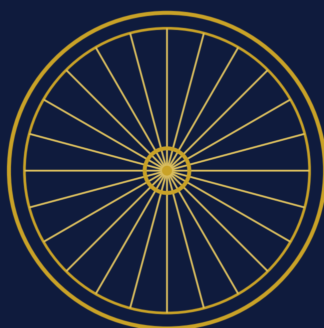
A WHEEL
THROUGH THE PRIMES

A Concrete Tour of
Analytic Number Theory



the wheel modulo 210

BOOK ONE

companion to *A Spectrum Through the Primes* and *Scaling the Wall*

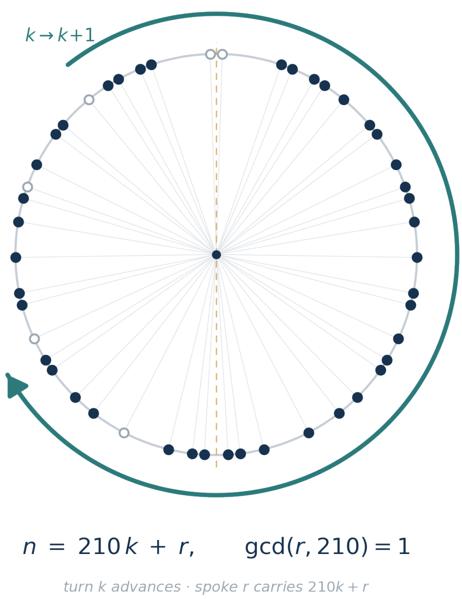
JOHN MAGGIO

Efficient Intelligence, LLC

A Wheel Through the Primes

A Concrete Tour of Analytic Number Theory

Book One of The Primes Trilogy



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Efficient Intelligence, LLC

2026

Published by Efficient Intelligence, LLC

Paperback ISBN: 979-8-9967325-0-0
Hardcover ISBN: 979-8-9967325-1-7
Ebook ISBN: 979-8-9967325-2-4

First edition, 2026 Printed in the United States of America

Preface

This is a book about the prime numbers, told through a single object: a wheel of forty-eight spokes, the residues coprime to $210 = 2 \cdot 3 \cdot 5 \cdot 7$. Everything in it — the way primes spread across the spokes, the faint bias in their race, the mirror that pairs each spoke with another, the diffraction grating that sifts them, the zeros of the zeta function reappearing as their spectrum, the gas of prime "modes," the climb up the primorials, the twin slots — is standard analytic number theory. The wheel is not a new theory. It is a place to stand where the standard theory can be seen at human scale.

Two things are worth saying at the outset. The book promises that the machinery is real and that you will meet it concretely, not in outline: Dirichlet characters as the harmonics of the wheel, the explicit formula as genuine wave interference, the Hardy–Littlewood singular series as fifteen surviving slots out of 210. And it promises the opposite of a triumphal ending. The tour does not solve the Riemann Hypothesis, prove the twin-prime conjecture, or make factoring easy. It ends, deliberately, at those three walls — because that is where the subject actually ends, and because seeing exactly where a wall stands, and why, is worth more than a story that pretends to pass it.

The wheel earns its place by being small. 210 is the smallest modulus that resolves the first four primes at once, so it is the smallest wheel on which the whole early sieve is visible in a single turn. That legibility is the entire source of its specialness. The structures it reveals belong to the integers, not to 210, and the book closes by saying so plainly.

How to read this

The chapters build, but loosely. Chapter 1 sets up the wheel and is assumed everywhere after it; beyond that, each part can be sampled on its own. A reader who wants the spectral heart of the subject can go from Chapter 1 straight to Part III; a reader who wants the elementary combinatorics can stay in Parts I and IV. The figures are meant to be read, not skipped — much of the argument lives in them. Each figure is accompanied in the text by the short Python program that produced it, printed in full so it can be rerun rather than trusted; this book ships no separate code files. A glossary at the back collects the recurring vocabulary.

Contents

Part I

The object

Chapter 1

The Mod-210 Wheel

This is the opening chapter of a book that studies the prime numbers through a single concrete object: the wheel modulo $210 = 2 \cdot 3 \cdot 5 \cdot 7$*. The book claims no new result; its aim is unity and clarity — to gather a scattered body of classical number theory onto one stage, so that each idea can be seen as a feature of the same picture. Here we build the stage. The integers coprime to* 210 *form a group of* $\varphi(210) = 48$ *residue classes — the "spokes" of the wheel — and every prime larger than* 7 *lands on one of them. We record the structure of this group,* $(\mathbb{Z}/210\mathbb{Z})^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/6$*; the wheel as a sieve, with the surviving density* $\varphi(M)/M = \prod_{p|M}(1 - 1/p)$*; and the reflection* $r \mapsto 210 - r$ *that returns in every later chapter.*

The later chapters then act on this stage. Dirichlet's theorem spreads the primes evenly across the 48 *spokes, while a closer look exposes the persistent Chebyshev tilt — a prime race among the classes. The wheel's seven real characters arrange into a conductor lattice split by parity, and the reflection* $r \mapsto 210 - r$ *is unmasked as the elementary shadow of the functional equation* $s \mapsto 1 - s$*. The primes are then read off the zeros of the zeta function as a diffraction pattern; the zeros themselves repel like the eigenvalues of a random matrix; and the squarefree integers assemble into a supersymmetric "primon gas" whose bookkeeping is the Möbius function. Climbing the primorials yields a conservation law and Mertens' theorem; the repeating decimals of the wheel — a structure already in Gauss — split the spokes into two cyclic families; and the book closes on the twin primes, where the same wheel that organizes everything so cleanly runs headlong into the parity barrier that no elementary method has crossed.*

Everything is elementary and classical. The wheel is a lens, not a law: it renders a great deal of known mathematics visible at once, and it is candid about where the difficulty actually lives — not in the structure of the wheel, which is settled, but in how the primes distribute across it, where the oldest questions remain open. This first note fixes the stage cleanly, before the others act on it.

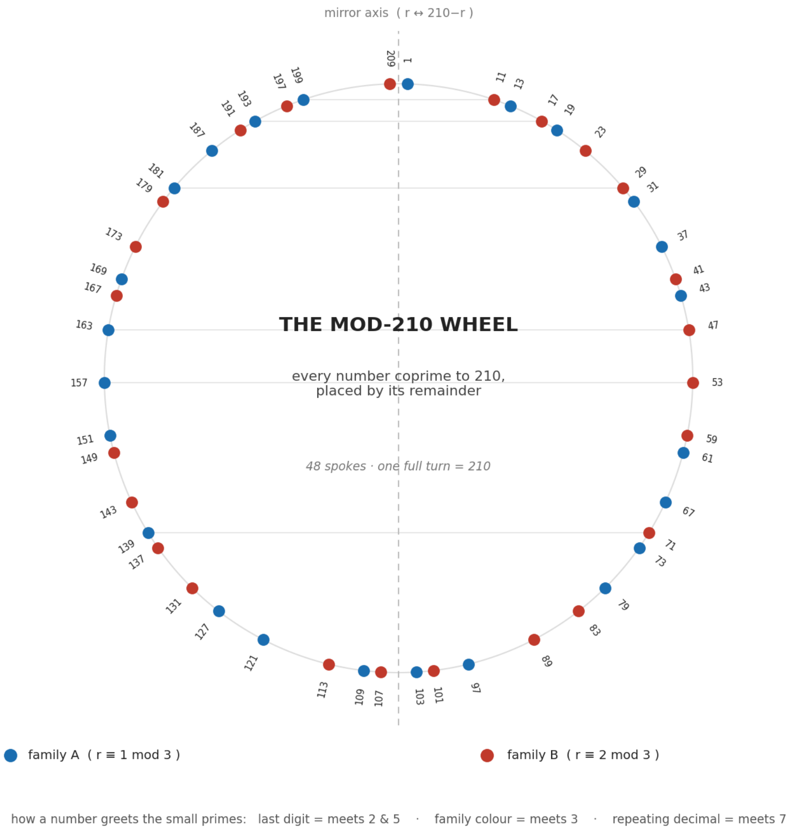

Figure 1.1: The whole subject on a single face. Every number coprime to 210 is placed by its remainder; color marks the family ($r \equiv 1$ versus $r \equiv 2 \bmod 3$), and the vertical line is the mirror $r \mapsto 210 - r$, across which the two families swap. The line at the foot decodes the rest: a number's last digit records how it meets 2 and 5, its color how it meets 3, and its repeating decimal how it meets 7. Everything in this chapter and the ones that follow is a closer look at one feature of this one picture.

1.1 Scope

This chapter is foundational and entirely classical [1, 2]. It introduces no result; it fixes notation and the basic structure of the mod-210 wheel so that the chapters that follow — on the prime race, the real characters and their conductor lattice, the zeros as a diffraction pattern, and the supersymmetric primon gas — can each act on a stage that has already been described.

1.2 The modulus and its units

The modulus is

$$210 = 2 \cdot 3 \cdot 5 \cdot 7, \tag{1.1}$$

the product of the first four primes — the fourth *primorial*. The residues modulo 210 that are coprime to it form the unit group $(\mathbb{Z}/210\mathbb{Z})^\times$, whose size is Euler's totient

$$\varphi(210) = \varphi(2)\,\varphi(3)\,\varphi(5)\,\varphi(7) = 1 \cdot 2 \cdot 4 \cdot 6 = 48. \tag{1.2}$$

These 48 classes are the spokes of the wheel (Fig. 1.2). By the Chinese Remainder Theorem the group factors as

$$(\mathbb{Z}/210\mathbb{Z})^\times \;\cong\; (\mathbb{Z}/2)^\times \times (\mathbb{Z}/3)^\times \times (\mathbb{Z}/5)^\times \times (\mathbb{Z}/7)^\times \;\cong\; \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/6, \tag{1.3}$$

of order $2 \cdot 4 \cdot 6 = 48$. It is not cyclic — 210 has no primitive root — which is why the wheel carries several independent quadratic characters rather than one; that multiplicity is the subject of the conductor-lattice note.

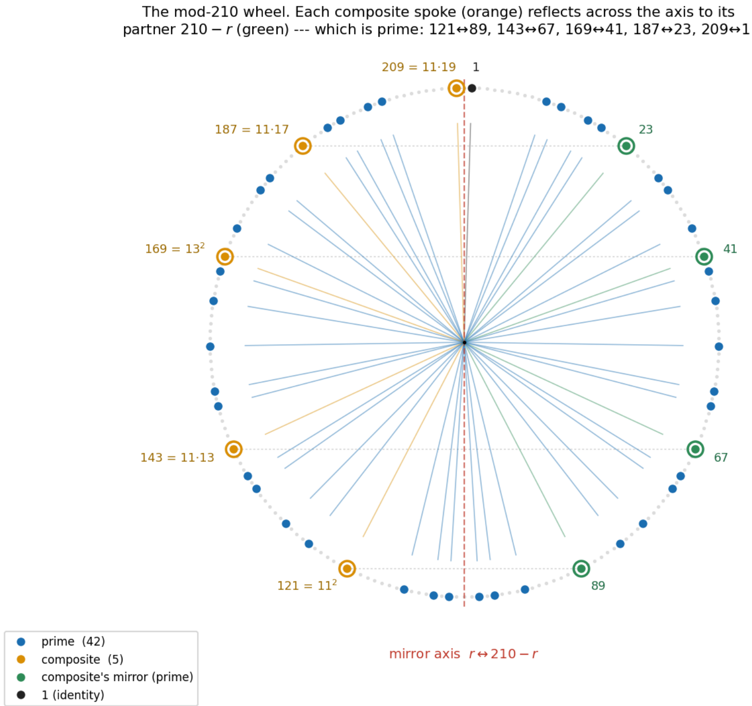

Figure 1.2: The mod-210 wheel. The 48 residues coprime to 210 are the spokes; the remaining positions (multiples of 2, 3, 5, or 7) are the faint dots on the rim. The spokes are colored by type: of the 48, one is the identity, 42 are primes, and five are composite (orange)—the products of two primes ≥ 11 that remain below 210, namely $121 = 11^2$, $143 = 11 \cdot 13$, $169 = 13^2$, $187 = 11 \cdot 17$, and $209 = 11 \cdot 19$. Every prime greater than 7 is a spoke, but so is each of these composites. The dashed axis is the reflection $r \mapsto 210 - r$ of Sec. 1.4. Reflecting each composite across it gives its partner $210 - r$ (green, with faint chords): $121 \leftrightarrow 89$, $143 \leftrightarrow 67$, $169 \leftrightarrow 41$, $187 \leftrightarrow 23$, and $209 \leftrightarrow 1$. Every one of those partners is prime (or the identity) — the reflection keeps a residue coprime to 210 but does not keep it composite.

1.3 The wheel as a sieve

The reason the primes live here is immediate: a prime $p > 7$ is divisible by none of $2, 3, 5, 7$, so $\gcd(p, 210) = 1$ and p occupies one of the 48 unit classes. The first four primes $2, 3, 5, 7$ are exactly the teeth of the sieve — they are the primes *removed* to build the wheel — so apart from those four, the entire prime sequence is distributed among the 48 spokes. This is the elementary content of *wheel factorization*: to hunt for primes one need only examine the residues coprime to 210, a little under a quarter of all integers.

The spokes are not all prime, and it would misrepresent the wheel to draw them as if they were. Of the 48, one is the identity 1, forty-two are the primes between 11 and 199, and five are composite: the products $121 = 11^2$, $143 = 11 \cdot 13$, $169 = 13^2$,

$187 = 11 \cdot 17$, $209 = 11 \cdot 19$ of two primes at least 11 that stay below 210 (orange in Fig. 1.2). The wheel selects every residue coprime to 210 whatever its factorization; in this range the primes are simply the large majority.

The exact fraction is the density of coprime residues,

$$\frac{\varphi(210)}{210} = \frac{48}{210} = \frac{8}{35} \approx 0.229 = \prod_{p|210}\left(1 - \frac{1}{p}\right) = \left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{3}\right)\left(1 - \tfrac{1}{5}\right)\left(1 - \tfrac{1}{7}\right). \quad (1.4)$$

Adding more primes to the primorial thins the wheel further (Fig. 1.3); by Mertens' theorem the density $\prod_{p \le y}(1 - 1/p)$ decays like $e^{-\gamma}/\log y$, slowly but without bound. The choice of 210 is a pragmatic sweet spot: large enough that the four smallest primes are swept away and the structure is rich, small enough that all 48 classes fit on a single page.

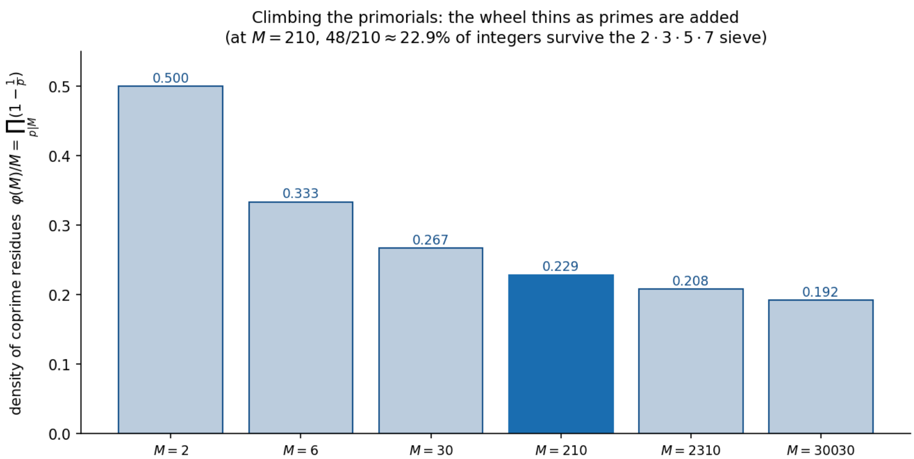

Figure 1.3: The density $\varphi(M)/M = \prod_{p|M}(1 - 1/p)$ of residues coprime to successive primorials. At $M = 210$ (highlighted) just under 23% of integers survive the $2 \cdot 3 \cdot 5 \cdot 7$ sieve. The decay continues without bound, at the slow Mertens rate.

1.4 The mirror

One symmetry of the wheel deserves to be named at the outset, because it recurs in every later chapter. The reflection

$$r \longmapsto 210 - r \pmod{210} \quad (1.5)$$

is an involution that preserves coprimality, since $\gcd(210 - r, 210) = \gcd(r, 210)$. Its only fixed points are $r = 0$ and $r = 105$, and neither is a unit; so the 48 spokes split cleanly into 24 mirror pairs $\{r, 210 - r\}$. In Fig. 1.2 this is the reflection across the dashed vertical axis.

The reflection keeps a residue coprime to 210, but it need not keep it prime: it sends each of the five composite spokes to a prime, $121 \leftrightarrow 89$, $143 \leftrightarrow 67$, $169 \leftrightarrow 41$,

$187 \leftrightarrow 23$, and $209 \leftrightarrow 1$. The mirror is a fact about residues, not about factorization — a distinction worth keeping in view, since the later notes lean on it.

This single reflection wears several costumes downstream. In the prime-race note it is the axis along which Chebyshev's bias is measured, separating the quadratic residues from the non-residues. In the character note it is the parity $\chi(-1) = \pm 1$, because $\chi(210 - r) = \chi(-r) = \chi(-1)\chi(r)$, sorting the characters into even and odd. And in the deepest reading it is the shadow of the functional equation's reflection $s \mapsto 1 - s$. Seeing it first as a plain geometric fold on the wheel makes its later appearances easier to recognize.

1.5 The wheel turning

The wheel of Fig. 1.2 is a single turn. The integers do not stop at 210: the spoke at residue r carries the number $210k + r$ on turn k, so the same 48 spokes are reused turn after turn, with only the leading multiple of 210 advancing. The later turns are worth a look, because the pattern on the spokes both persists and quietly changes.

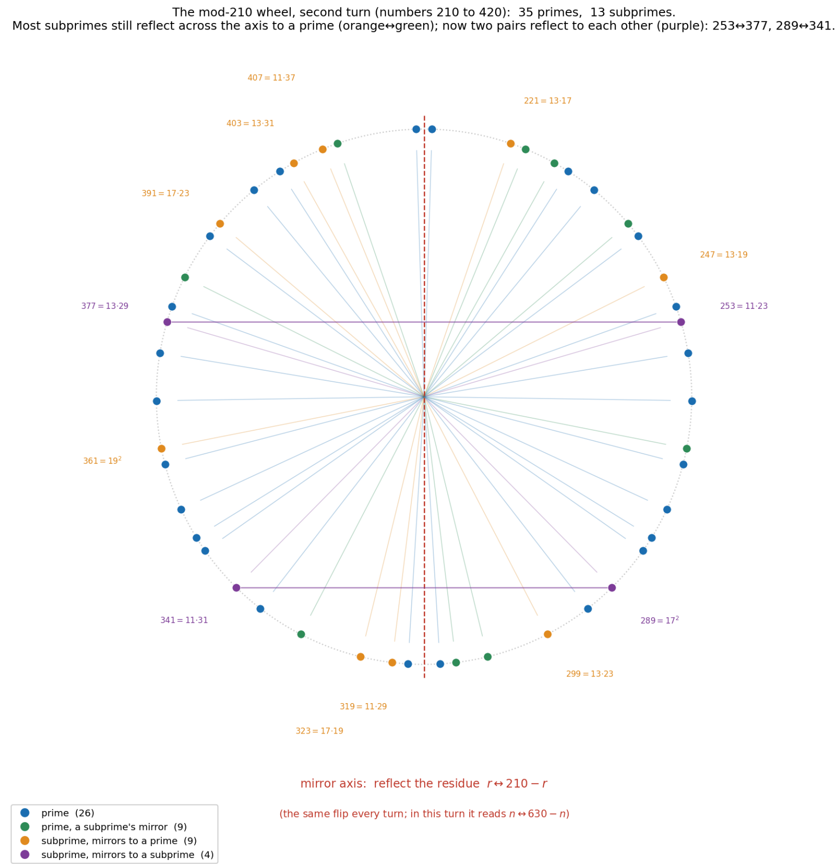

Figure 1.4: The wheel's second turn, the numbers 210 to 420 (the spoke at residue r now carries $210 + r$). The primes have thinned to 35 and the subprimes risen to 13. As on the first turn most subprimes still reflect across the axis onto a prime (orange onto green); the two purple pairs, $253 \leftrightarrow 377$ and $289 \leftrightarrow 341$, are the first composite spokes whose mirror is *also* composite.

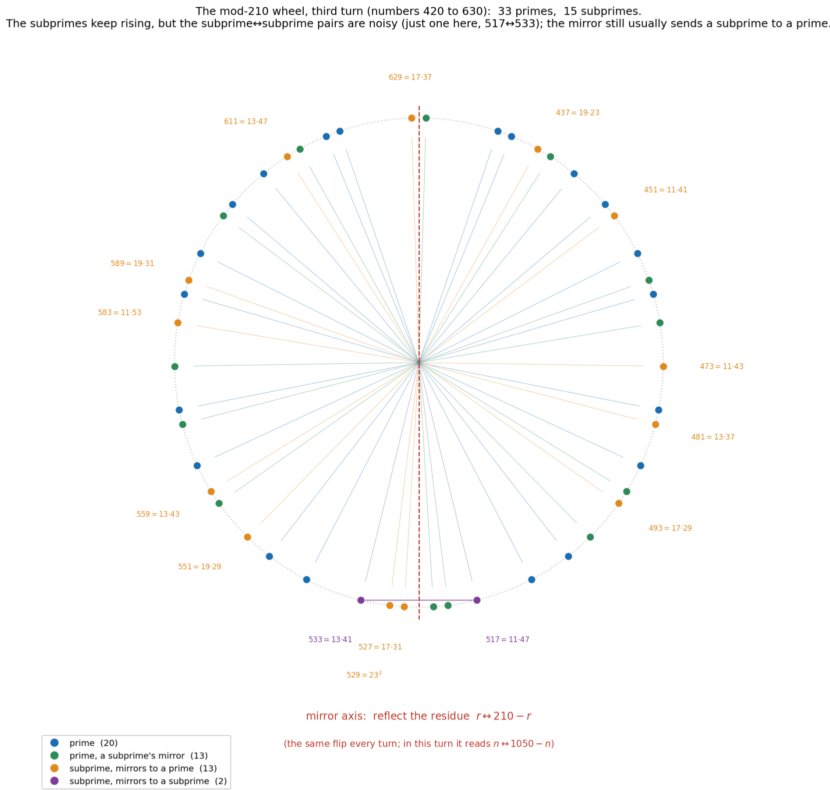

Figure 1.5: The third turn, 420 to 630: 33 primes and 15 subprimes. The thinning continues, while the subprime-onto-subprime pairs are noisy from turn to turn (just one here, $517 \leftrightarrow 533$). The mirror still usually carries a subprime to a prime.

Two things happen as the wheel turns. First, the primes thin: 42 primes on the first turn, 35 on the second, 33 on the third (Figs. 1.4 and 1.5), with the subprimes filling the vacated spokes. This is the prime number theorem seen one turn at a time, the prime density falling like $1/\log$. Second, the mirror of Sec. 1.4 keeps acting, but its character clarifies. On the first turn every composite spoke reflected onto a prime, which can make the reflection look as though it inverts compositeness. From the second turn onward, pairs of composite spokes begin reflecting onto *each other* (Fig. 1.4). The mirror is not inverting primality and never was; it is simply indifferent to it, sometimes sending a composite to a prime and sometimes to another composite. Primality is not among its invariants.

Beneath the thinning lies a strictly periodic mechanism. Fix a spoke r and ask when a prime $p \geq 11$ divides the number sitting on it. Since $p \nmid 210$, we have $210 \equiv c_p \pmod{p}$ for some nonzero c_p, so the residue $210k + r \bmod p$ advances by c_p each turn and reaches 0 once every p turns. Each prime is therefore a metronome with its own period: 11 strikes a given spoke every 11 turns, 13 every 13, 17 every 17. Placing the spokes at their true values straightens these beats into exact diagonals of slope $-c_p$, namely -1, -2, and -6 for 11, 13, and 17 (Fig. 1.6). A spoke is composite on a given turn exactly when at least one metronome strikes it, and prime only when it dodges every one up to $\sqrt{210k + r}$.

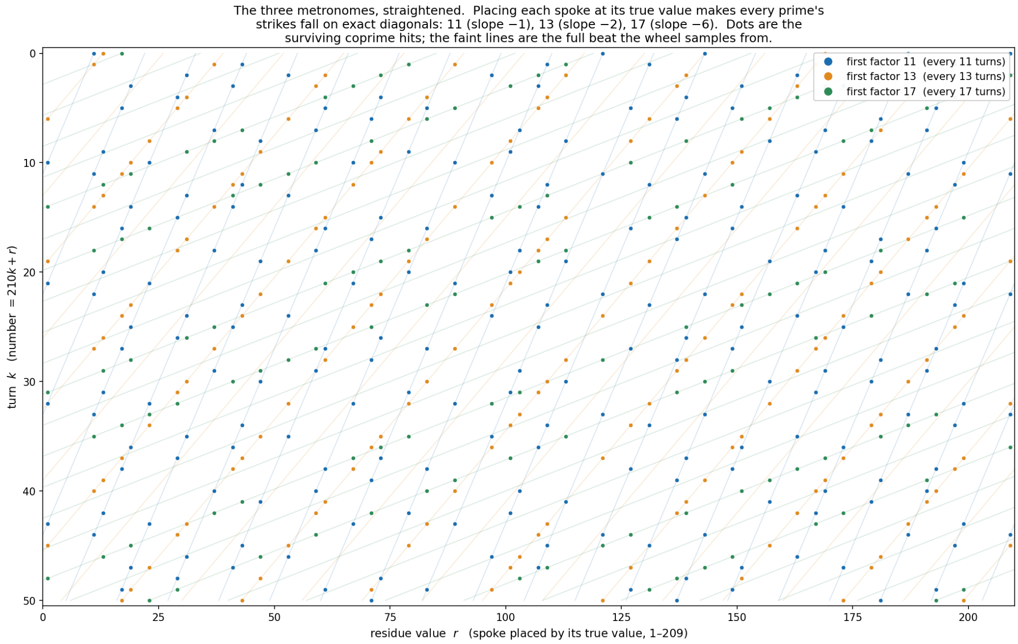

Figure 1.6: The three smallest metronomes across fifty turns. With each spoke placed at its true residue value, the multiples of 11, 13, and 17 fall on exact diagonals of slope $-(210 \bmod p)$. The dots are the coprime hits the wheel keeps; the faint lines are the full periodic beat they are sampled from. The blank space between the stripes is where the primes live.

Pulled apart one metronome to a panel (Fig. 1.7), the same beats become unmistakable. Each prime rides a single family of parallel diagonals, and the larger its step c_p, the faster it sweeps across all 48 spokes: 17, stepping by 6, lays down flatter and more tightly spaced stripes than 11, which steps by 1. Each panel is one prime's entire contribution to the sieve, drawn in isolation.

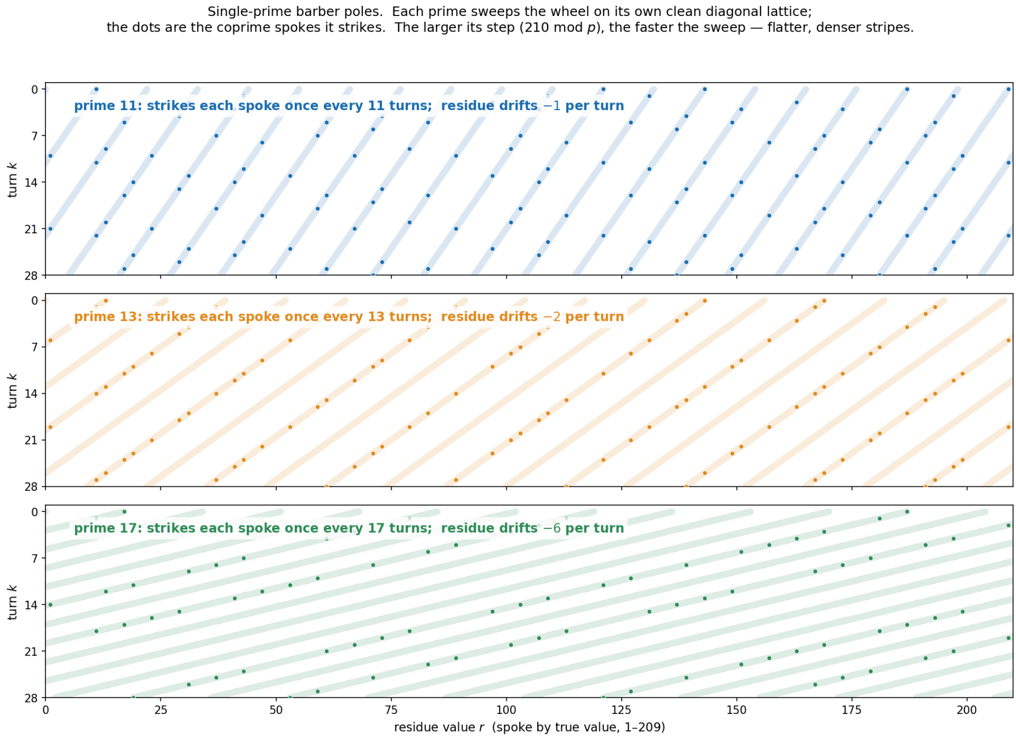

Figure 1.7: The three smallest metronomes separated one to a panel. Each prime p strikes a coprime spoke (the dots) once every p turns, drifting by $-(210 \bmod p)$ spokes per turn, so its hits ride a single family of parallel diagonals — the faint bands are the full periodic beat they are sampled from. The step grows with p, namely -1, -2, and -6 for 11, 13, and 17, so the larger primes sweep the wheel faster: flatter, more crowded stripes.

This is the boundary the wheel keeps drawing, now laid out in time. Any *fixed* set of metronomes is perfectly periodic: 11 and 13 together repeat every $11 \cdot 13 = 143$ turns, and admitting one more prime is precisely the step from one primorial modulus to the next, $210 \to 2310 \to 30030$, which is the subject of the chapter on climbing the primorials. But genuine primality requires every metronome up to $\sqrt{n}$, and that set never stops growing; a new, incommensurate beat keeps joining, so the full pattern never closes into a finite period. The composites are pure clockwork — one could mark every one of them in advance from the metronomes alone. The primes are the spokes no metronome ever claims, and that leftover is the one thing all this periodic machinery cannot itself put a period to. The wheel turns systematically; what it leaves behind does not.

1.6 Two sides of one coin

Every spoke on the wheel, in every turn, carries exactly one number, and that number is either prime or a product of primes that are themselves at least 11; call the latter the *subprimes* of the wheel. The two sets are exact complements: a spoke is prime precisely when it is not a subprime, so the primes are nothing more than the gaps the subprimes leave, and naming either set names the other (Fig. 1.8). This is what a sieve is — Eratosthenes finds the primes by marking every composite and keeping what survives. The only number on the wheel that is neither is the unit 1, sitting alone on the first spoke.

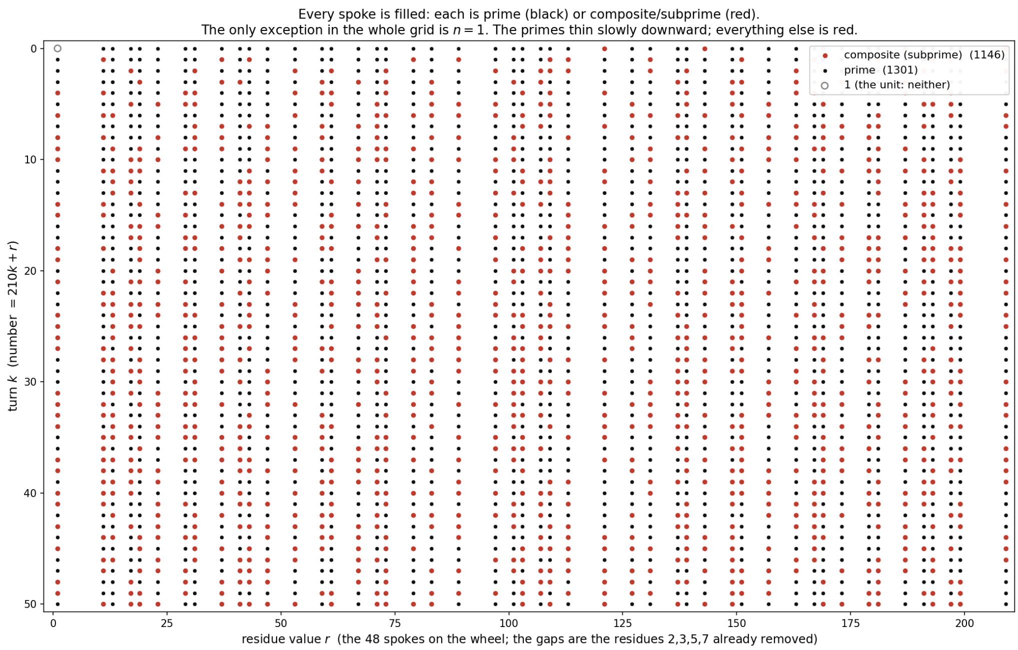

Figure 1.8: Fifty turns of the wheel, every spoke filled: prime (black) or subprime (red). The vertical bands of empty space are not missing spokes; they are the residues divisible by 2, 3, 5, or 7, removed before the wheel begins. Along the 48 true spokes nothing is ever blank, and $1301 + 1146 + 1 = 2448 = 48 \times 51$ accounts for every number. Red gains slowly on black going down: subprimes grow more common as the supply of small prime factors widens, the same $1/\ln n$ thinning of the primes seen from the other side.

The two faces of this coin are not peers, however. The subprimes are *generated*: each is a product of smaller wheel-primes, so the whole red set can be built forwards by multiplication (11^2, $11 \cdot 13$, $13^2, \ldots$), which is exactly the periodic clockwork of the metronomes above. The primes carry no such rule; they are defined only negatively, as the numbers that are *not* such products. The red side is therefore built out of the black side — the primes are the source and the subprimes their shadow. That lopsidedness is the familiar asymmetry of the subject: forming a subprime is a moment's multiplication, while recovering the primes a given subprime was built from is integer factorization, for which no efficient method is known. The duality is perfect; the direction is not.

Stripped of its cheap structure, the leftover looks startlingly like noise, and this too is a precise phenomenon. The primes are *pseudorandom*: fully deterministic, yet passing most statistical tests one can put to them. Cramér made the resemblance quantitative by modeling each integer n as prime independently with probability $1/\ln n$ [3]; on the wheel, where the residues coprime to 210 are already $210/48 = 4.375$ times denser in primes, the matched rate is $(210/48)/\ln n$. Figure 1.9 sets the real wheel beside a single draw of this coin-flip model at that rate. The two are hard to tell apart, which is the point: the model knows nothing about which numbers are actually prime, only their density, yet reproduces their texture. What it cannot reproduce is their rigidity. The real primes are not random — they are forced — and the gap between *behaves like a coin flip* and *is a coin flip* is precisely where the Riemann Hypothesis and the twin-prime conjecture live, and why they remain open.

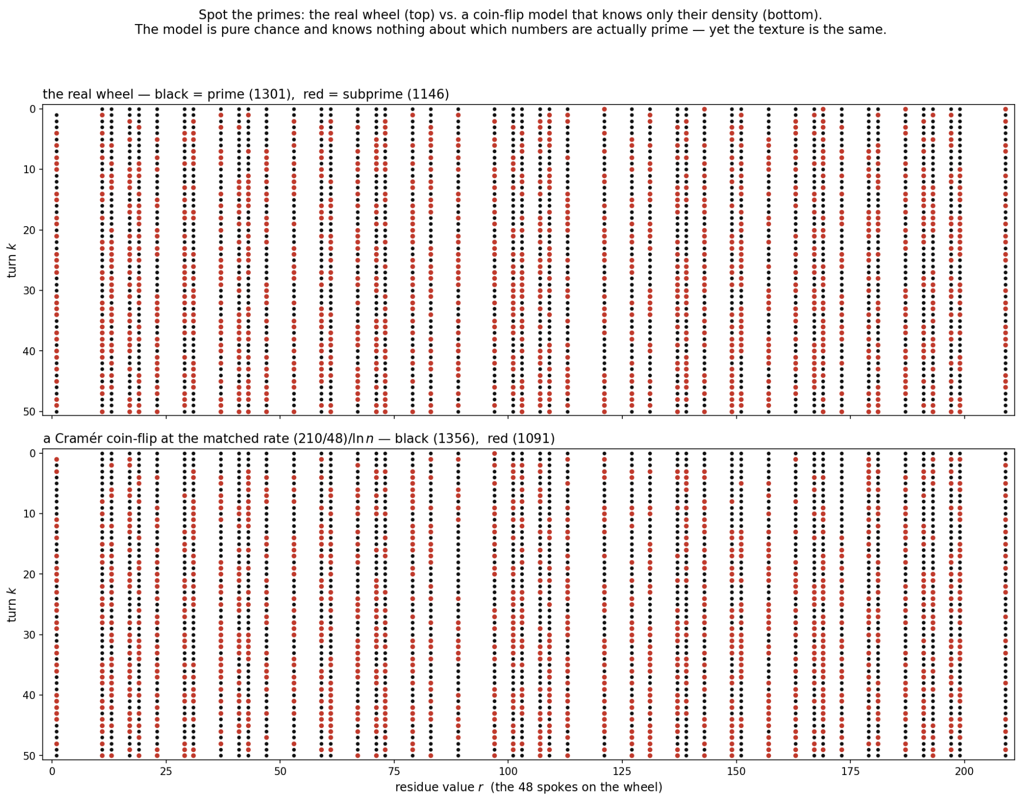

Figure 1.9: The real wheel (top) beside one draw of a Cramér coin-flip (bottom), each number coloured prime with probability $(210/48)/\ln n$ and capped at 1. The model produced 1356 primes against the wheel's 1301 — a match in density, and to the eye a match in texture, though the model is pure chance and the wheel is wholly determined. The visible difference is statistical, not structural; the model's one genuine failure is that it cannot know the primes are not, in fact, being flipped.

This same density reaches further than the two counts. Applied to the wheel's mirror pairs (Sec. 1.4), it fixes how the two reflected residues fall by primality, turn after turn—a small effective model, with the prime number theorem as its only input, set out in the chapter on the mirror-pair populations.

1.7 What this sets up

With the stage fixed, each later chapter puts something in motion on it. The prime race watches the primes distribute themselves across the 48 spokes and measures the slight, persistent bias in that distribution. The conductor-lattice chapter studies the quadratic characters the non-cyclic group carries, and organizes them by their conductors. The diffraction chapter passes to the spectral side, drawing the Riemann and Dirichlet zeros and showing their Fourier transform lands on the primes. The supersymmetric-gas chapter recovers the Möbius function as the partner of those same L-functions. The mirror-pair chapter stays closest to this stage, carrying the Cramér density just above into a count of how the reflection's prime–prime, prime–subprime, and subprime–subprime pairs populate as the primes thin. Every one of them returns, sooner or later, to the 48 spokes and the one mirror described here.

Reproducing the figures

```
from math import gcd
M = 210
units = [r for r in range(M) if gcd(r, M) == 1]
print(len(units))                        # 48
print(units[:6], units[-3:])             # [1, 11, 13, 17, 19, 23] ... [197, 199, 209]

# density of coprime residues for the primorials
prod = 1.0
for p in [2, 3, 5, 7, 11, 13]:
    prod *= (1 - 1/p)
    print(p, round(prod, 4))             # 0.5, 0.3333, 0.2667, 0.2286, 0.2078, 0.1918

# the mirror pairs
pairs = sorted({tuple(sorted((r, (M - r) % M))) for r in units})
print(len(pairs))                        # 24 mirror pairs

# metronome diagonals: the coprime spokes a prime p strikes on turn k
def strikes(p, k):
    step = M % p                         # 11 -> 1, 13 -> 2, 17 -> 6
    return [r for r in units if (r + step * k) % p == 0]
for p in (11, 13, 17):
    print(p, strikes(p, 0))              # 11 [11,121,143,187,209]; 13 [13,143,169]

# Cramer coin-flip model, matched to the wheel's prime density
import math, random
random.seed(7)
density = 210 / 48                       # = 4.375; primes are this much denser
def model_prime(n):
    return random.random() < min(1.0, density / math.log(n))
nums = [210*k + r for k in range(51) for r in units if 210*k + r > 1]
print(sum(model_prime(n) for n in nums))   # 1356 (seed 7)
```

Continue the Tour

You have just read Chapter 1 of

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The wheel keeps turning: the prime race, the conductor lattice,
the zeros as a diffraction pattern, the supersymmetric primon gas,
and the twin slots — through to the three walls where the subject ends.

The Primes Trilogy

Book One — A Wheel Through the Primes
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