Introduction

Why Differential Equations?

The universe speaks in differential equations.

When Newton wrote $F = ma$, he was really writing $F = m\frac{d^2x}{dt^2}$—a differential equation relating force to the second derivative of position. When engineers model circuits, the relationship between voltage and current involves derivatives. When biologists track population growth, the rate of change depends on the current population. When economists model markets, rates of change appear everywhere.

A **differential equation** is an equation involving an unknown function and its derivatives. Instead of asking "what is y?", we ask "what function $y(x)$ satisfies this relationship between y and its derivatives?"

For example, the equation:

$$\frac{dy}{dx} = 2y$$

asks: "What function equals twice its own derivative?" The answer is $y = Ce^{2x}$—the exponential function, which grows at a rate proportional to its current value.

Differential equations are the bridge between **rates of change** (derivatives) and **accumulated quantities** (functions). They let us predict how systems evolve over time, whether that system is a falling object, a swinging pendulum, a chemical reaction, or an electrical circuit.

What This Book Is

This is a **problem-solving cookbook** for ordinary differential equations.

The goal is simple: given a differential equation, find its solution. We focus on **techniques**—the substitutions, patterns, and methods that transform difficult equations into tractable ones. Each chapter follows a consistent structure:

1. **Core concept** — What is the method? When does it apply?
2. **Step-by-step procedure** — A clear algorithm you can follow
3. **Worked examples** — 10-20 problems solved in detail, showing every step and explaining the reasoning
4. **Common mistakes** — Errors to watch for and how to avoid them
5. **Summary** — Key formulas and decision points at a glance
6. **Exercises** — 20-25 problems for practice, ranging from routine to challenging

There's a certain satisfaction in solving a differential equation by hand. You examine it, try a substitution, watch the terms rearrange—and suddenly it simplifies into something you can integrate. It's like picking a lock. This book teaches you to pick those locks.

What This Book Is Not

This is not a theory book. We won't prove existence and uniqueness theorems or develop the abstract theory of differential operators. The focus is on solving equations, not proving theorems about them.

This is not a survey course. We go deep on the techniques that work, with enough examples to build genuine fluency. Breadth without depth produces students who recognize equation types but can't actually solve them.

This is not a cookbook without understanding. While we emphasize methods, we explain *why* they work. Understanding the reasoning helps you adapt techniques to new situations and catch your own errors.

The Programmer's Perspective

This book is written for people who think like programmers—who value clear procedures, explicit algorithms, and concrete examples over abstract generality.

Programmers know that understanding a concept means being able to implement it. The same principle applies here: understanding a solution technique means being able to execute it reliably on a variety of problems. That's why this book emphasizes:

- **Explicit steps**: Methods are presented as clear procedures, not vague guidelines
- **Pattern recognition**: Learning to identify which technique applies to which equation
- **Verification**: Checking your answer by substituting back into the original equation
- **Edge cases**: Understanding where methods break down and what to do about it

The appendix includes numerical methods implemented in Rust, allowing you to verify solutions computationally and visualize behavior when closed-form solutions aren't available.

Structure of the Book

Part 1: Calculus Toolkit

Chapter 1 reviews differentiation—the foundation you'll use constantly.

If your calculus is solid, skim this chapter or use it as a reference. If you're rusty, work through it carefully—weak calculus skills are the #1 obstacle to solving differential equations.

Part 2: Integration

Chapters 2-7 cover integration techniques essential for solving differential equations.

- Integration Basics — Antiderivatives and the fundamental techniques
- The Substitution Method — The most common technique, reversing the chain rule
- Integration by Parts — For products involving polynomials, exponentials, and trig functions
- Partial Fractions — Decomposing rational functions for integration
- Trigonometric Integrals — Handling powers and products of sine and cosine
- Trigonometric Substitution — When square roots of quadratics appear

Part 3: First-Order Differential Equations

Chapters 8-12 cover equations involving only first derivatives.

- What is a Differential Equation? — Terminology, classification, and the concept of solutions
- Separable Equations — When variables can be isolated: $f(y), dy = g(x), dx$
- Linear Equations — The integrating factor method for $y' + P(x)y = Q(x)$
- Exact Equations — When $M, dx + N, dy = 0$ comes from a potential function
- Substitution Techniques — Transformations that convert difficult equations into familiar forms
  - Homogeneous Equations — When $y' = f(y/x)$
  - Bernoulli Equations — When $y' + Py = Qy^n$
  - Riccati Equations — When $y' = P + Qy + Ry^2$

Part 4: Second-Order Differential Equations

Chapters 13-15 cover equations involving second derivatives.

- Constant Coefficients — The workhorse method for $ay'' + by' + cy = f(x)$, using characteristic equations
- Reduction of Order — Finding a second solution when you know one
- Variation of Parameters — A general method for nonhomogeneous equations

Part 5: Special Methods

Chapters 16-17 cover powerful techniques for harder problems.

- Power Series Solutions — When coefficients aren't constant, expand the solution as a series
- Laplace Transforms — Transform differential equations into algebraic ones, especially useful for discontinuous forcing

Part 6: Systems

Chapters 18-19 extend the theory to multiple equations and geometric understanding.

- Systems of Equations — Solving coupled equations using eigenvalues and eigenvectors
- Phase Portraits — Visualizing solutions geometrically, classifying equilibrium behavior

Appendices

The appendices provide essential reference material and tools:

- Appendix A: Problem Solving Loop — A flowchart and decision tables to help you identify the right technique when you're stuck
- Appendix B: Common ODEs and Their Solutions — A catalog of frequently-encountered differential equations with their standard solutions
- Appendix C: Laplace Transform Table — Comprehensive reference for Laplace transforms and inverse transforms

- Appendix D: Essential Integrals — The integrals that appear most frequently when solving differential equations
- Appendix E: Numerical Methods in Rust — When analytical methods fail, numerical methods provide the answer (Euler, Runge-Kutta, visualization)
- Appendix F: Solutions to Odd-Numbered Exercises — Complete worked solutions to check your work and study techniques
- Appendix G: Index by Application — Find the right chapter based on your real-world problem (physics, biology, circuits, etc.)

> **Getting Started:** If you're new to differential equations or feeling overwhelmed, start with Appendix A: Problem Solving Loop. It provides a roadmap for identifying equation types and choosing the right technique—a useful reference to keep open as you work through the book.

Notation

We use standard notation throughout:

Symbol	Meaning
y, x	Dependent and independent variables
$y' = \frac{dy}{dx}$	First derivative
$y'' = \frac{d^2y}{dx^2}$	Second derivative
$\dot{y} = \frac{dy}{dt}$	Derivative with respect to time
C, C_1, C_2	Arbitrary constants
y_h, y_p	Homogeneous and particular solutions
$\mathbf{x}, \mathbf{v}$	Vectors (bold)
A, J	Matrices (capital letters)

The independent variable is usually x for general equations or t for time-dependent problems. The dependent variable is usually y for scalar equations or $\mathbf{x}$ for systems.

We use both Leibniz notation ($\frac{dy}{dx}$) and prime notation (y') interchangeably, choosing whichever is clearer in context.

How to Use This Book

Active Reading

Read with pencil in hand. Mathematics is not a spectator sport.

Before reading a worked example, try the problem yourself. When you do read the solution, follow along actively—verify each step, don't just skim. Ask yourself:

- *Why did they try that substitution?*
- *What pattern did they recognize?*
- *Could I have seen that on my own?*

When you finish an example, close the book and redo it from scratch. If you can't, you haven't learned it yet.

The Learning Loop

For each technique:

1. **Read the method** — Understand the steps and when they apply
2. **Study the examples** — See how the steps play out in practice
3. **Try the exercises** — Apply the method yourself
4. **Check your work** — Verify by substitution or compare with solutions
5. **Review mistakes** — Understand where you went wrong

Resist the temptation to look at solutions too quickly. Struggling with a problem—even failing—builds understanding that passive reading cannot.

Building Pattern Recognition

Differential equations are solved by recognizing patterns. “This looks like a linear equation.” “That substitution might simplify things.” “The characteristic equation will have complex roots.”

This pattern recognition comes only from practice. Work many problems. When you encounter a new equation, your brain should automatically cycle through: *Is it separable? Linear? Exact? What substitution might help?*

The common mistakes sections highlight where pattern recognition fails—equations that look like one type but aren’t, or techniques that seem applicable but don’t work.

Verification

Always verify your solutions. Substitute your answer back into the original equation. This catches errors and builds confidence.

For initial value problems, also check that your solution satisfies the initial conditions.

Numerical verification provides an additional check. If your analytical solution disagrees with a numerical approximation, one of them is wrong (usually the analytical one—numerical methods make small errors, not large ones).

Prerequisites

This book assumes you know:

- **Single-variable calculus**: Derivatives, integrals, the chain rule, basic integration techniques
- **Basic algebra**: Manipulating equations, completing the square, working with complex numbers
- **Elementary linear algebra** (for later chapters): Matrices, determinants, eigenvalues, eigenvectors

If you're shaky on calculus, Part 1 provides a review. If you're shaky on linear algebra, review the basics before tackling Chapters 18-19.

A Note on Difficulty

The exercises in each chapter are roughly ordered by difficulty:

- **Early exercises**: Direct application of the technique
- **Middle exercises**: Variations requiring adaptation
- **Later exercises**: Challenging problems requiring insight or combining techniques

Don't get discouraged if later problems are hard. They're meant to be. Work through what you can, and return to difficult problems after you've built more experience.

Let's Begin

Differential equations are one of the great achievements of mathematics—a language for describing change, a toolkit for understanding dynamics, a bridge between abstract mathematics and the physical world.

But you don't learn them by reading about them. You learn by solving them.

Turn the page, pick up your pencil, and let's get started.

What is a Differential Equation?

You've spent the first part of this book mastering two fundamental operations: differentiation and integration. Now it's time to put them to work.

A **differential equation** is an equation involving a function and its derivatives. Your job is to find the function. This is where all those techniques — substitution, integration by parts, partial fractions — become essential tools.

📋 **Quick Reference: Differential Equation Basics**

Vocabulary:

- **Order:** Highest derivative present (1st, 2nd, etc.)
- **Linear:** y and derivatives appear to first power only, not multiplied together
- **General solution:** Contains arbitrary constants (n constants for nth-order)
- **Particular solution:** Specific solution satisfying initial conditions

First-order types preview:

Type	Form	Method
Direct integration	$y' = f(x)$	Integrate
Separable	$y' = f(x)g(y)$	Ch 9
Linear	$y' + P(x)y = Q(x)$	Ch 10
Exact	$M, dx + N, dy = 0$	Ch 11

From Integration to Differential Equations

Consider the simplest case. You know that:

$$\frac{d}{dx}[x^2] = 2x$$

Now flip the question: **What function has derivative $2x$?**

This is a differential equation:

$$\frac{dy}{dx} = 2x$$

You already know how to solve it — integrate both sides:

$$y = \int 2x, dx = x^2 + C$$

That's it. You've just solved your first differential equation. The solution $y = x^2 + C$ is called the **general solution** — a family of curves, one for each value of C.

Example 1: Direct Integration

Problem: Solve $\frac{dy}{dx} = 3x^2 - 4x + 1$

Step 1: Recognize the form.

The right side depends only on x, so we can integrate directly.

Step 2: Integrate both sides.

$$y = \int (3x^2 - 4x + 1), dx$$

Step 3: Apply integration rules.

$$y = x^3 - 2x^2 + x + C$$

$$\boxed{y = x^3 - 2x^2 + x + C}$$

Step 4: Verify.

$$\frac{dy}{dx} = 3x^2 - 4x + 1 \quad ✓$$

Example 2: Requiring Substitution

Problem: Solve $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 1}}$

Step 1: Recognize the form.

Right side depends only on x. Integrate directly — but we'll need substitution.

Step 2: Set up the integral.

$$y = \int \frac{x}{\sqrt{x^2 + 1}} , dx$$

Step 3: Use substitution.

Let $u = x^2 + 1$, so $du = 2x, dx$, meaning $x, dx = \frac{1}{2}du$.

$$y = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2} , du = \frac{1}{2} \int u^{-1/2} , du$$

$$= \frac{1}{2} \cdot 2u^{1/2} + C = \sqrt{u} + C$$

Step 4: Substitute back.

$$y = \sqrt{x^2 + 1} + C$$

$$\boxed{y = \sqrt{x^2 + 1} + C}$$

Step 5: Verify.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}} \quad ✓$$

Example 3: Requiring Integration by Parts

Problem: Solve $\frac{dy}{dx} = xe^x$

Step 1: Recognize the form.

Right side depends only on x. We need integration by parts.

Step 2: Set up the integral.

$$y = \int xe^x, dx$$

Step 3: Apply integration by parts.

Let $u = x$ and $dv = e^x, dx$, so $du = dx$ and $v = e^x$.

$$y = xe^x - \int e^x, dx = xe^x - e^x + C$$

Step 4: Simplify.

$$y = e^x(x-1) + C$$

$$\boxed{y = e^x(x-1) + C}$$

Step 5: Verify.

$$\frac{dy}{dx} = e^x(x-1) + e^x \cdot 1 = e^x(x-1+1) = xe^x \quad ✓$$

The Constant of Integration Matters

Why do we always write $+C$? Because differentiation destroys information about vertical shifts.

The functions $y = x^2$, $y = x^2 + 1$, and $y = x^2 - 7$ all have the same derivative: $2x$.

When we solve $\frac{dy}{dx} = 2x$, we get ALL of these:

$$y = x^2 + C$$

This is the **general solution** — an infinite family of curves.

Initial Conditions

To pick out a specific curve, we need extra information. An **initial condition** tells us the value of y at some point.

Example 4: Initial Value Problem

Problem: Solve $\dfrac{dy}{dx} = \cos x$ with $y(0) = 2$

Step 1: Find the general solution.

$$y = \int \cos x, dx = \sin x + C$$

Step 2: Apply the initial condition.

We need $y(0) = 2$:

$$y(0) = \sin(0) + C = 0 + C = 2$$

So $C = 2$.

Step 3: Write the particular solution.

$$\boxed{y = \sin x + 2}$$

Step 4: Verify.

- $\frac{dy}{dx} = \cos x$ ✓
- $y(0) = \sin(0) + 2 = 2$ ✓

Example 5: Initial Value Problem with Substitution

Problem: Solve $\frac{dy}{dx} = \frac{1}{1 + x^2}$ with $y(1) = 0$

Step 1: Find the general solution.

$$y = \int \frac{1}{1 + x^2} , dx = \arctan x + C$$

Step 2: Apply the initial condition.

We need $y(1) = 0$:

$$y(1) = \arctan(1) + C = \frac{\pi}{4} + C = 0$$

So $C = -\frac{\pi}{4}$.

Step 3: Write the particular solution.

$$\boxed{y = \arctan x - \frac{\pi}{4}}$$

But What If y Appears on the Right Side?

The examples above were easy because $\frac{dy}{dx}$ depended only on x. We could integrate directly.

But what about:

$$\frac{dy}{dx} = y$$

This says: "find a function whose derivative equals itself."

You might recognize the answer: $y = e^x$. But what about $y = 2e^x$ or $y = -5e^x$?

Check: $\frac{d}{dx}[Ce^x] = Ce^x$. Yes!

So the general solution is:

$$y = Ce^x$$

This equation — where y appears on the right — requires new techniques. That's what the rest of this book is about.

Classification of Differential Equations

Before diving into solution techniques, let's establish vocabulary.

Order

The **order** is the highest derivative that appears.

Equation	Order
$\frac{dy}{dx} = y$	First
$\frac{d^2y}{dx^2} + y = 0$	Second

Equation	Order
$y'''-3y''+2y'=x$	Third
$y^{(4)}+y=\sin x$	Fourth

Linearity

An equation is **linear** if:

1. The dependent variable (y) and its derivatives appear only to the first power
2. They are not multiplied together
3. No transcendental functions of y appear (like $\sin y$, e^y, etc.)

Equation	Linear?	Why?
$y'+2y=x$	Yes	y and y' to first power
$y''+3y'+2y=e^x$	Yes	All terms linear in y
$y'=y^2$	No	y^2 — second power
$yy'=x$	No	y times y'
$(y')^2+y=0$	No	$(y')^2$ — derivative squared
$y'=\sin y$	No	Transcendental function of y
$y'=x\sin x$	Yes	$\sin x$ is fine — it's the x, not y

Degree

The **degree** is the power of the highest derivative, after the equation is rationalized.

Equation	Degree
$y'=x+y$	1
$(y')^2=y$	2
$(y'')^3+y'=x$	3

ODE vs PDE

- **ODE** (Ordinary Differential Equation): One independent variable
- **PDE** (Partial Differential Equation): Multiple independent variables

$$\frac{dy}{dx} = y \quad \text{(ODE — one variable x)}$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{(PDE — two variables x and t)}$$

This book focuses exclusively on ODEs.

Standard Forms

We write first-order equations in several equivalent forms:

Explicit Form

$$\frac{dy}{dx} = f(x, y)$$

The derivative is isolated on one side.

Differential Form

$$M(x, y), dx + N(x, y), dy = 0$$

Both differentials appear symmetrically.

Conversion Between Forms

From explicit to differential:

$$\frac{dy}{dx} = f(x, y) \quad \Longrightarrow \quad dy = f(x, y), dx \quad \Longrightarrow \quad f(x, y), dx - dy = 0$$

From differential to explicit (if $N \neq 0$):

$$M, dx + N, dy = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{M}{N}$$

Example 6: Converting Forms

Problem: Write $x, dx + y, dy = 0$ in explicit form.

Solution:

$$\frac{dy}{dx} = -\frac{M}{N} = -\frac{x}{y}$$

This should look familiar — it's the implicit derivative of $x^2 + y^2 = C$ (circles)!

The General Solution

For an nth-order ODE, the general solution typically contains n arbitrary constants.

Order	Constants	Conditions Needed
1st	1	1 initial condition
2nd	2	2 initial conditions
3rd	3	3 initial conditions

Example 7: Second-Order Equation

Problem: Solve $\frac{d^2y}{dx^2} = 6x$ with $y(0) = 1$ and $y'(0) = 2$

Step 1: Integrate once to find y'.

$$\frac{dy}{dx} = \int 6x, dx = 3x^2 + A$$

Step 2: Apply condition on y'.

$y'(0) = 2$: $3(0)^2 + A = 2$, so $A = 2$.

$$\frac{dy}{dx} = 3x^2 + 2$$

Step 3: Integrate again to find y.

$$y = \int (3x^2 + 2), dx = x^3 + 2x + B$$

Step 4: Apply condition on y.

$y(0) = 1$: $0 + 0 + B = 1$, so $B = 1$.

$$\boxed{y = x^3 + 2x + 1}$$

Step 5: Verify.

- $y' = 3x^2 + 2$, so $y'' = 6x$ ✓
- $y(0) = 1$ ✓
- $y'(0) = 2$ ✓

Example 8: Third-Order Equation

Problem: Solve $y'`' = 0$ with $y(0) = 1, y'(0) = 2, y''(0) = 3$

Step 1: Integrate repeatedly.

$$y'' = \int 0, dx = A$$

$$y' = \int A, dx = Ax + B$$

$$y = \int (Ax + B), dx = \frac{A}{2}x^2 + Bx + C$$

Step 2: Apply conditions.

- $y''(0) = 3$: $A = 3$
- $y'(0) = 2$: $A(0) + B = 2$, so $B = 2$
- $y(0) = 1$: $0 + 0 + C = 1$, so $C = 1$

$$\boxed{y = \frac{3}{2}x^2 + 2x + 1}$$

What's Ahead

The rest of this book develops systematic techniques for solving differential equations where y appears on the right side.

First-Order Methods (This Part)

Type	Form	Method
Separable	$\frac{dy}{dx} = f(x)g(y)$	Separate and integrate
Linear	$\frac{dy}{dx} + P(x)y = Q(x)$	Integrating factor
Exact	$M, dx + N, dy = 0$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	Find potential function

Type	Form	Method
Homogeneous	$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	Substitute $v = y/x$
Bernoulli	$\frac{dy}{dx} + P(x)y = Q(x)y^n$	Substitute $v = y^{1-n}$

Second-Order Methods (Part 3)

Type	Form	Method
Constant coefficients	$ay'' + by' + cy = 0$	Characteristic equation
Nonhomogeneous	$ay'' + by' + cy = f(x)$	Variation of parameters
Reduction of order	One solution known	Reduce to first-order

Special Methods (Part 4)

- Power series solutions
- Laplace transforms

Each technique builds on your differentiation and integration skills. The patterns will become familiar with practice.

Verifying Solutions

Always verify your answer by:

1. **Substituting back** into the original equation
2. **Checking initial conditions** (if given)

A solution that doesn't satisfy the equation isn't a solution!

Example 9: Verification

Claim: $y = e^{-x} + x - 1$ solves $y' + y = x$ with $y(0) = 0$.

Step 1: Check the equation.

$$y' = -e^{-x} + 1$$

$$y' + y = (-e^{-x} + 1) + (e^{-x} + x - 1) = x \quad ✓$$

Step 2: Check the initial condition.

$$y(0) = e^{0} + 0 - 1 = 1 - 1 = 0 \quad ✓$$

The solution is verified.

Common Mistakes

1. **Forgetting the constant.** The general solution of $y' = 2x$ is $y = x^2 + C$, not $y = x^2$.
2. **Wrong number of constants.** A second-order equation needs two constants: $y'' = 0$ gives $y = Ax + B$, not $y = Ax$.
3. **Not verifying.** Always substitute back to check your answer.
4. **Confusing linear/nonlinear.** $y' = y^2$ is nonlinear (y squared), but $y' = x^2$ is linear (x squared is fine).
5. **Forgetting to apply ALL conditions.** For second-order problems, you need both $y(x_0)$ and $y'(x_0)$.

🔗 **Connections**

- **Integration (Part 2):** Every technique you learned—substitution, parts, partial fractions—will be needed.

- **Classification guides method:** Recognizing the type (separable, linear, exact) tells you which technique to use.
- **Physics:** DEs describe motion ($F = ma$), circuits ($V = L\frac{dI}{dt}$), population dynamics, and more.
- **Constants = degrees of freedom:** An nth-order equation needs n initial conditions to determine a unique solution.

🔧 **Troubleshooting**

Problem	Likely Cause	Fix
Solution doesn't verify	Algebra or integration error	Substitute back; derivative of answer should satisfy DE
Wrong number of constants	Miscounted order	nth-order → n constants
IC gives wrong constant	Applied IC to wrong equation	Apply IC to final general solution, not intermediate steps
"Linear" confusion	Looked at x instead of y	Linear refers to how y and its derivatives appear
Can't integrate $f(x)$	Need technique from Part 2	Review substitution, parts, partial fractions

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Summary

Concept	Meaning
Differential equation	Equation involving derivatives
Order	Highest derivative present
Linear	y and derivatives to first power only
General solution	Contains arbitrary constants

Concept	Meaning
Particular solution	Specific solution satisfying initial conditions
Initial condition	Value of solution at a specific point

Exercises

Classification

Classify each equation by order and linearity:

1. $\frac{dy}{dx} = x^2 + y$
2. $y'' + 4y = 0$
3. $y' = y^2 + 1$
4. $y''' - 2y'' + y = e^x$
5. $(y')^2 + y^2 = 1$
6. $\frac{dy}{dx} = \sin x \cos y$
7. $xy'' + y' = x$
8. $yy' = x$

Direct Integration

Find the general solution:

9. $\frac{dy}{dx} = 4x^3 - 2x$

10. $y' = e^{2x}$

11. $\frac{dy}{dx} = \frac{1}{x}$

12. $y' = \sec^2 x$

13. $\frac{dy}{dx} = \frac{x}{x^2 + 1}$

14. $y' = x \sin(x^2)$

15. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

16. $y' = x^2 e^{x^3}$

Initial Value Problems

Solve:

17. $\frac{dy}{dx} = 2x + 3$, $y(0) = 1$

18. $y' = \sin x$, $y(\pi) = 0$

19. $\frac{dy}{dx} = e^x$, $y(0) = 2$

20. $y' = \frac{1}{x}$, $y(1) = 0$

Higher-Order

21. $y'' = 12x^2$, find general solution

22. $y'' = \cos x$, $y(0) = 0$, $y'(0) = 1$

23. $y''' = 6$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$

Verification

24. Verify that $y = Ce^{2x}$ solves $y' = 2y$.

25. Verify that $y = \sin x + \cos x$ solves $y'' + y = 0$.

26. Verify that $y = xe^{x}$ solves $y'' - 2y' + y = 0$.