients

Undetermined coefficients

We have already seen a simple example of the method of undetermined coefficients for second order systems in § 14.6. This method is essentially the same as undetermined coefficients for first order systems. There are some simplifications that we can make, as we did in § 14.6. Let the equation be

$$\vec{x}'' = A\vec{x} + \vec{F}(t),$$

where A is a constant matrix. If $\vec{F}(t)$ is of the form $\vec{F}_0 \cos(\omega t)$, then as two derivatives of cosine is again cosine we can try a solution of the form

$$\vec{x}_p = \vec{c} \cos(\omega t),$$

and we do not need to introduce sines.

If the $\vec{F}$ is a sum of cosines, note that we still have the superposition principle. If $\vec{F}(t) = \vec{F}_0 \cos(\omega_0 t) + \vec{F}_1 \cos(\omega_1 t)$, then we would try $\vec{a} \cos(\omega_0 t)$ for the problem $\vec{x}'' = A\vec{x} + \vec{F}_0 \cos(\omega_0 t)$, and we would try $\vec{b} \cos(\omega_1 t)$ for the problem $\vec{x}'' = A\vec{x} + \vec{F}_1 \cos(\omega_1 t)$. Then we sum the solutions.

However, if there is duplication with the complementary solution, or the equation is of the form $\vec{x}'' = A\vec{x}' + B\vec{x} + \vec{F}(t)$, then we need to do the same thing as we do for first order systems.

You will never go wrong with putting in more terms than needed into your guess. You will find that the extra coefficients will turn out to be zero. But it is useful to save some time and effort.

Eigenvector decomposition

If we have the system

$$\vec{x}'' = A\vec{x} + \vec{f}(t),$$

we can do *eigenvector decomposition*, just like for first order systems.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues and $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be eigenvectors. Again form the matrix $E = [\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_n]$. Write

$$\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \dots + \vec{v}_n \xi_n(t).$$

Decompose $\vec{f}$ in terms of the eigenvectors

$$\vec{f}(t) = \vec{v}_1 g_1(t) + \vec{v}_2 g_2(t) + \dots + \vec{v}_n g_n(t),$$

where, again, $\vec{g} = E^{-1}\vec{f}$.

We plug in and as before we obtain

$$\begin{aligned} \overbrace{\vec{v}_1 \xi_1'' + \vec{v}_2 \xi_2'' + \dots + \vec{v}_n \xi_n''}^{\vec{x}''} &= \overbrace{A(\vec{v}_1 \xi_1 + \vec{v}_2 \xi_2 + \dots + \vec{v}_n \xi_n)}^{A\vec{x}} + \overbrace{\vec{v}_1 g_1 + \vec{v}_2 g_2 + \dots + \vec{v}_n g_n}^{\vec{f}} \\ &= A\vec{v}_1 \xi_1 + A\vec{v}_2 \xi_2 + \dots + A\vec{v}_n \xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \dots + \vec{v}_n g_n \\ &= \vec{v}_1 \lambda_1 \xi_1 + \vec{v}_2 \lambda_2 \xi_2 + \dots + \vec{v}_n \lambda_n \xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \dots + \vec{v}_n g_n \\ &= \vec{v}_1 (\lambda_1 \xi_1 + g_1) + \vec{v}_2 (\lambda_2 \xi_2 + g_2) + \dots + \vec{v}_n (\lambda_n \xi_n + g_n). \end{aligned}$$

We identify the coefficients of the eigenvectors to get the equations

$$\xi_1'' = \lambda_1 \xi_1 + g_1,$$

$$\xi_2'' = \lambda_2 \xi_2 + g_2,$$

$$\vdots$$

$$\xi_n'' = \lambda_n \xi_n + g_n.$$

Each one of these equations is independent of the others. We solve each equation using the methods of [chapter 7](#). We write $\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \cdots + \vec{v}_n \xi_n(t)$, and we are done; we have a particular solution. We find the general solutions for ξ_1 through ξ_n , and again $\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \cdots + \vec{v}_n \xi_n(t)$ is the general solution (and not just a particular solution).

Example 14.9.5: Let us do the example from [§ 14.6](#) using this method. The equation is

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(3t).$$

The eigenvalues are -1 and -4 , with eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore $E = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ and $E^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Therefore,

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1} \vec{f}(t) = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \cos(3t) \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \cos(3t) \\ -\frac{2}{3} \cos(3t) \end{bmatrix}.$$

So after the whole song and dance of plugging in, the equations we get are

$$\xi_1'' = -\xi_1 + \frac{2}{3} \cos(3t), \quad \xi_2'' = -4 \xi_2 - \frac{2}{3} \cos(3t).$$

For each equation we use the method of undetermined coefficients. We try $C_1 \cos(3t)$ for the first equation and $C_2 \cos(3t)$ for the second equation. We plug in to get

$$\begin{aligned} -9C_1 \cos(3t) &= -C_1 \cos(3t) + \frac{2}{3} \cos(3t), \\ -9C_2 \cos(3t) &= -4C_2 \cos(3t) - \frac{2}{3} \cos(3t). \end{aligned}$$

We solve each of these equations separately. We get $-9C_1 = -C_1 + 2/3$ and $-9C_2 = -4C_2 - 2/3$. And hence $C_1 = -1/12$ and $C_2 = 2/15$. So our particular solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(\frac{-1}{12} \cos(3t) \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(\frac{2}{15} \cos(3t) \right) = \begin{bmatrix} 1/20 \\ -3/10 \end{bmatrix} \cos(3t).$$

This solution matches what we got previously in [§ 14.6](#).

14.9.4 Exercises

Exercise 14.9.4: Find a particular solution to $x' = x + 2y + 2t$, $y' = 3x + 2y - 4$,

- a) using integrating factor method,
- b) using eigenvector decomposition,
- c) using undetermined coefficients.

Exercise 14.9.5: Find the general solution to $x' = 4x + y - 1$, $y' = x + 4y - e^t$,

- a) using integrating factor method,
- b) using eigenvector decomposition,
- c) using undetermined coefficients.

Exercise 14.9.6: Find the general solution to $x_1'' = -6x_1 + 3x_2 + \cos(t)$, $x_2'' = 2x_1 - 7x_2 + 3 \cos(t)$,

- a) using eigenvector decomposition,
- b) using undetermined coefficients.

Exercise 14.9.7: Find the general solution to $x_1'' = -6x_1 + 3x_2 + \cos(2t)$, $x_2'' = 2x_1 - 7x_2 + 3 \cos(2t)$,

- a) using eigenvector decomposition,
- b) using undetermined coefficients.

Exercise 14.9.8: Take the equation $\vec{x}' = \begin{bmatrix} \frac{1}{t} & -1 \\ 1 & \frac{1}{t} \end{bmatrix} \vec{x} + \begin{bmatrix} t^2 \\ -t \end{bmatrix}$.

a) Check that $\vec{x}_c = c_1 \begin{bmatrix} t \sin t \\ -t \cos t \end{bmatrix} + c_2 \begin{bmatrix} t \cos t \\ t \sin t \end{bmatrix}$ is the complementary solution.

b) Use variation of parameters to find a particular solution.

Exercise 14.9.101: Find a particular solution to $x' = 5x + 4y + t$, $y' = x + 8y - t$,

a) using integrating factor method,

b) using eigenvector decomposition,

c) using undetermined coefficients.

Exercise 14.9.102: Find a particular solution to $x' = y + e^t$, $y' = x + e^t$,

a) using integrating factor method,

b) using eigenvector decomposition,

c) using undetermined coefficients.

Exercise 14.9.103: Solve $x_1' = x_2 + t$, $x_2' = x_1 + t$ with initial conditions $x_1(0) = 1$, $x_2(0) = 2$, using eigenvector decomposition.

Exercise 14.9.104: Solve $x_1'' = -3x_1 + x_2 + t$, $x_2'' = 9x_1 + 5x_2 + \cos(t)$ with initial conditions $x_1(0) = 0$, $x_2(0) = 0$, $x_1'(0) = 0$, $x_2'(0) = 0$, using eigenvector decomposition.

Appendix A

Appendix: Linear Algebra

A.1 Vectors, Mappings, and Matrices

In real life, there is most often more than one variable. We wish to organize dealing with multiple variables in a consistent manner, and in particular organize dealing with linear equations and linear mappings, as those are both rather useful and rather easy to handle. Mathematicians joke that “to an engineer every problem is linear, and everything is a matrix.” And well, they (the engineers) are not wrong. Quite often, solving an engineering problem is figuring out the right finite-dimensional linear problem to solve, which is then solved with some matrix manipulation. Most importantly, linear problems are the ones that we know how to solve, and we have many tools to solve them. For engineers, mathematicians, physicists, and anybody else in a technical field, it is absolutely vital to learn linear algebra.

As motivation, suppose we wish to solve

$$x - y = 2,$$

$$2x + y = 4,$$

for x and y . That is, we desire numbers x and y such that the two equations are satisfied. Let us perhaps start by adding the equations together to find

$$x + 2x - y + y = 2 + 4, \quad \text{or} \quad 3x = 6.$$

In other words, $x = 2$. Once we have that, we plug $x = 2$ into the first equation to find $2 - y = 2$, so $y = 0$. OK, that was easy. What is all this fuss about linear equations. Well, try doing this if you have 5000 unknowns[†]. Also, we may have such equations not just of numbers, but of functions and derivatives of functions in differential equations. Clearly we need a systematic way of doing things. A nice consequence of making things systematic and simpler to write down is that it becomes easier to have computers do the work for us. Computers are rather stupid, they do not think, but are very good at doing lots of repetitive tasks precisely, as long as we figure out a systematic way for them to perform the tasks.

A.1.1 Vectors and Operations on Vectors

Consider n real numbers as an n -tuple:

$$(x_1, x_2, \dots, x_n).$$

The set of such n -tuples is the so-called n -dimensional space, often denoted by $\mathbb{R}^n$. Sometimes we call this the n -dimensional euclidean space[‡]. In two dimensions, $\mathbb{R}^2$ is called the cartesian plane[§]. Each such n -tuple

[†]One of the downsides of making everything look like a linear problem is that the number of variables tends to become huge.

[‡]Named after the ancient Greek mathematician [Euclid of Alexandria](#) (around 300 BC), possibly the most famous of mathematicians; even small towns often have Euclid Street or Euclid Avenue.

[§]Named after the French mathematician [René Descartes](#) (1596–1650). It is “cartesian” as his name in Latin is Renatus Cartesius.

represents a point in the n -dimensional space. For example, the point $(1, 2)$ in the plane $\mathbb{R}^2$ is one unit to the right and two units up from the origin.

When we do algebra with these n -tuples of numbers we call them *vectors**. Mathematicians are keen on separating what is a vector and what is a point of the space or in the plane, and it turns out to be an important distinction, however, for the purposes of linear algebra we can think of everything being represented by a vector. A way to think of a vector, which is especially useful in calculus and differential equations, is an arrow. It is an object that has a *direction* and a *magnitude*. For instance, the vector $(1, 2)$ is the arrow from the origin to the point $(1, 2)$ in the plane. The magnitude is the length of the arrow. See Figure A.1. If we think of vectors as arrows, the arrow does not always have to start at the origin. If we do move it around, however, it should always keep the same direction and the same magnitude.

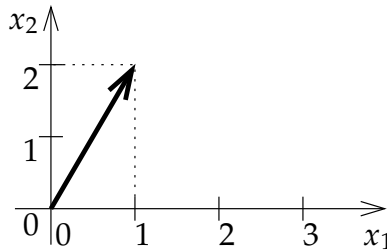


Figure A.1: The vector $(1, 2)$ drawn as an arrow from the origin to the point $(1, 2)$.

As vectors are arrows, when we want to give a name to a vector, we draw a little arrow above it:

$$\vec{x}$$

Another popular notation is a bold $\mathbf{x}$, although we will use the little arrows. It may be easy to write a bold letter in a book, but it is not so easy to write it by hand on paper or on the board. Mathematicians often do not even write the arrows. A mathematician would write x and remember that x is a vector and not a number. Just like you remember that Bob is your uncle, and you don't have to keep repeating "Uncle Bob" and you can just say "Bob." In this book, however, we will call Bob "Uncle Bob" and write vectors with the little arrows.

The *magnitude* can be computed using the Pythagorean theorem. The vector $(1, 2)$ drawn in the figure has magnitude $\sqrt{1^2 + 2^2} = \sqrt{5}$. The magnitude is denoted by $\|\vec{x}\|$, and, in any number of dimensions, it can be computed in the same way:

$$\|\vec{x}\| = \|(x_1, x_2, \dots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

For reasons that will become clear in the next section, we often write vectors as so-called *column vectors*:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Don't worry. It is just a different way of writing the same thing. For example, the vector $(1, 2)$ can be written as

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

*A common notation to distinguish vectors from points is to write $(1, 2)$ for the point and $\langle 1, 2 \rangle$ for the vector. We write both as $(1, 2)$.

The fact that we write arrows above vectors allows us to write several vectors $\vec{x}_1, \vec{x}_2$, etc., without confusing these with the components of some other vector $\vec{x}$.

So where is the *algebra* from *linear algebra*? Well, arrows can be added, subtracted, and multiplied by numbers. First we consider *addition*. If we have two arrows, we simply move along one, and then along the other. See Figure A.2.

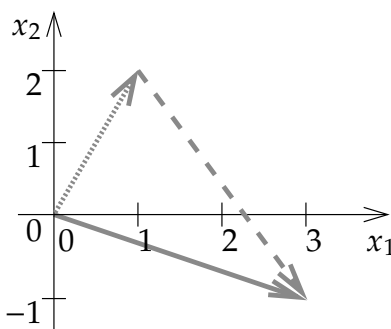


Figure A.2: Adding the vectors $(1, 2)$, drawn dotted, and $(2, -3)$, drawn dashed. The result, $(3, -1)$, is drawn as a solid arrow.

It is rather easy to see what it does to the numbers that represent the vectors. Suppose we want to add $(1, 2)$ to $(2, -3)$ as in the figure. We travel along $(1, 2)$ and then we travel along $(2, -3)$. What we did was travel one unit right, two units up, and then we travelled two units right, and three units down (the negative three). That means that we ended up at $(1 + 2, 2 + (-3)) = (3, -1)$. And that's how addition always works:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

Subtracting is similar. What $\vec{x} - \vec{y}$ means visually is that we first travel along $\vec{x}$, and then we travel backwards along $\vec{y}$. See Figure A.3. It is like adding $\vec{x} + (-\vec{y})$ where $-\vec{y}$ is the arrow we obtain by erasing the arrow head from one side and drawing it on the other side, that is, we reverse the direction. In terms of the numbers, we simply go backwards both horizontally and vertically, so we negate both numbers. For instance, if $\vec{y}$ is $(-2, 1)$, then $-\vec{y}$ is $(2, -1)$.

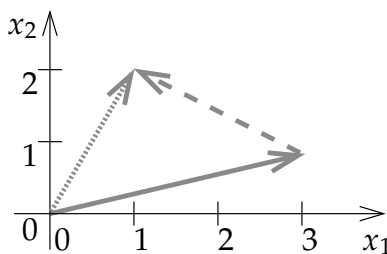


Figure A.3: Subtraction, the vector $(1, 2)$, drawn dotted, minus $(-2, 1)$, drawn dashed. The result, $(3, 1)$, is drawn as a solid arrow.

Another intuitive thing to do to a vector is to *scale* it. We represent this by multiplication of a number with a vector. Because of this, when we wish to distinguish between vectors and numbers, we call the numbers

scalars. For example, suppose we want to travel three times further. If the vector is $(1, 2)$, travelling 3 times further means going 3 units to the right and 6 units up, so we get the vector $(3, 6)$. We just multiply each number in the vector by 3. If α is a number, then

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}.$$

Scaling (by a positive number) multiplies the magnitude and leaves direction untouched. The magnitude of $(1, 2)$ is $\sqrt{5}$. The magnitude of 3 times $(1, 2)$, that is, $(3, 6)$, is $3\sqrt{5}$.

When the scalar is negative, then when we multiply a vector by it, the vector is not only scaled, but it also switches direction. Multiplying $(1, 2)$ by -3 means we should go 3 times further but in the opposite direction, so 3 units to the left and 6 units down, or in other words, $(-3, -6)$. As we mentioned above, $-\vec{y}$ is a reverse of $\vec{y}$, and this is the same as $(-1)\vec{y}$.

In Figure A.4, you can see a couple of examples of what scaling a vector means visually.

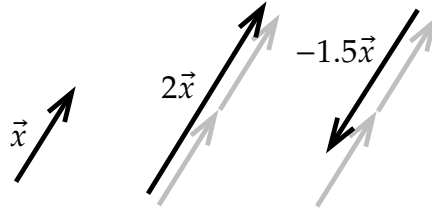


Figure A.4: A vector $\vec{x}$, the vector $2\vec{x}$ (same direction, double the magnitude), and the vector $-1.5\vec{x}$ (opposite direction, 1.5 times the magnitude).

We put all of these operations together to work out more complicated expressions. Let us compute a small example:

$$3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -4 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3(1) + 2(-4) - 3(-2) \\ 3(2) + 2(-1) - 3(2) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

As we said a vector is a direction and a magnitude. Magnitude is easy to represent, it is just a number. The *direction* is usually given by a vector with magnitude one. We call such a vector a *unit vector*. That is, $\vec{u}$ is a unit vector when $\|\vec{u}\| = 1$. For instance, the vectors $(1, 0)$, $(1/\sqrt{2}, 1/\sqrt{2})$, and $(0, -1)$ are all unit vectors.

To represent the direction of a vector $\vec{x}$, we need to find the unit vector in the same direction. To do so, we simply rescale $\vec{x}$ by the reciprocal of the magnitude, that is $\frac{1}{\|\vec{x}\|}\vec{x}$, or more concisely $\frac{\vec{x}}{\|\vec{x}\|}$.

As an example, the unit vector in the direction of $(1, 2)$ is the vector

$$\frac{1}{\sqrt{1^2 + 2^2}}(1, 2) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right).$$

A.1.2 Linear Mappings and Matrices

A *vector-valued function* F is a rule that takes a vector $\vec{x}$ and returns another vector $\vec{y}$. For example, F could be a scaling that doubles the size of vectors:

$$F(\vec{x}) = 2\vec{x}.$$

Applied to say $(1, 3)$ we get

$$F\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

If F is a mapping that takes vectors in $\mathbb{R}^2$ to $\mathbb{R}^2$ (such as the above), we write

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

The words *function* and *mapping* are used rather interchangeably, although more often than not, *mapping* is used when talking about a vector-valued function, and the word *function* is often used when the function is scalar-valued.

A beginning student of mathematics (and many a seasoned mathematician), that sees an expression such as

$$f(3x + 8y)$$

yearns to write

$$3f(x) + 8f(y).$$

After all, who hasn't wanted to write $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ or something like that at some point in their mathematical lives. Wouldn't life be simple if we could do that? Of course we cannot always do that (for example, not with the square roots!) But there are many other functions where we can do exactly the above. Such functions are called *linear*.

A mapping $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called *linear* if

$$F(\vec{x} + \vec{y}) = F(\vec{x}) + F(\vec{y}),$$

for any vectors $\vec{x}$ and $\vec{y}$, and also

$$F(\alpha\vec{x}) = \alpha F(\vec{x}),$$

for any scalar α . The F we defined above that doubles the size of all vectors is linear. Let us check:

$$F(\vec{x} + \vec{y}) = 2(\vec{x} + \vec{y}) = 2\vec{x} + 2\vec{y} = F(\vec{x}) + F(\vec{y}),$$

and also

$$F(\alpha\vec{x}) = 2\alpha\vec{x} = \alpha 2\vec{x} = \alpha F(\vec{x}).$$

We also call a linear function a *linear transformation*. If you want to be really fancy and impress your friends, you can call it a *linear operator*. When a mapping is linear we often do not write the parentheses. We write simply

$$F\vec{x}$$

instead of $F(\vec{x})$. We do this because linearity means that the mapping F behaves like multiplying $\vec{x}$ by "something." That something is a matrix.

A *matrix* is an $m \times n$ array of numbers (m rows and n columns). A 3×5 matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}.$$

The numbers a_{ij} are called *elements* or *entries*.

A column vector is simply an $m \times 1$ matrix. Similarly to a column vector there is also a *row vector*, which is a $1 \times n$ matrix. If we have an $n \times n$ matrix, then we say that it is a *square matrix*.

Now how does a matrix A relate to a linear mapping? Well a matrix tells you where certain special vectors go. Let's give a name to those certain vectors. The *standard basis vectors* of $\mathbb{R}^n$ are

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

In $\mathbb{R}^3$ these vectors are

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

You may recall from calculus of several variables that these are sometimes called $\vec{i}, \vec{j}, \vec{k}$.

The reason these are called a *basis* is that every other vector can be written as a *linear combination* of them. For example, in $\mathbb{R}^3$ the vector $(4, 5, 6)$ can be written as

$$4\vec{e}_1 + 5\vec{e}_2 + 6\vec{e}_3 = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

So how does a matrix represent a linear mapping? Well, the columns of the matrix are the vectors where A as a linear mapping takes $\vec{e}_1, \vec{e}_2$, etc. For instance, consider

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

As a linear mapping $M: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ takes $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. In other words,

$$M\vec{e}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad M\vec{e}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

More generally, if we have an $n \times m$ matrix A , that is, we have n rows and m columns, then the mapping $A: \mathbb{R}^m \rightarrow \mathbb{R}^n$ takes $\vec{e}_j$ to the j^{th} column of A . For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

represents a mapping from $\mathbb{R}^5$ to $\mathbb{R}^3$ that does

$$A\vec{e}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \quad A\vec{e}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \quad A\vec{e}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}, \quad A\vec{e}_4 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}, \quad A\vec{e}_5 = \begin{bmatrix} a_{15} \\ a_{25} \\ a_{35} \end{bmatrix}.$$

What about another vector $\vec{x}$ that is not in the standard basis? Where does it go? We use linearity. First, we write the vector as a linear combination of the standard basis vectors:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4 + x_5\vec{e}_5.$$

Then

$$A\vec{x} = A(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4 + x_5\vec{e}_5) = x_1A\vec{e}_1 + x_2A\vec{e}_2 + x_3A\vec{e}_3 + x_4A\vec{e}_4 + x_5A\vec{e}_5.$$

If we know where A takes all the basis vectors, we know where it takes all vectors.

Suppose M is the 2×2 matrix from above, then

$$M \begin{bmatrix} -2 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 0.1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0.1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1.8 \\ -5.6 \end{bmatrix}.$$

Every linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ can be represented by an $n \times m$ matrix. You just figure out where it takes the standard basis vectors. Conversely, every $n \times m$ matrix represents a linear mapping. Hence, we may think of matrices being linear mappings, and linear mappings being matrices.

Or can we? In this book we study mostly linear differential operators, and linear differential operators are linear mappings, although they are not acting on $\mathbb{R}^n$, but on an infinite-dimensional space of functions:

$$Lf = g.$$

For a function f we get a function g , and L is linear in the sense that

$$L(f + h) = Lf + Lh, \quad \text{and} \quad L(\alpha f) = \alpha Lf.$$

for any number (scalar) α and all functions f and h .

So the answer is not really. But if we consider vectors in finite-dimensional spaces $\mathbb{R}^n$ then yes, every linear mapping is a matrix. We have mentioned at the beginning of this section, that we can “make everything a vector.” That’s not strictly true, but it is true approximately. Those “infinite-dimensional” spaces of functions can be approximated by a finite-dimensional space, and then linear operators are just matrices. So approximately, this is true. And as far as actual computations that we can do on a computer, we can work only with finitely many dimensions anyway. If you ask a computer or your calculator to plot a function, it samples the function at finitely many points and then connects the dots*. It does not actually give you infinitely many values. The way that you have been using the computer or your calculator so far has already been a certain approximation of the space of functions by a finite-dimensional space.

To end the section, we notice how $A\vec{x}$ can be written more succinctly. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot (-1) \\ 3 \cdot 2 + 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

That is, you take the entries in a row of the matrix, you multiply them by the entries in your vector, you add things up, and that’s the corresponding entry in the resulting vector.

A.1.3 Exercises

Exercise A.1.1: On a piece of graph paper draw the vectors:

a) $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

b) $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$

c) $(3, -4)$

Exercise A.1.2: On a piece of graph paper draw the vector $(1, 2)$ starting at (based at) the given point:

a) based at $(0, 0)$

b) based at $(1, 2)$

c) based at $(0, -1)$

Exercise A.1.3: On a piece of graph paper draw the following operations. Draw and label the vectors involved in the operations as well as the result:

a) $\begin{bmatrix} 1 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

c) $3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Exercise A.1.4: Compute the magnitude of

a) $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

c) $(1, 3, -4)$

*If you have ever used Matlab, you may have noticed that to plot a function, we take a vector of inputs, ask Matlab to compute the corresponding vector of values of the function, and then we ask it to plot the result.

Exercise A.1.5: Compute

a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ -8 \end{bmatrix}$

b) $\begin{bmatrix} -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 6 \\ -4 \end{bmatrix}$

c) $-\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

d) $4\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

e) $5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

f) $3\begin{bmatrix} 1 \\ -8 \end{bmatrix} - 2\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Exercise A.1.6: Find the unit vector in the direction of the given vector

a) $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

b) $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

c) $(3, 1, -2)$

Exercise A.1.7: If $\vec{x} = (1, 2)$ and $\vec{y}$ are added together, we find $\vec{x} + \vec{y} = (0, 2)$. What is $\vec{y}$?

Exercise A.1.8: Write $(1, 2, 3)$ as a linear combination of the standard basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$.

Exercise A.1.9: If the magnitude of $\vec{x}$ is 4, what is the magnitude of

a) $0\vec{x}$

b) $3\vec{x}$

c) $-\vec{x}$

d) $-4\vec{x}$

e) $\vec{x} + \vec{x}$

f) $\vec{x} - \vec{x}$

Exercise A.1.10: Suppose a linear mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ takes $(1, 0)$ to $(2, -1)$ and it takes $(0, 1)$ to $(3, 3)$. Where does it take

a) $(1, 1)$

b) $(2, 0)$

c) $(2, -1)$

Exercise A.1.11: Suppose a linear mapping $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ takes $(1, 0, 0)$ to $(2, 1)$, it takes $(0, 1, 0)$ to $(3, 4)$, and it takes $(0, 0, 1)$ to $(5, 6)$. Write down the matrix representing the mapping F .

Exercise A.1.12: Suppose that a mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ takes $(1, 0)$ to $(1, 2)$, $(0, 1)$ to $(3, 4)$, and $(1, 1)$ to $(0, -1)$. Explain why F is not linear.

Exercise A.1.13 (challenging): Let $\mathbb{R}^3$ represent the space of quadratic polynomials in t : a point (a_0, a_1, a_2) in $\mathbb{R}^3$ represents the polynomial $a_0 + a_1t + a_2t^2$. Consider the derivative $\frac{d}{dt}$ as a mapping of $\mathbb{R}^3$ to $\mathbb{R}^3$, and note that $\frac{d}{dt}$ is linear. Write down $\frac{d}{dt}$ as a 3×3 matrix.

Exercise A.1.101: Compute the magnitude of

a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

c) $(-2, 1, -2)$

Exercise A.1.102: Find the unit vector in the direction of the given vector

a) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

c) $(2, -5, 2)$

Exercise A.1.103: Compute

a) $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \end{bmatrix}$

b) $\begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

c) $-\begin{bmatrix} -5 \\ 3 \end{bmatrix}$

d) $2\begin{bmatrix} -2 \\ 4 \end{bmatrix}$

e) $3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

f) $2\begin{bmatrix} 2 \\ -3 \end{bmatrix} - 6\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Exercise A.1.104: If the magnitude of $\vec{x}$ is 5, what is the magnitude of

a) $4\vec{x}$

b) $-2\vec{x}$

c) $-4\vec{x}$

Exercise A.1.105: Suppose a linear mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ takes $(1, 0)$ to $(1, -1)$ and it takes $(0, 1)$ to $(2, 0)$. Where does it take

a) $(1, 1)$

b) $(0, 2)$

c) $(1, -1)$

A.2 Matrix Algebra

Note: 2–3 lectures

A.2.1 One-By-One Matrices

Let us motivate what we want to achieve with matrices. Real-valued linear mappings of the real line, linear functions that eat numbers and spit out numbers, are just multiplications by a number. Consider a mapping defined by multiplying by a number. Let's call this number α . The mapping then takes x to αx . We can *add* such mappings: If we have another mapping β , then

$$\alpha x + \beta x = (\alpha + \beta)x.$$

We get a new mapping $\alpha + \beta$ that multiplies x by, well, $\alpha + \beta$. If D is a mapping that doubles its input, $Dx = 2x$, and T is a mapping that triples, $Tx = 3x$, then $D + T$ is a mapping that multiplies by 5, $(D + T)x = 5x$.

Similarly we can *compose* such mappings, that is, we could apply one and then the other. We take x , we run it through the first mapping α to get αx , then we run αx through the second mapping β . In other words,

$$\beta(\alpha x) = (\beta\alpha)x.$$

We just multiply those two numbers. Using our doubling and tripling mappings, if we double and then triple, that is $T(Dx)$ then we obtain $3(2x) = 6x$. The composition TD is the mapping that multiplies by 6. For larger matrices, composition also ends up being a kind of multiplication.

A.2.2 Matrix Addition and Scalar Multiplication

The mappings that multiply numbers by numbers are just 1×1 matrices. The number α above could be written as a matrix $[\alpha]$. Perhaps we would want to do to all matrices the same things that we did to those 1×1 matrices at the start of this section above. First, let us add matrices. If we have a matrix A and a matrix B that are of the same size, say $m \times n$, then they are mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. The mapping $A + B$ should also be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, and it should do the following to vectors:

$$(A + B)\vec{x} = A\vec{x} + B\vec{x}.$$

It turns out you just add the matrices element-wise: If the ij^{th} entry of A is a_{ij} , and the ij^{th} entry of B is b_{ij} , then the ij^{th} entry of $A + B$ is $a_{ij} + b_{ij}$. If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}.$$

Let us illustrate on a more concrete example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & -1 \end{bmatrix} = \begin{bmatrix} 1+7 & 2+8 \\ 3+9 & 4+10 \\ 5+11 & 6-1 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 12 & 14 \\ 16 & 5 \end{bmatrix}.$$

Let's check that this does the right thing to a vector. Let's use some of the vector algebra that we already know, and regroup things:

$$\begin{aligned}
 \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} &= \left(2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right) + \left(2 \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} - \begin{bmatrix} 8 \\ 10 \\ -1 \end{bmatrix} \right) \\
 &= 2 \left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} \right) - \left(\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \\ -1 \end{bmatrix} \right) \\
 &= 2 \begin{bmatrix} 1+7 \\ 3+9 \\ 5+11 \end{bmatrix} - \begin{bmatrix} 2+8 \\ 4+10 \\ 6-1 \end{bmatrix} = 2 \begin{bmatrix} 8 \\ 12 \\ 16 \end{bmatrix} - \begin{bmatrix} 10 \\ 14 \\ 5 \end{bmatrix} \\
 &= \begin{bmatrix} 8 & 10 \\ 12 & 14 \\ 16 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2(8)-10 \\ 2(12)-14 \\ 2(16)-5 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 27 \end{bmatrix}.
 \end{aligned}$$

If we replaced the numbers by letters that would constitute a proof! Notice that we did not really have to compute what the result is to convince ourselves that the two expressions were equal.

If the sizes of the matrices do not match, then addition is undefined. If A is 3×2 and B is 2×5 , then we cannot add the matrices. We do not know what that could possibly mean.

It is also useful to have a matrix that when added to any other matrix does nothing. This is the zero matrix, the matrix of all zeros:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We often denote the zero matrix by 0 without specifying size. We would then just write $A + 0$, where we just assume that 0 is the zero matrix of the same size as A .

There are really two things we can multiply matrices by. We can multiply matrices by scalars or we can multiply by other matrices. Let us first consider multiplication by scalars. For a matrix A and a scalar α , we want αA to be the matrix that accomplishes

$$(\alpha A)\vec{x} = \alpha(A\vec{x}).$$

That is just scaling the result by α . If you think about it, scaling every term in A by α achieves just that: If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad \text{then} \quad \alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{bmatrix}.$$

For example,

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Let us list some properties of matrix addition and scalar multiplication. Denote by 0 the zero matrix, by α, β scalars, and by A, B, C matrices. Then:

$$\begin{aligned}
 A + 0 &= A = 0 + A, \\
 A + B &= B + A, \\
 (A + B) + C &= A + (B + C), \\
 \alpha(A + B) &= \alpha A + \alpha B, \\
 (\alpha + \beta)A &= \alpha A + \beta A.
 \end{aligned}$$

These rules should look very familiar.

A.2.3 Matrix Multiplication

As we mentioned above, composition of linear mappings is also a multiplication of matrices. Suppose A is an $m \times n$ matrix, that is, A takes $\mathbb{R}^n$ to $\mathbb{R}^m$, and B is an $n \times p$ matrix, that is, B takes $\mathbb{R}^p$ to $\mathbb{R}^n$. The composition AB should work as follows

$$AB\vec{x} = A(B\vec{x}).$$

First, a vector $\vec{x}$ in $\mathbb{R}^p$ gets taken to the vector $B\vec{x}$ in $\mathbb{R}^n$. Then the mapping A takes it to the vector $A(B\vec{x})$ in $\mathbb{R}^m$. In other words, the composition AB should be an $m \times p$ matrix. In terms of sizes we should have

$$“ \quad [m \times n][n \times p] = [m \times p]. \quad ”$$

Notice how the middle size must match.

OK, now we know what sizes of matrices we should be able to multiply, and what the product should be. Let us see how to actually compute matrix multiplication. We start with the so-called *dot product* (or *inner product*) of two vectors. Usually this is a row vector multiplied with a column vector of the same size. Dot product multiplies each pair of entries from the first and the second vector and sums these products. The result is a single number. For example,

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3.$$

And similarly for larger (or smaller) vectors. A dot product is really a product of two matrices: a $1 \times n$ matrix and an $n \times 1$ matrix resulting in a 1×1 matrix, that is, a number.

Armed with the dot product we define the *product of matrices*. We denote by $\text{row}_i(A)$ the i^{th} row of A and by $\text{column}_j(A)$ the j^{th} column of A . For an $m \times n$ matrix A and an $n \times p$ matrix B we can compute the product AB : The matrix AB is an $m \times p$ matrix whose ij^{th} entry is the dot product

$$\text{row}_i(A) \cdot \text{column}_j(B).$$

For example, given a 2×3 and a 3×2 matrix we should end up with a 2×2 matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}, \quad (\text{A.1})$$

or with some numbers:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -7 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-1) + 2 \cdot (-7) + 3 \cdot 1 & 1 \cdot 2 + 2 \cdot 0 + 3 \cdot (-1) \\ 4 \cdot (-1) + 5 \cdot (-7) + 6 \cdot 1 & 4 \cdot 2 + 5 \cdot 0 + 6 \cdot (-1) \end{bmatrix} = \begin{bmatrix} -12 & -1 \\ -33 & 2 \end{bmatrix}.$$

A useful consequence of the definition is that the evaluation $A\vec{x}$ for a matrix A and a (column) vector $\vec{x}$ is also matrix multiplication. That is really why we think of vectors as column vectors, or $n \times 1$ matrices. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot (-1) \\ 3 \cdot 2 + 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

If you look at the last section, that is precisely the last example we gave.

You should stare at the computation of multiplication of matrices AB and the previous definition of $A\vec{y}$ as a mapping for a moment. What we are doing with matrix multiplication is applying the mapping A to the columns of B . This is usually written as follows. Suppose we write the $n \times p$ matrix $B = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p]$, where $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_p$ are the columns of B . Then for an $m \times n$ matrix A ,

$$AB = A[\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_p] = [A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_p].$$

The columns of the $m \times p$ matrix AB are the vectors $A\vec{b}_1, A\vec{b}_2, \dots, A\vec{b}_p$. For example, in (A.1), the columns of

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

are

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix}.$$

This is a very useful way to understand what matrix multiplication is. It should also make it easier to remember how to perform matrix multiplication.

A.2.4 Some Rules of Matrix Algebra

For multiplication we want an analogue of a 1. That is, we desire a matrix that just leaves everything as it found it. This analogue is the so-called *identity matrix*. The identity matrix is a square matrix with 1s on the main diagonal and zeros everywhere else. It is usually denoted by I . For each size we have a different identity matrix and so sometimes we may denote the size as a subscript. For example, I_3 is the 3×3 identity matrix

$$I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us see how the matrix works on a smaller example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} \cdot 1 + a_{12} \cdot 0 & a_{11} \cdot 0 + a_{12} \cdot 1 \\ a_{21} \cdot 1 + a_{22} \cdot 0 & a_{21} \cdot 0 + a_{22} \cdot 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Multiplication by the identity from the left looks similar, and also does not touch anything.

We have the following rules for matrix multiplication. Suppose that A, B, C are matrices of the correct sizes so that the following make sense. Let α denote a scalar (number). Then

$$\begin{aligned} A(BC) &= (AB)C && \text{(associative law),} \\ A(B + C) &= AB + AC && \text{(distributive law),} \\ (B + C)A &= BA + CA && \text{(distributive law),} \\ \alpha(AB) &= (\alpha A)B = A(\alpha B), \\ IA &= A = AI && \text{(identity).} \end{aligned}$$

Example A.2.1: Let us demonstrate a couple of these rules. For example, the associative law:

$$\underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_A \underbrace{\left(\underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_B \underbrace{\begin{bmatrix} -1 & 4 \\ 5 & 2 \end{bmatrix}}_C \right)}_{BC} = \underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 16 & 24 \\ -16 & -2 \end{bmatrix}}_{BC} = \underbrace{\begin{bmatrix} -96 & -78 \\ 64 & 52 \end{bmatrix}}_{A(BC)},$$

and

$$\left(\underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_B \right) \underbrace{\begin{bmatrix} -1 & 4 \\ 5 & 2 \end{bmatrix}}_C = \underbrace{\begin{bmatrix} -9 & -21 \\ 6 & 14 \end{bmatrix}}_{AB} \underbrace{\begin{bmatrix} -1 & 4 \\ 5 & 2 \end{bmatrix}}_C = \underbrace{\begin{bmatrix} -96 & -78 \\ 64 & 52 \end{bmatrix}}_{(AB)C}.$$

Or how about multiplication by scalars:

$$10 \left(\underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_B \right) = 10 \underbrace{\begin{bmatrix} -9 & -21 \\ 6 & 14 \end{bmatrix}}_{AB} = \underbrace{\begin{bmatrix} -90 & -210 \\ 60 & 140 \end{bmatrix}}_{10(AB)},$$

$$\underbrace{\left(10 \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}\right)}_A \underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_B = \underbrace{\begin{bmatrix} -30 & 30 \\ 20 & -20 \end{bmatrix}}_{10A} \underbrace{\begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}}_B = \underbrace{\begin{bmatrix} -90 & -210 \\ 60 & 140 \end{bmatrix}}_{(10A)B},$$

and

$$\underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_A \underbrace{\left(10 \begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}\right)}_B = \underbrace{\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 40 & 40 \\ 10 & -30 \end{bmatrix}}_{10B} = \underbrace{\begin{bmatrix} -90 & -210 \\ 60 & 140 \end{bmatrix}}_{A(10B)}.$$

A multiplication rule, one you have used since primary school on numbers, is quite conspicuously missing for matrices. That is, matrix multiplication is not commutative. Firstly, just because AB makes sense, it may be that BA is not even defined. For example, if A is 2×3 , and B is 3×4 , then we can multiply AB but not BA .

Even if AB and BA are both defined, does not mean that they are equal. For example, take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$:

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = BA.$$

A.2.5 Inverse

A couple of other algebra rules you know for numbers do not quite work on matrices:

(i) $AB = AC$ does not necessarily imply $B = C$, even if A is not 0.

(ii) $AB = 0$ does not necessarily mean that $A = 0$ or $B = 0$.

For example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

To make these rules hold, we do not just need one of the matrices to not be zero, we would need to “divide” by a matrix. This is where the *matrix inverse* comes in. Suppose that A and B are $n \times n$ matrices such that

$$AB = I = BA.$$

Then we call B the inverse of A and we denote B by A^{-1} . Perhaps not surprisingly, $(A^{-1})^{-1} = A$, since if the inverse of A is B , then the inverse of B is A . If the inverse of A exists, then we say A is *invertible*. If A is not invertible, we say A is *singular*.

If $A = [a]$ is a 1×1 matrix, then A^{-1} is $a^{-1} = \frac{1}{a}$. That is where the notation comes from. The computation is not nearly as simple when A is larger.

The proper formulation of the cancellation rule is:

$$\text{If } A \text{ is invertible, then } AB = AC \text{ implies } B = C.$$

The computation is what you would do in regular algebra with numbers, but you have to be careful never to commute matrices:

$$\begin{aligned} AB &= AC, \\ A^{-1}AB &= A^{-1}AC, \\ IB &= IC, \\ B &= C. \end{aligned}$$

And similarly for cancellation on the right:

$$\text{If } A \text{ is invertible, then } BA = CA \text{ implies } B = C.$$

The rule says, among other things, that the inverse of a matrix is unique if it exists: If $AB = I = AC$, then A is invertible and $B = C$.

We will see later how to compute an inverse of a matrix in general. For now, let us note that there is a simple formula for the inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For example:

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 4 - 1 \cdot 2} \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix}.$$

Let's try it:

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Just as we cannot divide by every number, not every matrix is invertible. In the case of matrices however we may have singular matrices that are not zero. For example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

is a singular matrix. But didn't we just give a formula for an inverse? Let us try it:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 2 - 1 \cdot 2} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = ?$$

We get into a bit of trouble; we are trying to divide by zero.

So a 2×2 matrix A is invertible whenever

$$ad - bc \neq 0$$

and otherwise it is singular. The expression $ad - bc$ is called the *determinant* and we will look at it more carefully in a later section. There is a similar expression for a square matrix of any size.

A.2.6 Diagonal Matrices

A simple (and surprisingly useful) type of a square matrix is a so-called *diagonal matrix*. It is a matrix whose entries are all zero except those on the main diagonal from top left to bottom right. For example a 4×4 diagonal matrix is of the form

$$\begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}.$$

Such matrices have nice properties when we multiply by them. If we multiply them by a vector, they multiply the k^{th} entry by d_k . For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 \\ 2 \cdot 5 \\ 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 18 \end{bmatrix}.$$

Similarly, when they multiply another matrix from the left, they multiply the k^{th} row by d_k . For example,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix}.$$

On the other hand, multiplying on the right, they multiply the columns:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 2 & 3 & -1 \\ 2 & 3 & -1 \end{bmatrix}.$$

And it is really easy to multiply two diagonal matrices together—we multiply the entries:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 & 0 & 0 \\ 0 & 2 \cdot 3 & 0 \\ 0 & 0 & 3 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

For this last reason, they are easy to invert, you simply invert each diagonal element:

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}^{-1} = \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \end{bmatrix}.$$

Let us check an example

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}^{-1}}_{A^{-1}} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_I.$$

It is no wonder that the way we solve many problems in linear algebra (and in differential equations) is to try to reduce the problem to the case of diagonal matrices.

A.2.7 Transpose

Vectors do not always have to be column vectors, that is just a convention. Swapping rows and columns is from time to time needed. The operation that swaps rows and columns is the so-called *transpose*. The transpose of A is denoted by A^T . Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Transpose takes an $m \times n$ matrix to an $n \times m$ matrix.

A key feature of the transpose is that if the product AB makes sense, then $B^T A^T$ also makes sense, at least from the point of view of sizes. In fact, we get precisely the transpose of AB . That is:

$$(AB)^T = B^T A^T.$$

For example,

$$\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & -2 \end{bmatrix} \right)^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

It is left to the reader to verify that computing the matrix product on the left and then transposing is the same as computing the matrix product on the right.

If we have a column vector $\vec{x}$ to which we apply a matrix A and we transpose the result, then the row vector $\vec{x}^T$ applies to A^T from the left:

$$(A\vec{x})^T = \vec{x}^T A^T.$$

Another place where transpose is useful is when we wish to apply the dot product* to two column vectors:

$$\vec{x} \cdot \vec{y} = \vec{y}^T \vec{x}.$$

That is the way that one often writes the dot product in software.

We say a matrix A is *symmetric* if $A = A^T$. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

is a symmetric matrix. Notice that a symmetric matrix is always square, that is, $n \times n$. Symmetric matrices have many nice properties†, and come up quite often in applications.

A.2.8 Exercises

Exercise A.2.1: Add the following matrices

$$a) \begin{bmatrix} -1 & 2 & 2 \\ 5 & 8 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 3 \\ 8 & 3 & 5 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -8 & -3 \\ 3 & 1 & 0 \\ 6 & -4 & 1 \end{bmatrix}$$

Exercise A.2.2: Compute

$$a) 3 \begin{bmatrix} 0 & 3 \\ -2 & 2 \end{bmatrix} + 6 \begin{bmatrix} 1 & 5 \\ -1 & 5 \end{bmatrix}$$

$$b) 2 \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} - 3 \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

Exercise A.2.3: Multiply the following matrices

$$a) \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 3 & -1 & 3 & 1 \\ 8 & 3 & 2 & -3 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & 7 \\ 1 & 2 & 3 & -1 \\ 1 & -1 & 3 & 0 \end{bmatrix}$$

$$c) \begin{bmatrix} 4 & 1 & 6 & 3 \\ 5 & 6 & 5 & 0 \\ 4 & 6 & 6 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \\ 3 & 5 \\ 5 & 6 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 & 1 & 4 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 6 & 4 \end{bmatrix}$$

Exercise A.2.4: Compute the inverse of the given matrices

$$a) \begin{bmatrix} -3 \end{bmatrix}$$

$$b) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$

$$d) \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$$

Exercise A.2.5: Compute the inverse of the given matrices

$$a) \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b) \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

Exercise A.2.101: Add the following matrices

$$a) \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

$$b) \begin{bmatrix} 6 & -2 & 3 \\ 7 & 3 & 3 \\ 8 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -1 & -3 \\ 6 & 7 & 3 \\ -9 & 4 & -1 \end{bmatrix}$$

Exercise A.2.102: Compute

$$a) 2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 3 \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$$

$$b) 3 \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} - 2 \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

*As a side note, mathematicians write $\vec{y}^T \vec{x}$ and physicists write $\vec{x}^T \vec{y}$. Shhh...don't tell anyone, but the physicists are probably right on this.

†Although so far we have not learned enough about matrices to really appreciate them.

Exercise A.2.103: Multiply the following matrices

$$a) \begin{bmatrix} 2 & 1 & 4 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & 3 \\ 3 & 5 \end{bmatrix}$$

$$b) \begin{bmatrix} 0 & 3 & 3 \\ 2 & -2 & 1 \\ 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 6 & 2 \\ 4 & 6 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$$c) \begin{bmatrix} 3 & 4 & 1 \\ 2 & -1 & 0 \\ 4 & -1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 2 & 5 & 0 \\ 2 & 0 & 5 & 2 \\ 3 & 6 & 1 & 6 \end{bmatrix}$$

$$d) \begin{bmatrix} -2 & -2 \\ 5 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 3 \end{bmatrix}$$

Exercise A.2.104: Compute the inverse of the given matrices

$$a) \begin{bmatrix} 2 \end{bmatrix}$$

$$b) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$$d) \begin{bmatrix} 4 & 2 \\ 4 & 4 \end{bmatrix}$$

Exercise A.2.105: Compute the inverse of the given matrices

$$a) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$b) \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$c) \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

A.3 Elimination

Note: 2–3 lectures

A.3.1 Linear Systems of Equations

One application of matrices is to solve systems of linear equations*. Consider the following system of linear equations

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + x_2 + 3x_3 &= 5, \\ x_1 + 4x_2 + x_3 &= 10. \end{aligned} \tag{A.2}$$

There is a systematic procedure called *elimination* to solve such a system. In this procedure, we attempt to eliminate each variable from all but one equation. We want to end up with equations such as $x_3 = 2$, where we can just read off the answer.

We write a system of linear equations as a matrix equation:

$$A\vec{x} = \vec{b}.$$

The system (A.2) is written as

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}}_{\vec{b}}.$$

If we knew the inverse of A , then we would be done; we would simply solve the equation:

$$\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}.$$

Well, but that is part of the problem, we do not know how to compute the inverse for matrices bigger than 2×2 . We will see later that to compute the inverse we are really solving $A\vec{x} = \vec{b}$ for several different $\vec{b}$. In other words, we will need to do elimination to find A^{-1} . In addition, we may wish to solve $A\vec{x} = \vec{b}$ if A is not invertible, or perhaps not even square.

Let us return to the equations themselves and see how we can manipulate them. There are a few operations we can perform on the equations that do not change the solution. First, perhaps an operation that may seem stupid, we can swap two equations in (A.2):

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 5, \\ 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + 4x_2 + x_3 &= 10. \end{aligned}$$

Clearly these new equations have the same solutions x_1, x_2, x_3 . A second operation is that we can multiply an equation by a nonzero number. For example, we multiply the third equation in (A.2) by 3:

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + x_2 + 3x_3 &= 5, \\ 3x_1 + 12x_2 + 3x_3 &= 30. \end{aligned}$$

*Although perhaps we have this backwards, quite often we solve a linear system of equations to find out something about matrices, rather than vice versa.

Finally, we can add a multiple of one equation to another equation. For instance, we add 3 times the third equation in (A.2) to the second equation:

$$\begin{array}{rrcr} 2x_1 + & 2x_2 + & 2x_3 = & 2, \\ (1+3)x_1 + & (1+12)x_2 + & (3+3)x_3 = & 5+30, \\ x_1 + & 4x_2 + & x_3 = & 10. \end{array}$$

The same x_1, x_2, x_3 should still be solutions to the new equations. These were just examples; we did not get any closer to the solution. We must do these three operations in some more logical manner, but it turns out these three operations suffice to solve every linear equation.

The first thing is to write the equations in a more compact manner. Given

$$A\vec{x} = \vec{b},$$

we write down the so-called *augmented matrix*

$$[A \mid \vec{b}],$$

where the vertical line is just a marker for us to know where the “right-hand side” of the equation starts. For the system (A.2) the augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right].$$

The entire process of elimination, which we will describe, is often applied to any sort of matrix, not just an augmented matrix. Simply think of the matrix as the 3×4 matrix

$$\left[\begin{array}{cccc} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right].$$

A.3.2 Row Echelon Form and Elementary Operations

We apply the three operations above to the matrix. We call these the *elementary operations* or *elementary row operations*. Translating the operations to the matrix setting, the operations become:

- (i) Swap two rows.
- (ii) Multiply a row by a nonzero number.
- (iii) Add a multiple of one row to another row.

We run these operations until we get into a state where it is easy to read off the answer, or until we get into a contradiction indicating no solution.

More specifically, we run the operations until we obtain the so-called *row echelon form*. Let us call the first (from the left) nonzero entry in each row the *leading entry*. A matrix is in *row echelon form* if the following conditions are satisfied:

- (i) The leading entry in any row is strictly to the right of the leading entry of the row above.
- (ii) Any zero rows are below all the nonzero rows.
- (iii) All leading entries are 1.

A matrix is in *reduced row echelon form* if furthermore the following condition is satisfied.

(iv) All the entries above a leading entry are zero.

Note that the definition applies to matrices of any size.

Example A.3.1: The following matrices are in row echelon form. The leading entries are marked:

$$\begin{bmatrix} \boxed{1} & 2 & 9 & 3 \\ 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & -1 & -3 \\ 0 & \boxed{1} & 5 \\ 0 & 0 & \boxed{1} \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \boxed{1} & -5 & 2 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

None of the matrices above are in *reduced* row echelon form. For example, in the first matrix none of the entries above the second and third leading entries are zero; they are 9, 3, and 5. The following matrices are in reduced row echelon form. The leading entries are marked:

$$\begin{bmatrix} \boxed{1} & 3 & 0 & 8 \\ 0 & 0 & \boxed{1} & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The procedure we will describe to find a reduced row echelon form of a matrix is called *Gauss–Jordan elimination*. The first part of it, which obtains a row echelon form, is called *Gaussian elimination* or *row reduction*. For some problems, a row echelon form is sufficient, and it is a bit less work to only do this first part.

To attain the row echelon form we work systematically. We go column by column, starting at the first column. We find topmost entry in the first column that is not zero, and we call it the *pivot*. If there is no nonzero entry we move to the next column. We swap rows to put the row with the pivot as the first row. We divide the first row by the pivot to make the pivot entry be a 1. Now look at all the rows below and subtract the correct multiple of the pivot row so that all the entries below the pivot become zero.

After this procedure we forget that we had a first row (it is now fixed), and we forget about the column with the pivot and all the preceding zero columns. Below the pivot row, all the entries in these columns are just zero. Then we focus on the smaller matrix and we repeat the steps above.

It is best shown by example, so let us go back to the example from the beginning of the section. We keep the vertical line in the matrix, even though the procedure works on any matrix, not just an augmented matrix. We start with the first column and we locate the pivot, in this case the first entry of the first column.

$$\left[\begin{array}{ccc|c} \boxed{2} & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right]$$

We multiply the first row by $1/2$.

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right]$$

We subtract the first row from the second and third row (two elementary operations).

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 9 \end{array} \right]$$

We are done with the first column and the first row for now. We almost pretend the matrix does not have the first column and the first row.

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & 0 & 2 & 4 \\ * & 3 & 0 & 9 \end{array} \right]$$

OK, look at the second column, and notice that now the pivot is in the third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & \boxed{3} & 0 & 9 \end{array} \right]$$

We swap rows.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{3} & 0 & 9 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

And we divide the pivot row by 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

We do not need to subtract anything as everything below the pivot is already zero. We move on, we again start ignoring the second row and second column and focus on

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & 2 & 4 \end{array} \right].$$

We find the pivot, then divide that row by 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & 4 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right].$$

The matrix is now in row echelon form.

The equation corresponding to the last row is $x_3 = 2$. We know x_3 and we could substitute it into the first two equations to get equations for x_1 and x_2 . Then we could do the same thing with x_2 , until we solve for all 3 variables. This procedure is called *backsubstitution* and we can achieve it via elementary operations. We start from the lowest pivot (leading entry in the row echelon form) and subtract the right multiple from the row above to make all the entries above this pivot zero. Then we move to the next pivot and so on. After we are done, we will have a matrix in reduced row echelon form.

We continue our example. Subtract the last row from the first to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right].$$

The entry above the pivot in the second row is already zero. So we move onto the next pivot, the one in the second row. We subtract this row from the top row to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right].$$

The matrix is in reduced row echelon form.

If we now write down the equations for x_1, x_2, x_3 , we find

$$x_1 = -4, \quad x_2 = 3, \quad x_3 = 2.$$

In other words, we have solved the system.

A.3.3 Non-Unique Solutions and Inconsistent Systems

It is possible that the solution of a linear system of equations is not unique, or that no solution exists. Suppose for a moment that the row echelon form we found was

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

Then we have an equation $0 = 1$ coming from the last row. That is impossible and the equations are what we call *inconsistent*. There is no solution to $A\vec{x} = \vec{b}$.

On the other hand, if we find a row echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

then there is no issue with finding solutions. In fact, we will find way too many. Let us continue with backsubstitution (subtracting 3 times the third row from the first) to find the reduced row echelon form and let's mark the pivots.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & -5 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The last row is all zeros; it just says $0 = 0$ and we ignore it. The two remaining equations are

$$x_1 + 2x_2 = -5, \quad x_3 = 3.$$

Let us solve for the variables that corresponded to the pivots, that is x_1 and x_3 as there was a pivot in the first column and in the third column:

$$\begin{aligned} x_1 &= -2x_2 - 5, \\ x_3 &= 3. \end{aligned}$$

The variable x_2 can be anything you wish and we still get a solution. The x_2 is called a *free variable*. There are infinitely many solutions, one for every choice of x_2 . If we pick $x_2 = 0$, then $x_1 = -5$, and $x_3 = 3$ give a solution. But we also get a solution by picking say $x_2 = 1$, in which case $x_1 = -9$ and $x_3 = 3$, or by picking $x_2 = -5$ in which case $x_1 = 5$ and $x_3 = 3$.

The general idea is that if any row has all zeros in the columns corresponding to the variables, but a nonzero entry in the column corresponding to the right-hand side $\vec{b}$, then the system is inconsistent and has no solutions. In other words, the system is inconsistent if you find a pivot on the right side of the vertical line drawn in the augmented matrix. Otherwise, the system is *consistent*, and at least one solution exists.

Suppose the system is consistent (at least one solution exists):

- (i) If every column corresponding to a variable has a pivot element, then the solution is unique.
- (ii) If there are columns corresponding to variables with no pivot, then those are *free variables* that can be chosen arbitrarily, and there are infinitely many solutions.

When $\vec{b} = \vec{0}$, we have a so-called *homogeneous matrix equation*

$$A\vec{x} = \vec{0}.$$

There is no need to write an augmented matrix in this case. As the elementary operations do not do anything to a zero column, it always stays a zero column. Moreover, $A\vec{x} = \vec{0}$ always has at least one solution, namely

$\vec{x} = \vec{0}$. Such a system is always consistent. It may have other solutions: If you find any free variables, then you get infinitely many solutions.

The set of solutions of $A\vec{x} = \vec{0}$ comes up quite often so people give it a name. It is called the *nullspace* or the *kernel* of A . One place where the kernel comes up is invertibility of a square matrix A . If the kernel of A contains a nonzero vector, then it contains infinitely many vectors (there was a free variable). But then it is impossible to invert $\vec{0}$, since infinitely many vectors go to $\vec{0}$, so there is no unique vector that A takes to $\vec{0}$. So if the kernel is nontrivial, that is, if there are any nonzero vectors in the kernel, in other words, if there are any free variables, or in yet other words, if the row echelon form of A has columns without pivots, then A is not invertible. We will return to this idea later.

A.3.4 Linear Independence and Rank

If rows of a matrix correspond to equations, it may be good to find out how many equations we really need to find the same set of solutions. Similarly, if we find a number of solutions to a linear equation $A\vec{x} = \vec{0}$, we may ask if we found enough so that all other solutions can be formed out of the given set. The concept we want is that of linear independence. That same concept is useful for differential equations, for example in [chapter 7](#).

Given row or column vectors $\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n$, a *linear combination* is an expression of the form

$$\alpha_1 \vec{y}_1 + \alpha_2 \vec{y}_2 + \dots + \alpha_n \vec{y}_n,$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are all scalars. For example, $3\vec{y}_1 + \vec{y}_2 - 5\vec{y}_3$ is a linear combination of $\vec{y}_1, \vec{y}_2$, and $\vec{y}_3$.

We have seen linear combinations before. The expression

$$A\vec{x}$$

is a linear combination of the columns of A , while

$$\vec{x}^T A = (A^T \vec{x})^T$$

is a linear combination of the rows of A .

The way linear combinations come up in our study of differential equations is similar to the following computation. Suppose that $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are solutions to $A\vec{x}_1 = \vec{0}, A\vec{x}_2 = \vec{0}, \dots, A\vec{x}_n = \vec{0}$. Then the linear combination

$$\vec{y} = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \dots + \alpha_n \vec{x}_n$$

is a solution to $A\vec{y} = \vec{0}$:

$$\begin{aligned} A\vec{y} &= A(\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \dots + \alpha_n \vec{x}_n) = \\ &= \alpha_1 A\vec{x}_1 + \alpha_2 A\vec{x}_2 + \dots + \alpha_n A\vec{x}_n = \alpha_1 \vec{0} + \alpha_2 \vec{0} + \dots + \alpha_n \vec{0} = \vec{0}. \end{aligned}$$

So if you have found enough solutions, you have them all. The question is, when did we find enough of them?

We say the vectors $\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n$ are *linearly independent* if the only solution to

$$\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \dots + \alpha_n \vec{x}_n = \vec{0}$$

is $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$. Otherwise, we say the vectors are *linearly dependent*.

For example, the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent. Let's try:

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 2\alpha_1 + \alpha_2 \end{bmatrix} = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $\alpha_1 = 0$, and then it is clear that $\alpha_2 = 0$ as well. In other words, the two vectors are linearly independent.

If a set of vectors is linearly dependent, that is, some of the α_j s are nonzero, then we can solve for one vector in terms of the others. Suppose $\alpha_1 \neq 0$. Since $\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \cdots + \alpha_n \vec{x}_n = \vec{0}$, then

$$\vec{x}_1 = \frac{-\alpha_2}{\alpha_1} \vec{x}_2 - \frac{-\alpha_3}{\alpha_1} \vec{x}_3 + \cdots + \frac{-\alpha_n}{\alpha_1} \vec{x}_n.$$

For example,

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and so

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

You may have noticed that solving for those α_j s is just solving linear equations, and so you may not be surprised that to check if a set of vectors is linearly independent we use row reduction.

Given a set of vectors, we may not be interested in just finding if they are linearly independent or not, we may be interested in finding a linearly independent subset. Or perhaps we may want to find some other vectors that give the same linear combinations and are linearly independent. The way to figure this out is to form a matrix out of our vectors. If we have row vectors we consider them as rows of a matrix. If we have column vectors we consider them columns of a matrix. The set of all linear combinations of a set of vectors is called their *span*.

$$\text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\} = \{\text{Set of all linear combinations of } \vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}.$$

Given a matrix A , the maximal number of linearly independent rows is called the *rank* of A , and we write “rank A ” for the rank. For example,

$$\text{rank} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix} = 1.$$

The second and third row are multiples of the first one. We cannot choose more than one row and still have a linearly independent set. But what is

$$\text{rank} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = ?$$

That seems to be a tougher question to answer. The first two rows are linearly independent (neither is a multiple of the other), so the rank is at least two. If we would set up the equations for the α_1 , α_2 , and α_3 , we would find a system with infinitely many solutions. One solution is

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} - 2 \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

So the set of all three rows is linearly dependent, the rank cannot be 3. Therefore the rank is 2.

But how can we do this in a more systematic way? We find the row echelon form!

$$\text{Row echelon form of } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The elementary row operations do not change the set of linear combinations of the rows (that was one of the main reasons for defining them as they were). In other words, the span of the rows of the A is the same as the span of the rows of the row echelon form of A . In particular, the number of linearly independent rows

is the same. And in the row echelon form, all nonzero rows are linearly independent. This is not hard to see. Consider the two nonzero rows in the example above. Suppose we tried to solve for the α_1 and α_2 in

$$\alpha_1 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Since the first column of the row echelon matrix has zeros except in the first row means that $\alpha_1 = 0$. For the same reason, α_2 is zero. We only have two nonzero rows, and they are linearly independent, so the rank of the matrix is 2.

The span of the rows is called the *row space*. The row space of A and the row echelon form of A are the same. In the example,

$$\begin{aligned} \text{row space of } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} &= \text{span} \left\{ \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 7 & 8 & 9 \end{bmatrix} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \right\}. \end{aligned}$$

Similarly to row space, the span of columns is called the *column space*.

$$\text{column space of } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}.$$

So it may also be good to find the number of linearly independent columns of A . One way to do that is to find the number of linearly independent rows of A^T . It is a tremendously useful fact that the number of linearly independent columns is always the same as the number of linearly independent rows:

Theorem A.3.1. $\text{rank } A = \text{rank } A^T$

In particular, to find a set of linearly independent columns we need to look at where the pivots were. If you recall above, when solving $A\vec{x} = \vec{0}$ the key was finding the pivots, any non-pivot columns corresponded to free variables. That means we can solve for the non-pivot columns in terms of the pivot columns. Let's see an example. First we reduce some random matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix}.$$

We find a pivot and reduce the rows below:

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & -1 & -2 \\ 3 & 6 & 7 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix}.$$

We find the next pivot, make it one, and rinse and repeat:

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{-1} & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The final matrix is the row echelon form of the matrix. Consider the pivots that we marked. The pivot columns are the first and the third column. All other columns correspond to free variables when solving $A\vec{x} = \vec{0}$, so all other columns can be solved in terms of the first and the third column. In other words

$$\text{column space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right\}.$$

We could perhaps use another pair of columns to get the same span, but the first and the third are guaranteed to work because they are pivot columns.

The discussion above could be expanded into a proof of the theorem if we wanted. As each nonzero row in the row echelon form contains a pivot, then the rank is the number of pivots, which is the same as the maximal number of linearly independent columns.

The idea also works in reverse. Suppose we have a bunch of column vectors and we just need to find a linearly independent set. For example, suppose we started with the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}.$$

These vectors are not linearly independent as we saw above. In particular, the span of $\vec{v}_1$ and $\vec{v}_3$ is the same as the span of all four of the vectors. So $\vec{v}_2$ and $\vec{v}_4$ can both be written as linear combinations of $\vec{v}_1$ and $\vec{v}_3$. A common thing that comes up in practice is that one gets a set of vectors whose span is the set of solutions of some problem. But perhaps we get way too many vectors, we want to simplify. For example above, all vectors in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ can be written $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 + \alpha_4 \vec{v}_4$ for some numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. But it is also true that every such vector can be written as $a\vec{v}_1 + b\vec{v}_3$ for two numbers a and b . And one has to admit, that looks much simpler. Moreover, these numbers a and b are unique. More on that in the next section.

To find this linearly independent set we simply take our vectors and form the matrix $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4]$, that is, the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix}.$$

We crank up the row-reduction machine, feed this matrix into it, find the pivot columns, and pick those. In this case, $\vec{v}_1$ and $\vec{v}_3$.

A.3.5 Computing the Inverse

If the matrix A is square and there exists a unique solution $\vec{x}$ to $A\vec{x} = \vec{b}$ for any $\vec{b}$ (there are no free variables), then A is invertible. This is equivalent to the $n \times n$ matrix A being of rank n .

In particular, if $A\vec{x} = \vec{b}$ then $\vec{x} = A^{-1}\vec{b}$. Now we just need to compute what A^{-1} is. We can surely do elimination every time we want to find $A^{-1}\vec{b}$, but that would be ridiculous. The mapping A^{-1} is linear and hence given by a matrix, and we have seen that to figure out the matrix we just need to find where A^{-1} takes the standard basis vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$.

That is, to find the first column of A^{-1} , we solve $A\vec{x} = \vec{e}_1$, because then $A^{-1}\vec{e}_1 = \vec{x}$. To find the second column of A^{-1} , we solve $A\vec{x} = \vec{e}_2$. And so on. It is really just n eliminations that we need to do. But it gets even easier. If you think about it, the elimination is the same for everything on the left side of the augmented matrix. Doing n eliminations separately we would redo most of the computations. Best is to do all at once.

Therefore, to find the inverse of A , we write an $n \times 2n$ augmented matrix $[A \mid I]$, where I is the identity matrix, whose columns are precisely the standard basis vectors. We then perform row reduction until we arrive at the reduced row echelon form. If A is invertible, then pivots can be found in every column of A , and so the reduced row echelon form of $[A \mid I]$ looks like $[I \mid A^{-1}]$. We then just read off the inverse A^{-1} . If you do not find a pivot in every one of the first n columns of the augmented matrix, then A is not invertible.

This is best seen by example. Suppose we wish to invert the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}.$$

We write the augmented matrix and we start reducing:

$$\begin{aligned}
 & \left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -5 & -2 & 1 & 0 \\ 0 & -5 & -9 & -3 & 0 & 1 \end{array} \right] \rightarrow \\
 & \rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & 5/4 & 1/2 & 1/4 & 0 \\ 0 & -5 & -9 & -3 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & 5/4 & 1/2 & 1/4 & 0 \\ 0 & 0 & -11/4 & -1/2 & -5/4 & 1 \end{array} \right] \rightarrow \\
 & \rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & 5/11 & -5/11 & 12/11 \\ 0 & \boxed{1} & 0 & 3/11 & -9/11 & 5/11 \\ 0 & 0 & \boxed{1} & 2/11 & 5/11 & -4/11 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & 5/11 & -5/11 & 12/11 \\ 0 & \boxed{1} & 0 & 3/11 & -9/11 & 5/11 \\ 0 & 0 & \boxed{1} & 2/11 & 5/11 & -4/11 \end{array} \right] \rightarrow \\
 & \rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & -1/11 & 3/11 & 2/11 \\ 0 & \boxed{1} & 0 & 3/11 & -9/11 & 5/11 \\ 0 & 0 & \boxed{1} & 2/11 & 5/11 & -4/11 \end{array} \right].
 \end{aligned}$$

So

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -1/11 & 3/11 & 2/11 \\ 3/11 & -9/11 & 5/11 \\ 2/11 & 5/11 & -4/11 \end{bmatrix}.$$

Not too terrible, no? Perhaps harder than inverting a 2×2 matrix for which we had a simple formula, but not too bad. Really in practice this is done efficiently by a computer.

A.3.6 Exercises

Exercise A.3.1: Compute the reduced row echelon form for the following matrices:

$$\begin{array}{llll}
 a) \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} & b) \begin{bmatrix} 3 & 3 \\ 6 & -3 \end{bmatrix} & c) \begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix} & d) \begin{bmatrix} 6 & 6 & 7 & 7 \\ 1 & 1 & 0 & 1 \end{bmatrix} \\
 e) \begin{bmatrix} 9 & 3 & 0 & 2 \\ 8 & 6 & 3 & 6 \\ 7 & 9 & 7 & 9 \end{bmatrix} & f) \begin{bmatrix} 2 & 1 & 3 & -3 \\ 6 & 0 & 0 & -1 \\ -2 & 4 & 4 & 3 \end{bmatrix} & g) \begin{bmatrix} 6 & 6 & 5 \\ 0 & -2 & 2 \\ 6 & 5 & 6 \end{bmatrix} & h) \begin{bmatrix} 0 & 2 & 0 & -1 \\ 6 & 6 & -3 & 3 \\ 6 & 2 & -3 & 5 \end{bmatrix}
 \end{array}$$

Exercise A.3.2: Compute the inverse of the given matrices

$$\begin{array}{lll}
 a) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & b) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} & c) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}
 \end{array}$$

Exercise A.3.3: Solve (find all solutions), or show no solution exists

$$\begin{array}{ll}
 a) \begin{array}{l} 4x_1 + 3x_2 = -2 \\ -x_1 + x_2 = 4 \end{array} & \begin{array}{l} x_1 + 5x_2 + 3x_3 = 7 \\ 8x_1 + 7x_2 + 8x_3 = 8 \\ 4x_1 + 8x_2 + 6x_3 = 4 \end{array} \\
 c) \begin{array}{l} 4x_1 + 8x_2 + 2x_3 = 3 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 4x_1 + 8x_2 = 2 \end{array} & \begin{array}{l} x + 2y + 3z = 4 \\ 2x - y + 3z = 1 \\ 3x + y + 6z = 6 \end{array}
 \end{array}$$

Exercise A.3.4: By computing the inverse, solve the following systems for $\vec{x}$.

$$\begin{array}{ll}
 a) \begin{bmatrix} 4 & 1 \\ -1 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 13 \\ 26 \end{bmatrix} & b) \begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}
 \end{array}$$

Exercise A.3.5: Compute the rank of the given matrices

$$a) \begin{bmatrix} 6 & 3 & 5 \\ 1 & 4 & 1 \\ 7 & 7 & 6 \end{bmatrix}$$

$$b) \begin{bmatrix} 5 & -2 & -1 \\ 3 & 0 & 6 \\ 2 & 4 & 5 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}$$

Exercise A.3.6: For the matrices in [Exercise A.3.5](#), find a linearly independent set of row vectors that span the row space (they do not need to be rows of the matrix).

Exercise A.3.7: For the matrices in [Exercise A.3.5](#), find a linearly independent set of columns that span the column space. That is, find the pivot columns of the matrices.

Exercise A.3.8: Find a linearly independent subset of the following vectors that has the same span.

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

Exercise A.3.101: Compute the reduced row echelon form for the following matrices:

$$a) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 & -3 & 1 \\ 4 & 6 & -2 \\ -2 & 6 & -2 \end{bmatrix}$$

$$e) \begin{bmatrix} 2 & 2 & 5 & 2 \\ 1 & -2 & 4 & -1 \\ 0 & 3 & 1 & -2 \end{bmatrix}$$

$$f) \begin{bmatrix} -2 & 6 & 4 & 3 \\ 6 & 0 & -3 & 0 \\ 4 & 2 & -1 & 1 \end{bmatrix}$$

$$g) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$h) \begin{bmatrix} 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

Exercise A.3.102: Compute the inverse of the given matrices

$$a) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$c) \begin{bmatrix} 2 & 4 & 0 \\ 2 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$$

Exercise A.3.103: Solve (find all solutions), or show no solution exists

$$a) \begin{aligned} 4x_1 + 3x_2 &= -1 \\ 5x_1 + 6x_2 &= 4 \end{aligned}$$

$$a + b + c = -1$$

$$c) \begin{aligned} a + 5b + 6c &= -1 \\ -2a + 5b + 6c &= 8 \end{aligned}$$

$$5x + 6y + 5z = 7$$

$$b) 6x + 8y + 6z = -1$$

$$5x + 2y + 5z = 2$$

$$-2x_1 + 2x_2 + 8x_3 = 6$$

$$d) \begin{aligned} x_2 + x_3 &= 2 \end{aligned}$$

$$x_1 + 4x_2 + x_3 = 7$$

Exercise A.3.104: By computing the inverse, solve the following systems for $\vec{x}$.

$$a) \begin{bmatrix} -1 & 1 \\ 3 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$b) \begin{bmatrix} 2 & 7 \\ 1 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Exercise A.3.105: Compute the rank of the given matrices

$$a) \begin{bmatrix} 7 & -1 & 6 \\ 7 & 7 & 7 \\ 7 & 6 & 2 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$c) \begin{bmatrix} 0 & 3 & -1 \\ 6 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix}$$

Exercise A.3.106: For the matrices in [Exercise A.3.105](#), find a linearly independent set of row vectors that span the row space (they do not need to be rows of the matrix).

Exercise A.3.107: For the matrices in [Exercise A.3.105](#), find a linearly independent set of columns that span the column space. That is, find the pivot columns of the matrices.

Exercise A.3.108: Find a linearly independent subset of the following vectors that has the same span.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$$

A.4 Subspaces, Dimension, and the Kernel

Note: 1 lecture

A.4.1 Subspaces, Basis, and Dimension

We often find ourselves looking at the set of solutions of a linear equation $L\vec{x} = \vec{0}$ for some matrix L , that is, we are interested in the kernel of L . The set of all such solutions has a nice structure: It looks and acts a lot like some euclidean space $\mathbb{R}^k$.

We say that a set S of vectors in $\mathbb{R}^n$ is a *subspace* if whenever $\vec{x}$ and $\vec{y}$ are members of S and α is a scalar, then

$$\vec{x} + \vec{y}, \quad \text{and} \quad \alpha\vec{x}$$

are also members of S . That is, we can add and multiply by scalars and we still land in S . So every linear combination of vectors of S is still in S . That is really what a subspace is. It is a subset where we can take linear combinations and still end up being in the subset. Consequently the span of a number of vectors is automatically a subspace.

Example A.4.1: If we let $S = \mathbb{R}^n$, then this S is a subspace of $\mathbb{R}^n$. Adding any two vectors in $\mathbb{R}^n$ gets a vector in $\mathbb{R}^n$, and so does multiplying by scalars.

The set $S' = \{\vec{0}\}$, that is, the set of the zero vector by itself, is also a subspace of $\mathbb{R}^n$. There is only one vector in this subspace, so we only need to verify the definition for that one vector, and everything checks out: $\vec{0} + \vec{0} = \vec{0}$ and $\alpha\vec{0} = \vec{0}$.

The set S'' of all the vectors of the form (a, a) for any real number a , such as $(1, 1)$, $(3, 3)$, or $(-0.5, -0.5)$ is a subspace of $\mathbb{R}^2$. Adding two such vectors, say $(1, 1) + (3, 3) = (4, 4)$ again gets a vector of the same form, and so does multiplying by a scalar, say $8(1, 1) = (8, 8)$.

If S is a subspace and we can find k linearly independent vectors in S

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k,$$

such that every other vector in S is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$, then the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is called a *basis* of S . In other words, S is the span of $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$. We say that S has dimension k , and we write

$$\dim S = k.$$

Theorem A.4.1. If $S \subset \mathbb{R}^n$ is a subspace and S is not the trivial subspace $\{\vec{0}\}$, then there exists a unique positive integer k (the dimension) and a (not unique) basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$, such that every $\vec{w}$ in S can be uniquely represented by

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_k\vec{v}_k,$$

for some scalars $\alpha_1, \alpha_2, \dots, \alpha_k$.

Just as a vector in $\mathbb{R}^k$ is represented by a k -tuple of numbers, so is a vector in a k -dimensional subspace of $\mathbb{R}^n$ represented by a k -tuple of numbers. At least once we have fixed a basis. A different basis would give a different k -tuple of numbers for the same vector.

We should reiterate that while k is unique (a subspace cannot have two different dimensions), the set of basis vectors is not at all unique. There are lots of different bases for any given subspace. Finding just the right basis for a subspace is a large part of what one does in linear algebra. In fact, that is what we spend a lot of time on in linear differential equations, although at first glance it may not seem like that is what we are doing.

Example A.4.2: The standard basis

$$\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n,$$

is a basis of $\mathbb{R}^n$, (hence the name). So as expected

$$\dim \mathbb{R}^n = n.$$

On the other hand the subspace $\{\vec{0}\}$ is of dimension 0.

The subspace S'' from [Example A.4.1](#), that is, the set of vectors (a, a) , is of dimension 1. One possible basis is simply $\{(1, 1)\}$, the single vector $(1, 1)$: every vector in S'' can be represented by $a(1, 1) = (a, a)$. Similarly another possible basis would be $\{(-1, -1)\}$. Then the vector (a, a) would be represented as $(-a)(1, 1)$.

Row and column spaces of a matrix are also examples of subspaces, as they are given as the span of vectors. We can use what we know about rank, row spaces, and column spaces from the previous section to find a basis.

Example A.4.3: In the last section, we considered the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix}.$$

Using row reduction to find the pivot columns, we found

$$\text{column space of } A \left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{bmatrix} \right) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right\}.$$

What we did was we found the basis of the column space. The basis has two elements, and so the column space of A is two-dimensional. Notice that the rank of A is two.

We would have followed the same procedure if we wanted to find the basis of the subspace X spanned by

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}.$$

We would have simply formed the matrix A with these vectors as columns and repeated the computation above. The subspace X is then the column space of A .

Example A.4.4: Consider the matrix

$$L = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}.$$

Conveniently, the matrix is in reduced row echelon form. The matrix is of rank 3. The column space is the span of the pivot columns. It is the 3-dimensional space

$$\text{column space of } L = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3.$$

The row space is the 3-dimensional space

$$\text{row space of } L = \text{span} \{ [1 \ 2 \ 0 \ 0 \ 3], [0 \ 0 \ 1 \ 0 \ 4], [0 \ 0 \ 0 \ 1 \ 5] \}.$$

As these vectors have 5 components, we think of the row space of L as a subspace of $\mathbb{R}^5$.

The way the dimensions worked out in the examples is not an accident. Since the number of vectors that we needed to take was always the same as the number of pivots, and the number of pivots is the rank, we get the following result.

Theorem A.4.2 (Rank). *The dimension of the column space and the dimension of the row space of a matrix A are both equal to the rank of A .*

A.4.2 Kernel

The set of solutions of a linear equation $L\vec{x} = \vec{0}$, the kernel of L , is a subspace: If $\vec{x}$ and $\vec{y}$ are solutions, then

$$L(\vec{x} + \vec{y}) = L\vec{x} + L\vec{y} = \vec{0} + \vec{0} = \vec{0}, \quad \text{and} \quad L(\alpha\vec{x}) = \alpha L\vec{x} = \alpha\vec{0} = \vec{0}.$$

So $\vec{x} + \vec{y}$ and $\alpha\vec{x}$ are solutions. The dimension of the kernel is called the *nullity* of the matrix.

The same sort of idea governs the solutions of linear differential equations. We try to describe the kernel of a linear differential operator, and as it is a subspace, we look for a basis of this kernel. Much of this book is dedicated to finding such bases.

The kernel of a matrix is the same as the kernel of its reduced row echelon form. For a matrix in reduced row echelon form, the kernel is rather easy to find. If a vector $\vec{x}$ is applied to a matrix L , then each entry in $\vec{x}$ corresponds to a column of L , the column that the entry multiplies. To find the kernel, pick a non-pivot column make a vector that has a -1 in the entry corresponding to this non-pivot column and zeros at all the other entries corresponding to the other non-pivot columns. Then for all the entries corresponding to pivot columns make it precisely the value in the corresponding row of the non-pivot column to make the vector be a solution to $L\vec{x} = \vec{0}$. This procedure is best understood by example.

Example A.4.5: Consider

$$L = \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 5 \end{bmatrix}.$$

This matrix is in reduced row echelon form, the pivots are marked. There are two non-pivot columns, so the kernel has dimension 2, that is, it is the span of 2 vectors. Let us find the first vector. We look at the first non-pivot column, the 2nd column, and we put a -1 in the 2nd entry of our vector. We put a 0 in the 5th entry as the 5th column is also a non-pivot column:

$$\begin{bmatrix} ? \\ -1 \\ ? \\ ? \\ 0 \end{bmatrix}.$$

Let us fill the rest. When this vector hits the first row, we get a -2 and 1 times whatever the first question mark is. So make the first question mark 2. For the second and third rows, it is sufficient to make it the question marks zero. We are really filling in the non-pivot column into the remaining entries. Let us check while marking which numbers went where:

$$\begin{bmatrix} 1 & \boxed{2} & 0 & 0 & 3 \\ 0 & \boxed{0} & 1 & 0 & 4 \\ 0 & \boxed{0} & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} \boxed{2} \\ -1 \\ \boxed{0} \\ \boxed{0} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Yay! How about the second vector. We start with

$$\begin{bmatrix} ? \\ 0 \\ ? \\ ? \\ -1 \end{bmatrix}$$

We set the first question mark to 3, the second to 4, and the third to 5. Let us check, marking things as

previously,

$$\begin{bmatrix} 1 & 2 & 0 & 0 & \boxed{3} \\ 0 & 0 & 1 & 0 & \boxed{4} \\ 0 & 0 & 0 & 1 & \boxed{5} \end{bmatrix} \begin{bmatrix} \boxed{3} \\ 0 \\ \boxed{4} \\ \boxed{5} \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

There are two non-pivot columns, so we only need two vectors. We have found the basis of the kernel. So,

$$\text{kernel of } L = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \\ 5 \\ -1 \end{bmatrix} \right\}$$

What we did in finding a basis of the kernel is we expressed all solutions of $L\vec{x} = \vec{0}$ as a linear combination of some given vectors.

The procedure to find the basis of the kernel of a matrix L :

- (i) Find the reduced row echelon form of L .
- (ii) Write down the basis of the kernel as above, one vector for each non-pivot column.

The rank of a matrix is the dimension of the column space, and that is the span on the pivot columns, while the kernel is the span of vectors one for each non-pivot column. So the two numbers must add to the number of columns.

Theorem A.4.3 (Rank–Nullity). *If a matrix A has n columns, rank r , and nullity k (dimension of the kernel), then*

$$n = r + k.$$

The theorem is immensely useful in applications. It allows one to compute the rank r if one knows the nullity k and vice versa, without doing any extra work.

Let us consider an example application, a simple version of the so-called *Fredholm alternative*. A similar result is true for differential equations. Consider

$$A\vec{x} = \vec{b},$$

where A is a square $n \times n$ matrix. There are then two mutually exclusive possibilities:

- (i) A nonzero solution $\vec{x}$ to $A\vec{x} = \vec{0}$ exists.
- (ii) The equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for every $\vec{b}$.

How does the Rank–Nullity theorem come into the picture? Well, if A has a nonzero solution $\vec{x}$ to $A\vec{x} = \vec{0}$, then the nullity k is positive. But then the rank $r = n - k$ must be less than n . It means that the column space of A is of dimension less than n , so it is a subspace that does not include everything in $\mathbb{R}^n$. So $\mathbb{R}^n$ has to contain some vector $\vec{b}$ not in the column space of A . In fact, most vectors in $\mathbb{R}^n$ are not in the column space of A .

A.4.3 Exercises

Exercise A.4.1: *For the following sets of vectors, find a basis for the subspace spanned by the vectors, and find the dimension of the subspace.*

$$a) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$c) \begin{bmatrix} -4 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$e) \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$f) \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ -2 \end{bmatrix}$$

Exercise A.4.2: For the following matrices, find a basis for the kernel (nullspace).

$$a) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 5 \\ 1 & 1 & -4 \end{bmatrix}$$

$$b) \begin{bmatrix} 2 & -1 & -3 \\ 4 & 0 & -4 \\ -1 & 1 & 2 \end{bmatrix}$$

$$c) \begin{bmatrix} -4 & 4 & 4 \\ -1 & 1 & 1 \\ -5 & 5 & 5 \end{bmatrix}$$

$$d) \begin{bmatrix} -2 & 1 & 1 & 1 \\ -4 & 2 & 2 & 2 \\ 1 & 0 & 4 & 3 \end{bmatrix}$$

Exercise A.4.3: Suppose a 5×5 matrix A has rank 3. What is the nullity?

Exercise A.4.4: Suppose that X is the set of all the vectors of $\mathbb{R}^3$ whose third component is zero. Is X a subspace? And if so, find a basis and the dimension.

Exercise A.4.5: Consider a square matrix A , and suppose that $\vec{x}$ is a nonzero vector such that $A\vec{x} = \vec{0}$. What does the Fredholm alternative say about invertibility of A .

Exercise A.4.6: Consider

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & ? & ? \\ -1 & ? & ? \end{bmatrix}.$$

If the nullity of this matrix is 2, fill in the question marks. Hint: What is the rank?

Exercise A.4.101: For the following sets of vectors, find a basis for the subspace spanned by the vectors, and find the dimension of the subspace.

$$a) \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$c) \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$$

$$d) \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$$

$$e) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$f) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Exercise A.4.102: For the following matrices, find a basis for the kernel (nullspace).

$$a) \begin{bmatrix} 2 & 6 & 1 & 9 \\ 1 & 3 & 2 & 9 \\ 3 & 9 & 0 & 9 \end{bmatrix}$$

$$b) \begin{bmatrix} 2 & -2 & -5 \\ -1 & 1 & 5 \\ -5 & 5 & -3 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & -5 & -4 \\ 2 & 3 & 5 \\ -3 & 5 & 2 \end{bmatrix}$$

$$d) \begin{bmatrix} 0 & 4 & 4 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{bmatrix}$$

Exercise A.4.103: Suppose the column space of a 9×5 matrix A of dimension 3. Find

a) Rank of A .

b) Nullity of A .

c) Dimension of the row space of A .

d) Dimension of the nullspace of A .

e) Size of the maximum subset of linearly independent rows of A .

A.5 Inner Product and Projections

Note: 1–2 lectures

A.5.1 Inner Product and Orthogonality

To do basic geometry, we need length, and we need angles. We have already seen the euclidean length, so let us figure out how to compute angles. Mostly, we are worried about the right angle*.

Given two (column) vectors in $\mathbb{R}^n$, we define the (standard) *inner product* as the dot product:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \vec{y}^T \vec{x} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^n x_i y_i.$$

Why do we seemingly give a new notation for the dot product? Because there are other possible inner products, which are not the dot product, although we will not worry about others here. An inner product can even be defined on spaces of functions as we do in chapter ??:

$$\langle f(t), g(t) \rangle = \int_a^b f(t)g(t) dt.$$

But we digress.

The inner product satisfies the following rules:

- (i) $\langle \vec{x}, \vec{x} \rangle \geq 0$, and $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = 0$,
- (ii) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$,
- (iii) $\langle a\vec{x}, \vec{y} \rangle = \langle \vec{x}, a\vec{y} \rangle = a\langle \vec{x}, \vec{y} \rangle$,
- (iv) $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$ and $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$.

Anything that satisfies the properties above can be called an inner product, although in this section we are concerned with the standard inner product in $\mathbb{R}^n$.

The standard inner product gives the euclidean length:

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

How does it give angles?

You may recall from multivariable calculus, that in two or three dimensions, the standard inner product (the dot product) gives you the angle between the vectors:

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos \theta.$$

That is, θ is the angle that $\vec{x}$ and $\vec{y}$ make when they are based at the same point.

In $\mathbb{R}^n$ (any dimension), we are simply going to say that θ from the formula is what the angle is. This makes sense as any two vectors based at the origin lie in a 2-dimensional plane (subspace), and the formula works in 2 dimensions. In fact, one could even talk about angles between functions this way, and we do in chapter ??, where we talk about orthogonal functions (functions at right angle to each other).

To compute the angle we compute

$$\cos \theta = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \|\vec{y}\|}.$$

Our angles are always in radians. We are computing the cosine of the angle, which is really the best we can do. Given two vectors at an angle θ , we can give the angle as $-\theta$, $2\pi - \theta$, etc., see Figure A.5 on the next

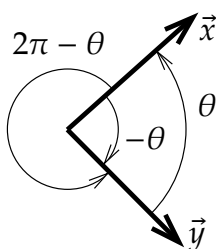


Figure A.5: Angle between vectors.

page. Fortunately, $\cos \theta = \cos(-\theta) = \cos(2\pi - \theta)$. If we solve for θ using the inverse cosine $\cos^{-1}$, we can just decree that $0 \leq \theta \leq \pi$.

Example A.5.1: Let us compute the angle between the vectors $(3, 0)$ and $(1, 1)$ in the plane. Compute

$$\cos \theta = \frac{\langle (3, 0), (1, 1) \rangle}{\|(3, 0)\| \|(1, 1)\|} = \frac{3 + 0}{3\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Therefore $\theta = \pi/4$.

As we said, the most important angle is the right angle. A right angle is $\pi/2$ radians, and $\cos(\pi/2) = 0$, so the formula is particularly easy in this case. We say vectors $\vec{x}$ and $\vec{y}$ are *orthogonal* if they are at right angles, that is if

$$\langle \vec{x}, \vec{y} \rangle = 0.$$

The vectors $(1, 0, 0, 1)$ and $(1, 2, 3, -1)$ are orthogonal. So are $(1, 1)$ and $(1, -1)$. However, $(1, 1)$ and $(1, 2)$ are not orthogonal as their inner product is 3 and not 0.

A.5.2 Orthogonal Projection

A typical application of linear algebra is to take a difficult problem, write everything in the right basis, and in this new basis the problem becomes simple. A particularly useful basis is an orthogonal basis, that is a basis where all the basis vectors are orthogonal. When we draw a coordinate system in two or three dimensions, we almost always draw our axes as orthogonal to each other.

Generalizing this concept to functions, it is particularly useful in chapter ?? to express a function using a particular orthogonal basis, the Fourier series.

To express one vector in terms of an orthogonal basis, we need to first *project* one vector onto another. Given a nonzero vector $\vec{v}$, we define the *orthogonal projection* of $\vec{w}$ onto $\vec{v}$ as

$$\text{proj}_{\vec{v}}(\vec{w}) = \left(\frac{\langle \vec{w}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \right) \vec{v}.$$

For the geometric idea, see Figure A.6 on the facing page. That is, we find the “shadow of $\vec{w}$ ” on the line spanned by $\vec{v}$ if the direction of the sun’s rays were exactly perpendicular to the line. Another way of thinking about it is that the tip of the arrow of $\text{proj}_{\vec{v}}(\vec{w})$ is the closest point on the line spanned by $\vec{v}$ to the tip of the arrow of $\vec{w}$. In terms of euclidean distance, $\vec{u} = \text{proj}_{\vec{v}}(\vec{w})$ minimizes the distance $\|\vec{w} - \vec{u}\|$ among all vectors $\vec{u}$ that are multiples of $\vec{v}$. Because of this, this projection comes up often in applied mathematics in all sorts of contexts we cannot solve a problem exactly: We cannot always solve “Find $\vec{w}$ as a multiple of $\vec{v}$,” but $\text{proj}_{\vec{v}}(\vec{w})$ is the best “solution.”

*When Euclid defined angles in his *Elements*, the only angle he ever really defined was the right angle.

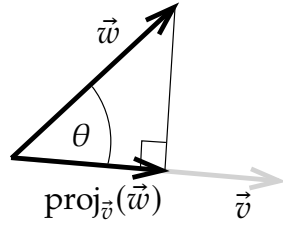


Figure A.6: Orthogonal projection.

The formula follows from basic trigonometry. The length of $\text{proj}_{\vec{v}}(\vec{w})$ should be $\cos \theta$ times the length of $\vec{w}$, that is $(\cos \theta)\|\vec{w}\|$. We take the unit vector in the direction of $\vec{v}$, that is, $\frac{\vec{v}}{\|\vec{v}\|}$ and we multiply it by the length of the projection. In other words,

$$\text{proj}_{\vec{v}}(\vec{w}) = (\cos \theta)\|\vec{w}\| \frac{\vec{v}}{\|\vec{v}\|} = \frac{(\cos \theta)\|\vec{w}\|\|\vec{v}\|}{\|\vec{v}\|^2} \vec{v} = \frac{\langle \vec{w}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}.$$

Example A.5.2: Suppose we wish to project the vector $(3, 2, 1)$ onto the vector $(1, 2, 3)$. Compute

$$\begin{aligned} \text{proj}_{(1,2,3)}((3, 2, 1)) &= \frac{\langle (3, 2, 1), (1, 2, 3) \rangle}{\langle (1, 2, 3), (1, 2, 3) \rangle} (1, 2, 3) = \frac{3 \cdot 1 + 2 \cdot 2 + 1 \cdot 3}{1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3} (1, 2, 3) \\ &= \frac{10}{14} (1, 2, 3) = \left(\frac{5}{7}, \frac{10}{7}, \frac{15}{7} \right). \end{aligned}$$

Let us double check that the projection is orthogonal. That is $\vec{w} - \text{proj}_{\vec{v}}(\vec{w})$ ought to be orthogonal to $\vec{v}$, see the right angle in Figure A.6. That is,

$$(3, 2, 1) - \text{proj}_{(1,2,3)}((3, 2, 1)) = \left(3 - \frac{5}{7}, 2 - \frac{10}{7}, 1 - \frac{15}{7} \right) = \left(\frac{16}{7}, \frac{4}{7}, \frac{-8}{7} \right)$$

ought to be orthogonal to $(1, 2, 3)$. We compute the inner product and we had better get zero:

$$\left\langle \left(\frac{16}{7}, \frac{4}{7}, \frac{-8}{7} \right), (1, 2, 3) \right\rangle = \frac{16}{7} \cdot 1 + \frac{4}{7} \cdot 2 - \frac{8}{7} \cdot 3 = 0.$$

A.5.3 Orthogonal Basis

As we said, a basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is an *orthogonal basis* if all vectors in the basis are orthogonal to each other, that is, if

$$\langle \vec{v}_j, \vec{v}_k \rangle = 0$$

for all choices of j and k where $j \neq k$ (a nonzero vector cannot be orthogonal to itself). A basis is furthermore called an *orthonormal basis* if all the vectors in a basis are also unit vectors, that is, if all the vectors have magnitude 1. For example, the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is an orthonormal basis of $\mathbb{R}^3$: Any pair is orthogonal, and each vector is of unit magnitude.

The reason why we are interested in orthogonal (or orthonormal) bases is that they make it really simple to represent a vector (or a projection onto a subspace) in the basis. The simple formula for the orthogonal projection onto a vector gives us the coefficients. In chapter ??, we use the same idea by finding the correct orthogonal basis for the set of solutions of a differential equation. We are then able to find any particular solution by simply applying the orthogonal projection formula, which is just a couple of a inner products.

Let us come back to linear algebra. Suppose that we have a subspace and an orthogonal basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. We wish to express $\vec{x}$ in terms of the basis. If $\vec{x}$ is not in the span of the basis (when it is not in the given subspace), then of course it is not possible, but the following formula gives us at least the orthogonal projection onto the subspace, or in other words, the best approximation in the subspace.

First suppose that $\vec{x}$ is in the span. Then it is the sum of the orthogonal projections:

$$\vec{x} = \text{proj}_{\vec{v}_1}(\vec{x}) + \text{proj}_{\vec{v}_2}(\vec{x}) + \dots + \text{proj}_{\vec{v}_n}(\vec{x}) = \frac{\langle \vec{x}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{x}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 + \dots + \frac{\langle \vec{x}, \vec{v}_n \rangle}{\langle \vec{v}_n, \vec{v}_n \rangle} \vec{v}_n.$$

In other words, if we want to write $\vec{x} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n$, then

$$a_1 = \frac{\langle \vec{x}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}, \quad a_2 = \frac{\langle \vec{x}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}, \quad \dots, \quad a_n = \frac{\langle \vec{x}, \vec{v}_n \rangle}{\langle \vec{v}_n, \vec{v}_n \rangle}.$$

Another way to derive this formula is to work in reverse. Suppose that $\vec{x} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n$. Take an inner product with $\vec{v}_j$, and use the properties of the inner product:

$$\begin{aligned} \langle \vec{x}, \vec{v}_j \rangle &= \langle a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n, \vec{v}_j \rangle \\ &= a_1 \langle \vec{v}_1, \vec{v}_j \rangle + a_2 \langle \vec{v}_2, \vec{v}_j \rangle + \dots + a_n \langle \vec{v}_n, \vec{v}_j \rangle. \end{aligned}$$

As the basis is orthogonal, then $\langle \vec{v}_k, \vec{v}_j \rangle = 0$ whenever $k \neq j$. That means that only one of the terms, the j^{th} one, on the right-hand side is nonzero and we get

$$\langle \vec{x}, \vec{v}_j \rangle = a_j \langle \vec{v}_j, \vec{v}_j \rangle.$$

Solving for a_j we find $a_j = \frac{\langle \vec{x}, \vec{v}_j \rangle}{\langle \vec{v}_j, \vec{v}_j \rangle}$ as before.

Example A.5.3: The vectors $(1, 1)$ and $(1, -1)$ form an orthogonal basis of $\mathbb{R}^2$. Suppose we wish to represent $(3, 4)$ in terms of this basis, that is, we wish to find a_1 and a_2 such that

$$(3, 4) = a_1(1, 1) + a_2(1, -1).$$

We compute:

$$a_1 = \frac{\langle (3, 4), (1, 1) \rangle}{\langle (1, 1), (1, 1) \rangle} = \frac{7}{2}, \quad a_2 = \frac{\langle (3, 4), (1, -1) \rangle}{\langle (1, -1), (1, -1) \rangle} = \frac{-1}{2}.$$

So

$$(3, 4) = \frac{7}{2}(1, 1) + \frac{-1}{2}(1, -1).$$

If the basis is orthonormal rather than orthogonal, then all the denominators are one. It is easy to make a basis orthonormal—divide all the vectors by their size. If you want to decompose many vectors, it may be better to find an orthonormal basis. In the example above, the orthonormal basis we would thus create is

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right).$$

Then the computation would have been

$$\begin{aligned} (3, 4) &= \left\langle (3, 4), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\rangle \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) + \left\langle (3, 4), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) \right\rangle \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) \\ &= \frac{7}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) + \frac{-1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right). \end{aligned}$$

Maybe the example is not so awe inspiring, but given vectors in $\mathbb{R}^{20}$ rather than $\mathbb{R}^2$, then surely one would much rather do 20 inner products (or 40 if we did not have an orthonormal basis) rather than solving a system of twenty equations in twenty unknowns using row reduction of a 20×21 matrix.

As we said above, the formula still works even if $\vec{x}$ is not in the subspace, although then it does not get us the vector $\vec{x}$ but its projection. More concretely, suppose that S is a subspace that is the span of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ and $\vec{x}$ is any vector. Let $\text{proj}_S(\vec{x})$ be the vector in S that is the closest to $\vec{x}$. Then

$$\text{proj}_S(\vec{x}) = \frac{\langle \vec{x}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{x}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 + \dots + \frac{\langle \vec{x}, \vec{v}_n \rangle}{\langle \vec{v}_n, \vec{v}_n \rangle} \vec{v}_n.$$

Of course, if $\vec{x}$ is in S , then $\text{proj}_S(\vec{x}) = \vec{x}$, as the closest vector in S to $\vec{x}$ is $\vec{x}$ itself. But true utility is obtained when $\vec{x}$ is not in S . In much of applied mathematics, we cannot find an exact solution to a problem, but we try to find the best solution out of a small subset (subspace). The partial sums of Fourier series from chapter ?? are one example. Another example is least square approximation to fit a curve to data. Yet another example is given by the most commonly used numerical methods to solve partial differential equations, the finite element methods.

Example A.5.4: The vectors $(1, 2, 3)$ and $(3, 0, -1)$ are orthogonal, and so they are an orthogonal basis of a subspace S :

$$S = \text{span}\{(1, 2, 3), (3, 0, -1)\}.$$

Let us find the vector in S that is closest to $(2, 1, 0)$. That is, let us find $\text{proj}_S((2, 1, 0))$.

$$\begin{aligned} \text{proj}_S((2, 1, 0)) &= \frac{\langle (2, 1, 0), (1, 2, 3) \rangle}{\langle (1, 2, 3), (1, 2, 3) \rangle} (1, 2, 3) + \frac{\langle (2, 1, 0), (3, 0, -1) \rangle}{\langle (3, 0, -1), (3, 0, -1) \rangle} (3, 0, -1) \\ &= \frac{2}{7} (1, 2, 3) + \frac{3}{5} (3, 0, -1) \\ &= \left(\frac{73}{35}, \frac{4}{7}, \frac{9}{35} \right). \end{aligned}$$

A.5.4 The Gram-Schmidt Process

Before leaving orthogonal bases, let us note a procedure for manufacturing them out of any old basis. It may not be difficult to come up with an orthogonal basis for a 2-dimensional subspace, but for a 20-dimensional subspace, it seems a daunting task. Fortunately, the orthogonal projection can be used to “project away” the bits of the vectors that are making them not orthogonal. It is called the *Gram-Schmidt process*.

We start with a basis of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. We construct an orthogonal basis $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ as follows.

$$\begin{aligned} \vec{w}_1 &= \vec{v}_1, \\ \vec{w}_2 &= \vec{v}_2 - \text{proj}_{\vec{w}_1}(\vec{v}_2), \\ \vec{w}_3 &= \vec{v}_3 - \text{proj}_{\vec{w}_1}(\vec{v}_3) - \text{proj}_{\vec{w}_2}(\vec{v}_3), \\ \vec{w}_4 &= \vec{v}_4 - \text{proj}_{\vec{w}_1}(\vec{v}_4) - \text{proj}_{\vec{w}_2}(\vec{v}_4) - \text{proj}_{\vec{w}_3}(\vec{v}_4), \\ &\vdots \\ \vec{w}_n &= \vec{v}_n - \text{proj}_{\vec{w}_1}(\vec{v}_n) - \text{proj}_{\vec{w}_2}(\vec{v}_n) - \dots - \text{proj}_{\vec{w}_{n-1}}(\vec{v}_n). \end{aligned}$$

What we do is at the k^{th} step, we take $\vec{v}_k$ and we subtract the projection of $\vec{v}_k$ to the subspace spanned by $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_{k-1}$.

Example A.5.5: Consider the vectors $(1, 2, -1)$, and $(0, 5, -2)$ and call S the span of the two vectors. Let us find an orthogonal basis of S :

$$\begin{aligned} \vec{w}_1 &= (1, 2, -1), \\ \vec{w}_2 &= (0, 5, -2) - \text{proj}_{(1, 2, -1)}((0, 5, -2)) \\ &= (0, 5, -2) - \frac{\langle (0, 5, -2), (1, 2, -1) \rangle}{\langle (1, 2, -1), (1, 2, -1) \rangle} (1, 2, -1) = (0, 5, -2) - 2(1, 2, -1) = (-2, 1, 0). \end{aligned}$$

So $(1, 2, -1)$ and $(-2, 1, 0)$ span S and are orthogonal. Let us check: $\langle (1, 2, -1), (-2, 1, 0) \rangle = 0$.

Suppose we wish to find an orthonormal basis, not just an orthogonal one. Well, we simply make the vectors into unit vectors by dividing them by their magnitude. The two vectors making up the orthonormal basis of S are:

$$\frac{1}{\sqrt{6}}(1, 2, -1) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right), \quad \frac{1}{\sqrt{5}}(-2, 1, 0) = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right).$$

A.5.5 Exercises

Exercise A.5.1: Find the s that makes the following vectors orthogonal: $(1, 2, 3), (1, 1, s)$.

Exercise A.5.2: Find the angle θ between $(1, 3, 1), (2, 1, -1)$.

Exercise A.5.3: Given that $\langle \vec{v}, \vec{w} \rangle = 3$ and $\langle \vec{v}, \vec{u} \rangle = -1$ compute

$$a) \langle \vec{u}, 2\vec{v} \rangle \quad b) \langle \vec{v}, 2\vec{w} + 3\vec{u} \rangle \quad c) \langle \vec{w} + 3\vec{u}, \vec{v} \rangle$$

Exercise A.5.4: Suppose $\vec{v} = (1, 1, -1)$. Find

$$a) \text{proj}_{\vec{v}}((1, 0, 0)) \quad b) \text{proj}_{\vec{v}}((1, 2, 3)) \quad c) \text{proj}_{\vec{v}}((1, -1, 0))$$

Exercise A.5.5: Consider the vectors $(1, 2, 3), (-3, 0, 1), (1, -5, 3)$.

- a) Check that the vectors are linearly independent and so form a basis. b) Check that the vectors are mutually orthogonal, and are therefore an orthogonal basis.
- c) Represent $(1, 1, 1)$ as a linear combination of this basis. d) Make the basis orthonormal.

Exercise A.5.6: Let S be the subspace spanned by $(1, 3, -1), (1, 1, 1)$. Find an orthogonal basis of S by the Gram-Schmidt process.

Exercise A.5.7: Starting with $(1, 2, 3), (1, 1, 1), (2, 2, 0)$, follow the Gram-Schmidt process to find an orthogonal basis of $\mathbb{R}^3$.

Exercise A.5.8: Find an orthogonal basis of $\mathbb{R}^3$ such that $(3, 1, -2)$ is one of the vectors. Hint: First find two extra vectors to make a linearly independent set.

Exercise A.5.9: Using cosines and sines of θ , find a unit vector $\vec{u}$ in $\mathbb{R}^2$ that makes angle θ with $\vec{v} = (1, 0)$. What is $\langle \vec{v}, \vec{u} \rangle$?

Exercise A.5.101: Find the s that makes the following vectors orthogonal: $(1, 1, 1), (1, s, 1)$.

Exercise A.5.102: Find the angle θ between $(1, 2, 3), (1, 1, 1)$.

Exercise A.5.103: Given that $\langle \vec{v}, \vec{w} \rangle = 1$ and $\langle \vec{v}, \vec{u} \rangle = -1$ and $\|\vec{v}\| = 3$ and

$$a) \langle 3\vec{u}, 5\vec{v} \rangle \quad b) \langle \vec{v}, 2\vec{w} + 3\vec{u} \rangle \quad c) \langle \vec{w} + 3\vec{v}, \vec{v} \rangle$$

Exercise A.5.104: Suppose $\vec{v} = (1, 0, -1)$. Find

$$a) \text{proj}_{\vec{v}}((0, 2, 1)) \quad b) \text{proj}_{\vec{v}}((1, 0, 1)) \quad c) \text{proj}_{\vec{v}}((4, -1, 0))$$

Exercise A.5.105: The vectors $(1, 1, -1), (2, -1, 1), (1, -5, 3)$ form an orthogonal basis. Represent the following vectors in terms of this basis:

$$a) (1, -8, 4) \quad b) (5, -7, 5) \quad c) (0, -6, 2)$$

Exercise A.5.106: Let S be the subspace spanned by $(2, -1, 1), (2, 2, 2)$. Find an orthogonal basis of S by the Gram-Schmidt process.

Exercise A.5.107: Starting with $(1, 1, -1), (2, 3, -1), (1, -1, 1)$, follow the Gram-Schmidt process to find an orthogonal basis of $\mathbb{R}^3$.

A.6 Determinant

Note: 1 lecture

For square matrices we define a useful quantity called the *determinant*. Define the determinant of a 1×1 matrix as the value of its only entry

$$\det([a]) \stackrel{\text{def}}{=} a.$$

For a 2×2 matrix, define

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \stackrel{\text{def}}{=} ad - bc.$$

Before defining the determinant for larger matrices, we note the meaning of the determinant. An $n \times n$ matrix gives a mapping of the n -dimensional euclidean space $\mathbb{R}^n$ to itself. So a 2×2 matrix A is a mapping of the plane to itself. The determinant of A is the factor by which the area of objects changes. If we take the unit square (square of side 1) in the plane, then A takes the square to a parallelogram of area $|\det(A)|$. The sign of $\det(A)$ denotes a change of orientation (negative if the axes get flipped). For example, let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then $\det(A) = 1 + 1 = 2$. Let us see where A sends the unit square—the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. The point $(0, 0)$ gets sent to $(0, 0)$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The image of the square is another square with vertices $(0, 0)$, $(1, -1)$, $(1, 1)$, and $(2, 0)$. The image square has a side of length $\sqrt{2}$, and it is therefore of area 2. See [Figure A.7](#).

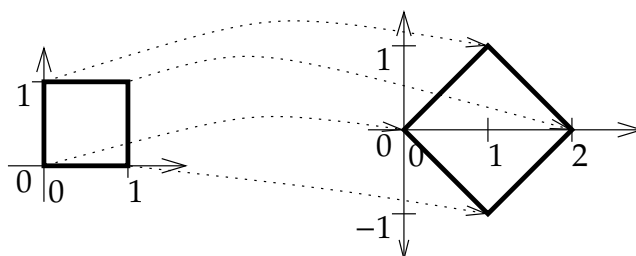


Figure A.7: Image of the unit square via the mapping A .

In general, the image of a square is going to be a parallelogram. In high school geometry, you may have seen a formula for computing the area of a parallelogram with vertices $(0, 0)$, (a, c) , (b, d) and $(a + b, c + d)$. The area is

$$\left| \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \right| = |ad - bc|.$$

The vertical lines above mean absolute value. The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ carries the unit square to the given parallelogram.

There are a number of ways to define the determinant for an $n \times n$ matrix. Let us use the so-called *cofactor expansion*. We define A_{ij} as the matrix A with the i^{th} row and the j^{th} column deleted. For example, if

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{then } A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \quad \text{and} \quad A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}.$$

We now define the determinant recursively

$$\det(A) \stackrel{\text{def}}{=} \sum_{j=1}^n (-1)^{1+j} a_{1j} \det(A_{1j}),$$

or in other words

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) - \dots \begin{cases} +a_{1n} \det(A_{1n}) & \text{if } n \text{ is odd,} \\ -a_{1n} \det(A_{1n}) & \text{if } n \text{ even.} \end{cases}$$

For a 3×3 matrix, we get $\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13})$. For example,

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= 1 \cdot \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0. \end{aligned}$$

It turns out that we did not have to necessarily use the first row. That is for any i ,

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

It is sometimes useful to use a row other than the first. In the following example it is more convenient to expand along the second row. Notice that for the second row we are starting with a negative sign.

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix} &= -0 \cdot \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 5 \cdot \det \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} \\ &= 0 + 5(1 \cdot 9 - 3 \cdot 7) + 0 = -60. \end{aligned}$$

Let us check if it is really the same as expanding along the first row,

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix} &= 1 \cdot \det \begin{pmatrix} 5 & 0 \\ 8 & 9 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 0 & 0 \\ 7 & 9 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 0 & 5 \\ 7 & 8 \end{pmatrix} \\ &= 1(5 \cdot 9 - 0 \cdot 8) - 2(0 \cdot 9 - 0 \cdot 7) + 3(0 \cdot 8 - 5 \cdot 7) = -60. \end{aligned}$$

In computing the determinant, we alternately add and subtract the determinants of the submatrices A_{ij} multiplied by a_{ij} for a fixed i and all j . The numbers $(-1)^{i+j} \det(A_{ij})$ are called *cofactors* of the matrix. And that is why this method of computing the determinant is called the *cofactor expansion*.

Similarly we do not need to expand along a row, we can expand along a column. For any j ,

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

A related fact is that

$$\det(A) = \det(A^T).$$

A matrix is *upper triangular* if all elements below the main diagonal are 0. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

is upper triangular. Similarly a *lower triangular* matrix is one where everything above the diagonal is zero. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}.$$

The determinant for triangular matrices is very simple to compute. Consider the lower triangular matrix. If we expand along the first row, we find that the determinant is 1 times the determinant of the lower triangular matrix $\begin{bmatrix} 5 & 0 \\ 8 & 9 \end{bmatrix}$. So the determinant is just the product of the diagonal entries:

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}\right) = 1 \cdot 5 \cdot 9 = 45.$$

Similarly for upper triangular matrices

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}\right) = 1 \cdot 5 \cdot 9 = 45.$$

In general, if A is triangular, then

$$\det(A) = a_{11}a_{22} \cdots a_{nn}.$$

If A is diagonal, then it is also triangular (upper and lower), so same formula applies. For example,

$$\det\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right) = 2 \cdot 3 \cdot 5 = 30.$$

In particular, the identity matrix I is diagonal, and the diagonal entries are all 1. Thus,

$$\det(I) = 1.$$

The determinant is telling you how geometric objects scale. If B doubles the sizes of geometric objects and A triples them, then AB (which applies B to an object and then it applies A) should make size go up by a factor of 6. This is true in general:

Theorem A.6.1.

$$\det(AB) = \det(A) \det(B).$$

This property is one of the most useful, and it is employed often to actually compute determinants. A particularly interesting consequence is to note what it means for the existence of inverses. Take A and B to be inverses, that is $AB = I$. Then

$$\det(A) \det(B) = \det(AB) = \det(I) = 1.$$

Neither $\det(A)$ nor $\det(B)$ can be zero. This fact is an extremely useful property of the determinant, and one which is used often in this book:

Theorem A.6.2. *An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.*

In fact, $\det(A^{-1}) \det(A) = 1$ says that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

So we know what the determinant of A^{-1} is without computing A^{-1} .

Let us return to the formula for the inverse of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Notice the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the denominator of the fraction. The formula only works if the determinant is nonzero, otherwise we are dividing by zero.

A common notation for the determinant is a pair of vertical lines:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).$$

Personally, I find this notation confusing as vertical lines usually mean a positive quantity, while determinants can be negative. Also think about how to write the absolute value of a determinant. This notation is not used in this book.

A.6.1 Exercises

Exercise A.6.1: Compute the determinant of the following matrices:

$$\begin{array}{llll} a) [3] & b) \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} & c) \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} & d) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \\ e) \begin{bmatrix} 2 & 1 & 0 \\ -2 & 7 & -3 \\ 0 & 2 & 0 \end{bmatrix} & f) \begin{bmatrix} 2 & 1 & 3 \\ 8 & 6 & 3 \\ 7 & 9 & 7 \end{bmatrix} & g) \begin{bmatrix} 0 & 2 & 5 & 7 \\ 0 & 0 & 2 & -3 \\ 3 & 4 & 5 & 7 \\ 0 & 0 & 2 & 4 \end{bmatrix} & h) \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 2 & 1 \\ 2 & -1 & -2 & 3 \end{bmatrix} \end{array}$$

Exercise A.6.2: For which x are the following matrices singular (not invertible).

$$\begin{array}{llll} a) \begin{bmatrix} 2 & 3 \\ 2 & x \end{bmatrix} & b) \begin{bmatrix} 2 & x \\ 1 & 2 \end{bmatrix} & c) \begin{bmatrix} x & 1 \\ 4 & x \end{bmatrix} & d) \begin{bmatrix} x & 0 & 1 \\ 1 & 4 & 2 \\ 1 & 6 & 2 \end{bmatrix} \end{array}$$

Exercise A.6.3: Compute

$$\det \left(\begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & 8 & 6 & 5 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \right)$$

without computing the inverse.

Exercise A.6.4: Suppose

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 7 & \pi & 1 & 0 \\ 2^8 & 5 & -99 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 5 & 9 & 1 & -\sin(1) \\ 0 & 1 & 88 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $A = LU$. Compute $\det(A)$ in a simple way, without computing what is A . Hint: First read off $\det(L)$ and $\det(U)$.

Exercise A.6.5: Consider the linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 1 & x \\ 2 & 1 \end{bmatrix}$ for some number x . You wish to make A such that it doubles the area of every geometric figure. What are the possibilities for x (there are two answers).

Exercise A.6.6: Suppose A and S are $n \times n$ matrices, and S is invertible. Suppose that $\det(A) = 3$. Compute $\det(S^{-1}AS)$ and $\det(SAS^{-1})$. Justify your answer using the theorems in this section.

Exercise A.6.7: Let A be an $n \times n$ matrix such that $\det(A) = 1$. Compute $\det(xA)$ given a number x . Hint: First try computing $\det(xI)$, then note that $xA = (xI)A$.

Exercise A.6.101: Compute the determinant of the following matrices:

$$a) \begin{bmatrix} -2 \end{bmatrix}$$

$$b) \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$$

$$c) \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$d) \begin{bmatrix} 2 & 9 & -11 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

$$e) \begin{bmatrix} 2 & 1 & 0 \\ -2 & 7 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$f) \begin{bmatrix} 5 & 1 & 3 \\ 4 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$$

$$g) \begin{bmatrix} 3 & 2 & 5 & 7 \\ 0 & 0 & 2 & 0 \\ 0 & 4 & 5 & 0 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

$$h) \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 2 & -3 & 4 \\ 5 & 6 & -7 & 8 \\ 1 & 2 & 3 & -2 \end{bmatrix}$$

Exercise A.6.102: For which x are the following matrices singular (not invertible).

$$a) \begin{bmatrix} 1 & 3 \\ 1 & x \end{bmatrix}$$

$$b) \begin{bmatrix} 3 & x \\ 1 & 3 \end{bmatrix}$$

$$c) \begin{bmatrix} x & 3 \\ 3 & x \end{bmatrix}$$

$$d) \begin{bmatrix} x & 1 & 0 \\ 1 & 4 & 0 \\ 1 & 6 & 2 \end{bmatrix}$$

Exercise A.6.103: Compute

$$\det \left(\begin{bmatrix} 3 & 4 & 7 & 12 \\ 0 & -1 & 9 & -8 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-1} \right)$$

without computing the inverse.

Exercise A.6.104 (challenging): Find all the x that make the matrix inverse

$$\begin{bmatrix} 1 & 2 \\ 1 & x \end{bmatrix}^{-1}$$

have only integer entries (no fractions). Note that there are two answers.

Appendix B

Appendix: Table of Laplace Transforms

On the next page is a brief table of common Laplace transforms.

Table of Laplace Transforms

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	(1) $\delta(t)$	1 (19)
$f'(t)$	$sF(s) - f(0)$	(2) $\delta(t - t_0)$	e^{-st_0} (20)
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	(3) $\frac{e^{at} - e^{bt}}{a - b}$	$\frac{1}{(s - a)(s - b)}$ (21)
$f^{(n)}(t)$	$s^n F(s) - s^{(n-1)}f(0) - \dots - f^{(n-1)}(0)$	$\frac{ae^{at} - be^{bt}}{a - b}$	$\frac{s}{(s - a)(s - b)}$ (22)
$e^{at}f(t)$	$F(s - a)$	(5) $\sinh(kt)$	$\frac{k}{s^2 - k^2}$ (23)
$f(t - a)\mathcal{U}(t - a)$	$e^{-as}F(s)$	(6) $\cosh(kt)$	$\frac{s}{s^2 - k^2}$ (24)
$f(t)\mathcal{U}(t - a)$	$e^{-as}F(a + s)$	(7) te^{at}	$\frac{1}{(s - a)^2}$ (25)
$tf(t)$	$-\frac{d}{ds}F(s)$	(8) $t^n e^{at}$	$\frac{n!}{(s - a)^{n+1}}$ (26)
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	(9) $e^{at} \sin(kt)$	$\frac{k}{(s - a)^2 + k^2}$ (27)
$\int_0^t f(\tau)g(t - \tau)d\tau$	$F(s)G(s)$	(10) $e^{at} \cos(kt)$	$\frac{s - a}{(s - a)^2 + k^2}$ (28)
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	(11) $e^{at} \sinh(kt)$	$\frac{k}{(s - a)^2 - k^2}$ (29)
1	$\frac{1}{s}$	(12) $e^{at} \cosh(kt)$	$\frac{s - a}{(s - a)^2 - k^2}$ (30)
$t^n (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$	(13) $t \sin(kt)$	$\frac{2ks}{(s^2 + k^2)^2}$ (31)
$t^a (a > -1)$	$\frac{\Gamma(a + 1)}{s^{a+1}}$	(14) $t \cos(kt)$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ (32)
$\sin(kt)$	$\frac{k}{s^2 + k^2}$	(15) $t \sinh(kt)$	$\frac{2ks}{(s^2 - k^2)^2}$ (33)
$\cos(kt)$	$\frac{s}{s^2 + k^2}$	(16) $t \cosh(kt)$	$\frac{s^2 - k^2}{(s^2 - k^2)^2}$ (34)
e^{at}	$\frac{1}{s - a}$	(17) $\frac{\sin(kt)}{t}$	$\arctan\left(\frac{k}{s}\right)$ (35)
$\mathcal{U}(t - a)$	$\frac{e^{-as}}{s}$	(18)	

FACTS: • $\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt$; $\Gamma(a + 1) = a\Gamma(a)$; $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ • $\cosh(t) = \frac{e^t + e^{-t}}{2}$; $\sinh(t) = \frac{e^t - e^{-t}}{2}$

Further Reading

- [BM] Paul W. Berg and James L. McGregor, *Elementary Partial Differential Equations*, Holden-Day, San Francisco, CA, 1966.
- [BD] William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 11th edition, John Wiley & Sons Inc., New York, NY, 2017.
- [EP] C.H. Edwards and D.E. Penney, *Differential Equations and Boundary Value Problems: Computing and Modeling*, 5th edition, Pearson, 2014.
- [F] Stanley J. Farlow, *An Introduction to Differential Equations and Their Applications*, McGraw-Hill, Inc., Princeton, NJ, 1994. (Published also by Dover Publications, 2006.)
- [I] E.L. Ince, *Ordinary Differential Equations*, Dover Publications, Inc., New York, NY, 1956.
- [T] William F. Trench, *Elementary Differential Equations with Boundary Value Problems*. Books and Monographs. Book 9. 2013. <https://digitalcommons.trinity.edu/mono/9>

Solutions to Selected Exercises

0.1.101: Compute $x' = -2e^{-2t}$ and $x'' = 4e^{-2t}$. Then $(4e^{-2t}) + 4(-2e^{-2t}) + 4(e^{-2t}) = 0$.

0.1.102: Yes.

0.1.103: $y = x^r$ is a solution for $r = 0$ and $r = 2$.

0.1.104: $C_1 = 100, C_2 = -90$

0.1.105: $\varphi = -9e^{8s}$

0.1.106: a) $x = 9e^{-4t}$ b) $x = \cos(2t) + \sin(2t)$ c) $p = 4e^{3q}$ d) $T = 3 \sinh(2x)$

0.2.101: a) PDE, equation, second order, linear, nonhomogeneous, constant coefficient.

b) ODE, equation, first order, linear, nonhomogeneous, not constant coefficient, not autonomous.

c) ODE, equation, seventh order, linear, homogeneous, constant coefficient, autonomous.

d) ODE, equation, second order, linear, nonhomogeneous, constant coefficient, autonomous.

e) ODE, system, second order, nonlinear.

f) PDE, equation, second order, nonlinear.

0.2.102: equation: $a(x)y = b(x)$, solution: $y = \frac{b(x)}{a(x)}$.

0.2.103: $k = 0$ or $k = 1$

4.1.101: $y = e^x + \frac{x^2}{2} + 9$

4.1.102: $x = (3t - 2)^{1/3}$

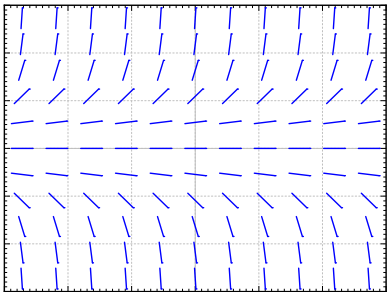
4.1.103: $x = \sin^{-1}(t + 1/\sqrt{2})$

4.1.104: 170

4.1.105: If $n \neq 1$, then $y = ((1 - n)x + 1)^{1/(1-n)}$. If $n = 1$, then $y = e^x$.

4.1.106: The equation is $r' = -C$ for some constant C . The snowball will be completely melted in 25 minutes from time $t = 0$.

4.1.107: $y = Ax^3 + Bx^2 + Cx + D$, so 4 constants.

4.2.101:  $y = 0$ is a solution such that $y(0) = 0$.

4.2.102: Yes a solution exists. The equation is $y' = f(x, y)$ where $f(x, y) = xy$. The function $f(x, y)$ is continuous and $\frac{\partial f}{\partial y} = x$, which is also continuous near $(0, 0)$. So a solution exists and is unique. (In fact, $y = 0$ is the solution.)

4.2.103: No, the equation is not defined at $(x, y) = (1, 0)$.

4.2.104: a) $y' = \cos y$, b) $y' = y \cos(x)$, c) $y' = \sin x$. Justification left to reader.

4.2.105: Picard does not apply as f is not continuous at $y = 0$. The equation does not have a continuously differentiable solution. Suppose it did. Notice that $y'(0) = 1$. By the first derivative test, $y(x) > 0$ for small positive x . But then for those x , we have $y'(x) = f(y(x)) = 0$. It is not possible for y' to be continuous, $y'(0) = 1$ and $y'(x) = 0$ for arbitrarily small positive x .

4.2.106: The solution is $y(x) = \int_{x_0}^x f(s) ds + y_0$, and this does indeed exist for every x .

4.3.101: $y = Ce^{x^2}$

4.3.102: $x = e^{t^3} + 1$

4.3.103: $x^3 + x = t + 2$

4.3.104: $y = \frac{1}{1 - \ln x}$

4.3.105: $\sin(y) = -\cos(x) + C$

4.3.106: The range is approximately 7.45 to 12.15 minutes.

4.3.107: a) $x = \frac{1000e^t}{e^t + 24}$. b) 102 rabbits after one month, 861 after 5 months, 999 after 10 months, 1000 after 15 months.

4.4.101: $y = Ce^{-x^3} + 1/3$

4.4.102: $y = 2e^{\cos(2x)+1} + 1$

4.4.103: 250 grams

4.4.104: $P(5) = 1000e^{2 \times 5 - 0.05 \times 5^2} = 1000e^{8.75} \approx 6.31 \times 10^6$

4.4.105: $Ah' = I - kh$, where k is a constant with units m^2/s .

4.5.101: $y = \frac{2}{3x-2}$

4.5.102: $y = \frac{3-x^2}{2x}$

4.5.103: $y = (7e^{3x} + 3x + 1)^{1/3}$

4.5.104: $y = \pm \sqrt{x^2 - \ln(C - x)}$

4.6.101: a) 0, 1, 2 are critical points. b) $x = 0$ is unstable (semistable), $x = 1$ is stable, and $x = 2$ is unstable. c) 1

4.6.102: a) There are no critical points. b) ∞

4.6.103: a) $\frac{dx}{dt} = kx(M - x) + A$ b) $\frac{kM + \sqrt{(kM)^2 + 4Ak}}{2k}$

4.6.104: a) α is a stable critical point, β is an unstable one. b) α , c) α , d) ∞ or DNE.

4.7.101: Approximately: 1.0000, 1.2397, 1.3829

4.7.102: a) 0, 8, 12 b) $x(4) = 16$, so errors are: 16, 8, 4. c) Factors are 0.5, 0.5, 0.5.

4.7.103: a) 0, 0, 0 b) $x = 0$ is a solution so errors are: 0, 0, 0.

4.7.104: a) Improved Euler: $y(1) \approx 3.3897$ for $h = 1/4$, $y(1) \approx 3.4237$ for $h = 1/8$, b) Standard Euler: $y(1) \approx 2.8828$ for $h = 1/4$, $y(1) \approx 3.1316$ for $h = 1/8$, c) $y = 2e^x - x - 1$, so $y(2)$ is approximately 3.4366. d) Approximate errors for improved Euler: 0.046852 for $h = 1/4$, and 0.012881 for $h = 1/8$. For standard Euler: 0.55375 for $h = 1/4$, and 0.30499 for $h = 1/8$. Factor is approximately 0.27 for improved Euler, and 0.55 for standard Euler.

7.1.101: Yes. To justify try to find a constant A such that $\sin(x) = Ae^x$ for all x .

7.1.102: No. $e^{x+2} = e^2 e^x$.

7.1.103: $y = 5$

7.1.104: $y = C_1 \ln(x) + C_2$

7.1.105: $y'' - 3y' + 2y = 0$

$$7.2.101: y = C_1 e^{(-2+\sqrt{2})x} + C_2 e^{(-2-\sqrt{2})x}$$

$$7.2.102: y = C_1 e^{3x} + C_2 x e^{3x}$$

$$7.2.103: y = e^{-x/4} \cos((\sqrt{7}/4)x) - \sqrt{7} e^{-x/4} \sin((\sqrt{7}/4)x)$$

$$7.2.104: y = \frac{2(a-b)}{5} e^{-3x/2} + \frac{3a+2b}{5} e^x$$

$$7.2.105: z(t) = 2e^{-t} \cos(t)$$

$$7.2.106: y = \frac{a\beta-b}{\beta-\alpha} e^{\alpha x} + \frac{b-a\alpha}{\beta-\alpha} e^{\beta x}$$

$$7.2.107: y'' - y' - 6y = 0$$

$$7.3.101: y = C_1 e^x + C_2 x^3 + C_3 x^2 + C_4 x + C_5$$

$$7.3.102: \text{a) } r^3 - 3r^2 + 4r - 12 = 0 \quad \text{b) } y''' - 3y'' + 4y' - 12y = 0 \quad \text{c) } y = C_1 e^{3x} + C_2 \sin(2x) + C_3 \cos(2x)$$

$$7.3.103: y = 0$$

$$7.3.104: \text{No. } e^1 e^x - e^{x+1} = 0.$$

7.3.105: Yes. (Hint: First note that $\sin(x)$ is bounded. Then note that x and $x \sin(x)$ cannot be multiples of each other.)

$$7.3.106: y''' - y'' + y' - y = 0$$

$$7.4.101: k = 8/9 \text{ (and larger)}$$

$$7.4.102: \text{a) } 0.05I'' + 0.1I' + (1/5)I = 0 \quad \text{b) } I = C e^{-t} \cos(\sqrt{3}t - \gamma) \quad \text{c) } I = 10e^{-t} \cos(\sqrt{3}t) + \frac{10}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t)$$

$$7.4.103: \text{a) } k = 500000 \quad \text{b) } \frac{1}{5\sqrt{2}} \approx 0.141 \quad \text{c) } 45000 \text{ kg} \quad \text{d) } 11250 \text{ kg}$$

$$7.4.104: m_0 = \frac{1}{3}. \text{ If } m < m_0, \text{ then the system is overdamped and will not oscillate.}$$

$$7.5.101: y = \frac{-16 \sin(3x) + 6 \cos(3x)}{73}$$

$$7.5.102: \text{a) } y = \frac{2e^x + 3x^3 - 9x}{6} \quad \text{b) } y = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x) + \frac{2e^x + 3x^3 - 9x}{6}$$

$$7.5.103: y(x) = x^2 - 4x + 6 + e^{-x}(x - 5)$$

$$7.5.104: y = \frac{2xe^x - (e^x + e^{-x}) \ln(e^{2x} + 1)}{4}$$

$$7.5.105: y = \frac{-\sin(x+c)}{3} + C_1 e^{\sqrt{2}x} + C_2 e^{-\sqrt{2}x}$$

$$7.5.106: y = x^2 + 2x + 3$$

$$7.6.101: \omega = \frac{\sqrt{31}}{4\sqrt{2}} \approx 0.984 \quad C(\omega) = \frac{16}{3\sqrt{7}} \approx 2.016$$

$$7.6.102: x_{sp} = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \sin(\omega t) + \frac{A}{k}, \text{ where } p = \frac{c}{2m} \text{ and } \omega_0 = \sqrt{\frac{k}{m}}.$$

$$7.6.103: \text{a) } \omega = 2 \quad \text{b) } 25$$

10.1.101: Yes. Radius of convergence is 10.

10.1.102: Yes. Radius of convergence is e .

$$10.1.103: \frac{1}{1-x} = -\frac{1}{1-(2-x)} \text{ so } \frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n, \text{ which converges for } 1 < x < 3.$$

$$10.1.104: \sum_{n=7}^{\infty} \frac{1}{(n-7)!} x^n$$

10.1.105: $f(x) - g(x)$ is a polynomial. Hint: Use Taylor series.

10.2.101: $a_2 = 0, a_3 = 0, a_4 = 0$, recurrence relation (for $k \geq 5$): $a_k = \frac{-2a_{k-5}}{k(k-1)}$, so:

$$y(x) = a_0 + a_1 x - \frac{a_0}{10} x^5 - \frac{a_1}{15} x^6 + \frac{a_0}{450} x^{10} + \frac{a_1}{825} x^{11} - \frac{a_0}{47250} x^{15} - \frac{a_1}{99000} x^{16} + \dots$$

10.2.102: a) $a_2 = \frac{1}{2}$, and for $k \geq 1$ we have $a_k = \frac{a_{k-3}+1}{k(k-1)}$, so

$$y(x) = a_0 + a_1x + \frac{1}{2}x^2 + \frac{a_0+1}{6}x^3 + \frac{a_1+1}{12}x^4 + \frac{3}{40}x^5 + \frac{a_0+2}{30}x^6 + \frac{a_1+2}{42}x^7 + \frac{5}{112}x^8 + \frac{a_0+3}{72}x^9 + \frac{a_1+3}{90}x^{10} + \dots$$

b) $y(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 + \frac{1}{15}x^6 + \frac{1}{21}x^7 + \frac{5}{112}x^8 + \frac{1}{24}x^9 + \frac{1}{30}x^{10} + \dots$

10.2.103: Applying the method of this section directly we obtain $a_k = 0$ for all k and so $y(x) = 0$ is the only solution we find.

10.3.101: a) ordinary, b) singular but not regular singular, c) regular singular, d) regular singular, e) ordinary.

10.3.102: $y = Ax^{\frac{1+\sqrt{5}}{2}} + Bx^{\frac{1-\sqrt{5}}{2}}$

10.3.103: $y = x^{3/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+2)!} x^k$ (Note that for convenience we did not pick $a_0 = 1$.)

10.3.104: $y = Ax + Bx \ln(x)$

12.1.101: $\frac{8}{s^3} + \frac{8}{s^2} + \frac{4}{s}$

12.1.102: $2t^2 - 2t + 1 - e^{-2t}$

12.1.103: $\frac{1}{(s+1)^2}$

12.1.104: $\frac{1}{s^2+2s+2}$

12.2.101: $f(t) = (t-1)(u(t-1) - u(t-2)) + u(t-2)$

12.2.102: $x(t) = (2e^{t-1} - t^2 - 1)u(t-1) - \frac{1}{2}e^{-t} + \frac{3}{2}e^t$

12.2.103: $H(s) = \frac{1}{s+1}$

12.2.104: $F(s) = \frac{4}{s}$, $X(s) = \frac{1}{s} - \frac{s}{s^2+4}$, $H(s) = \frac{1}{s^2+4}$.

12.3.101: $\frac{1}{2}(\cos t + \sin t - e^{-t})$

12.3.102: $5t - 5 \sin t$

12.3.103: $\frac{1}{2}(\sin t - t \cos t)$

12.3.104: $\int_0^t f(\tau)(1 - \cos(t - \tau)) d\tau$

12.4.101: $x(t) = t$

12.4.102: $x(t) = e^{-at}$

12.4.103: $x(t) = (e^t) * (e^t \sin(t)) = e^t(1 - \cos(t))$

12.4.104: $\delta(t) - \sin(t)$

12.4.105: $3\delta(t-1) + 2t$

14.1.101: $y_1 = C_1 e^{3x}$, $y_2 = y(x) = C_2 e^x + \frac{C_1}{2} e^{3x}$, $y_3 = y(x) = C_3 e^x + \frac{C_1}{2} e^{3x}$

14.1.102: $x = \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t}$, $y = \frac{5}{3}e^{2t} + \frac{4}{3}e^{-t}$

14.1.103: $x'_1 = x_2$, $x'_2 = x_3$, $x'_3 = x_1 + t$

14.1.104: $y'_3 + y_1 + y_2 = t$, $y'_4 + y_1 - y_2 = t^2$, $y'_1 = y_3$, $y'_2 = y_4$

14.1.105: $x_1 = x_2 = at$. Explanation of the intuition is left to reader.

14.1.106: a) Left to reader. b) $x'_1 = \frac{t-s}{\sqrt{t}}x_2 - \frac{t}{\sqrt{t}}x_1$, $x'_2 = \frac{t}{\sqrt{t}}(x_1 - x_2)$. c) As t goes to infinity, both x_1 and x_2 go to zero, explanation is left to reader.

14.2.101: -15

14.2.102: -2

14.2.103: $\vec{x} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$

14.2.104: a) $\begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$ b) $\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$

14.3.101: Yes.

14.3.102: No. $2 \begin{bmatrix} \cosh(t) \\ 1 \end{bmatrix} - \begin{bmatrix} e^t \\ 1 \end{bmatrix} - \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} = \vec{0}$

14.3.103: $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$

14.3.104: a) $\vec{x}' = \begin{bmatrix} 0 & 2t \\ 0 & 2t \end{bmatrix} \vec{x}$ b) $\vec{x} = \begin{bmatrix} C_2 e^{t^2} + C_1 \\ C_2 e^{t^2} \end{bmatrix}$

14.4.101: a) Eigenvalues: 4, 0, -1 Eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$

b) $\vec{x} = C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{4t} + C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} e^{-t}$

14.4.102: a) Eigenvalues: $\frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$, Eigenvectors: $\begin{bmatrix} -2 \\ 1-\sqrt{3}i \end{bmatrix}, \begin{bmatrix} -2 \\ 1+\sqrt{3}i \end{bmatrix}$

b) $\vec{x} = C_1 e^{t/2} \begin{bmatrix} -2 \cos(\frac{\sqrt{3}t}{2}) \\ \cos(\frac{\sqrt{3}t}{2}) + \sqrt{3} \sin(\frac{\sqrt{3}t}{2}) \end{bmatrix} + C_2 e^{t/2} \begin{bmatrix} -2 \sin(\frac{\sqrt{3}t}{2}) \\ \sin(\frac{\sqrt{3}t}{2}) - \sqrt{3} \cos(\frac{\sqrt{3}t}{2}) \end{bmatrix}$

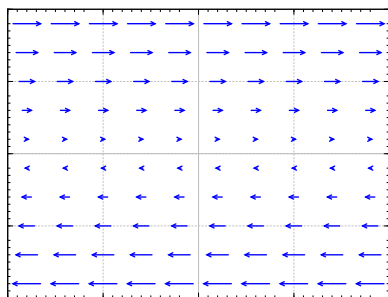
14.4.103: $\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

14.4.104: $\vec{x} = C_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$

14.5.101: a) Two eigenvalues: $\pm\sqrt{2}$ so the behavior is a saddle. b) Two eigenvalues: 1 and 2, so the behavior is a source. c) Two eigenvalues: $\pm 2i$, so the behavior is a center (ellipses). d) Two eigenvalues: -1 and -2, so the behavior is a sink. e) Two eigenvalues: 5 and -3, so the behavior is a saddle.

14.5.102: Spiral source.

14.5.103:



The solution does not move anywhere if $y = 0$. When y is positive, the solution moves (with constant speed) in the positive x direction. When y is negative, the solution moves (with constant speed) in the negative x direction. It is not one of the behaviors we saw.

Note that the matrix has a double eigenvalue 0 and the general solution is $x = C_1 t + C_2$ and $y = C_1$, which agrees with the description above.

14.6.101: $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_1 \cos(\sqrt{3}t) + b_1 \sin(\sqrt{3}t)) + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} (a_2 \cos(\sqrt{2}t) + b_2 \sin(\sqrt{2}t)) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (a_3 \cos(t) + b_3 \sin(t)) + \begin{bmatrix} -1 \\ 1/2 \\ 2/3 \end{bmatrix} \cos(2t)$

14.6.102: $\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \vec{x}'' = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \vec{x}$. Solution: $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} (a_1 \cos(\sqrt{3k/m}t) + b_1 \sin(\sqrt{3k/m}t)) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} (a_2 \cos(\sqrt{k/m}t) + b_2 \sin(\sqrt{k/m}t)) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (a_3 + b_3)$.

14.6.103: $x_2 = (2/5) \cos(\sqrt{1/6}t) - (2/5) \cos(t)$

14.7.101: a) 3, 0, 0 b) No defects. c) $\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

14.7.102: a) 1, 1, 2

b) Eigenvalue 1 has a defect of 1

c) $\vec{x} = C_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^t + C_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) e^t + C_3 \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} e^{2t}$

14.7.103: a) 2, 2, 2

b) Eigenvalue 2 has a defect of 2

$$c) \vec{x} = C_1 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} e^{2t} + C_2 \left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right) e^{2t} + C_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right) e^{2t}$$

$$14.7.104: A = \begin{bmatrix} 5 & 5 \\ 0 & 5 \end{bmatrix}$$

$$14.8.101: e^{tA} = \begin{bmatrix} \frac{e^{3t} + e^{-t}}{2} & \frac{e^{-t} - e^{3t}}{2} \\ \frac{e^{-t} - e^{3t}}{2} & \frac{e^{3t} + e^{-t}}{2} \end{bmatrix}$$

$$14.8.102: e^{tA} = \begin{bmatrix} 2e^{3t} - 4e^{2t} + 3e^t & \frac{3e^t}{2} - \frac{3e^{3t}}{2} & -e^{3t} + 4e^{2t} - 3e^t \\ 2e^t - 2e^{2t} & e^t & 2e^{2t} - 2e^t \\ 2e^{3t} - 5e^{2t} + 3e^t & \frac{3e^t}{2} - \frac{3e^{3t}}{2} & -e^{3t} + 5e^{2t} - 3e^t \end{bmatrix}$$

$$14.8.103: a) e^{tA} = \begin{bmatrix} (t+1)e^{2t} & -te^{2t} \\ te^{2t} & (1-t)e^{2t} \end{bmatrix} \quad b) \vec{x} = \begin{bmatrix} (1-t)e^{2t} \\ (2-t)e^{2t} \end{bmatrix}$$

$$14.8.104: \begin{bmatrix} 1+2t+5t^2 & 3t+6t^2 \\ 2t+4t^2 & 1+2t+5t^2 \end{bmatrix} e^{0.1A} \approx \begin{bmatrix} 1.25 & 0.36 \\ 0.24 & 1.25 \end{bmatrix}$$

$$14.8.105: a) \begin{bmatrix} 5(3^n) - 2^{n+2} & 4(3^n) - 2^{n+2} \\ 5(2^n) - 5(3^n) & 5(2^n) - 4(3^n) \end{bmatrix} \quad b) \begin{bmatrix} 3 - 2(3^n) & 2(3^n) - 2 \\ 3 - 3^{n+1} & 3^{n+1} - 2 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ if } n \text{ is even, and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ if } n \text{ is odd.}$$

14.9.101: The general solution is (particular solutions should agree with one of these):

$$x(t) = C_1 e^{9t} + 4C_2 e^{4t} - t/3 - 5/54, \quad y(t) = C_1 e^{9t} - C_2 e^{4t} + t/6 + 7/216$$

14.9.102: The general solution is (particular solutions should agree with one of these):

$$x(t) = C_1 e^t + C_2 e^{-t} + t e^t, \quad y(t) = C_1 e^t - C_2 e^{-t} + t e^t$$

$$14.9.103: \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{5}{2} e^t - t - 1 \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{-1}{2} e^{-t}$$

$$14.9.104: \vec{x} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \left(\left(\frac{1}{140} + \frac{1}{120\sqrt{6}} \right) e^{\sqrt{6}t} + \left(\frac{1}{140} + \frac{1}{120\sqrt{6}} \right) e^{-\sqrt{6}t} - \frac{t}{60} - \frac{\cos(t)}{70} \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(\frac{-9}{80} \sin(2t) + \frac{1}{30} \cos(2t) + \frac{9t}{40} - \frac{\cos(t)}{30} \right)$$

$$A.1.101: a) \sqrt{10} \quad b) \sqrt{14} \quad c) 3$$

$$A.1.102: a) \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad b) \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \quad c) \left(\frac{2}{\sqrt{33}}, \frac{-5}{\sqrt{33}}, \frac{2}{\sqrt{33}} \right)$$

$$A.1.103: a) \begin{bmatrix} 9 \\ -2 \end{bmatrix} \quad b) \begin{bmatrix} -3 \\ 3 \end{bmatrix} \quad c) \begin{bmatrix} 5 \\ -3 \end{bmatrix} \quad d) \begin{bmatrix} -4 \\ 8 \end{bmatrix} \quad e) \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad f) \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

$$A.1.104: a) 20 \quad b) 10 \quad c) 20$$

$$A.1.105: a) (3, -1) \quad b) (4, 0) \quad c) (-1, -1)$$

$$A.2.101: a) \begin{bmatrix} 7 & 4 & 4 \\ 2 & 3 & 4 \end{bmatrix} \quad b) \begin{bmatrix} 5 & -3 & 0 \\ 13 & 10 & 6 \\ -1 & 3 & 1 \end{bmatrix}$$

$$A.2.102: a) \begin{bmatrix} -1 & 13 \\ 9 & 14 \end{bmatrix} \quad b) \begin{bmatrix} 2 & -5 \\ 5 & 5 \end{bmatrix}$$

$$A.2.103: a) \begin{bmatrix} 22 & 31 \\ 42 & 44 \end{bmatrix} \quad b) \begin{bmatrix} 18 & 18 & 12 \\ 6 & 0 & 8 \\ 34 & 48 & -2 \end{bmatrix} \quad c) \begin{bmatrix} 11 & 12 & 36 & 14 \\ -2 & 4 & 5 & -2 \\ 13 & 38 & 20 & 28 \end{bmatrix} \quad d) \begin{bmatrix} -2 & -12 \\ 3 & 24 \\ 1 & 9 \end{bmatrix}$$

$$A.2.104: a) [1/2] \quad b) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad c) \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \quad d) \begin{bmatrix} 1/2 & -1/4 \\ -1/2 & 1/2 \end{bmatrix}$$

A.2.105: a) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$ b) $\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ c) $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

A.3.101: a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 0 & 0 & 77/15 \\ 0 & 1 & 0 & -2/15 \\ 0 & 0 & 1 & -8/5 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
 g) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ h) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A.3.102: a) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 5/2 & 1 & -3 \\ -1 & -1/2 & 3/2 \\ -1 & 0 & 1 \end{bmatrix}$

A.3.103: a) $x_1 = -2, x_2 = 7/3$ b) no solution c) $a = -3, b = 10, c = -8$ d) x_3 is free, $x_1 = -1 + 3x_3$, $x_2 = 2 - x_3$

A.3.104: a) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ b) $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

A.3.105: a) 3 b) 1 c) 2

A.3.106: a) $[1 \ 0 \ 0], [0 \ 1 \ 0], [0 \ 0 \ 1]$ b) $[1 \ 1 \ 1]$ c) $[1 \ 0 \ 1/3], [0 \ 1 \ -1/3]$

A.3.107: a) $\begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ c) $\begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$

A.3.108: $\begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$

A.4.101: a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ dimension 2, b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ dimension 2, c) $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$ dimension 3, d) $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

dimension 2, e) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ dimension 1, f) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ dimension 2

A.4.102: a) $\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -1 \end{bmatrix}$ b) $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ d) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

A.4.103: a) 3 b) 2 c) 3 d) 2 e) 3

A.5.101: $s = -2$

A.5.102: $\theta \approx 0.3876$

A.5.103: a) -15 b) -1 c) 28

A.5.104: a) $(-1/2, 0, \frac{1}{2})$ b) $(0, 0, 0)$ c) $(2, 0, -2)$

A.5.105: a) $(1, 1, -1) - (2, -1, 1) + 2(1, -5, 3)$ b) $2(2, -1, 1) + (1, -5, 3)$ c) $2(1, 1, -1) - 2(2, -1, 1) + 2(1, -5, 3)$

A.5.106: $(2, -1, 1), (2/3, 8/3, 4/3)$

A.5.107: $(1, 1, -1), (0, 1, 1), (4/3, -2/3, 2/3)$

A.6.101: a) -2 b) 8 c) 0 d) -6 e) -3 f) 28 g) 16 h) -24

A.6.102: a) 3 b) 9 c) 3 d) $1/4$

A.6.103: 12

A.6.104: 1 and 3

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