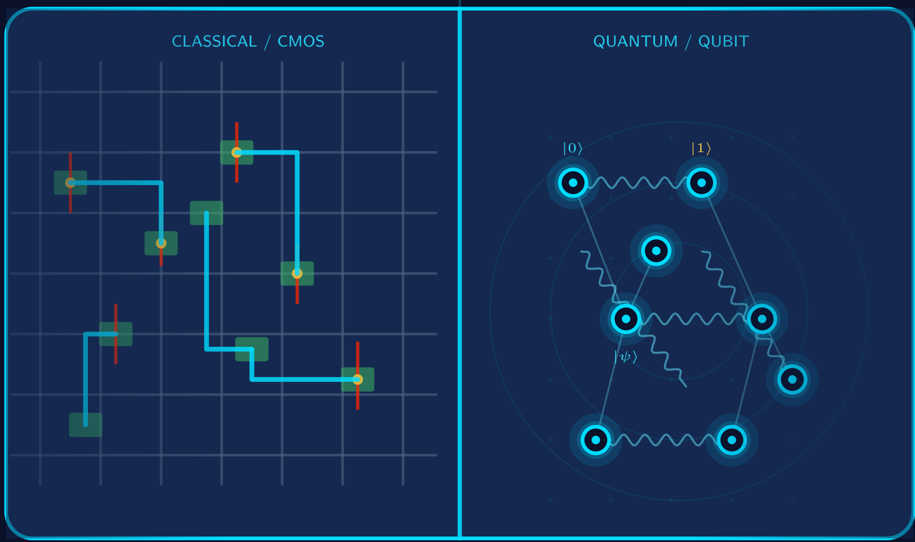
FROM SILICON TO QUBITS

An Engineer's Guide to Quantum Computing



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Why Engineers Should Care About Quantum Computing

Abstract

This book bridges the gap between classical digital design and quantum computing by mapping familiar architectural concepts to the quantum stack. Engineers already understand layered abstractions in classical computing; quantum systems follow a similar hierarchy but operate under fundamentally different physical rules. Understanding these differences—reversibility, measurement dynamics, and hybrid architectures—is essential for engineers entering quantum computing.

Reality Check

Quantum computers do **not** provide general speedup over classical systems. For most engineering workloads (code simulation, compilation, verification, databases), classical hardware will remain superior.

Quantum advantage appears only in **narrow problem classes** such as:

- Integer factorization (Shor's algorithm)
- Quantum system simulation

1.1 The Opportunity for Classical Design Engineers

Classical digital design has reached fundamental physical limits. Transistor scaling follows Moore's Law, but quantum effects and heat dissipation now dominate chip design. **Quantum computers offer a complementary approach: instead of building faster classical processors, they solve specific classes of problems using quantum mechanics itself.**

For you as an RTL/Classical design engineer transitioning to quantum:

- **Same abstraction philosophy:** Quantum systems use layered architectures just like classical computers—from physical qubits to high-level algorithms.
- **New design rules:** The primitives change. Quantum gates must be reversible, measurements collapse states, and error correction becomes central to hardware architecture.
- **Hybrid systems are emerging:** Real quantum systems rarely operate alone. Classical processors orchestrate quantum hardware in feedback loops, creating new co-design challenges similar to CPU+GPU or CPU+FPGA systems you may already know.

Understanding the quantum stack prepares you to design the next generation of hybrid classical-quantum processors.

1.2 The Layered Architecture Model

Modern computing systems are built on a **layered architecture**. As a digital/classical design engineer, you're familiar with this hierarchy, where each layer abstracts the complexity of the one below it, enabling complex systems to be designed and reasoned about independently.

A simplified classical computing stack looks like:

Classical Computing Stack

```
Algorithms / Software Code
↓
Compiler
↓
Instruction Set Architecture (ISA)
↓
Microarchitecture
↓
Logic Gates
↓
Transistors
```

Each layer provides a well-defined abstraction that allows engineers to design complex systems without managing every physical detail simultaneously.

Quantum computers follow a **similar layered design philosophy**, but the underlying physics changes the behavior and constraints at every level. The quantum computing stack can be represented as:

Quantum Computing Stack

```
Quantum Algorithms (Python via Qiskit, Cirq)
↓
Quantum Compiler
↓
Quantum Assembly
↓
Quantum Processor Architecture
↓
Logical Qubits & Error Correction
↓
Quantum Circuits
↓
Quantum Gates
↓
Physical Qubits
```

Key Insight

Although the stack structure appears similar, the rules governing each layer are **fundamentally different** from classical computing. The physical constraints at the bottom ripple up through the entire system.

Important Practical Note

This layered model is a **conceptual abstraction**, not a clean separation in real systems.
In practice:

- Quantum compilers must be aware of hardware noise and connectivity
- Circuit design is constrained by physical qubit layout
- Error correction influences gate scheduling and architecture

Unlike classical systems, layers in quantum computing are **tightly coupled**, not cleanly isolated.

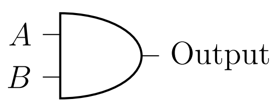
1.3 Fundamental Differences: How Quantum Layers Differ from Classical

Difference 1: Reversibility and Unitary Operations

Classical Logic: Information Loss

Most classical logic operations are **irreversible**—they destroy information. Consider an AND gate:

$$\texttt{AND Gate:} \quad A, B \rightarrow A \& B$$

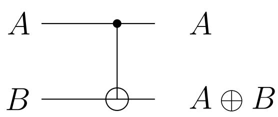


Once you have the output, you cannot recover the original inputs. If the output is 0, you don't know if (A, B) was $(0, 0)$, $(0, 1)$, or $(1, 0)$. **Information is lost.**

Reversible Classical Logic: The CNOT Gate

Reversible classical logic does exist. A simple example is the **CNOT gate** (controlled-NOT), which flips the target bit only when the control bit is 1:

$$\texttt{CNOT Gate:} \quad (A, B) \rightarrow (A, A \oplus B)$$



The key point: Given the outputs $(A, A \oplus B)$, you can uniquely recover the inputs (A, B), so no information is lost—the operation is reversible. (In later chapters, we will explore CNOT and other quantum gates in much more detail.)

Quantum Operations: Unitarity Requirement

Quantum time evolution must be described by a **unitary operator**, which is mathematically reversible:

$$U^\dagger U = I \quad \text{(Unitary property)}$$

where $U^\dagger$ is the conjugate transpose and I is the identity.

This is not a design choice—it's a postulate of quantum mechanics. **Any quantum circuit evolution (excluding measurement) must correspond to a unitary transformation.** This requirement fundamentally changes how circuits are designed:

- **All quantum gates must be reversible** (unlike most classical gates)
- **Circuit design is constrained** to ensure unitarity
- **Error correction must preserve reversibility**

Critical Design Rule

Quantum circuits must respect unitary constraints at every layer. This is why quantum compiler and architecture design are fundamentally different from classical—the entire stack must preserve this mathematical property.

Difference 2: Measurement Collapses Quantum Information

Engineering Reality

Measurement is not the only limitation. In current hardware:

- Decoherence destroys quantum states within microseconds
- Gate operations are noisy and imperfect

This means real quantum circuits must be:

- **Short (low depth)**
- **Error-aware**

Many theoretically valid quantum algorithms are **not runnable** on current devices.

Classical Circuits: Non-Invasive Observation

In classical digital circuits, you can probe any signal at any time without affecting circuit behavior. This is a fundamental feature of classical logic—observability is non-invasive.

Quantum Systems: Measurement Collapses Superposition

Quantum systems behave completely differently. When you measure a quantum state in superposition, the state **collapses** into a classical outcome, permanently destroying the superposition and all quantum correlations.

Example: A qubit in superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ contains quantum information. Once measured, it becomes either $|0\rangle$ or $|1\rangle$ with probability 50%. The superposition is gone—measurement is destructive.

This constraint fundamentally affects circuit design:

- **Intermediate measurements are avoided** within a single quantum computation, because they collapse the quantum state early and waste quantum advantage.
- **Information extraction happens at the end:** Most quantum algorithms apply all quantum gates first, then measure final results.
- **Hybrid algorithms are an exception:** Modern variational algorithms (VQE, QAOA) use mid-circuit measurement in feedback loops where measurements feed classical optimization, not quantum circuits.

Design Implication

While classical circuits are built to be highly observable, quantum circuits are designed to *delay* observation until the very end. This shifts the mental model: quantum computing requires **deferred measurement and probabilistic reasoning** rather than deterministic signal flow.

1.4 Stack Comparison: Classical vs. Quantum

The conceptual mapping between classical and quantum computing layers can be summarized as follows. Note the addition of the **Logical Qubit / Error Correction** layer, which is absent in classical computing but essential in quantum systems:

Layer	Classical	Quantum
Algorithm	Classical Algorithm (C, Python)	Quantum Algorithms (Qiskit, Cirq)
Compilation	Compiler	Quantum Compiler
ISA	Instruction Set	Quantum Assembly (QASM)
Processor	CPU Microarchitecture	Quantum Processor Architecture
Error Correction	*(Not primary layer)*	**Logical Qubits & Error Correction**
Circuits	Digital Logic Circuits	Quantum Circuits
Gates	AND, OR, XOR, NOT	Hadamard, CNOT, Toffoli, Pauli
Hardware	Transistors	Physical Qubits

Why Error Correction Is a Distinct Layer

In classical computing, error correction is typically handled within microarchitecture (parity checks, Hamming codes). In quantum computing, error correction requires an **entire architectural layer** because:

1. **Physical qubits are fragile** and suffer decoherence quickly (milliseconds to microseconds).
2. **Quantum error correction is expensive:** One logical qubit requires ~ 1000 physical qubits (current state-of-art), due to the need for multi-qubit entanglement to detect and correct errors without measuring the encoded information directly.
3. **Circuit overhead is substantial:** Error correction codes add significant gate depth, affecting algorithm runtime and coherence requirements.

This makes error correction a **primary design concern at the architecture level**—not an afterthought.

1.5 Examples: How Layers Interact Differently

1.5.1 Example 1: Hardware Representation

Classical: Bits

Classical computing uses binary states:

$$\text{bit} \in \{0, 1\}$$

Quantum: Qubits and Superposition

Quantum systems use quantum bits (qubits), which can exist in superpositions:

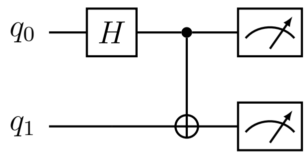
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \text{where } |\alpha|^2 + |\beta|^2 = 1$$

A qubit can be in a superposition of both states simultaneously. This is not "unknown which state"—it's genuinely in both states with quantum amplitudes α and β.

1.5.2 Example 2: Basic Quantum Gates

To understand quantum circuits, we need to see quantum gates in action. Here's a Bell state circuit, which entangles two qubits:

Creating a Bell State with Hadamard + CNOT

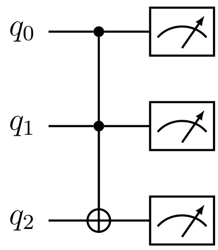


What happens:

1. **Hadamard gate (H):** Creates superposition on qubit 0: $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
2. **CNOT gate:** Entangles qubit 1 to qubit 0's state, creating the Bell state: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
3. **Measurement:** Both qubits are measured, always returning correlated outcomes (either both 0 or both 1)

Now let's look at a Toffoli gate—the quantum universal reversible gate:

Quantum Toffoli (CCNOT) Gate



Behavior: If both q_0 and q_1 are 1 (in classical sense), the target q_2 is flipped. In the quantum version, this applies to all basis states in superposition simultaneously, leveraging quantum parallelism.

1.5.3 Example 3: Circuit Behavior

Classical circuits: Propagate deterministic signals through logic gates. Output is always the same for a given input.

Quantum circuits: Propagate probability amplitudes, not classical values. The final measurement outcome is probabilistic, but the circuit evolution itself is deterministic (unitary).

Key Difference

Classical: Deterministic signals → Deterministic outputs
Quantum: Unitary amplitude evolution → Probabilistic measurement outcomes
The *circuit* is deterministic, but *measurement* is probabilistic.

1.5.4 Example 4: Hardware Constraint (Connectivity)

Real quantum processors have **limited qubit connectivity**. Two-qubit gates can only be applied between physically connected qubits.

Implication:

- Additional SWAP gates are required for distant qubits
- Circuit depth increases significantly

This is similar to routing congestion in VLSI design, where placement directly impacts performance.

1.6 Why Classical Computers Struggle With Certain Problems

Quantum computing is not intended to replace classical computing. Instead, it accelerates specific types of problems that are fundamentally difficult for classical machines.

Example 1: Integer Factorization

Large integer factorization underlies modern cryptography (RSA). The security of RSA relies on the fact that factoring is computationally hard:

Classical approach: The best known algorithms (General Number Field Sieve) scale super-polynomially, roughly $O(e^{1.9 \ln N})$. For a 2048-bit number, this requires billions of years.

Quantum approach: Shor's algorithm reduces the complexity to polynomial time, roughly $O((\ln N)^3)$. A 2048-bit number could theoretically be factored in hours on a sufficiently large quantum computer.

Why the advantage? Quantum computers can explore many candidate factors in superposition, then use quantum interference to amplify correct answers. This quantum parallelism, combined with interference, is the key speedup.

Example 2: Quantum System Simulation

Simulating molecules or quantum materials requires tracking the quantum states of many particles.

Classical problem: A system with N quantum particles requires 2^N classical bits to represent all possible states. The memory grows exponentially. For a 300-particle system, $2^{300} \sim 10^{90}$ states exceed the number of atoms in the observable universe.

Important clarification: Quantum computers do **not** give direct access to all 2^N states. Measurement yields only one outcome. Algorithms must use **interference** to extract useful information.

Quantum solution: Quantum computers naturally represent these states using N qubits, which can be in superposition of all 2^N basis states simultaneously. No exponential classical simulation is needed.

Example 3: Combinatorial Optimization

Problems like traveling salesman, graph coloring, and portfolio optimization are NP-hard classically. Quantum approaches (like QAOA) offer potential speedup by exploring many candidate solutions in superposition.

The Takeaway

Quantum computers excel at problems where quantum parallelism and interference can be exploited: factorization, simulation, and optimization. For everyday computing (word processing, web browsing, databases), classical computers remain superior and always will.

1.7 Hybrid Classical-Quantum Systems: The Real Architecture

In practice, quantum computers rarely operate independently. Most real-world systems use **hybrid architectures** where classical processors orchestrate quantum hardware through a feedback loop. The classical component manages the workflow: sending quantum circuits to the processor, collecting results, and adjusting parameters.

A simplified hybrid system stack is:

Hybrid System Stack

```
Classical CPU / GPU
        ↓
  Quantum Runtime
        ↓
 Quantum Compiler
        ↓
 Quantum Processor
```

The classical component manages the workflow: sending quantum circuits to the processor, collecting results, and adjusting parameters. This is similar to how a classical CPU orchestrates a GPU or FPGA.

System Bottleneck

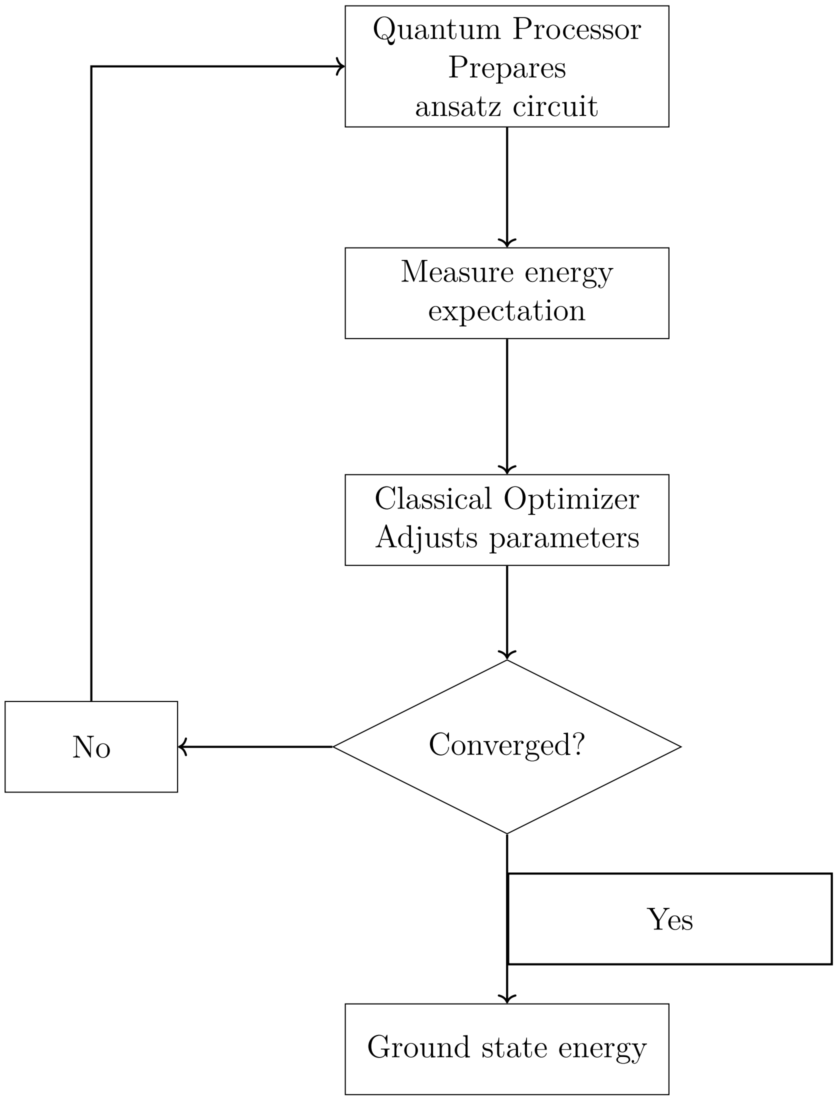
Hybrid systems introduce significant overhead:

- Classical-quantum communication latency
- Repeated circuit executions (thousands of shots)

In many current systems, the **classical feedback loop dominates runtime**, not the quantum computation itself.

1.7.1 Example 1: Variational Quantum Eigensolver (VQE)

VQE is a hybrid algorithm that estimates the ground state energy of a quantum system:



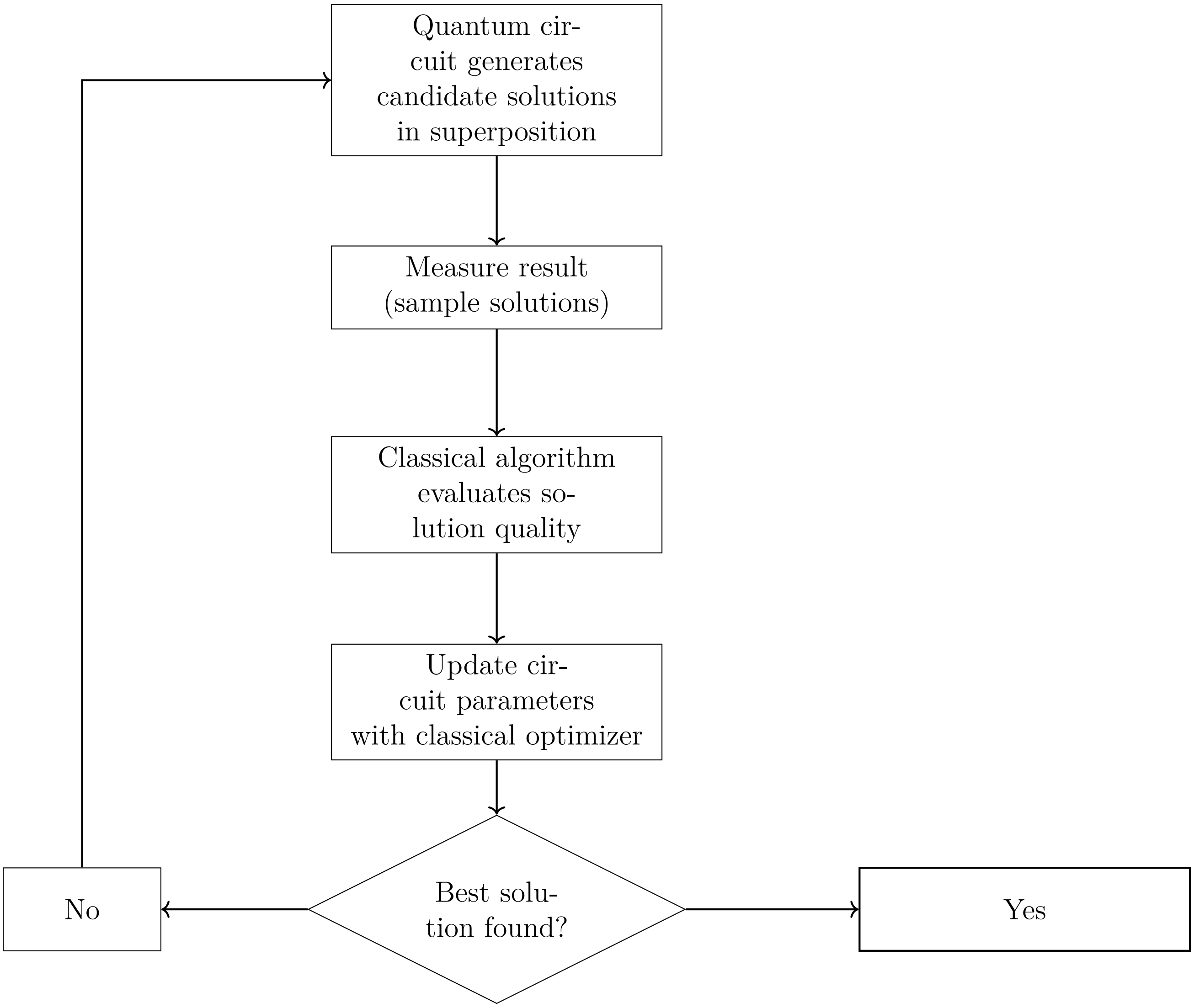
Breakdown:

1. **Step 1:** Quantum processor evaluates the energy of a parameterized quantum circuit for a given set of parameters
2. **Step 2:** Measure energy expectation value (extract classical information)
3. **Step 3:** Classical optimizer analyzes result and adjusts circuit parameters
4. **Step 4:** Repeat until convergence

The **quantum part** handles superposition and entanglement; the **classical part** handles optimization logic. Neither alone suffices—they're deeply coupled.

1.7.2 Example 2: Quantum Approximate Optimization Algorithm (QAOA)

QAOA solves combinatorial optimization problems by encoding them as quantum circuits:



Breakdown:

1. **Step 1:** Quantum circuit encodes the optimization problem and generates candidate solutions in superposition
2. **Step 2:** Measure (sample) candidate solutions from the quantum superposition
3. **Step 3:** Classical algorithm evaluates solution quality for each candidate
4. **Step 4:** Update parameters to amplify good solutions, iterate

Hybrid Systems Are Here

Both VQE and QAOA demonstrate that practical quantum computing is **fundamentally hybrid**. The quantum and classical components are inseparable. Understanding both is essential for designing next-generation quantum systems.

1.8 Implications for Engineers Transitioning to Quantum

1.9 What Transfers from Classical Design

- **Abstraction thinking:** Layered architectures are universal. Quantum systems use the same philosophy.
- **Circuit design patterns:** Decomposing complex operations into basic gates is familiar; quantum gate synthesis is similar.
- **Hardware-software co-design:** Optimizing for specific hardware constraints is central to both classical and quantum design.
- **Error and reliability thinking:** Both systems require error correction, though the mechanisms differ fundamentally.

1.10 What's Fundamentally New

- **Reversibility as a constraint:** All quantum operations must be unitary. This is a hard requirement, not an optimization.
- **Measurement collapses information:** You cannot probe quantum states without destroying them. Defer measurement to the end.
- **Probabilistic reasoning:** Quantum algorithms produce probabilistic outcomes. Design for amplification of correct answers, not deterministic paths.
- **Error correction as architecture:** Quantum error correction requires an entire design layer. Physical-to-logical qubit scaling is a primary concern.
- **Hybrid systems are essential:** Classical-quantum feedback loops are not optional; they're the standard operating model.

Conclusion

The quantum computing stack mirrors classical architecture in structure but differs fundamentally in every layer's operational rules. **Reversibility, measurement dynamics, and error correction** are not peripheral concerns—they define the entire design space.

For engineers transitioning into quantum computing, this means:

1. Reuse abstraction thinking, but learn new physics constraints.
2. Understand that quantum advantage comes from specific problem classes (factorization, simulation, optimization), not general speedup.
3. Recognize that real quantum systems are hybrid—classical and quantum components are deeply coupled.

4. Accept that error correction is a primary architectural concern, not a patch applied later.

5. Know when **not** to use quantum computing:

   - Deterministic workloads (e.g., digital logic simulation)
   - Data-heavy applications (e.g., databases, ML inference)

The quantum computing field is still in its infancy. Hardware, software, and algorithms are all under active development. **This is your opportunity to shape the future of computing architecture from the ground up.**

Final Thought

Quantum computing is not the future of all computing—it's the future of specific computing problems. But for those problems, it's revolutionary. Your job is to build the bridges between classical and quantum worlds, ensuring that the hybrid systems of tomorrow work seamlessly.

Quantum Foundations: Key Terms and Notation

Chapter Overview

This chapter is a reference guide for quantum terminology and notation you'll encounter throughout the book. Some concepts receive only brief treatment here—full derivations appear in later chapters. Skim this now, then return when you encounter unfamiliar terms or notation.

Think of this chapter like a dictionary or quick-reference card. You don't need to memorize everything on first read.

2.1 Dirac Notation (Bra-Ket)

Quantum mechanics uses **Dirac notation** (also called bra-ket notation) to represent quantum states and operations. You've already seen $|\psi\rangle$ in the previous chapter. Here's the complete reference.

A **ket** $|\psi\rangle$ represents a quantum state as a column vector:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha |0\rangle + \beta |1\rangle$$

The basis states are:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A **bra** $\langle\psi|$ represents the conjugate transpose (row vector):

$$\langle\psi| = |\psi\rangle^\dagger = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$$

The † symbol means conjugate transpose (take the transpose, then complex conjugate all entries).

The **inner product** $\langle\phi\rangle\,\psi$ is a complex number:

$$\langle\phi\rangle\,\psi = \begin{pmatrix} \alpha_\phi^* & \beta_\phi^* \end{pmatrix} \begin{pmatrix} \alpha_\psi \\ \beta_\psi \end{pmatrix} = \alpha_\phi^*\alpha_\psi + \beta_\phi^*\beta_\psi$$

This measures the overlap between two states. If $|\langle\phi\rangle\,\psi|^2 = 1$, the states are identical. If $\langle\phi\rangle\,\psi = 0$, they are orthogonal (distinguishable).

The **outer product** $|\psi\rangle\langle\phi|$ is a matrix:

$$|\psi\rangle\langle\phi| = \begin{pmatrix} \alpha_\psi \\ \beta_\psi \end{pmatrix} \begin{pmatrix} \alpha_\phi^* & \beta_\phi^* \end{pmatrix} = \begin{pmatrix} \alpha_\psi\alpha_\phi^* & \alpha_\psi\beta_\phi^* \\ \beta_\psi\alpha_\phi^* & \beta_\psi\beta_\phi^* \end{pmatrix}$$

This creates a projection operator.

The **expectation value** $\langle O \rangle$ of an operator O is:

$$\langle O \rangle = \langle \psi | O | \psi \rangle \tag{1}$$

This is the average value you'd measure if you prepared $|\psi\rangle$ many times and measured O each time.

Example 1: Inner Product Calculation

Compute the inner product between $|0\rangle$ and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Step 1: Write $|0\rangle$ and $|+\rangle$ as column vectors.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 2: Form the bra $\langle 0|$.

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Step 3: Compute the product.

$$\begin{aligned} \langle 0 \rangle + &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}(1 \cdot 1 + 0 \cdot 1) \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

Step 4: Compute the overlap probability.

$$|\langle 0 \rangle +|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

If you prepare $|+\rangle$ and measure in the computational basis, you get 0 with probability 1/2.

2.2 Hilbert Space

A **Hilbert space** is the mathematical framework for quantum states. For engineers, think of it as the state space—the set of all possible states a system can occupy.

A single qubit lives in a **2-dimensional complex vector space**, written $\mathbb{C}^2$. Any qubit state is a linear combination of two basis vectors:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad \alpha, \beta \in \mathbb{C}$$

A two-qubit system lives in $\mathbb{C}^4$ (4 basis states: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$).

An n-qubit system lives in $\mathbb{C}^{2^n}$. The dimension grows **exponentially** with the number of qubits. This is why quantum systems can hold so much information—but also why they're exponentially hard to simulate classically.

Classical Analogy

In control theory, a system's state lives in a state space. For a 2D pendulum, the state space is 4D (position x, y and velocity $\dot{x}$, $\dot{y}$).

Similarly, a qubit's state lives in $\mathbb{C}^2$. The difference: quantum state spaces require **complex** numbers and have strict normalization ($|\alpha|^2 + |\beta|^2 = 1$).

The basis states $\{|0\rangle, |1\rangle\}$ form an **orthonormal basis**:

$$\langle 0\rangle\, 0 = 1, \quad \langle 1\rangle\, 1 = 1$$
$$\langle 0\rangle\, 1 = 0, \quad \langle 1\rangle\, 0 = 0$$

Any state can be expressed in this basis uniquely.

2.3 Hermitian Operators and Observables

In quantum mechanics, measurements are described by **Hermitian operators** (also called observables). A Hermitian operator H satisfies:

$$H^\dagger = H \tag{2}$$

The † symbol means conjugate transpose.

Why Hermitian? Two critical properties:

Property 1: Eigenvalues are real.
Measurement outcomes must be real numbers (you can't measure "$3 + 2i$ volts"). Hermitian operators guarantee real eigenvalues.

Property 2: Eigenvectors are orthogonal.
After measurement, the system is in one of the operator's eigenstates. Orthogonal eigenstates mean measurement outcomes are distinguishable.

Every Hermitian operator has a **spectral decomposition**:

$$O = \sum_i \lambda_i |e_i\rangle \langle e_i| \tag{3}$$

where λ_i are the eigenvalues (possible measurement outcomes) and $|e_i\rangle$ are the eigenstates (states after measurement if you get outcome λ_i).

Example 2: Verifying a Hermitian Operator

Check if the Pauli Z gate is Hermitian.

Step 1: Write the matrix.

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Step 2: Compute the conjugate transpose.
Transpose:

$$Z^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Complex conjugate (no complex entries here, so no change):

$$Z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Step 3: Compare.

$$Z^\dagger = Z \quad \checkmark$$

The Z operator is Hermitian. Its eigenvalues are $+1$ and -1 (the measurement outcomes), and its eigenstates are $|0\rangle$ and $|1\rangle$.

Key Insight

The Pauli matrices X, Y, Z are all Hermitian. They represent measurements along different axes of the Bloch sphere (covered in Chapter 9). Each has eigenvalues ± 1.

2.4 Unitary Operators

Quantum gates are **unitary operators**. A unitary operator U satisfies:

$$U^\dagger U = UU^\dagger = I \tag{4}$$

This means $U^\dagger = U^{-1}$ (the inverse equals the conjugate transpose).

Why unitary? Unitarity preserves the norm of quantum states:

$$\langle\psi\rangle\, \psi = 1 \quad \Rightarrow \quad \langle\psi| U^\dagger U |\psi\rangle = \langle\psi\rangle\, \psi = 1$$

Since the norm $|\alpha|^2 + |\beta|^2$ represents total probability, unitarity ensures **probability is conserved**. You can't create or destroy probability.

For a 2×2 unitary matrix, the determinant always has unit magnitude:

$$|\det(U)| = 1 \quad \Rightarrow \quad \det(U) = e^{i\phi}$$

for some real phase ϕ. This is analogous to rotation matrices in 3D, which also have $\det = 1$.

Classical Analogy

Rotation matrices in 3D are unitary (in the real-valued case, this is called orthogonal). A rotation preserves the length of vectors:

$$\mathbf{v}^T\mathbf{v} = (R\mathbf{v})^T(R\mathbf{v})$$

Similarly, quantum gates preserve the "length" (probability) of quantum states.

Example 3: Verifying Unitarity

Verify that the Hadamard gate is unitary.

Step 1: Write the matrix.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Step 2: Compute $H^\dagger$.
Transpose:

$$H^T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Complex conjugate (no complex entries):

$$H^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H$$

Notice H is Hermitian: $H^\dagger = H$.

Step 3: Compute $H^\dagger H = H^2$.

$$\begin{aligned}H^2 &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}1+1 & 1-1\\ 1-1 & 1+1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I\end{aligned}$$

Thus $H^\dagger H = I$, so H is unitary. Additionally, $H^2 = I$ means applying H twice returns you to the original state.

2.5 Pure State vs Mixed State

Quantum states come in two fundamental types: **pure states** and **mixed states**.

A **pure state** is a single, definite quantum state $|\psi\rangle$. When you prepare a qubit in a well-defined superposition, it's in a pure state. Pure states represent maximal knowledge about the system.

A **mixed state** is a statistical ensemble—a collection of pure states where you don't know which one you have, only probabilities:

$$\text{Mixed state: } \begin{cases} |\psi_1\rangle & \text{with probability } p_1 \\ |\psi_2\rangle & \text{with probability } p_2 \\ \vdots \end{cases}$$

Mixed states represent **classical uncertainty** about which pure state you have.

Common Confusion

Superposition ($|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$) is NOT the same as a mixed state.

- $|+\rangle$: A **pure state**. Quantum coherence exists. Can exhibit interference.
- 50/50 mix of $|0\rangle$ and $|1\rangle$: A **mixed state**. No coherence. No interference.

Mixed states result from decoherence, noise, or incomplete knowledge. They cannot be written as $|\psi\rangle$.

Bloch Sphere Representation:

- Pure states lie on the **surface** of the Bloch sphere.
- Mixed states lie **inside** the sphere.
- The maximally mixed state (50/50 $|0\rangle/|1\rangle$ with no coherence) is at the **center**.

When a qubit decoheres due to noise, it transitions from a pure state (on the surface) to a mixed state (moving toward the center). This is why maintaining coherence is critical in quantum computing.

2.6 Density Matrix ρ

The **density matrix** ρ is the universal formalism that describes both pure and mixed states.

For a **pure state** $|\psi\rangle$:

$$\rho = |\psi\rangle \langle\psi| \tag{5}$$

For a **mixed state** (classical probabilities p_i of being in state $|\psi_i\rangle$):

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| \tag{6}$$

Key properties of density matrices:

1. ρ is Hermitian: $\rho^\dagger = \rho$
2. Trace equals 1: $\text{Tr}(\rho) = 1$ (total probability)
3. ρ is positive semidefinite (all eigenvalues ≥ 0)
4. Purity test: $\text{Tr}(\rho^2) = 1$ if and only if ρ is a pure state; $\text{Tr}(\rho^2) < 1$ for mixed states

The purity $\text{Tr}(\rho^2)$ quantifies how "quantum" the state is. Pure states have purity 1. Maximally mixed states (complete randomness) have purity $1/d$ where d is the dimension (1/2 for a qubit).

Example 4: Pure State Density Matrix

Compute the density matrix for $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and verify it's pure.

Step 1: Write the state as a column vector.

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 2: Form the bra.

$$\langle +| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Step 3: Compute the outer product.

$$\begin{aligned} \rho &= |+\rangle \langle +| \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

Step 4: Verify trace equals 1.

$$\text{Tr}(\rho) = \frac{1}{2}(1 + 1) = 1 \quad \checkmark$$

Step 5: Compute purity $\text{Tr}(\rho^2)$.

$$\begin{aligned} \rho^2 &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \rho \end{aligned}$$

$$\text{Tr}(\rho^2) = \text{Tr}(\rho) = 1 \quad \checkmark$$

Since $\text{Tr}(\rho^2) = 1$, this is a pure state.

Example 5: Mixed State Density Matrix

Compute the density matrix for a 50/50 classical mixture: 50% chance of $|0\rangle$, 50% chance of $|1\rangle$. Verify it's mixed.

Step 1: Compute the pure-state density matrices.

$$\rho_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rho_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Step 2: Form the mixed state.

$$\begin{aligned} \rho &= \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1 \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \end{aligned}$$

Step 3: Verify trace equals 1.

$$\mathrm{Tr}(\rho) = \frac{1}{2} + \frac{1}{2} = 1 \quad \checkmark$$

Step 4: Compute purity.

$$\begin{aligned} \rho^2 &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \end{aligned}$$

$$\mathrm{Tr}(\rho^2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1$$

Since $\mathrm{Tr}(\rho^2) < 1$, this is a **mixed state**. Notice the off-diagonal elements are zero—there's no quantum coherence.

2.7 Fidelity

Fidelity quantifies how close two quantum states are. It's the quantum analog of distance between vectors.

For two **pure states** $|\psi\rangle$ and $|\phi\rangle$:

$$F(|\psi\rangle, |\phi\rangle) = |\langle\psi\rangle\,\phi|^2 \tag{7}$$

Fidelity ranges from 0 (orthogonal states) to 1 (identical states).

For **density matrices** ρ and σ, the general formula is:

$$F(\rho, \sigma) = \left(\text{Tr}\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right)^2 \tag{8}$$

This simplifies when one state is pure. If $\rho = |\psi\rangle\langle\psi|$:

$$F(|\psi\rangle, \sigma) = \langle\psi|\,\sigma\,|\psi\rangle$$

In quantum hardware, **gate fidelity** measures how well a physical gate matches the ideal unitary. A "99% fidelity" gate means the average fidelity over all input states is 0.99.

Hardware Specs Warning

When you see "99% gate fidelity" in hardware specifications, this typically refers to **average gate fidelity**—the fidelity averaged over all possible input states.

This is **different** from state fidelity between two specific states. Don't confuse the two.

Example 6: State Fidelity Calculation

Compute the fidelity between $|0\rangle$ and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Step 1: Compute the inner product.

$$\langle 0\rangle + = \langle 0|\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{\sqrt{2}}\langle 0\rangle\, 0 + \frac{1}{\sqrt{2}}\langle 0\rangle\, 1 = \frac{1}{\sqrt{2}}$$

Step 2: Compute fidelity.

$$F(|0\rangle, |+\rangle) = |\langle 0\rangle +|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

The fidelity is 0.5. These states have 50% overlap—they're neither identical nor orthogonal. This makes sense: measuring $|+\rangle$ in the computational basis gives $|0\rangle$ with 50% probability.

2.8 Tensor Products

Multi-qubit systems are built using the **tensor product** (also called the Kronecker product), denoted $\otimes$.

If qubit A is in state $|\psi\rangle$ and qubit B is in state $|\phi\rangle$, the joint system is:

$$|\psi\rangle \otimes |\phi\rangle \tag{9}$$

The $\otimes$ symbol is often omitted: $|\psi\rangle|\phi\rangle$ or $|\psi\phi\rangle$.

For basis states:

$$|0\rangle \otimes |0\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|1\rangle \otimes |0\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|1\rangle \otimes |1\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The dimension grows exponentially: 2 qubits $\to 2^2 = 4$ dimensions, n qubits $\to 2^n$ dimensions.

Classical Analogy

Tensor products are like joint probability distributions. If you flip two independent coins, the joint state space has 4 outcomes: HH, HT, TH, TT.

Similarly, two independent qubits have 4 basis states: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

Important: Not all two-qubit states can be written as $|\psi\rangle \otimes |\phi\rangle$. States that **cannot** be factored are called **entangled**. For example:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

cannot be written as $|\psi\rangle \otimes |\phi\rangle$ for any single-qubit states $|\psi\rangle$ and $|\phi\rangle$. This is the Bell state—maximally entangled. We'll cover entanglement in Chapter 13.

Example 7: Tensor Product Calculation

Compute $|+\rangle \otimes |1\rangle$ where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Step 1: Expand the tensor product.

$$\begin{aligned} |+\rangle \otimes |1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle \\ &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle) \\ &= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \end{aligned}$$

Step 2: Write as a column vector.

$$|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|+\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

This is a two-qubit state with the second qubit definitely in $|1\rangle$ and the first qubit in superposition.

2.9 Expectation Value

The **expectation value** $\langle O \rangle$ is the average outcome you'd measure if you repeated the experiment many times.

For a **pure state** $|\psi\rangle$ and observable O:

$$\langle O \rangle = \langle \psi | O | \psi \rangle \tag{10}$$

For a **mixed state** with density matrix ρ:

$$\langle O \rangle = \mathrm{Tr}(\rho O) \tag{11}$$

The trace formula works for both pure and mixed states (it reduces to the first formula when $\rho = |\psi\rangle \langle \psi|$).

Expectation values connect to the **Born rule**. If $O = \sum_i \lambda_i |e_i\rangle \langle e_i|$ (spectral decomposition), then:

$$\langle O \rangle = \sum_i \lambda_i \, P(\text{outcome } \lambda_i)$$

This is just the weighted average of measurement outcomes.

Example 8: Expectation Value of Pauli Z

Compute the expectation value of Z for the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Step 1: Write the operator and state.

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 2: Compute $Z|+\rangle$.

$$\begin{aligned} Z|+\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

Step 3: Compute $\langle + | Z | + \rangle$.

$$\begin{aligned} \langle Z \rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{1}{2}(1 \cdot 1 + 1 \cdot (-1)) \\ &= \frac{1}{2}(1 - 1) = 0 \end{aligned}$$

The expectation value is zero. This makes sense: $|+\rangle$ has equal probability of measuring +1 (for $|0\rangle$) and -1 (for $|1\rangle$), so the average is zero.

2.10 Commutator $[A, B]$

The **commutator** of two operators A and B is:

$$[A, B] = AB - BA \tag{12}$$

If $[A, B] = 0$, the operators **commute**. Commuting operators can be measured simultaneously without disturbing each other. They share a common set of eigenstates.

If $[A, B] \neq 0$, the operators are **incompatible**. Measuring A disturbs the state in a way that makes B uncertain, and vice versa. This is the essence of the Heisenberg uncertainty principle.

For qubits, the Pauli matrices have well-known commutation relations:

$$\begin{aligned} [X, Y] &= 2iZ \\ [Y, Z] &= 2iX \\ [Z, X] &= 2iY \end{aligned}$$

Because these don't commute, you cannot simultaneously know the X, Y, and Z components of a qubit's state. This is why a qubit isn't just three classical bits (one for each axis)—the axes are incompatible.

Example 9: Computing a Commutator

Compute $[X, Z]$ where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Step 1: Compute XZ.

$$\begin{aligned} XZ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Step 2: Compute ZX.

$$ZX = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Step 3: Compute the commutator.

$$[X, Z] = XZ - ZX$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

Step 4: Simplify using the Pauli Y matrix.
Recall $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, so:

$$2iY = 2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Wait, our result is $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$. Let me recalculate:

$$-2iY = -2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2i \cdot (-i) \\ -2i \cdot i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

Thus:

$$[X, Z] = -2iY$$

Or equivalently, $[Z, X] = 2iY$. The Pauli matrices form a closed algebra under commutation.

Geometric Interpretation

On the Bloch sphere, the Pauli operators represent rotations/measurements along the X, Y, Z axes. Their non-commutativity means you cannot simultaneously know all three coordinates—knowing one axis disturbs the others.

This is why a qubit state is specified by **two** angles (θ, ϕ), not three independent coordinates. The third degree of freedom is constrained by the uncertainty relations.

Practice Problems

1. Compute the inner product $\langle 1\rangle -$ where $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. What is the probability of measuring $|1\rangle$ if the qubit is in state $|-\rangle$?

2. A two-qubit system is in state $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. What is the dimension of its Hilbert space? Verify that $|\psi\rangle$ is normalized.

3. The Pauli X gate is $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Verify that X is both Hermitian and unitary.

4. Consider the density matrix $\rho = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.3 \end{pmatrix}$. Compute $\text{Tr}(\rho)$ and $\text{Tr}(\rho^2)$. Is this a pure state or mixed state?

5. Compute the fidelity between $|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

6. Compute the tensor product $|0\rangle \otimes |-\rangle$ and write it as a 4-dimensional column vector.

7. Compute the expectation value $\langle X \rangle$ for the state $|0\rangle$, where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

8. Compute the commutator $[Y, Z]$ where $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Express your answer in terms of Pauli matrices.

Solutions

Problem 1:

Step 1: Write the states as vectors.

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Step 2: Compute the inner product.

$$\begin{aligned} \langle 1 \rangle - &= \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}(0 \cdot 1 + 1 \cdot (-1)) \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

Step 3: Compute the probability.

$$P(1) = |\langle 1 \rangle -|^2 = \left| -\frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

The probability of measuring $|1\rangle$ is 50%.

Problem 2:

A two-qubit system lives in $\mathbb{C}^{2^2} = \mathbb{C}^4$ (dimension 4).

Verify normalization:

$$\begin{aligned} \langle \psi \rangle \psi &= \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 \\ &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1 \quad \checkmark \end{aligned}$$

The state is normalized.

Problem 3:
Check Hermitian: $X^\dagger = X^T$ (since X is real).

$$X^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \quad \checkmark$$

Check unitary: $X^\dagger X = X^2$.

$$\begin{aligned} X^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \checkmark \end{aligned}$$

X is both Hermitian and unitary.

Problem 4:
Compute trace:

$$\text{Tr}(\rho) = 0.7 + 0.3 = 1 \quad \checkmark$$

Compute ρ^2:

$$\begin{aligned} \rho^2 &= \begin{pmatrix} 0.7 & 0 \\ 0 & 0.3 \end{pmatrix} \begin{pmatrix} 0.7 & 0 \\ 0 & 0.3 \end{pmatrix} \\ &= \begin{pmatrix} 0.49 & 0 \\ 0 & 0.09 \end{pmatrix} \end{aligned}$$

$$\text{Tr}(\rho^2) = 0.49 + 0.09 = 0.58 < 1$$

Since $\text{Tr}(\rho^2) < 1$, this is a **mixed state**.

Problem 5:
Compute inner product:

$$\begin{aligned} \langle 1 \rangle - &= \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}(0 - 1) = -\frac{1}{\sqrt{2}} \end{aligned}$$

Fidelity:

$$F(|1\rangle, |-\rangle) = |\langle 1 \rangle -|^2 = \frac{1}{2}$$

Problem 6:
Expand:

$$\begin{aligned} |0\rangle \otimes |-\rangle &= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle) \end{aligned}$$

As a vector:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$$

Problem 7:
Compute $X|0\rangle$:

$$X|0\rangle = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$$

Compute expectation:

$$\langle X\rangle = \langle 0| X |0\rangle = \langle 0| |1\rangle = 0$$

The expectation value is 0. This makes sense: $|0\rangle$ is an eigenstate of Z, not X. Measuring X gives ± 1 with certain probabilities, averaging to 0 for the Z-basis states.

Problem 8:
Compute YZ:

$$\begin{aligned} YZ &= \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \\ &= \begin{pmatrix}0 & i\\i & 0\end{pmatrix}\end{aligned}$$

Compute ZY:

$$\begin{aligned} ZY &= \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \\ &= \begin{pmatrix}0 & -i\\-i & 0\end{pmatrix}\end{aligned}$$

Compute commutator:

$$\begin{aligned}[][Y, Z] &= YZ - ZY \\ &= \begin{pmatrix}0 & i\\i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\-i & 0\end{pmatrix} \\ &= \begin{pmatrix}0 & 2i\\2i & 0\end{pmatrix}\end{aligned}$$

Express in terms of Pauli X:

$$2iX = 2i\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 2i\\2i & 0\end{pmatrix}$$

Thus $[Y, Z] = 2iX$.

Key Takeaways

- **Dirac notation**: $|\psi\rangle$ = state, $\langle\psi|$ = conjugate transpose, $\langle\phi\rangle\,\psi$ = inner product (overlap), $|\psi\rangle\langle\phi|$ = outer product (projection).
- **Hilbert space**: Single qubit in $\mathbb{C}^2$, n qubits in $\mathbb{C}^{2^n}$. Dimension grows exponentially.
- **Hermitian operators**: Represent measurements. Eigenvalues are real (measurement outcomes). Spectral decomposition: $O = \sum_i \lambda_i |e_i\rangle\langle e_i|$.
- **Unitary operators**: Represent quantum gates. Preserve probability: $U^\dagger U = I$. All quantum evolution is unitary until measurement.
- **Pure vs mixed states**: Pure state = definite quantum state $|\psi\rangle$ (on Bloch sphere surface). Mixed state = classical probability distribution over pure states (inside Bloch sphere).
- **Density matrix** ρ: Universal formalism for both pure and mixed states. Purity test: $\text{Tr}(\rho^2) = 1$ iff pure.
- **Fidelity**: Quantifies how close two states are. $F = |\langle\psi\rangle\,\phi|^2$ for pure states. Ranges from 0 (orthogonal) to 1 (identical).
- **Tensor product** $\otimes$: Builds multi-qubit states. Two qubits: $|\psi\rangle \otimes |\phi\rangle$. Not all multi-qubit states factorize—those that don't are entangled.
- **Expectation value**: Average measurement outcome. $\langle O\rangle = \langle\psi| O |\psi\rangle$ for pure states, $\text{Tr}(\rho O)$ for mixed states.
- **Commutator** $[A, B]$: If $[A, B] = 0$, operators are compatible (can measure simultaneously). If $[A, B] \neq 0$, measuring one disturbs the other (Heisenberg uncertainty).

Qubit Physics

Chapter Overview

This chapter bridges the gap between classical bits and quantum bits. By the end, you'll understand what a qubit is mathematically, how it's physically built, and why quantum systems are so much more fragile than classical computers.

3.1 From Bits to Qubits

Classical computers store information using bits. A bit can exist in one of two states:

$$0 \quad \text{or} \quad 1$$

These correspond to physical states like voltage levels: 0V for 0, 5V for 1. The bit is always in one definite state.

Quantum computers use quantum bits, or **qubits**. A qubit can exist in a **superposition** of both states simultaneously. The quantum state of a qubit is written as:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{13}$$

The coefficients α and β are **probability amplitudes**. They are complex numbers satisfying:

$$|\alpha|^2 + |\beta|^2 = 1 \tag{14}$$

Critical Misconception

Superposition does **not** mean the qubit is "both 0 and 1 at the same time." It means the qubit is in a state where measurement will produce 0 with probability $|\alpha|^2$ and 1 with probability $|\beta|^2$.

When you measure a qubit, the superposition collapses to one of the basis states. You get a classical outcome: either 0 or 1.

Example 1: Classical-Like State

Consider the state:

$$|\psi\rangle = 1 \cdot |0\rangle + 0 \cdot |1\rangle$$

Calculate the measurement probabilities:

$$P(0) = |1|^2 = 1$$
$$P(1) = |0|^2 = 0$$

This qubit will always measure as 0. It behaves like a classical bit.

Example 2: Equal Superposition

Consider the state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

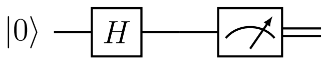
Calculate the probabilities:

$$P(0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$P(1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

This qubit has equal probability of measuring 0 or 1. This is the maximally uncertain state.

Creating and Measuring Superposition



Initial state	After H gate	Measurement
$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	0 or 1 (50% each)

Figure 1: Basic qubit operation: Initialize to $|0\rangle$, apply Hadamard gate to create equal superposition, then measure. Outcome is probabilistic—50% chance of 0, 50% chance of 1. After measurement, superposition is destroyed and qubit is in definite state.

Reading Circuit Diagrams

- Horizontal line = qubit wire (time flows left to right)
- $|0\rangle$ on left = initial state
- Box = quantum gate operation
- Meter symbol = measurement (quantum $\to$ classical)
- Double line = classical bit storing the measurement result

Example 3: Unequal Superposition with Full Calculation

Consider the state:

$$|\psi\rangle = 0.6|0\rangle + 0.8|1\rangle$$

Step 1: Verify normalization.

$$\begin{aligned}|\alpha|^2 + |\beta|^2 &= |0.6|^2 + |0.8|^2 \\ &= 0.36 + 0.64 \\ &= 1.00 \quad \checkmark\end{aligned}$$

Step 2: Calculate measurement probabilities.

$$P(0) = |0.6|^2 = 0.36 = 36\%$$
$$P(1) = |0.8|^2 = 0.64 = 64\%$$

If you measure this qubit 1000 times (preparing it fresh each time), you expect approximately 360 outcomes of 0 and 640 outcomes of 1.

3.2 Superposition

Superposition is the defining feature of quantum mechanics. It allows a qubit to encode information about multiple classical states at once—but with a critical caveat.

Classical analogy: Think of a coin spinning in the air. While spinning, it's neither heads nor tails—it's in a state where the outcome is probabilistic. When it lands (measurement), you get a definite result.

Key Insight

Superposition does **not** give you parallel classical processing. You cannot extract both α and β from a single measurement. Measurement gives you one outcome, and the superposition is destroyed.

The power of quantum computing comes from interference—constructively amplifying the amplitude of correct answers and destructively canceling wrong ones before measurement.

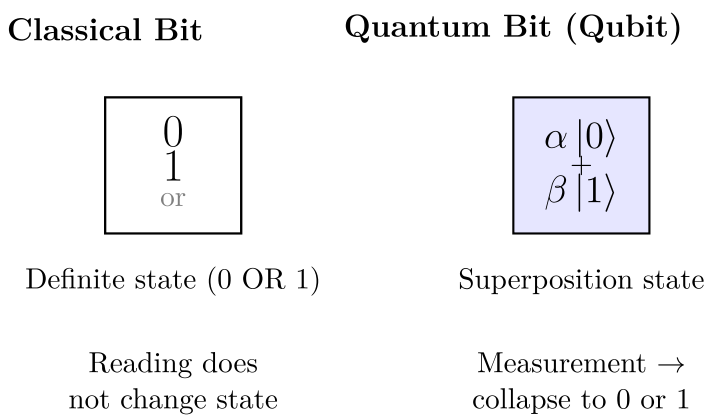


Figure 2: Classical bit vs qubit. A classical bit is always in a definite state (0 or 1) and can be read without disturbing it. A qubit can be in a superposition of both states, but measurement destroys the superposition and yields a single classical outcome.

Quantum gates manipulate the amplitudes α and β without measuring. This allows algorithms to explore multiple computational paths simultaneously. But to extract an answer, you must measure, which collapses the state.

Example 4: Why Superposition Isn't Two Bits of Information

A classical 2-bit system has 4 states: $00, 01, 10, 11$. To specify which state, you need 2 bits of information.

A single qubit has two complex amplitudes:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

It looks like you could store infinite information in α and β (they're continuous complex numbers). But you can't.

When you measure, you get only 1 classical bit: either 0 or 1. The amplitudes determine probabilities, not deterministic outcomes. Most of the information in α and β is inaccessible.

3.3 Physical Implementations of Qubits

Unlike classical bits (which can be built with almost any bistable system), qubits require carefully engineered quantum systems. The two-level quantum system must be:

- Isolated from the environment to prevent decoherence
- Controllable with high precision to apply gates
- Measurable to read out the final state

Several hardware platforms exist, each with trade-offs.

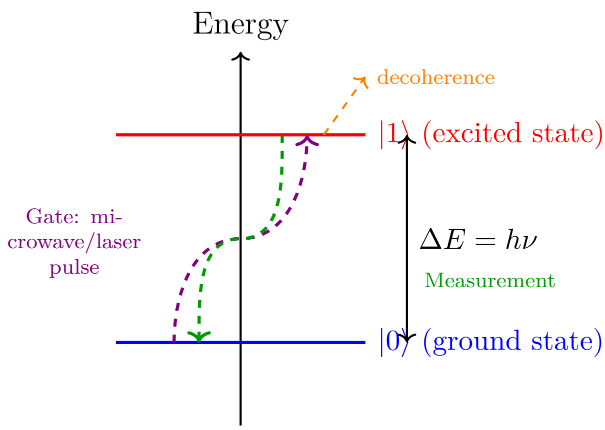


Figure 3: Two-level quantum system (qubit). Only the two lowest energy levels are used: $|0\rangle$ (ground) and $|1\rangle$ (excited). Gates drive transitions between levels via electromagnetic pulses. Decoherence (interaction with environment) destroys quantum information.

Platform	**Gate Speed**	**Coherence Time**	**Fidelity**
Superconducting	10–100 ns	10–200 μs	99–99.9%
Trapped Ion	1–100 μs	seconds	99.9%+
Photonic	sub-ns	N/A (flying qubits)	99%+
Topological	TBD	TBD (predicted: hours)	TBD

3.4 Superconducting Qubits

Superconducting qubits are the most mature platform. They are used by IBM, Google, and Rigetti.

How they work: A superconducting circuit (typically a Josephson junction) creates a quantized two-level system. The qubit is encoded in the lowest two energy levels. Microwave pulses drive transitions between $|0\rangle$ and $|1\rangle$.

Classical analogy: Think of an LC circuit (inductor + capacitor). In classical physics, the energy can be any value. In quantum mechanics, energy is quantized—only discrete levels are allowed. The qubit uses the ground state ($|0\rangle$) and first excited state ($|1\rangle$).

Advantages:

- Fast gate operations (10–100 nanoseconds)
- Fabricated using standard semiconductor processes
- Can integrate many qubits on a chip

Disadvantages:

- Require millikelvin temperatures ($\sim$15 mK, colder than outer space)
- Short coherence times (10–200 microseconds)
- Cross-talk between qubits on the same chip

Example 5: Gate Count Budget for Superconducting Qubits

Given:

- Coherence time $T_2 = 100\ \mu s$
- Gate time $t_{\text{gate}} = 50$ ns

Question: How many gates can you apply before decoherence destroys the quantum state?

Solution:

$$\begin{aligned} N_{\text{gates}} &= \frac{T_2}{t_{\text{gate}}} \\ &= \frac{100 \times 10^{-6}\ \text{s}}{50 \times 10^{-9}\ \text{s}} \\ &= \frac{100 \times 10^{-6}}{50 \times 10^{-9}} \\ &= 2000 \text{ gates} \end{aligned}$$

This is a theoretical maximum. In practice, gate errors accumulate, so useful circuit depth is much lower—typically 10–100 gates on current hardware.

3.5 Trapped Ion Qubits

Trapped ion qubits use individual atoms as qubits. They are used by IonQ and Quantinuum.

How they work: Ions (charged atoms) are trapped in vacuum using electric and magnetic fields. The qubit is encoded in two atomic energy levels (hyperfine or Zeeman states). Laser pulses manipulate the qubit state.

Classical analogy: Think of a tuning fork. It has discrete resonant frequencies. Similarly, an atom has discrete energy levels. A photon of the right frequency can excite the atom from one level to another.

Advantages:

- Long coherence times (seconds to minutes)
- High gate fidelities (99.9%+)
- All qubits are identical (they're the same atom)
- Full connectivity—any qubit can interact with any other

Disadvantages:

- Slow gate operations (microseconds to milliseconds)
- Hard to scale beyond 50–100 ions in one trap
- Requires complex laser systems and ultra-high vacuum

3.6 Noise and Decoherence

Quantum states are fragile. Any interaction with the environment destroys the coherence of the superposition. This process is called **decoherence**.

Two timescales characterize qubit quality:

- T_1 **(energy relaxation time):** How long before the qubit loses energy and decays from $|1\rangle$ to $|0\rangle$
- T_2 **(phase coherence time):** How long before the relative phase between $|0\rangle$ and $|1\rangle$ randomizes

Always: $T_2 \leq T_1$.

Classical analogy: Imagine a spinning top. T_1 is like friction slowly stopping the spin. T_2 is like wobbles that make the axis of rotation uncertain. Both destroy the useful motion, but in different ways.

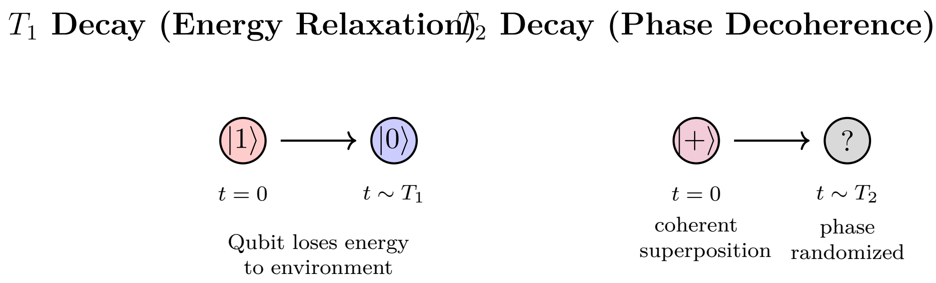


Figure 4: Two types of decoherence. T_1 **decay:** Excited state $|1\rangle$ loses energy and relaxes to ground state $|0\rangle$. T_2 **decay:** Relative phase between $|0\rangle$ and $|1\rangle$ components randomizes, destroying superposition without energy loss. Always $T_2 \leq T_1$.

Engineering Reality

Decoherence is the main obstacle to building large quantum computers. Current qubits have error rates of 0.1% to 1% per gate. Classical processors have error rates around 10^{-17}.

To run useful algorithms, we need error correction, which multiplies the number of physical qubits required.

Example 6: Error Accumulation Without Correction

Given:

- Gate fidelity: $F = 99\% = 0.99$
- Circuit depth: $d = 100$ gates

Question: What's the probability the computation completes without error?

Solution:

Assume errors are independent. The probability of no error on a single gate is 0.99. The probability of no error over 100 gates is:

$$\begin{aligned} P_{\text{success}} &= 0.99^{100} \\ &\approx 0.366 \\ &= 36.6\% \end{aligned}$$

Only about 1 in 3 runs will succeed. This is useless for reliable computation.

3.7 Why Error Correction Is Necessary

Classical computers have error rates so low ($\sim 10^{-17}$ per operation) that error correction is rarely needed at the gate level. Quantum computers cannot achieve this level of fidelity with current technology.

Error correction is mandatory for large-scale quantum computing. But it's hard because:

- The **no-cloning theorem** says you cannot copy a quantum state.
- Measurement destroys superposition, so you cannot simply "check" if an error occurred.

The solution: **quantum error correction codes**. These encode a single **logical qubit** using multiple **physical qubits**. Errors on physical qubits can be detected and corrected without measuring the logical state.

Example 7: Bit-Flip Code (Simplified)

The simplest quantum error correction code protects against bit-flip errors ($|0\rangle \leftrightarrow |1\rangle$).

Encoding:

$$|0\rangle_L \to |000\rangle$$
$$|1\rangle_L \to |111\rangle$$

A logical $|0\rangle$ is encoded as three physical qubits all in $|0\rangle$. A logical $|1\rangle$ is three qubits all in $|1\rangle$.

Error detection: If one qubit flips, the state becomes $|001\rangle$, $|010\rangle$, or $|100\rangle$. You detect this by measuring parity (without measuring the qubits directly). The majority vote tells you the intended state.

This code corrects 1 error but requires 3× physical qubits. Real codes (surface codes) require thousands of physical qubits per logical qubit.

3.8 Practice Problems

Problem 1: A qubit is prepared in the state $|\psi\rangle = 0.8\,|0\rangle + 0.6\,|1\rangle$. Calculate the probabilities $P(0)$ and $P(1)$ when measuring in the computational basis.

Problem 2: A superconducting qubit has $T_2 = 150\ \mu s$ and gate time $t_{\text{gate}} = 30$ ns. How many gates can theoretically be applied before decoherence dominates?

Problem 3: A qubit is in the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ with $|\alpha| = 0.5$. What are the possible values of $|\beta|$?

Problem 4: A quantum circuit has 200 gates, each with fidelity 99.5%. What is the probability that the entire circuit executes without error?

Problem 5: Explain why a classical AND gate cannot be directly implemented as a quantum gate.

3.8.1 Solutions

Solution 1:

$$P(0) = |0.8|^2 = 0.64 = 64\%$$
$$P(1) = |0.6|^2 = 0.36 = 36\%$$

Solution 2:

$$N_{\text{gates}} = \frac{T_2}{t_{\text{gate}}} = \frac{150 \times 10^{-6}}{30 \times 10^{-9}} = 5000 \text{ gates}$$

Solution 3:
From normalization: $|\alpha|^2 + |\beta|^2 = 1$

$$|\beta|^2 = 1 - |\alpha|^2 = 1 - 0.5^2 = 0.75$$
$$|\beta| = \sqrt{0.75} \approx 0.866$$

Solution 4:

$$\begin{aligned} P_{\text{success}} &= 0.995^{200} \\ &\approx 0.368 \\ &= 36.8\% \end{aligned}$$

Solution 5:
The AND gate maps multiple inputs to the same output (e.g., $00 \to 0$ and $01 \to 0$), so it's not reversible. Quantum gates must be unitary (reversible), so AND cannot be directly implemented. You need a reversible version like Toffoli gate.

Key Takeaways

- A qubit is described by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$
- Superposition does not mean "both at once"—it means probabilistic measurement outcomes
- Measurement collapses the state to $|0\rangle$ or $|1\rangle$ with probabilities $|\alpha|^2$ and $|\beta|^2$
- Superconducting qubits are fast but short-lived; trapped ions are slow but high-fidelity
- Decoherence limits circuit depth to hundreds or thousands of gates on current hardware
- Error correction is mandatory but requires large overhead in physical qubits

Bloch Sphere Geometry

Chapter Overview

The Bloch sphere is the single most important visualization tool for understanding single-qubit states and operations. By the end of this chapter, you'll be able to locate any qubit state on the sphere and understand how gates transform it.

4.1 Why We Need a Geometric Picture

A qubit is mathematically a vector in a two-dimensional complex Hilbert space:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

This representation is precise, but it's not intuitive. Complex numbers, amplitudes, phases—these are abstract concepts that don't give you a mental picture of what's happening.

The **Bloch sphere** provides a geometric representation where every single-qubit pure state corresponds to a point on the surface of a unit sphere.

Key Advantage

On the Bloch sphere:

- Every qubit state is a point on the surface
- Every quantum gate is a rotation
- Measurement axes are clearly visible

This turns abstract math into geometric intuition.

Two special states define the poles of the sphere:

- The state $|0\rangle$ is at the **north pole**
- The state $|1\rangle$ is at the **south pole**

All other pure states lie somewhere on the surface between these poles.

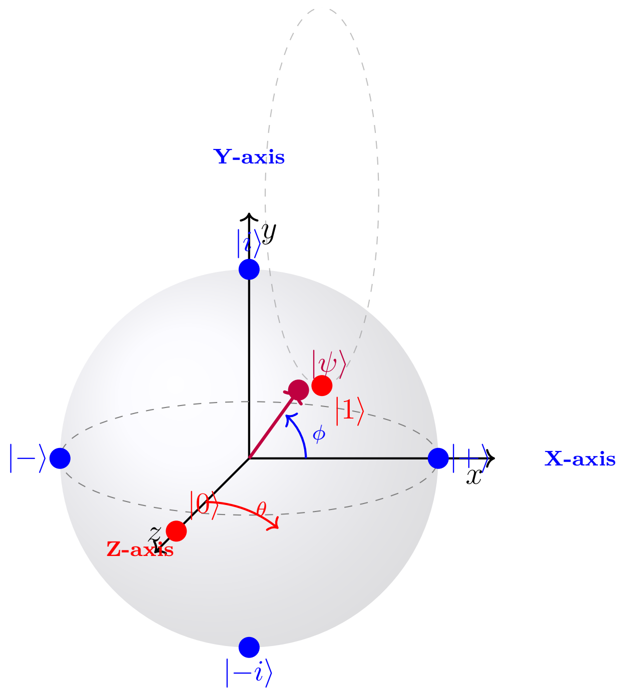


Figure 5: The Bloch sphere representation of a single qubit. North pole = $|0\rangle$, south pole = $|1\rangle$. Equator states are equal superpositions. The angles θ (polar) and ϕ (azimuthal) uniquely specify any pure qubit state.

Reading the Bloch Sphere

- **Poles:** $|0\rangle$ at top (north), $|1\rangle$ at bottom (south)
- **Equator:** All equal superposition states ($|\alpha|^2 = |\beta|^2 = 1/2$)
- **X-axis:** $|+\rangle$ and $|-\rangle$ states (real coefficients)
- **Y-axis:** $|i\rangle$ and $|-i\rangle$ states (imaginary coefficients)
- **Angles:** θ = how far from north pole, ϕ = rotation around Z-axis

4.2 General Qubit State

Any pure qubit state can be written in the form:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{15}$$

where:

- $0 \leq \theta \leq \pi$ is the polar angle (latitude)
- $0 \leq \phi < 2\pi$ is the azimuthal angle (longitude)

Why this form? Let's derive it from the general state.

Derivation: From General Form to Bloch Form

Start with the general qubit state:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

Step 1: Write amplitudes in polar form.

$$\alpha = |\alpha| e^{i\phi_0}$$
$$\beta = |\beta| e^{i\phi_1}$$

Step 2: Factor out the global phase $e^{i\phi_0}$.
Global phase is unobservable, so we can ignore it:

$$|\psi\rangle = e^{i\phi_0}\left(|\alpha| |0\rangle + |\beta| e^{i(\phi_1-\phi_0)} |1\rangle\right) \equiv |\alpha| |0\rangle + |\beta| e^{i\phi} |1\rangle$$

where $\phi = \phi_1 - \phi_0$ is the **relative phase**.

Step 3: Parameterize the magnitudes.
Since $|\alpha|^2 + |\beta|^2 = 1$, we can write:

$$|\alpha| = \cos\left(\frac{\theta}{2}\right)$$
$$|\beta| = \sin\left(\frac{\theta}{2}\right)$$

This automatically satisfies normalization:

$$\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1 \quad \checkmark$$

Result:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle$$

This is the Bloch sphere representation. Every qubit state is uniquely specified by two real numbers: θ and ϕ.

4.3 Meaning of the Angles

θ **(polar angle):** Controls how much $|0\rangle$ vs $|1\rangle$ character the state has.

- $\theta = 0 \Rightarrow |\psi\rangle = |0\rangle$ (north pole)
- $\theta = \pi/2 \Rightarrow$ equal superposition (equator)
- $\theta = \pi \Rightarrow |\psi\rangle = |1\rangle$ (south pole)

ϕ **(azimuthal angle):** Controls the relative phase between $|0\rangle$ and $|1\rangle$.

- $\phi = 0 \Rightarrow$ real positive coefficient on $|1\rangle$
- $\phi = \pi \Rightarrow$ real negative coefficient on $|1\rangle$
- $\phi = \pi/2 \Rightarrow$ imaginary positive coefficient on $|1\rangle$

Example 1: State at North Pole

$\theta = 0$, any ϕ:

$$\begin{aligned}|\psi\rangle &= \cos(0)\,|0\rangle + e^{i\phi}\sin(0)\,|1\rangle \\ &= 1 \cdot |0\rangle + 0 \cdot |1\rangle \\ &= |0\rangle\end{aligned}$$

At the poles, phase doesn't matter because one amplitude is zero.

Example 2: State on Equator with $\phi = 0$

$\theta = \pi/2$, $\phi = 0$:

$$\begin{aligned}|\psi\rangle &= \cos(\pi/4)\,|0\rangle + e^{i\cdot 0}\sin(\pi/4)\,|1\rangle \\ &= \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \\ &= |+\rangle\end{aligned}$$

This is the $|+\rangle$ state, lying on the positive X-axis.

Example 3: State on Equator with $\phi = \pi$

$\theta = \pi/2$, $\phi = \pi$:

$$\begin{aligned}|\psi\rangle &= \cos(\pi/4)\,|0\rangle + e^{i\pi}\sin(\pi/4)\,|1\rangle \\ &= \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle \\ &= |-\rangle\end{aligned}$$

This is the $|-\rangle$ state, on the negative X-axis.

Example 4: State with Imaginary Coefficient

$\theta = \pi/2$, $\phi = \pi/2$:

$$\begin{aligned}|\psi\rangle &= \cos(\pi/4)\,|0\rangle + e^{i\pi/2}\sin(\pi/4)\,|1\rangle \\ &= \frac{1}{\sqrt{2}}\,|0\rangle + i\frac{1}{\sqrt{2}}\,|1\rangle \\ &= |i\rangle\end{aligned}$$

This is the $|i\rangle$ state, on the positive Y-axis.

4.4 Axes of the Bloch Sphere

The Bloch sphere has three principal axes corresponding to three measurement bases.

Z-axis (vertical):

- $+Z$ direction: $|0\rangle$ (north pole)
- $-Z$ direction: $|1\rangle$ (south pole)
- Measurement in this basis is the standard computational basis measurement

X-axis (horizontal):

- $+X$ direction: $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$
- $-X$ direction: $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$
- This is the Hadamard basis

Y-axis (horizontal, perpendicular to X):

- $+Y$ direction: $|i\rangle = \frac{|0\rangle+i|1\rangle}{\sqrt{2}}$
- $-Y$ direction: $|-i\rangle = \frac{|0\rangle-i|1\rangle}{\sqrt{2}}$

Each axis corresponds to a Pauli operator: X, Y, Z.

4.5 Quantum Gates as Rotations

Every single-qubit gate corresponds to a rotation on the Bloch sphere. This is why unitarity (preserving probability) corresponds geometrically to rotations (preserving distance from the origin).

Fundamental Fact

A rotation by angle θ about axis $\hat{n}$ is represented by the unitary:

$$R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}}$$

where $\vec{\sigma} = (X, Y, Z)$ are the Pauli matrices.

Every single-qubit gate is a rotation. Every rotation is a unitary matrix.

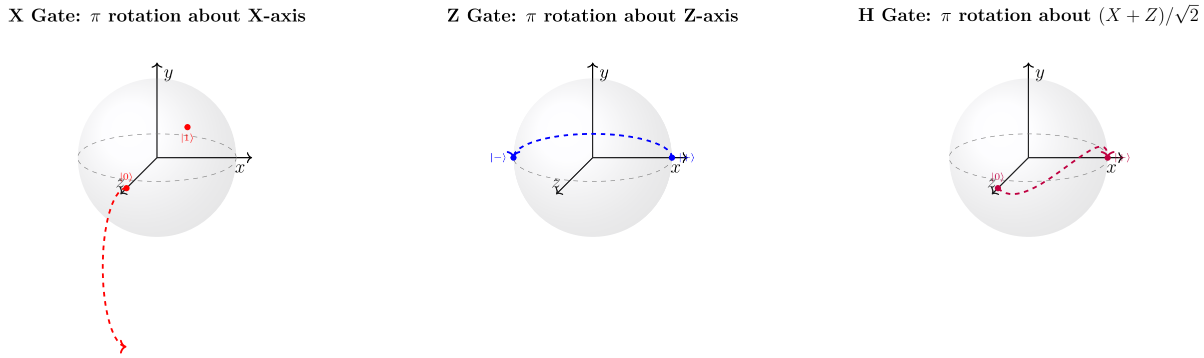


Figure 6: Common single-qubit gates visualized as rotations on the Bloch sphere. Each gate rotates by π radians about a specific axis.

Example 5: X Gate is π Rotation About X-Axis

The Pauli-X gate:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Apply to $|0\rangle$:

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

Geometrically: X rotates the Bloch vector by π radians about the X-axis. The north pole ($|0\rangle$) rotates to the south pole ($|1\rangle$).

Example 6: Z Gate is π Rotation About Z-Axis

The Pauli-Z gate:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Apply to $|+\rangle$:

$$\begin{aligned} Z|+\rangle &= Z\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\ &= \frac{Z|0\rangle + Z|1\rangle}{\sqrt{2}} \\ &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ &= |-\rangle \end{aligned}$$

Geometrically: Z rotates by π about the Z-axis. The $+X$ direction ($|+\rangle$) rotates to the $-X$ direction ($|-\rangle$), staying on the equator.

Example 7: Hadamard Gate is π Rotation About X+Z Axis

The Hadamard gate:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Apply to $|0\rangle$:

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = |+\rangle$$

Geometrically: H rotates by π about an axis halfway between X and Z. The north pole ($|0\rangle$) rotates to the $+X$ point on the equator ($|+\rangle$).

4.6 Superposition on the Bloch Sphere

Equator = equal superpositions. Any state on the equator has $\theta = \pi/2$, meaning:

$$|\alpha| = |\beta| = \frac{1}{\sqrt{2}}$$

So $P(0) = P(1) = 50\%$.

Between equator and poles = biased superpositions. States between the equator and north pole favor $|0\rangle$. States between the equator and south pole favor $|1\rangle$.

Example 8: Calculating Bloch Coordinates from State Vector

Given $|\psi\rangle = 0.6|0\rangle + 0.8|1\rangle$, find θ and ϕ.

Step 1: Identify magnitudes and phase.
Both coefficients are real, so $\phi = 0$ (or π if negative).

Step 2: Find θ from magnitudes.

$$\cos(\theta/2) = |0.6| = 0.6$$
$$\theta/2 = \arccos(0.6) \approx 0.927 \text{ rad}$$
$$\theta \approx 1.854 \text{ rad} \approx 106.3°$$

Verify:

$$\sin(\theta/2) = \sin(0.927) \approx 0.8 \quad \checkmark$$

Result: $\theta \approx 106.3°$, $\phi = 0$.
This state is slightly past the equator toward the south pole, on the X-Z plane.

4.7 Phase and Longitude

Phase has no classical analog. It determines where on the equator (or at a given latitude) the state lies.

Critical Distinction

Global phase: $e^{i\gamma}|\psi\rangle$ is physically identical to $|\psi\rangle$. It's not observable.

Relative phase: The phase *between* $|0\rangle$ and $|1\rangle$ (the ϕ in $e^{i\phi}$) *is* observable via interference.

Example 9: Phase Matters for Interference

Consider two states:

$$|\psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|\psi_2\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Both are on the equator with $\theta = \pi/2$. But $|\psi_1\rangle$ has $\phi = 0$ (positive sign), and $|\psi_2\rangle$ has $\phi = \pi$ (negative sign).

Apply Hadamard to both:

$$H|\psi_1\rangle = H|+\rangle = |0\rangle$$
$$H|\psi_2\rangle = H|-\rangle = |1\rangle$$

The phase difference causes completely different outcomes. This is quantum interference.

4.8 Practice Problems

Problem 1: Write the Bloch sphere form for $|1\rangle$. What are θ and ϕ?

Problem 2: A state has Bloch coordinates $\theta = \pi/3$, $\phi = \pi/6$. Write the state vector $|\psi\rangle$ explicitly.

Problem 3: What latitude (value of θ) on the Bloch sphere gives $P(0) = 0.75$?

Problem 4: Describe geometrically what the S gate does. (Hint: $S = \text{diag}(1, i)$.)

Problem 5: If a gate rotates the Bloch vector by $\pi/2$ about the Y-axis, what is the matrix representation?

4.8.1 Solutions

Solution 1:
$|1\rangle$ is at the south pole: $\theta = \pi$, ϕ is arbitrary (often taken as 0).

$$\begin{aligned}|\psi\rangle &= \cos(\pi/2)|0\rangle + e^{i\cdot 0}\sin(\pi/2)|1\rangle \\ &= 0 \cdot |0\rangle + 1 \cdot |1\rangle \\ &= |1\rangle\end{aligned}$$

Solution 2:

$$\begin{aligned}|\psi\rangle &= \cos(\pi/6)\,|0\rangle + e^{i\pi/6}\sin(\pi/6)\,|1\rangle \\ &= \frac{\sqrt{3}}{2}\,|0\rangle + \frac{1}{2}e^{i\pi/6}\,|1\rangle \\ &= \frac{\sqrt{3}}{2}\,|0\rangle + \frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)|1\rangle \\ &= \frac{\sqrt{3}}{2}\,|0\rangle + \frac{1}{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)|1\rangle\end{aligned}$$

Solution 3:
$P(0) = \cos^2(\theta/2) = 0.75$

$$\begin{aligned}\cos(\theta/2) &= \sqrt{0.75} = \frac{\sqrt{3}}{2} \\ \theta/2 &= \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \\ \theta &= \frac{\pi}{3} \approx 60^\circ\end{aligned}$$

Solution 4:
The S gate adds a phase of $\pi/2$ to $|1\rangle$:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Geometrically, S rotates by $\pi/2$ about the Z-axis. A state on the equator rotates by 90° longitude.

Solution 5:
A rotation by $\pi/2$ about Y is:

$$R_y(\pi/2) = e^{-i(\pi/4)Y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(This can be derived from the exponential of the Pauli-Y matrix.)

Key Takeaways

- Every single-qubit state maps to a point on the Bloch sphere surface
- θ determines $|0\rangle$ vs $|1\rangle$ probability; ϕ determines relative phase
- Poles are $|0\rangle$ (north) and $|1\rangle$ (south); equator states are equal superpositions
- Every single-qubit gate is a rotation on the Bloch sphere
- X, Y, Z gates are π rotations about their respective axes
- Relative phase (longitude ϕ) is observable via interference, unlike global phase

Measurement and Probabilities

Measurement is the most counterintuitive part of quantum mechanics. It's the moment where quantum information becomes classical—and quantum superposition is destroyed. This chapter makes measurement concrete with explicit calculations.

5.1 Measurement in Quantum Systems

In classical computing, reading a bit does not change its value. You can check if a classical bit is 0 or 1 as many times as you want—nothing changes.

In quantum computing, measurement is fundamentally different. When you measure a qubit:

1. The superposition **collapses** to one of the basis states
2. You get a classical outcome: 0 or 1
3. The quantum information is lost—you cannot "unmeasure"

Critical Fact

Measurement is **irreversible**. It's the only non-unitary operation in quantum mechanics.

Before measurement: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ (quantum superposition).
After measurement: $|0\rangle$ or $|1\rangle$ (classical bit).
You cannot reverse this process.

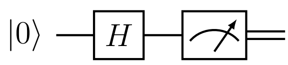


Figure 7: Measurement circuit example: Apply Hadamard to $|0\rangle$ to create $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, then measure in the Z-basis. Outcome: 0 or 1 with 50% probability each. The meter symbol represents measurement, and the double line represents a classical bit storing the outcome.

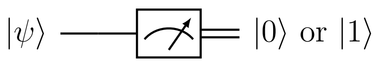
5.2 Measurement Probabilities

Given a qubit in state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

The probability of measuring outcome k is given by the **Born rule**:

$$P(k) = |\langle k|\psi\rangle|^2 \tag{16}$$



Measurement: Quantum → Classical

Input	Measurement	Output (Classical Bit)
$\vert\psi\rangle = \alpha\vert 0\rangle + \beta\vert 1\rangle$	↻	0 with prob. $\vert\alpha\vert^2$
(quantum superposition)	(collapse)	1 with prob. $\vert\beta\vert^2$

Figure 8: Measurement collapses a quantum superposition into a classical bit. The quantum state is destroyed in the process.

For computational basis measurement:

$$P(0) = |\langle 0|\psi\rangle|^2 = |\alpha|^2 \tag{17}$$
$$P(1) = |\langle 1|\psi\rangle|^2 = |\beta|^2 \tag{18}$$

These probabilities must sum to 1 due to normalization:

$$P(0) + P(1) = |\alpha|^2 + |\beta|^2 = 1$$

5.3 Examples of Measurement

Example 1: Measuring $|+\rangle$ in Z-Basis

The state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is measured in the computational (Z) basis.

Step 1: Identify amplitudes.

$$\alpha = \frac{1}{\sqrt{2}}$$
$$\beta = \frac{1}{\sqrt{2}}$$

Step 2: Calculate probabilities.

$$P(0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = 50\%$$
$$P(1) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = 50\%$$

Result: You get 0 or 1 with equal probability.

Example 2: Measuring $|0\rangle$ in Z-Basis

The state $|0\rangle$ is measured in the computational basis.

Step 1: Write state explicitly.

$$|0\rangle = 1 \cdot |0\rangle + 0 \cdot |1\rangle$$

Step 2: Calculate probabilities.

$$P(0) = |1|^2 = 1 = 100\%$$
$$P(1) = |0|^2 = 0 = 0\%$$

Result: You always get 0. The state was already in the measurement basis, so no probabilistic behavior.

Example 3: Measuring $|+\rangle$ in X-Basis

Now measure $|+\rangle$ in the X-basis (Hadamard basis), where the basis states are $|+\rangle$ and $|-\rangle$.

Step 1: Express the state in the X-basis.
Already done: $|\psi\rangle = |+\rangle = 1 \cdot |+\rangle + 0 \cdot |-\rangle$
Step 2: Calculate probabilities.

$$P(+) = |1|^2 = 1 = 100\%$$
$$P(-) = |0|^2 = 0 = 0\%$$

Result: You always get +. When measured in its own basis, the state gives a deterministic outcome.

Example 4: Measuring $|0\rangle$ in X-Basis

State $|0\rangle$ measured in X-basis.

Step 1: Express $|0\rangle$ in X-basis.
Recall:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Invert this:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$
$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

So:

$$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

Step 2: Calculate probabilities.

$$P(+) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = 50\%$$
$$P(-) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = 50\%$$

Result: Equal probability of + or −. Measuring in a different basis introduces uncertainty.

5.4 State Collapse

After measurement, the qubit is no longer in the original superposition. It is in the state corresponding to the outcome.

Before measurement:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

After measuring outcome 0:

$$|\psi\rangle_{\text{after}} = |0\rangle$$

After measuring outcome 1:

$$|\psi\rangle_{\text{after}} = |1\rangle$$

Example 5: State Before and After Measurement

Initial state:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

Step 1: Calculate measurement probabilities.

$$P(0) = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3} \approx 33.3\%$$
$$P(1) = \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3} \approx 66.7\%$$

Step 2: Suppose we measure and get outcome 1.
After measurement:

$$|\psi\rangle_{\text{after}} = |1\rangle$$

The superposition is gone. The qubit is now definitively in state $|1\rangle$.

5.5 Repeated Measurement

If you measure the same qubit multiple times without applying any gates between measurements, you get the same result every time (assuming measurement in the same basis).

Key Principle

First measurement: Probabilistic, governed by Born rule.
Subsequent measurements (same basis): Deterministic—you get the same outcome.

The first measurement collapses the state. After that, the qubit is in a definite state, so re-measuring gives the same answer.

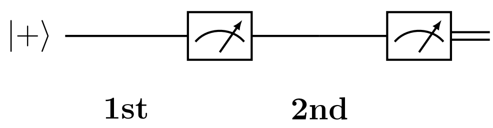


Figure 9: Repeated measurement in the same basis. First measurement: 0 or 1 with 50% probability (state collapses to $|0\rangle$ or $|1\rangle$). Second measurement: deterministic—same outcome as first.

Example 6: Measuring $|+\rangle$ Twice in Z-Basis

Initial state: $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

First measurement:

- Probability of 0: 50%
- Probability of 1: 50%

Suppose we get 0.

After first measurement: $|\psi\rangle = |0\rangle$

Second measurement (immediately after, no gates):
State is $|0\rangle$, so:

- Probability of 0: 100%
- Probability of 1: 0%

We get 0 again, with certainty.

Example 7: Measuring in Different Bases Gives Probabilistic Results

Initial state: $|+\rangle$

Step 1: Measure in Z-basis. Suppose we get 0.
State after measurement: $|0\rangle$

Step 2: Now measure in X-basis.
Express $|0\rangle$ in X-basis:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

Probabilities:

$$P(+) = 50\%$$
$$P(-) = 50\%$$

Result: Uncertainty is re-introduced because we changed measurement basis.

5.6 Measurement and the Bloch Sphere

Geometrically, measurement projects the Bloch vector onto the measurement axis.

Z-basis measurement: Projects onto the Z-axis.

- If state is near north pole, likely to measure $|0\rangle$
- If state is near south pole, likely to measure $|1\rangle$
- If state is on equator, 50/50 probability

X-basis measurement: Projects onto the X-axis.

- If state is near $+X$, likely to measure $|+\rangle$
- If state is near $-X$, likely to measure $|-\rangle$

Example 8: Geometric Interpretation

State: $|\psi\rangle = 0.8\,|0\rangle + 0.6\,|1\rangle$

Bloch coordinates:

$$\theta = 2\arccos(0.8) \approx 1.287 \text{ rad} \approx 73.7^\circ$$
$$\phi = 0$$

The state is in the X-Z plane, tilted 73.7° from the north pole.

Z-basis measurement:

The state is closer to the north pole than the south pole, so:

$$P(0) = 0.64 = 64\%$$
$$P(1) = 0.36 = 36\%$$

More likely to measure $|0\rangle$.

5.7 Importance of Measurement in Quantum Algorithms

Quantum algorithms must be designed so that the correct answer has high probability before measurement.

Quantum computation proceeds in three stages:

1. **Initialization:** Prepare qubits in a known state (usually $|0\rangle$)
2. **Unitary evolution:** Apply gates to manipulate amplitudes
3. **Measurement:** Extract classical information

The goal of the unitary evolution is to **amplify the amplitude** of the correct answer and **suppress the amplitude** of wrong answers. This is quantum interference.

Design Principle

A quantum algorithm must ensure that when you measure, the probability of getting the correct answer is close to 1.

Poor algorithm design leaves correct and incorrect answers with similar probabilities, making measurement useless.

Example 9: Amplitude Amplification Intuition

Suppose an algorithm searches for one marked item among 4 possibilities: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

Bad approach: Prepare equal superposition.

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Measure: $P(\text{each outcome}) = 25\%$. No advantage—you're just guessing.

Good approach (Grover's algorithm): Apply gates to increase the amplitude of the correct answer (say, $|10\rangle$):

$$|\psi\rangle \approx 0.95\,|10\rangle + (\text{small amplitudes on other states})$$

Measure: $P(10) \approx 90\%$. High probability of success.

5.8 Practice Problems

Problem 1: A qubit is in state $|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$. What are $P(0)$ and $P(1)$ when measuring in the computational basis?

Problem 2: A qubit in state $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ is measured in the X-basis. What is the outcome and post-measurement state?

Problem 3: A qubit is prepared in state $|+\rangle$, measured in Z-basis (result: 0), then measured again in Z-basis. What is the probability of getting 1 on the second measurement?

Problem 4: Express $|1\rangle$ in the X-basis and calculate $P(+)$ and $P(-)$ when measuring $|1\rangle$ in X-basis.

Problem 5: Why can't you "clone" a quantum state by measuring it and reconstructing it?

5.8.1 Solutions

Solution 1:

$$P(0) = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5} = 20\%$$
$$P(1) = \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5} = 80\%$$

Solution 2:
$|-\rangle$ is an eigenstate of the X-basis. When measured in X-basis:

- Outcome: $-$ (with 100% probability)
- Post-measurement state: $|-\rangle$

Solution 3:
After first measurement (result: 0), state is $|0\rangle$.
Second measurement in Z-basis on $|0\rangle$:

$$P(1) = 0\%$$

The qubit is already collapsed to $|0\rangle$, so you cannot get $|1\rangle$.
Solution 4:
Express $|1\rangle$ in X-basis:

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Probabilities:

$$P(+) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = 50\%$$
$$P(-) = \left|\frac{-1}{\sqrt{2}}\right|^2 = \frac{1}{2} = 50\%$$

(Note: The global phase on the coefficient of $|-\rangle$ doesn't affect probability.)

Solution 5:

Measurement collapses the state. If you measure $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, you get either 0 or 1—not both amplitudes. You lose the information about α and β, so you cannot reconstruct the original state. This is why the no-cloning theorem holds.

Key Takeaways

- Measurement is irreversible—it collapses superposition to a classical outcome
- Born rule: $P(k) = |\langle k|\psi\rangle|^2$ gives measurement probabilities
- Measuring in the state's own basis gives deterministic outcomes
- Measuring in a different basis introduces probabilistic uncertainty
- Repeated measurement in the same basis gives the same result
- Quantum algorithms must amplify correct answer amplitudes before measurement