

The Mathematics Masterclass

(FREE LEANPUB SAMPLE)

A Comprehensive Foundation for GCSE & IGCSE (Years
10-12)

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Contents

How to Use This Book

To the Student

Mathematics is not a spectator sport. To succeed at the GCSE, IGCSE, or early IB level, you must actively engage with the material. This textbook is designed with a unique pedagogical structure to guide you from foundational understanding to advanced mastery.

1. Topic Overviews

Before diving into the algebra of each chapter, you will find a **Topic Overview**. These sections lay out exactly what we are going to learn and why it's useful. I recommend reading them carefully so you understand the purpose of the maths before you start applying the methods.

2. Follow the Colour Coding

Mathematical notation can sometimes look a bit overwhelming. To help with this, the key steps in our working out are colour-coded. Whenever you see variables in **hot pink**, **navy blue**, or **crimson**, pay close attention. These colours show you exactly how terms expand, factorise, or cancel out from one line to the next.

3. Master the Graded Exercises

At the conclusion of each major concept, you will encounter the **Training Grounds**. These exercise sets are rigorously tiered:

- **Bronze & Silver:** Core competency and standard exam-style questions. Ensure you can solve these flawlessly before advancing.
- **Gold & Advanced:** Complex synthesis questions designed to stretch your reasoning. These mimic the most challenging questions found at the end of high-tier exam papers.

4. Utilize the Solutions Manual Constructively

The comprehensive worked solutions at the back of this book are not merely an answer key; they are a diagnostic tool. If you encounter a roadblock, do not simply copy the final line. Instead, study the step-by-step methodology to identify exactly where your logic deviated, then close the manual and re-attempt the problem independently.

Mastery of mathematics requires patience, precision, and persistence. Proceed with confidence.

Part I

The Number System

Introduction to Part I

Welcome to the course. To do well in maths, we need a solid foundation before we start tackling complex equations. In this first part, we are going to look closely at how numbers actually work. We will cover the number line, how to correctly write inequalities, and the core rules for primes, factors, and multiples. Getting these basics right will save you a lot of dropped marks later on.

Part II

Algebra

Introduction to Part II

Now that we are comfortable with our number skills, we can move on to algebra. In this section, we will look at expanding brackets, factorising polynomials, and solving quadratics. We will also look at how to set up equations from word problems. These techniques make up a huge portion of the GCSE syllabus, so it is vital we get confident with them.

Part III

Geometry

Introduction to Part III

In this part, we are going to link the algebra we've just learned with geometry. Coordinate geometry is all about taking equations and putting them onto a graph. We will look at how to find midpoints, calculate the gradient of a straight line, and find the equations of circles. This is a topic where drawing a quick sketch is often the best way to start your working out.

Chapter 1

Coordinate Geometry The Mathematics of Space

TOPIC OVERVIEW

> MODULE INTRODUCTION

Welcome to Coordinate Geometry. This chapter bridges the gap between pure algebra and pure geometry. Instead of simply solving for x , you will learn to visualize x and y as exact coordinates in a two-dimensional universe. You will acquire the tools to mathematically analyze points, calculate exact distances, prove the properties of shapes, and dictate the precise trajectory of straight lines and circles.

1.1 Geometry Terminology: Lines vs Segments

Before we begin calculating on the grid, we must be precise about what we are measuring. In geometry, there is a strict difference between a *Line* and a *Segment*.

- **Line Segment** $[AB]$: A finite piece of a line with a fixed start and end point. It has an exact length that we can measure.
- **Line** (AB) : An infinite path that passes through A and B and never stops in either direction. (You cannot measure its total length!)
- **Ray or Half-Line** $[AB)$: Starts exactly at A , passes through B , and continues infinitely in one direction.



1.2 Geometry Terminology: Understanding Length Ratios

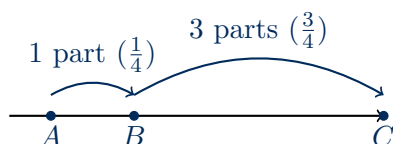
In exams, you will often see distances described as a **Ratio**.

For example: "The points A , B , and C lie on a line. Given that the ratio of lengths $AB : BC = 1 : 3 \dots$ "

The Mathematical Meaning (Fractions of a Whole):

A ratio means the line is divided into pieces. To understand it, find the **total parts**:

- The total line AC has $1 + 3 = 4$ equal parts.
- AB is exactly $\frac{1}{4}$ of the total line AC .
- BC is exactly $\frac{3}{4}$ of the total line AC .
- *Fast Exam Trick:* The step from B to C is simply $3\times$ larger than the step from A to B .



Exam Tip

Always find the **Total Parts** first. This works for any ratio, even tricky ones like $2 : 5$ (Total = 7 parts, so $\frac{2}{7}$ and $\frac{5}{7}$).

Geometry Terminology: More Ratio Examples

Let's translate a few more common exam phrases:

- $AB : BC = 1 : 1$

"The step from B to C is exactly the same as the step from A to B ."

→ This is just a fancy way of saying B is the **Midpoint** of A and C !

- $AB : BC = 2 : 5$

"For every 2 steps you take from A to B , you must take 5 steps to get from B to C ."

→ If the step from A to B is +4 spaces (which is 2×2), then the step from B to C is +10 spaces (5×2).

- $AB : AC = 1 : 4$

Careful! This compares a piece to the whole line!

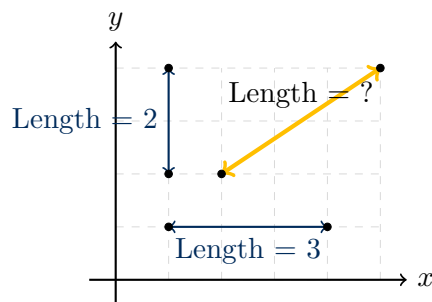
→ If the total line is 4 parts, and AB is 1 part, then BC must be 3 parts. This is mathematically identical to $AB : BC = 1 : 3$.

1.3 The Distance Problem

We already know how to measure distance on a grid if the line is **horizontal** or **vertical**.

The Limitation

But what happens if the line is diagonal? We cannot simply "count the squares" anymore because the line cuts through them at an angle!



1.4 The Pythagorean Solution

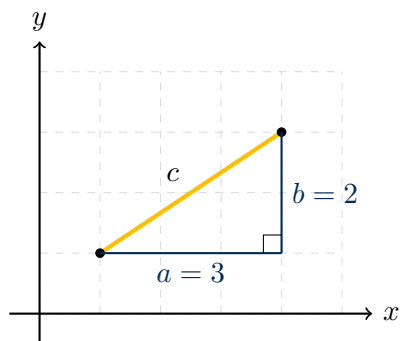
To find the length of a diagonal line, we treat it as the **hypotenuse** of a right-angled triangle.

Pythagoras Theorem

$$a^2 + b^2 = c^2$$

$$(\text{Horizontal})^2 + (\text{Vertical})^2 = (\text{Diagonal})^2$$

We can always draw a horizontal and vertical line to create our 90° angle!



1.5 The Distance Formula (Generalization)

Instead of drawing triangles every time, we can use algebra.

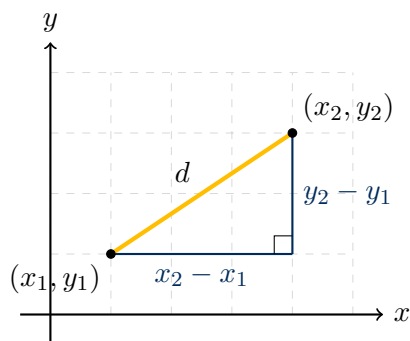
The Distance Formula

For any two points (x_1, y_1) and (x_2, y_2) :

$$a = x_2 - x_1$$

$$b = y_2 - y_1$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**Example 1: Calculating Distance****Example 1**

Calculate the exact distance between $A(2, 3)$ and $B(5, 7)$.

Step 1: Identify coordinates

$$(x_1, y_1) = (2, 3) \quad \text{and} \quad (x_2, y_2) = (5, 7)$$

Step 2: Apply the formula

$$d = \sqrt{(5 - 2)^2 + (7 - 3)^2}$$

Step 3: Solve

$$d = \sqrt{(3)^2 + (4)^2}$$

$$d = \sqrt{9 + 16}$$

$$d = \sqrt{25} = 5$$

Example 2: Distance with Negatives**Example 2: Watch out for negatives!**

Calculate the distance between $P(-3, 1)$ and $Q(2, -4)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(2 - (-3))^2 + (-4 - 1)^2} \quad \leftarrow \text{Note the double negative!}$$

$$d = \sqrt{(2+3)^2 + (-5)^2}$$

$$d = \sqrt{(5)^2 + 25}$$

$$d = \sqrt{25 + 25} = \sqrt{50}$$

(In GCSE exams, leaving the answer as a surd like $\sqrt{50}$ or simplifying to $5\sqrt{2}$ is perfect).

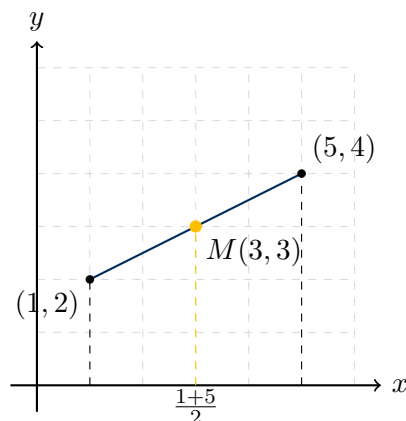
1.6 The Midpoint Formula

How do we find the exact center M of a line segment?

The Midpoint Formula

We simply take the **average** of the x values, and the **average** of the y values.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$



Midpoint Example 1

Example 3

Find the midpoint of the line connecting $A(-4, 2)$ and $B(6, -8)$.

Step 1: Average the x-coordinates

$$x_m = \frac{-4+6}{2} = \frac{2}{2} = 1$$

Step 2: Average the y-coordinates

$$y_m = \frac{2+(-8)}{2} = \frac{-6}{2} = -3$$

Final Answer:

The midpoint is $M(1, -3)$.

Midpoint Example 2: The Reverse Trap

Sometimes exams work **backwards**.

Question: The midpoint is $M(2, 5)$. Endpoint A is $(-1, 3)$. Find endpoint $B(x, y)$. **Algebraic**

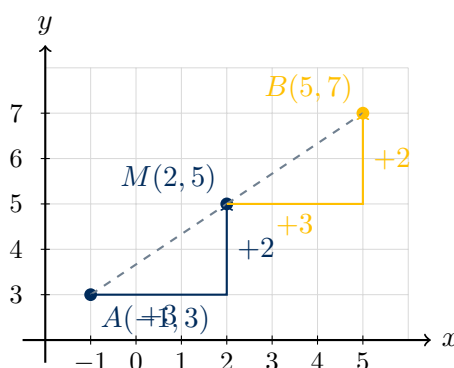
Method:

Set up equations for x and y :

$$\begin{aligned}\frac{-1 + x}{2} &= 2 \implies -1 + x = 4 \implies x = 5 \\ \frac{3 + y}{2} &= 5 \implies 3 + y = 10 \implies y = 7\end{aligned}$$

Answer: B is $(5, 7)$. **The "Jump" Method (Visual):**

From $A(-1, 3)$ to $M(2, 5)$, we jump $+3$ for x and $+2$ for y .
Do the exact same jump from M to find B !



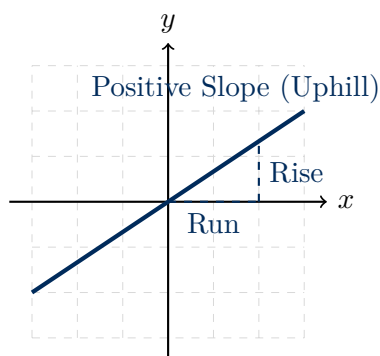
1.7 The Gradient (m): Steepness of Space

The **gradient** tells us how steep a line is, and which direction it is going.

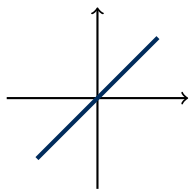
The Gradient Formula

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

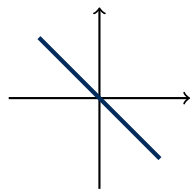
We calculate how much the line goes UP (y) for every unit it goes RIGHT (x).



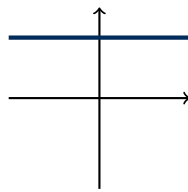
1.8 The Four Types of Gradients



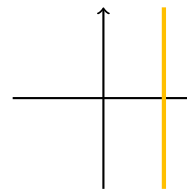
Positive (+)



Negative (-)



Zero (0)



Undefined

Gradient Example 1

Example 4

Find the gradient of the line passing through $A(1, 4)$ and $B(3, 10)$.

Step 1: Identify coordinates

$$(x_1, y_1) = (1, 4) \quad \text{and} \quad (x_2, y_2) = (3, 10)$$

Step 2: Apply formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 3: Solve

$$m = \frac{10 - 4}{3 - 1} = \frac{6}{2} = 3$$

This means for every 1 unit right, the line goes UP by 3 units.

Gradient Example 2: Horizontal Line

Example 5: A Zero Gradient

Find the gradient of the line passing through $C(-2, 5)$ and $D(8, 5)$.

Substitute into formula:

$$m = \frac{5 - 5}{8 - (-2)}$$

Solve:

$$m = \frac{0}{10} = 0$$

The y-coordinates did not change! This means there is NO rise. It is a perfectly flat, horizontal line.

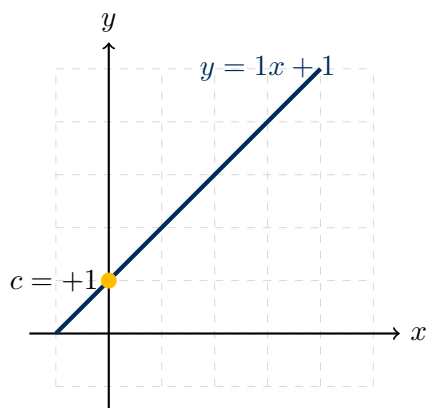
1.9 The Equation of a Straight Line

Every straight line has an algebraic "DNA" or equation that perfectly defines it.

The Master Formula

$$y = mx + c$$

- y and x represent every coordinate on the line.
- m is the **Gradient**.
- c is the **y-intercept** (where the line crosses the vertical y-axis).



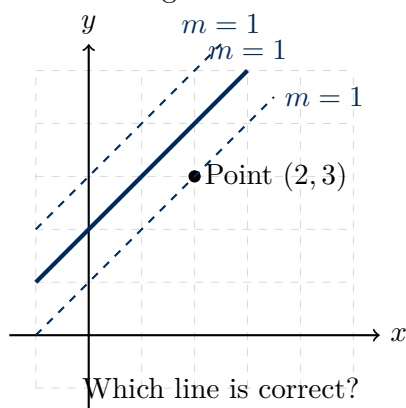
1.10 The Missing Intercept (c)

We know that every line follows the rule $y = mx + c$.

The Problem

What if we know the steepness (m) and one point the line passes through, but we don't know where it hits the y-axis (c)?

The Solution: We have a formula with 4 variables: y, m, x, c . If we know y, m , and x , we can use basic algebra to solve for c !



1.11 How to Sketch a Straight Line

If an exam asks you to draw $y = 2x - 3$, how do you actually do it? **Method 1: Table of**

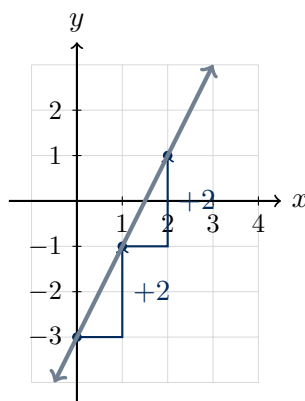
Values

Pick easy x numbers, find y , and plot the coordinates!

x	0	1	2
$y = 2x - 3$	-3	-1	1
Point	(0, -3)	(1, -1)	(2, 1)

Method 2: Gradient-Intercept

1. Start by plotting the y -intercept: $(0, -3)$
2. The gradient is $m = 2$. This means for every 1 step right, you go 2 steps up!

**Example 6: Finding c** **Example 6**

Find the equation of the line that has a gradient of 3 and passes through the point $(2, 10)$.

Step 1: Write down what you know

$$m = 3 \quad x = 2 \quad y = 10$$

Step 2: Substitute into $y = mx + c$

$$10 = (3)(2) + c$$

Step 3: Solve for c

$$10 = 6 + c \implies c = 4$$

Final Equation: $y = 3x + 4$

1.12 The Two-Point Equation

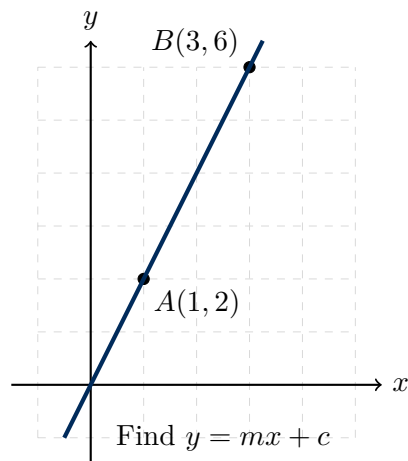
Sometimes, you are only given two coordinates and nothing else.

The Algorithm

Step 1: Find the gradient m using $\frac{y_2 - y_1}{x_2 - x_1}$.

Step 2: Pick **one** of the points to be your (x, y) .

Step 3: Substitute m, x, y into $y = mx + c$ to find c .



Example 7: Two-Point Equation

Example 7

Find the exact equation of the line passing through $A(1, 2)$ and $B(3, 6)$.

Step 1: Find m

$$m = \frac{6-2}{3-1} = \frac{4}{2} = 2$$

Step 2: Pick a point and substitute

Let's pick point $A(1, 2)$, so $x = 1, y = 2$.

$$y = mx + c \implies 2 = (2)(1) + c$$

Step 3: Solve for c

$$2 = 2 + c \implies c = 0$$

Final Equation:

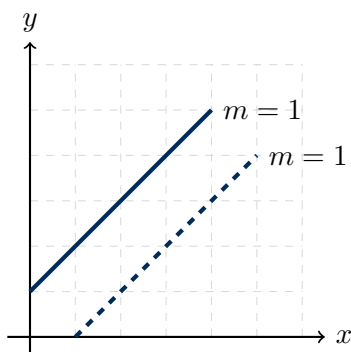
$$y = 2x \quad (\text{This means the line passes directly through the origin!})$$

1.13 Parallel Lines

Rule 1: Parallel Lines

Parallel lines never meet because they have the **exact same steepness**.

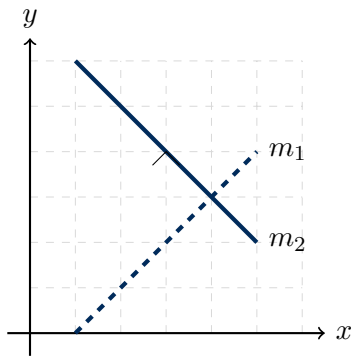
If two lines are parallel, their gradients are equal: $m_1 = m_2$



1.14 Perpendicular Lines

Rule 2: Perpendicular Lines

Perpendicular lines intersect at exactly 90° .
 If two lines are perpendicular, their gradients multiply to make -1.
 $m_1 \times m_2 = -1$ (They are negative reciprocals).



Example 8: Parallel Lines

Example 8

Find the exact equation of the line passing through $(2, 10)$ that is **parallel** to the line $y = 3x - 4$.

Step 1: Identify the Gradient (m)

The line is parallel to $y = 3x - 4$. Parallel lines have the same gradient.
 Therefore, $m = 3$.

Step 2: Use the given point to find c

We know $m = 3$, $x = 2$, and $y = 10$.
 $y = mx + c \implies 10 = (3)(2) + c$

Step 3: Solve for c

$$10 = 6 + c \implies c = 4$$

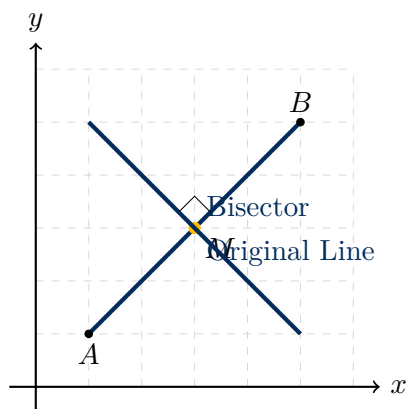
Final Equation: $y = 3x + 4$

Advanced Concept: Perpendicular Bisectors

A **perpendicular bisector** is a line that perfectly cuts another line segment in half, exactly at a 90° angle.

The 4-Step Algorithm

1. Find the **Midpoint** (where to cut).
2. Find the **Gradient** m_1 of original line.
3. Find the **Perpendicular Gradient** m_2 .
4. Substitute Midpoint and m_2 to find c .



Example 9: Perpendicular Bisector

Example 9

Find the equation of the perpendicular bisector of the line connecting $A(2, 2)$ and $B(6, 4)$.

1. **Find the Midpoint:** $M = \left(\frac{2+6}{2}, \frac{2+4}{2}\right) = (4, 3)$

2. **Find the Original Gradient (m_1):** $m_1 = \frac{4-2}{6-2} = \frac{2}{4} = \frac{1}{2}$

3. **Find the Perpendicular Gradient (m_2):** Flip it and reverse the sign! $m_2 = -2$

4. **Substitute $M(4, 3)$ and $m_2 = -2$ to find c :**

$$y = mx + c \implies 3 = (-2)(4) + c \implies 3 = -8 + c \implies c = 11$$

Final Equation: $y = -2x + 11$

1.15 The Standard Form: $ax + by + c = 0$

So far, we have always used the format $y = mx + c$.

The Standard Form

In IGCSE exams, you are often asked to give your final answer in the form $ax + by + c = 0$, where a, b , and c are integers (no fractions!).

To do this, we just use algebra to multiply out any denominators and move everything to one side of the equal sign! **Example: Convert $y = \frac{1}{2}x + 3$**

Step 1: Multiply EVERYTHING by 2 to remove the fraction.

$$2y = x + 6$$

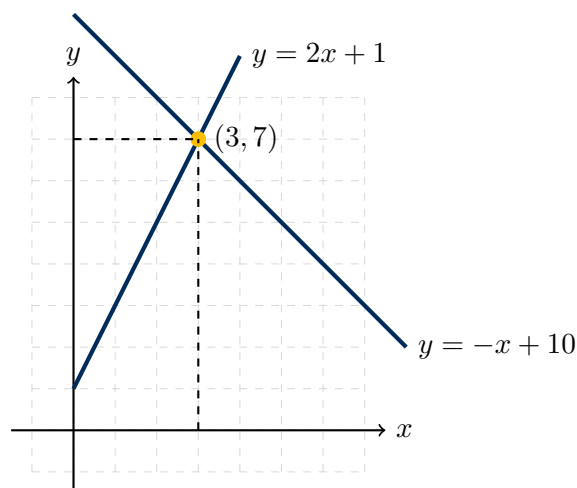
Step 2: Move everything to one side so it equals 0.

Subtract $2y$ from both sides:

$$0 = x - 2y + 6$$

Final Answer: $x - 2y + 6 = 0$

1.16 Finding Intersections: The Concept



of $y = 2x + 1$ and $y = -x + 10$?

Question: How do we find the exact intersection

Equating Lines

At the intersection point, both lines share the **exact same x and y** .
Because $y = \text{Line 1}$ and $y = \text{Line 2}$, we can just set them equal to each other!

1.17 Finding Intersections: The Calculation

Question: Find where $y = 2x + 1$ and $y = -x + 10$ intersect.

1. Set them equal:

$$2x + 1 = -x + 10$$

2. Solve for x :

$$3x + 1 = 10 \implies 3x = 9 \implies x = 3$$

3. Substitute x back in to find y :

$$y = 2(3) + 1 \implies y = 7$$

Intersection Point: $(3, 7)$

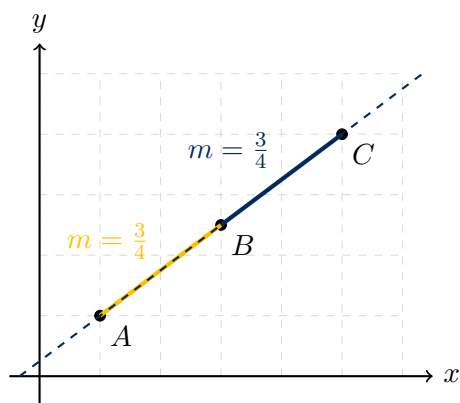
1.18 Collinear Points

Collinear points are 3 or more points that lie on the exact same straight line.

The Proof

To prove A , B , and C are collinear:

1. Find the gradient of AB .
2. Find the gradient of BC .
3. If the gradients are equal, and they share point B , they form a straight line!

**Graded Exercises: Interactive Bronze****Bronze Exercise**

1. **[Solution]** Find the exact distance between $A(1, 5)$ and $B(4, 9)$.

Graded Exercises: Interactive Silver 1**Silver Exercise 1**

1. **[Solution]** Find the equation of the line passing through $P(-2, 5)$ and $Q(2, 17)$.

Graded Exercises: Interactive Silver 2**Silver Exercise 2: Parallel Lines**

1. **[Solution]** Find the equation of the line that is parallel to $y = -4x + 9$ and passes through the point $(-1, 3)$.

Graded Exercises: Interactive Gold**Gold Exercise: Advanced Synthesis**

1. **[Solution]** Find the equation of the perpendicular bisector of the line segment connecting $A(0, 8)$ and $B(4, 0)$.

Graded Exercises: Interactive Platinum**Platinum Exercise: A* Level**

1. **[Solution]** Find the coordinates of the exact point where the line $y = 3x - 2$ intersects the line $y = -x + 6$.

Graded Exercises: Interactive Silver 3**Silver Exercise: Standard Form Parallel**

1. **[Solution]** Find the equation of the line parallel to $2x - 3y + 6 = 0$ that passes through $(3, 5)$. Write your final answer in the form $ax + by + c = 0$.

Graded Exercises: Interactive Silver 4**Silver Exercise: Intercept Distance**

1. **[Solution]** A straight line has the equation $4x + 3y = 12$. Calculate the exact length of the line segment that lies strictly between the x -axis and the y -axis.

Graded Exercises: Interactive Gold 2**Gold Exercise: The Missing Collinear Coordinate**

1. **[Solution]** The points $A(-2, 1)$, $B(2, 3)$, and $C(k, 8)$ form a straight line. Find the exact value of k .

Graded Exercises: Interactive Gold 3**Gold Exercise: The Intersecting Bisector**

1. **[Solution]** Find the exact coordinate of the intersection between the line $y = 3x - 10$ and the perpendicular bisector of the line segment joining $P(-1, 2)$ and $Q(3, -6)$.

Graded Exercises: Interactive Platinum 2**Platinum Exercise (A* Level): Advanced Area Synthesis**

1. **[Solution]** A triangle has vertices at $A(-2, 5)$, $B(4, 2)$, and $C(1, -4)$. Prove mathematically that it is a right-angled triangle, and calculate its exact area.

Graded Exercises: Interactive Platinum 2.5 (The Box Method)**Platinum Exercise: Area of a Non-Right Triangle**

1. Calculate the exact area of the triangle with vertices $A(1, 1)$, $B(5, 2)$, and $C(3, 5)$. *(Hint: The "Box Method". If a triangle isn't right-angled, $\frac{1}{2}bh$ is hard because we don't know the height! Draw a rectangle tightly around the triangle. Calculate the area of the rectangle, then subtract the areas of the 3 right-angled triangles in the corners!)* **[Solution]**

Graded Exercises: Interactive Platinum 3**Platinum Exercise (A* Level): Perpendicular Bisector Intersection**

1. **[Solution]** The points $A(-2, 4)$ and $B(6, 0)$ form a segment. Find the exact equation of their perpendicular bisector, and determine the coordinates where it crosses the x -axis.

Graded Exercises: Interactive Platinum 4**Platinum Exercise (A* Level): Collinear Ratio Expansion**

1. **[Solution]** Points $P(1, 2)$ and $Q(4, 8)$ lie on a straight line. The point R lies on the same line such that Q is between P and R . If the length of QR is exactly twice the length of PQ , find the coordinates of R .

Graded Exercises: Interactive Platinum 5**Platinum Exercise (A* Level): The Rhombus Proof**

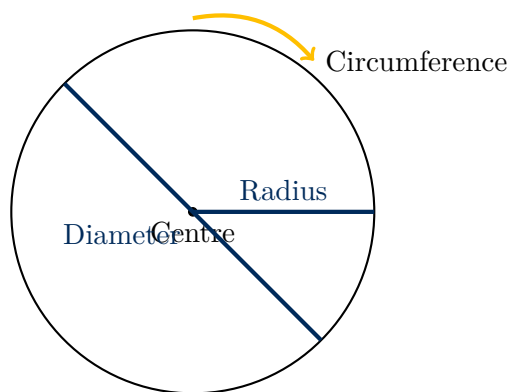
1. **[Solution]** The coordinates of a quadrilateral are $A(1, 1)$, $B(4, 5)$, $C(9, 5)$, and $D(6, 1)$. Prove algebraically that $ABCD$ is a rhombus.

1.19 Geometry of Circles

1.20 Anatomy of a Circle

Before we look at circles on a coordinate grid, we need to know the basic vocabulary!

- **Centre:** The exact middle point.
- **Radius:** The distance from the centre to the edge.
- **Diameter:** The distance from edge to edge, passing straight through the centre (Diameter = $2 \times$ Radius).
- **Circumference:** The mathematical name for the **perimeter** (the total distance around the outside edge).

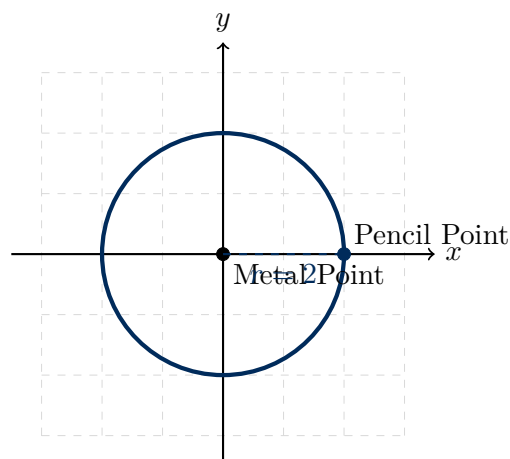


1.21 Drawing a Perfect Circle

In your GCSE exam, you will be expected to draw circles using a **Pair of Compasses**.

How to use compasses

1. Place the **sharp metal point** on the exact coordinate of the centre.
2. Set the width (the distance between the metal point and the pencil) to exactly the length of the **Radius**.
3. Keep the metal point firmly planted, and rotate the pencil around to draw the circumference!



1.22 Area and Circumference

To calculate the Area or Circumference of a circle, we must use the magic number π (Pi), which is approximately 3.14159...

The Essential Formulas

Area: $A = \pi r^2$

(Space inside the circle)

Circumference: $C = \pi d$ or $C = 2\pi r$

(Distance around the outside edge)

Example: A circle has a radius of 5cm. Find its exact Area and Circumference. **Area:**

$$A = \pi \times 5^2$$

$$A = 25\pi \text{ cm}^2$$

Circumference:

$$\text{Diameter} = 2 \times 5 = 10\text{cm}$$

$$C = \pi \times 10$$

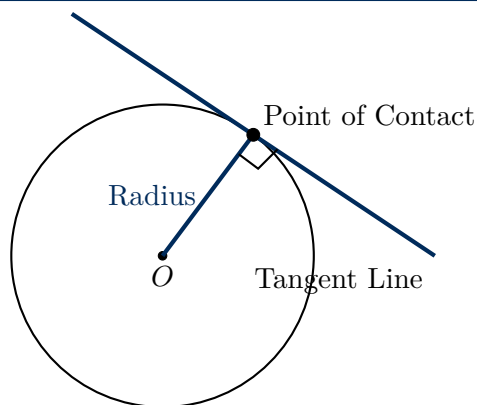
$$C = 10\pi \text{ cm}$$

1.23 What is a Tangent?

A **Tangent** is a straight line that perfectly **touches** the edge of a circle. It meets the circumference at exactly **one point** and never enters the inside of the circle.

The Golden Tangent Rule

A tangent line is ALWAYS exactly **perpendicular** (90°) to the radius at the point where it touches!



1.24 Coordinate Geometry: The Equation of a Circle

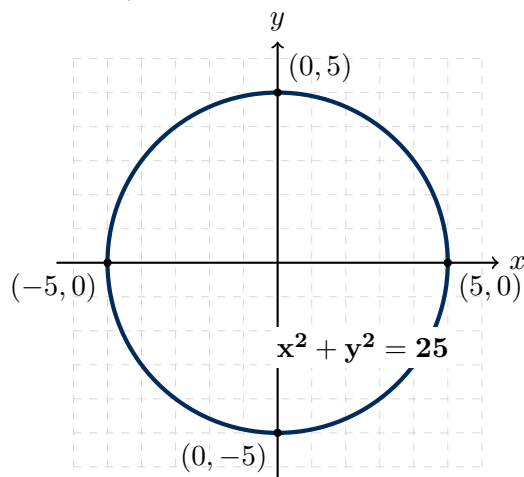
When we place a circle on an x, y coordinate grid, with its centre perfectly at the origin $(0, 0)$, it follows a very simple equation.

The Circle Equation

$$x^2 + y^2 = r^2$$

(where r is the radius)

For example, if the equation is $x^2 + y^2 = 25$
Then $r^2 = 25$, which means the radius $r = 5$.

**1.25 A-Level Extension: The Displaced Circle**

In GCSE, you only need to know circles centered at (0,0). But in A-Level Mathematics, the circle can be placed anywhere!

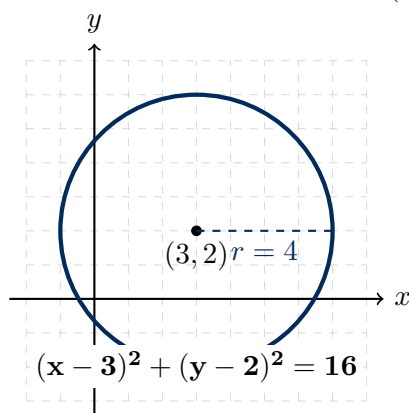
The General Circle Equation

$$(x - a)^2 + (y - b)^2 = r^2$$

Where the centre is (a, b) and the radius is r .

Example: $(x - 3)^2 + (y - 2)^2 = 16$

This is a circle with its centre at $(3, 2)$ and a radius of $r = 4$ (since $4^2 = 16$).



1.26 A-Level Extension: Inside, Outside, or On? (1/2)

We can use the circle equation to determine exactly where an object is located!

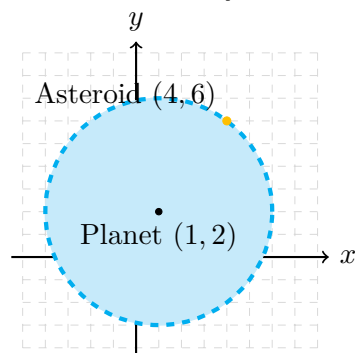
Scenario: A planet's forcefield is modeled by $(x - 1)^2 + (y - 2)^2 = 25$.
(Centre = $(1, 2)$, Radius = 5).

1. An Asteroid is at $(4, 6)$. Does it hit?

Substitute $x = 4, y = 6$ into the left side:

$$(4 - 1)^2 + (6 - 2)^2 = 3^2 + 4^2 = 9 + 16 = \mathbf{25}.$$

$25 = 25 \implies$ Exactly **ON** the forcefield!



1.27 A-Level Extension: Inside, Outside, or On? (2/2)

Scenario: A planet's forcefield is modeled by $(x - 1)^2 + (y - 2)^2 = 25$.
(Centre = $(1, 2)$, Radius = 5).

2. A Comet passes at $(6, 6)$.

Substitute $x = 6, y = 6$ into the left side:

$$(6 - 1)^2 + (6 - 2)^2 = 5^2 + 4^2 = 25 + 16 = \mathbf{41}.$$

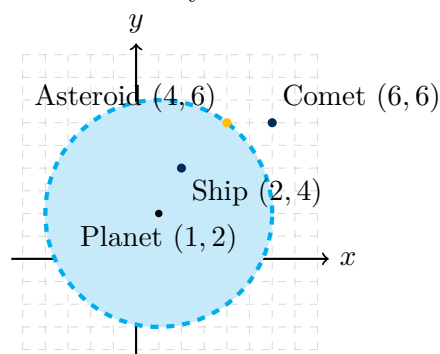
$41 > 25 \implies$ Safely **OUTSIDE**!

3. A Spaceship is at $(2, 4)$.

Substitute $x = 2, y = 4$ into the left side:

$$(2 - 1)^2 + (4 - 2)^2 = 1^2 + 2^2 = 1 + 4 = \mathbf{5}.$$

$5 < 25 \implies$ Safely **INSIDE**!



Advanced Tangent Proof

One of the hardest questions in the GCSE (Grade 9) is being asked to find the **Equation of a Tangent** at a specific point on a circle.

Because the tangent is a straight line, its equation will be $y = mx + c$.

The 3-Step Tangent Algorithm

1. **Find the Gradient of the Radius (m_1):** Find the gradient between the origin $(0, 0)$ and the point on the circle.
2. **Find the Perpendicular Gradient (m_2):** Because the tangent meets the radius at 90° , use the negative reciprocal $(-1/m_1)$.
3. **Find c :** Substitute the point's coordinates and m_2 into $y = mx + c$.

Example: Equation of a Tangent

Example 10: Tangent to a Circle

A circle has the equation $x^2 + y^2 = 20$.

Find the exact equation of the tangent to the circle at the point $(4, 2)$.

1. Gradient of the Radius (m_1):

The radius goes from $(0, 0)$ to $(4, 2)$.

$$m_1 = \frac{2-0}{4-0} = \frac{2}{4} = \frac{1}{2}.$$

2. Gradient of the Tangent (m_2):

The tangent is perpendicular! Negative reciprocal $\implies m_2 = -2$.

3. Find c :

Substitute the point $(4, 2)$ and $m = -2$ into $y = mx + c$:

$$2 = (-2)(4) + c \implies 2 = -8 + c \implies c = 10.$$

Final Equation: $y = -2x + 10$

Graded Exercises: Interactive Platinum 6

Platinum Exercise: Tangent Intersection Proof

1. **[Solution]** A circle has the equation $x^2 + y^2 = 45$.
A tangent meets the circle at the point $(-3, 6)$.
Find the exact coordinate where this tangent crosses the x -axis.

1.28 A-Level Mastery: Tangent to a Displaced Circle

Example 11: The A-Level Tangent

A circle has the equation $(x - 3)^2 + (y - 2)^2 = 25$.

Find the exact equation of the tangent at the point $(6, 6)$.

1. Gradient of the Radius (m_1):

The centre is $(3, 2)$. The point is $(6, 6)$.

$$m_1 = \frac{6-2}{6-3} = \frac{4}{3}.$$

2. Gradient of the Tangent (m_2):

The tangent is perpendicular! Negative reciprocal $\Rightarrow m_2 = -\frac{3}{4} = -0.75$.

3. Find c :

Substitute $(6, 6)$ and $m = -0.75$ into $y = mx + c$:

$$6 = (-0.75)(6) + c \Rightarrow 6 = -4.5 + c \Rightarrow c = 10.5.$$

Final Equation: $y = -0.75x + 10.5$

1.29 A-Level Mastery: Line and Circle Intersection

To find where a straight line intersects a circle, we must use **Simultaneous Equations** and solve a **Quadratic**!

Example 12: Intersection

Find the coordinates where $y = x - 1$ intersects $(x - 2)^2 + (y - 3)^2 = 10$.

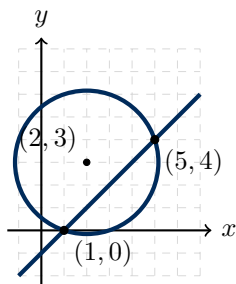
1. Substitute y into the circle: $(x - 2)^2 + ((x - 1) - 3)^2 = 10 \Rightarrow (x - 2)^2 + (x - 4)^2 = 10$

2. Expand the brackets: $(x^2 - 4x + 4) + (x^2 - 8x + 16) = 10$

3. Simplify into a Quadratic ($= 0$): $2x^2 - 12x + 20 = 10 \Rightarrow 2x^2 - 12x + 10 = 0 \Rightarrow x^2 - 6x + 5 = 0$

4. Factorise and solve for x and y : $(x - 1)(x - 5) = 0 \Rightarrow x = 1$ or $x = 5$.

$x = 1 \Rightarrow y = 0$. $x = 5 \Rightarrow y = 4$. **Coordinates:** $(1, 0)$, $(5, 4)$.



Graded Exercises: Platinum 7 (A-Level Edition)

Platinum 7: The A-Level Intersection

1. **[Solution]** Find the exact coordinates where the straight line $y = x + 2$ intersects the circle $x^2 + y^2 = 10$.

1.30 A-Level Mastery: The Ultimate Synthesis

Advanced Synthesis: Bringing It All Together

1. **[Solution]** A circle has a diameter with endpoints $A(-3, 2)$ and $B(5, -4)$. Find the exact equation of the circle.

This question tests **everything** you've learned.

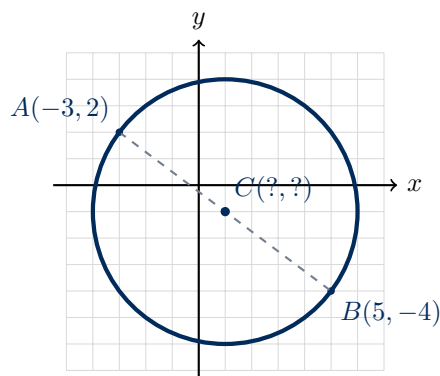
Hint 1: What do you need to write the equation of a circle?

A center coordinate (a, b) and a radius r ! **Hint 2:** How can you find the center of the circle if you

only have the diameter's endpoints?

The center is the **Midpoint** of the diameter! **Hint 3:** How do you find the radius?

Find the **Distance** from the center to an endpoint!



1.31 Exam Vault: Real GCSE Past Papers

Exam Vault

Real GCSE Past Paper Questions

Are you ready to test your A* skills?

1.32 Exam Vault: Past Paper Q1

Edexcel GCSE Higher Tier Style: Perpendicular Bisector

1. **[Solution]** A is the point with coordinates $(-5, 0)$ and B is the point with coordinates $(7, 3)$.

Find the equation of the perpendicular bisector of AB .

Give your answer in the form $ax + by = c$, where a , b , and c are integers.

1.33 Exam Vault: Past Paper Q2

Edexcel GCSE Higher Tier Style: Collinear Ratios

1. **[Solution]** The points A , B and C lie in order on a straight line.

The coordinates of A are $(2, 5)$.

The coordinates of B are $(4, p)$.

The coordinates of C are $(q, 17)$.

Given that the ratio of lengths $AB : BC = 1 : 3$, find the exact values of p and q .

1.34 Exam Vault: Past Paper Q3

AQA GCSE Higher Tier Style: Perpendicular Intersection

1. **[Solution]** Line L_1 has equation $3y = 2x - 6$.
Line L_2 is perpendicular to L_1 and passes through the point $(4, 1)$.
Line L_2 crosses the x -axis at the point P .
Find the coordinates of the point P .

Part IV

Graphs & Functions

Introduction to Part IV

We are now moving on to functions. Rather than just finding a single value for x , functions let us look at how one set of numbers (the domain) maps to another (the range). We will learn how to substitute into functions, find inverse functions, and work with composite functions. This is a step up in difficulty, but it is an essential topic if you are planning to take maths at A-Level.

Part V

Enrichment: Discrete Mathematics & Olympiad Challenges

Introduction to Part V

By this point, you have covered the core syllabus. In this final section, we are going to look at some extension topics. We will cover modular arithmetic—sometimes called ‘clock maths’—which is the foundation of computer cryptography. Finally, we will work through some Olympiad-style challenges. These questions are tough and require you to combine multiple topics, but they are fantastic practice for the hardest questions at the end of your exam paper.

Part VI

Appendices: Solutions Manuals

Coordinate Geometry (Solutions)

Question 1

[\[Back to Exercise\]](#)

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Solution

Solution:

Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Substitute: $d = \sqrt{(4 - 1)^2 + (9 - 5)^2}$

Simplify: $d = \sqrt{3^2 + 4^2}$

Solve: $d = \sqrt{9 + 16} = \sqrt{25} = 5.$

Question 2

[\[Back to Exercise\]](#)

[\[Back to Exercise\]](#)

Solution

Solution:

1. Find m: $m = \frac{17-5}{2-(-2)} = \frac{12}{4} = 3.$

2. Pick a point (Q is easier, no negatives): $x = 2, y = 17.$

3. Substitute into $y = mx + c$:

$$17 = (3)(2) + c \implies 17 = 6 + c \implies c = 11$$

Final Equation: $y = 3x + 11$

Question 3

[\[Back to Exercise\]](#)

[\[Back to Exercise\]](#)

Solution

Solution:

1. Identify m: Since it is parallel, it has the exact same gradient.
 $m = -4$.

2. Substitute point into $y = mx + c$: We know $m = -4, x = -1, y = 3$.
 $3 = (-4)(-1) + c$

3. Solve for c:
 $3 = 4 + c \implies c = -1$

Final Equation: $y = -4x - 1$

Question 4

[\[Back to Exercise\]](#)[\[Back to Exercise\]](#)

Solution

Solution:

1. Midpoint: $M = \left(\frac{0+4}{2}, \frac{8+0}{2}\right) = (2, 4)$.

2. Original Gradient (m_1): $m_1 = \frac{0-8}{4-0} = \frac{-8}{4} = -2$.

3. Perpendicular Gradient (m_2): Negative reciprocal of -2 is $\frac{1}{2}$.

4. Find c: Use $M(2, 4)$ and $m_2 = \frac{1}{2}$.
 $4 = \left(\frac{1}{2}\right)(2) + c$
 $4 = 1 + c \implies c = 3$.

Final Equation: $y = \frac{1}{2}x + 3$

Question 5

[\[Back to Exercise\]](#)[\[Back to Exercise\]](#)

Solution

Solution:

1. Set the lines equal to each other:
 $3x - 2 = -x + 6$

2. Solve for x:
 Add x to both sides: $4x - 2 = 6$

Add 2 to both sides: $4x = 8 \implies x = 2$

3. Substitute x back to find y:
 $y = 3(2) - 2 \implies y = 6 - 2 \implies y = 4$

Final Intersection Coordinate: $(2, 4)$

Question 6

[\[Back to Exercise\]](#)[\[Back to Exercise\]](#)

Solution

Solution:

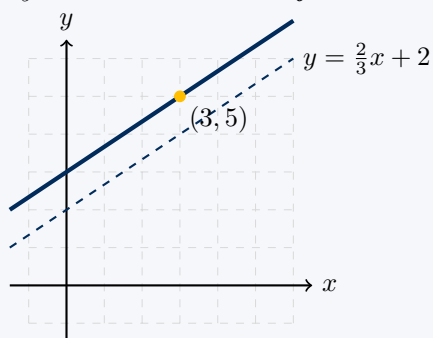
1. Find the gradient (m): Rearrange $2x - 3y + 6 = 0$
 $3y = 2x + 6 \implies y = \frac{2}{3}x + 2$. Gradient $m = \frac{2}{3}$.

2. Find c : Use $(3, 5)$ and $m = \frac{2}{3}$
 $5 = \frac{2}{3}(3) + c \implies 5 = 2 + c \implies c = 3$.

3. Write equation and convert to standard form:

$$y = \frac{2}{3}x + 3$$

$$3y = 2x + 9 \implies 2x - 3y + 9 = 0.$$



Question 7

[\[Back to Exercise\]](#)
[\[Back to Exercise\]](#)

Solution

Solution:

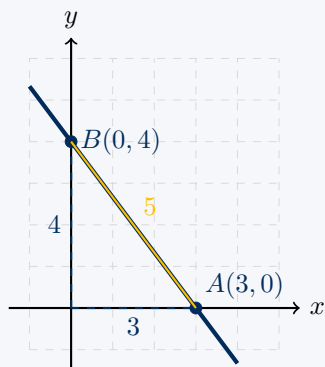
1. Find the x -intercept (set $y = 0$):
 $4x + 3(0) = 12 \implies 4x = 12 \implies x = 3$.
 Point A is $(3, 0)$.

2. Find the y -intercept (set $x = 0$):
 $4(0) + 3y = 12 \implies 3y = 12 \implies y = 4$.
 Point B is $(0, 4)$.

3. Calculate Distance AB :

$$d = \sqrt{(3-0)^2 + (0-4)^2}$$

$$d = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$



Question 8

[\[Back to Exercise\]](#)
[\[Back to Exercise\]](#)

Solution

Solution:**1. Find the Gradient of AB :**

$$m_{AB} = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}.$$

2. Find the Gradient of BC (in terms of k):

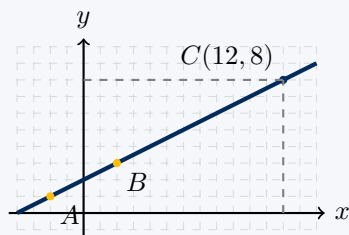
$$m_{BC} = \frac{8-3}{k-2} = \frac{5}{k-2}.$$

3. Equate Gradients (since they are collinear):

$$\frac{1}{2} = \frac{5}{k-2}$$

Cross-multiply: $1 \times (k - 2) = 2 \times 5$

$$k - 2 = 10 \implies k = 12.$$



Question 9

[\[Back to Exercise\]](#)[\[Back to Exercise\]](#)

Solution

Solution:**1. Bisector of PQ :** Midpoint $M(\frac{-1+3}{2}, \frac{2-6}{2}) = (1, -2)$.Gradient $m_{PQ} = \frac{-6-2}{3-(-1)} = -2$. Perpendicular $m = \frac{1}{2}$.

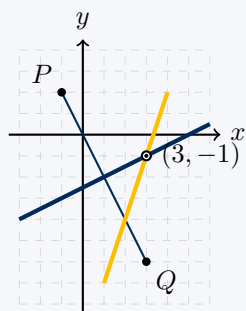
$$\text{Eq: } -2 = \frac{1}{2}(1) + c \implies c = -2.5. \text{ Bisector: } y = 0.5x - 2.5.$$

2. Find Intersection Point: Set equations equal

$$3x - 10 = 0.5x - 2.5$$

$$\text{Multiply by 2: } 6x - 20 = x - 5 \implies 5x = 15 \implies x = 3.$$

$$\text{Substitute } x: y = 3(3) - 10 = 9 - 10 \implies y = -1.$$

Intersection Point: $(3, -1)$.

Question 10

[\[Back to Exercise\]](#)[\[Back to Exercise\]](#)

Solution

Solution:

1. Prove Right Angle: Find gradients.

$$m_{AB} = \frac{2-5}{4-(-2)} = \frac{-3}{6} = -\frac{1}{2}. \quad m_{BC} = \frac{-4-2}{1-4} = \frac{-6}{-3} = 2.$$

Since $-\frac{1}{2} \times 2 = -1$, $AB \perp BC$. The triangle has a right angle at B .

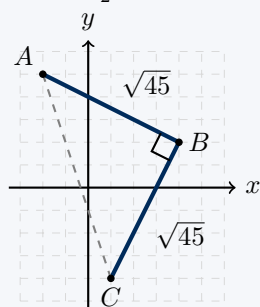
2. Find Lengths of Base and Height (AB and BC):

$$AB = \sqrt{(4 - (-2))^2 + (2 - 5)^2} = \sqrt{6^2 + (-3)^2} = \sqrt{45}.$$

$$BC = \sqrt{(1 - 4)^2 + (-4 - 2)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}.$$

3. Calculate Area:

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times \sqrt{45} \times \sqrt{45} = \frac{45}{2} = \mathbf{22.5}.$$

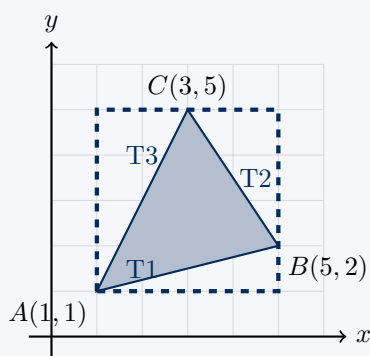


Calculate the exact area of the triangle with vertices $A(1, 1)$, $B(5, 2)$, and $C(3, 5)$. **The "Box Method" Trap:**

If a triangle isn't right-angled, $\frac{1}{2}bh$ is hard because we don't know the height!

Solution: Draw a **bounding rectangle** around it!

1. Area of full rectangle = $4 \times 4 = 16$.
2. Subtract the 3 outer right-angled triangles.



Solution

Step 1: Outer Rectangle Area

Width = $5 - 1 = 4$, Height = $5 - 1 = 4$.

Area = $4 \times 4 = 16$.

Step 2: Area of Outer Triangles ($\frac{1}{2}bh$)

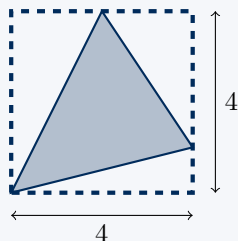
T1 (bottom): Base = 4, Height = 1. Area = $\frac{1}{2} \times 4 \times 1 = 2$.

T2 (right): Base = 2, Height = 3. Area = $\frac{1}{2} \times 2 \times 3 = 3$.

T3 (top left): Base = 2, Height = 4. Area = $\frac{1}{2} \times 2 \times 4 = 4$.

Step 3: Subtract from Total

Triangle Area = $16 - (2 + 3 + 4) = 16 - 9 = 7$.



Question 12

[\[Back to Exercise\]](#)
[\[Back to Exercise\]](#)

Solution

Solution:**1. Midpoint & Gradient of AB:**

$$M = \left(\frac{-2+6}{2}, \frac{4+0}{2} \right) = (2, 2).$$

$$m_{AB} = \frac{0-4}{6-(-2)} = \frac{-4}{8} = -\frac{1}{2}.$$

2. Gradient of the Bisector:

Perpendicular flip $\Rightarrow m = 2$.

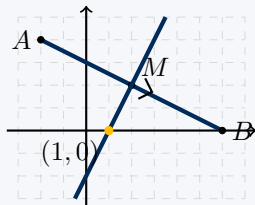
3. Build the Bisector Equation:

$$y = 2x + c \Rightarrow 2 = 2(2) + c \Rightarrow c = -2 \Rightarrow y = 2x - 2.$$

4. x-axis Intersection ($y = 0$):

$$0 = 2x - 2 \Rightarrow 2x = 2 \Rightarrow x = 1.$$

Intersection Coordinate: $(1, 0)$.



Question 13

[\[Back to Exercise\]](#)
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Solution

Solution:**1. Calculate the 'step' from P to Q:**

$$x\text{-step} = 4 - 1 = +3, \quad y\text{-step} = 8 - 2 = +6.$$

2. Scale the step for QR:

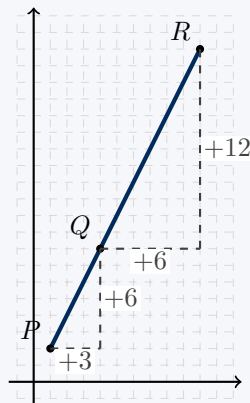
Since $QR = 2PQ$, double the step!

$$\text{New } x\text{-step} = +6, \quad \text{New } y\text{-step} = +12.$$

3. Apply the step from Q to find R:

$$R_x = 4 + 6 = 10, \quad R_y = 8 + 12 = 20.$$

Final Coordinate: R(10, 20).

**Question 14**
[\[Back to Exercise\]](#)
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Solution

Solution:**1. Prove Parallel Sides (Gradients):**

$$m_{AB} = \frac{4}{3} \quad m_{DC} = \frac{4}{3} \quad m_{AD} = 0 \quad m_{BC} = 0$$

Opposite sides are parallel $\Rightarrow$ Parallelogram.

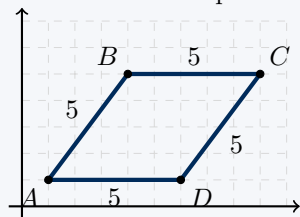
2. Prove Equal Sides (Distance):

Length $AD = 5$ and Length $BC = 5$ (horizontal lines).

$$\text{Length } AB = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

$$\text{Length } DC = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

Conclusion: 4 equal sides + parallel opposites = Rhombus.

**Question 15**
[\[Back to Exercise\]](#)

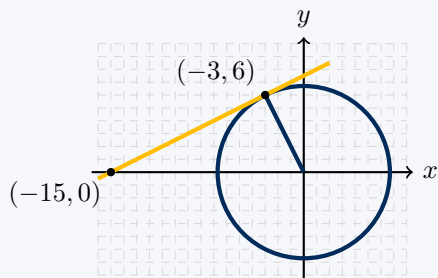
Solution

Solution: 1. Radius Gradient: $m_{rad} = \frac{6}{-3} = -2$.

2. Tangent Gradient: Perpendicular $\implies m_{tan} = \frac{1}{2}$.

3. Tangent Equation: Use $(-3, 6)$, $m = \frac{1}{2} \implies 6 = \frac{1}{2}(-3) + c \implies c = 7.5$. Equation: $y = 0.5x + 7.5$

4. x-axis intersection ($y = 0$): $0 = 0.5x + 7.5 \implies x = -15 \implies (-15, 0)$.



Question 16

[\[Back to Exercise\]](#)

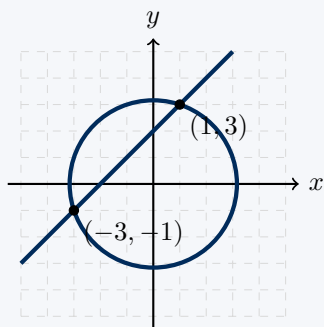
Solution

Solution: 1. Substitute y : $x^2 + (x + 2)^2 = 10$

2. Expand brackets: $x^2 + (x^2 + 4x + 4) = 10$

3. Simplify to Quadratic ($= 0$): $2x^2 + 4x + 4 = 10 \implies 2x^2 + 4x - 6 = 0 \implies x^2 + 2x - 3 = 0$

4. Factorise and solve: $(x + 3)(x - 1) = 0 \implies x = -3$ or $x = 1$.
 $x = -3 \implies y = -1$. $x = 1 \implies y = 3$. **Coordinates:** $(-3, -1)$, $(1, 3)$.



Question 18

[\[Back to Exercise\]](#)

Solution

Solution:**1. Midpoint of AB:**

$$M = \left(\frac{-5+7}{2}, \frac{0+3}{2} \right) = (1, 1.5)$$

2. Gradient of AB (m_1):

$$m_1 = \frac{3-0}{7-(-5)} = \frac{3}{12} = \frac{1}{4}$$

3. Perpendicular Gradient (m_2):

$$\text{Negative reciprocal} \implies \mathbf{m_2 = -4}$$

4. Equation ($y - y_1 = m(x - x_1)$):

$$y - 1.5 = -4(x - 1) \implies y - 1.5 = -4x + 4$$

$$4x + y = 5.5$$

5. Integer Format ($ax + by = c$):

Multiply by 2 to remove the decimal:

$$\mathbf{8x + 2y = 11}$$

Question 19
[\[Back to Exercise\]](#)

Solution

Solution:**1. Analyze the x -coordinates (Horizontal Step):**

$$A_x = 2, B_x = 4.$$

The step from A to B is $+2$.

Since $AB : BC = 1 : 3$, the step from B to C is exactly 3 times larger.

x -step for $BC = 3 \times 2 = +6$.

$$q = B_x + 6 = 4 + 6 \implies \mathbf{q = 10}.$$

2. Analyze the y -coordinates (Vertical Step):

$$A_y = 5, C_y = 17.$$

The total step from A to C is $17 - 5 = +12$.

The total ratio AC is $1 + 3 = 4$ parts.

Each part is worth $12 \div 4 = +3$.

Therefore, the vertical step from A to B (1 part) is $+3$.

$$p = A_y + 3 = 5 + 3 \implies \mathbf{p = 8}.$$

Final Answer: $\mathbf{p = 8, q = 10}.$

Question 20
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Solution

Solution:**1. Gradient of L_1 :**

$$3y = 2x - 6 \implies y = \frac{2}{3}x - 2 \implies \mathbf{m_1 = \frac{2}{3}}.$$

2. Gradient of L_2 :

$$\text{Negative reciprocal} \implies \mathbf{m_2 = -\frac{3}{2}}.$$

3. Equation of L_2 :

Passes through $(4, 1)$:

$$y - 1 = -\frac{3}{2}(x - 4) \implies y - 1 = -\frac{3}{2}x + 6 \implies \mathbf{y = -\frac{3}{2}x + 7}.$$

4. Finding Point P (x -axis intersection):

Crosses x -axis when $y = 0$.

$$0 = -\frac{3}{2}x + 7 \implies \frac{3}{2}x = 7 \implies 3x = 14 \implies \mathbf{x = \frac{14}{3}}.$$

Final Coordinate: $P\left(\frac{14}{3}, 0\right)$.

About the Author

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Dr. Zakaria ELHASNAOUI is a premier mathematics educator specializing in high-performance cognitive training for the GCSE, IGCSE, and IB Mathematics curricula.

With a philosophy that completely rejects the rote memorization of traditional schooling, Dr. Zakaria has engineered a "Advanced Bootcamp" approach. He believes that mathematics is not a set of arbitrary rules to be blindly followed, but rather a purely logical system waiting to be understood. His specialized instructional methods are designed to rescue failing grades and rapidly accelerate gifted students toward the elusive A* (Year 10) and beyond.

Elite Private Tutoring & Masterclasses

Are you a parent seeking a dramatic improvement in your child's mathematical confidence and examination results?

Or are you an ambitious student looking for an unfair advantage before stepping into the exam hall?

Dr. Zakaria offers exclusive, highly personalized 1-on-1 tutoring sessions and intensive tactical masterclasses. Availability is strictly limited.

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