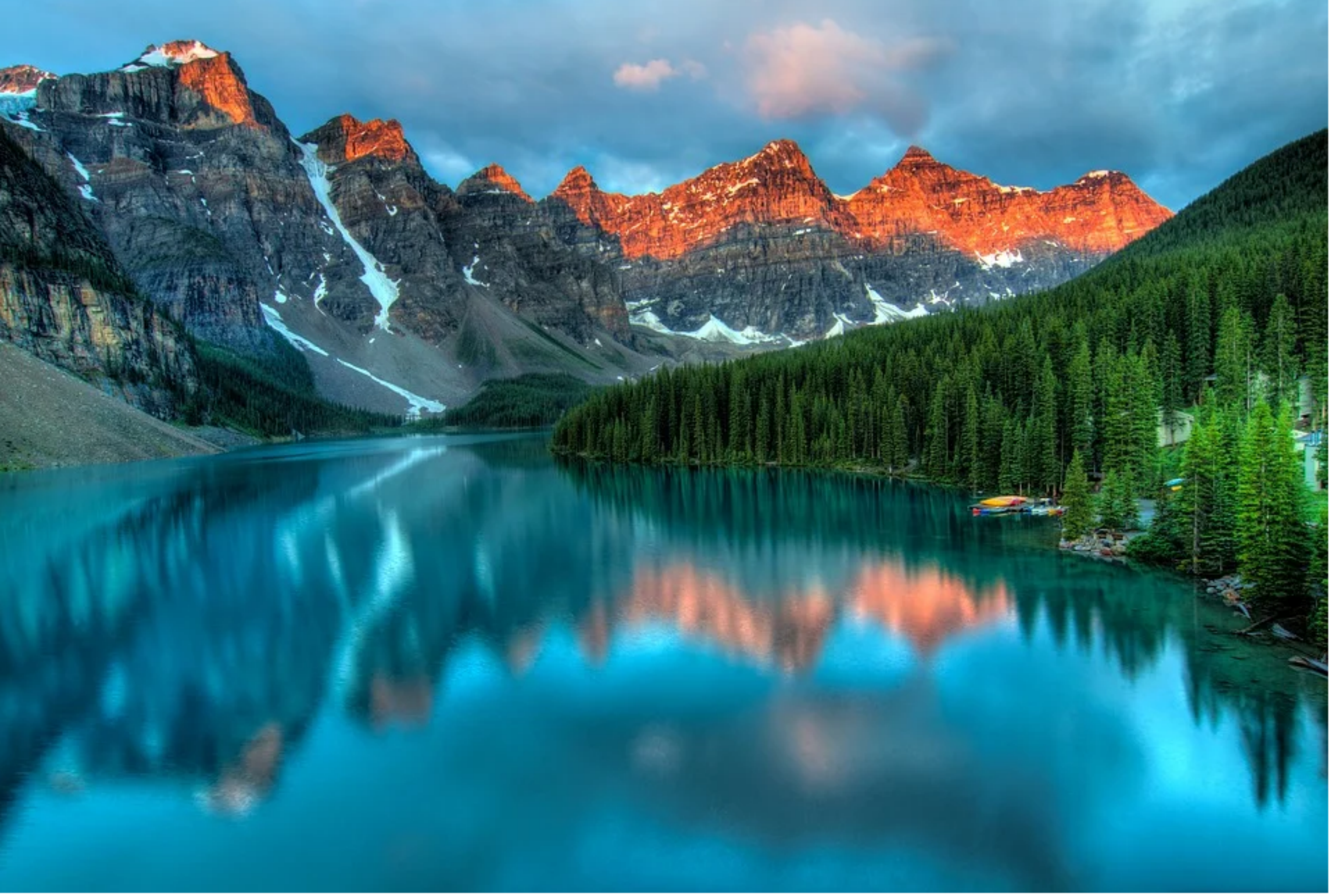

Advanced calculus II-2

Integration on higher dimensional spaces

Author: Teh, Jyh-Haur

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Contents of Advanced Calculus II-2

Contents of Advanced Calculus II-1

Preface

Advanced calculus occupies the most fundamental position in mathematics training. It is an essential path from elementary numerical calculation to higher-level abstract thinking. Advanced calculus is a two-semester course, four credits per semester, 200 minutes of lectures a week, plus 2 hours of recitation, weekly homework, and four exams in a semester in the mathematics department of National Tsing Hua University of Taiwan. It can be said that it is the most loaded undergraduate course. Many students are quite afraid of it.

The author is trying to write some books that may help students in understanding the materials. In these books, all proofs are explained in detail, easy to understand and complete, with many graphics and colors. Also, they have to be easy to read on mobile phones.

With these ideas in mind, the author produces a series of textbooks: Advanced Calculus I-1, I-2 and Advanced Calculus II-1, II-2.

These books stemmed from lecture notes for courses of advanced calculus that the author taught in the mathematics department of National Tsing Hua University of Taiwan. A dim hope for these books is that students will be more receptive and more willing to spend time in this course.

Mathematical knowledge and the ability of abstract thinking become more and more important in modern sciences and technology industries. Mathematics is indispensable from automated processes and big data processing. Knowledge of calculus is not enough for applied sciences. This may be a reason that regardless of its heavy loading, it attracts students from electrical engineering, computer science, financial engineering, management and medical school to take.

In order to facilitate the use on mobile phones, the author needs to make the files of the books small. The content of Advanced Calculus II is divided into two books: Advanced calculus II-1 and Advanced Calculus II-2. Each contains two midterm exams. Exams and practice exams are all attached to the books. Each section is accompanied by exercises. These books can be regarded as self-complete and suitable for self-study.

Main references of Advanced Calculus II-2 are

1. Real mathematical analysis by Pugh ([**P**]);

2. Elementary classical analysis by Marsden and Hoffman ([**MH**]);
3. Measure, integral and probability by Capinski and Kopp ([**CK**]);
4. Wikipedia.

Those beautiful pictures at the end of each chapter are free pictures from pixabay.com.

The latex documentclass "elegantbook"(https://github.com/ElegantLaTeX/ElegantBook) is used to edit this book.

Jyh-Haur Teh
Department of Mathematics
National Tsing Hua University of Taiwan
Hsinchu, Taiwan.
Website: https://www.math.nthu.edu.tw/∼jyhhaur

Chapter 5 Riemann integrals in higher dimensional spaces

5.1 Riemann integrals

Definition 5.1 (Rectangle)

A rectangle in $\mathbb{R}^n$ *is a set of the form*

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$$

for some real numbers $a_1 \leq b_1$, $a_2 \leq b_2$, ..., $a_n \leq b_n$. ♣

Example 5.1

1. $[a, a]$ is a rectangle in $\mathbb{R}$ for any $a \in \mathbb{R}$.
2. $[1, 3] \times [\sqrt{2}, \sqrt{3}]$ is a rectangle in $\mathbb{R}^2$.
3. $[-1, 4] \times [-3, -2] \times [1, 2]$ is a rectangle in $\mathbb{R}^3$.

Definition 5.2 (Partition)

A partition P *of a rectangle* $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ *is a set of points*

$$P = P_1 \times P_2 \times \cdots \times P_n$$

where

$$P_i = \{a_i = t_{i,0} < t_{i,1} < \cdots < t_{i,k_i} = b_i\}$$

is a partition of $[a_i, b_i]$, $k_i \in \mathbb{N}$ *for all* $i = 1, 2, ..., n$. ♣

Example 5.2 Let $R = [1, 2] \times [2, 5]$ be a rectangle in $\mathbb{R}^2$ and

$$P_1 := \{1, 1.1, 1.7, 2\}$$
$$P_2 := \{2, 2.8, 3.9, 5\}$$

Then

$$\begin{aligned} P := P_1 \times P_2 = \{&(1, 2), (1, 2.8), (1, 3.9), (1, 5), (1.1, 2), (1.1, 2.8), (1.1, 3.9), (1.1, 5) \\ &(1.7, 2), (1.7, 2.8), (1.7, 3.9), (1.7, 5), (2, 2), (2, 2.8), (2, 3.9), (2, 5)\} \end{aligned}$$

is a partition of R.

Definition 5.3 (Volume)

The volume of the rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ *is defined to be*

$$vol(R) := (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)$$

♣

Example 5.3 If $R = [1, 3] \times [2, 5] \times [3, 8]$, $\text{vol}(R) = (3 - 1)(5 - 2)(8 - 3) = 30$.

Definition 5.4 (Lower sum, upper sum and subrectangle)

Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ *be a rectangle in* $\mathbb{R}^n$ *and* $f : R \to \mathbb{R}$ *be a function. Given a partition*

$$P = P_1 \times P_2 \times \cdots \times P_n$$

of R *where*

$$P_i = \{a_i = t_{i,0} < t_{i,1} < \cdots < t_{i,k_i} = b_i\}$$

The set

$$R_{i_1,i_2,\ldots,i_n} := [t_{1,i_1-1}, t_{1,i_1}] \times [t_{2,i_2-1}, t_{2,i_2}] \times \cdots \times [t_{n,i_n-1}, t_{n,i_n}]$$

is called a subrectangle of R *with respect to* P*. Let*

$$m_{i_1,\ldots,i_n} = \inf_{x \in R_{i_1,\ldots,i_n}} \{f(x)\}$$
$$M_{i_1,\ldots,i_n} = \sup_{x \in R_{i_1,\ldots,i_n}} \{f(x)\}$$

for $1 \leq i_j \leq k_j, j = 1, \ldots, n$*.*

Define the lower sum of f *with respect to* P *to be*

$$L(f, P) := \sum_{i_1=1}^{k_1} \cdots \sum_{i_n=1}^{k_n} m_{i_1,i_2,\ldots,i_n} vol(R_{i_1,i_2\ldots,i_n})$$

and the upper sum of f *with respect to* P *to be*

$$U(f, P) := \sum_{i_1=1}^{k_1} \cdots \sum_{i_n=1}^{k_n} M_{i_1,i_2,\ldots,i_n} vol(R_{i_1,i_2,\ldots,i_n})$$

♣

Example 5.4 Let $R = [1,2] \times [2,4]$ and $f : R \to \mathbb{R}$ be defined by

$$f(x,y) = x + y$$

Fix $n \in \mathbb{N}$ and let

$$P_n = \{1 = t_{1,0} < t_{1,1} < \cdots < t_{1,n} = 2\} \times \{2 = t_{2,0} < t_{2,1} < \cdots < t_{2,n} = 4\}$$

where

$$t_{1,i} = 1 + \frac{i}{n}, \quad t_{2,j} = 2 + \frac{2j}{n}$$

Then

$$R_{i,j} = [t_{1,i-1}, t_{1,i}] \times [t_{2,j-1}, t_{2,j}] = [1 + \frac{i-1}{n}, 1 + \frac{i}{n}] \times [1 + \frac{2(j-1)}{n}, 1 + \frac{2j}{n}]$$

The volume

$$\text{vol}(R_{i,j}) = \frac{1}{n}\frac{2}{n} = \frac{2}{n^2}$$

We have

$$\inf_{x \in R_{i,j}} \{f(x)\} = (1 + \frac{i-1}{n}) + (2 + \frac{2(j-1)}{n}) = 3 + \frac{i+2j-3}{n}$$

$$\sup_{x \in R_{i,j}} \{f(x)\} = (1 + \frac{i}{n}) + (2 + \frac{2j}{n}) = 3 + \frac{i+2j}{n}$$

and

$$L(f, P_n) = \sum_{i=1}^{n}\sum_{j=1}^{n}(3 + \frac{i+2j-3}{n})(\frac{2}{n^2}) = 9 - \frac{3}{n}$$

$$U(f, P_n) = \sum_{i=1}^{n}\sum_{j=1}^{n}(3 + \frac{i+2j}{n})(\frac{2}{n^2}) = 9 + \frac{3}{n}$$

Definition 5.5 (Refinement)

A refinement of a partition P of a rectangle R is a partition P' of R such that $P \subseteq P'$. ♣

Example 5.5 Let $R = [1,2] \times [3,4]$ and $P = \{1, 1.2, 1.5, 2\} \times \{3, 4\}$. If

$$P' := P \cup \{(1, 3.5), (1.2, 3.5), (1.5, 3.5), (2, 3.5)\} = \{1, 1.2, 1.5, 2\} \times \{3, 3.5, 4\}$$

P' is a refinement of P.

Proposition 5.1

If $R \subset \mathbb{R}^n$ is a rectangle and $f : R \to \mathbb{R}$ is a bounded function, then

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$$

for any partitions $P \subseteq P'$ of R. ♠

Proof Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Suppose that

$$P = P_1 \times P_2 \times \cdots \times P_n$$

where $P_i = \{a_i = t_{i,0} < t_{i,1} < \cdots < t_{i,k_i} = b_i\}$ and

$$P' = P'_1 \times P'_2 \times \cdots \times P'_n$$

where $P'_i = \{a_i = t'_{i,0} < t'_{i,1} < \cdots < t'_{i,k'_i} = b_i\}$.

Let $R_1, ..., R_m$ be subrectangles of R with respect to P. Since P' is a refinement of P, for each R_i, there are subrectangles $S_{i,1}, ..., S_{i,\ell_i}$ of R with respect to P' such that

$$R_i = \bigcup_{j=1}^{\ell_i} S_{i,j}$$

Then

$$\begin{aligned} U(f, P') &= \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} \sup_{x \in S_{i,j}} \{f(x)\} \text{vol}(S_{i,j}) \\ &\leq \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} \sup_{x \in R_i} \{f(x)\} \text{vol}(S_{i,j}) \\ &= \sum_{i=1}^{m} \sup_{x \in R_i} \{f(x)\} \text{vol}(R_i) \\ &= U(f, P) \end{aligned}$$

Similar to the argument above, we have $L(f, P') \geq L(f, P)$. Since we always have

$$\inf_{x \in S_{i,j}} \{f(x)\} \leq \sup_{x \in S_{i,j}} \{f(x)\}$$

this gives us

$$L(f, P') \leq U(f, P')$$

Combining all the inequalities obtained above, we get the result.

Definition 5.6 (Lower integral and upper integral)

Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \to \mathbb{R}$ be a function. The lower integral of f over R is

$$\underline{I}(f) := \sup_P \{L(f, P)\}$$

The upper integral of f over R is

$$\overline{I}(f) := \inf_P \{U(f, P)\}$$

If $\underline{I}(f) = \overline{I}(f) \in \mathbb{R}$, we say that f is Riemann integrable on R and denote

$$\int_R f dV := \overline{I}(f) = \underline{I}(f)$$

♣

Example 5.6 We continue the calculation of Example 5.4. For any partition P of $[1, 2] \times [2, 4]$, there is a common refinement Q of P and P_n. Therefore by Proposition 5.1,

$$L(f, P) \leq U(f, Q) \leq U(f, P_n) = 9 + \frac{3}{n}$$

and

$$U(f, P) \geq U(f, Q) \geq L(f, P_n) = 9 - \frac{3}{n}$$

for any $n \in \mathbb{N}$.

Therefore

$$L(f, P) \leq 9 \quad \text{and} \quad U(f, P) \geq 9$$

This implies

$$\sup_P \{L(f, P)\} \leq 9 \quad \text{and} \quad \inf_P U(f, P) \geq 9$$

This shows that

$$\overline{I}(f) = \underline{I}(f) = 9$$

Exercise 5.1

1. Let $R = [1,3] \times [2,5]$ and $f : R \to \mathbb{R}$ be defined by $f(x,y) = xy$. Fix $n \in \mathbb{N}$. Suppose that
$$P = \{1 + \frac{2i}{n} | i = 0, 1, ..., n\} \times \{2 + \frac{3j}{n} | j = 0, ..., n\}$$
Find $L(f,P)$ and $U(f,P)$.

2. Let $f : [0,1] \times [2,3] \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} 1, & \text{if } 0 \le x \le y \le 1 \\ 3, & \text{otherwise} \end{cases}$$
Use the definition of Riemann integral to find
$$\int_{[0,1]\times[2,3]} f dV$$

3. Given a partition $P = P_1 \times \cdots P_n$ of a rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. Suppose that
$$P_i = \{a_i = t_{i,0} < t_{i,1} < \cdots < t_{i,k_i} = b_i\}$$
for $i = 1, ..., n$. We say that the partition P is an equal partition of R if there is a constant c such that
$$t_{i,j} - t_{i,j-1} = c$$
for all $j = 1, ..., k_i$ and $i = 1, ..., n$.
   (a). Let $R = [1,3] \times [1,4]$ and $P = \{1, 1.2, 3\} \times \{1, 2.7, 4\}$. Find a refinement P' of P such that P' is an equal partition.
   (b). Prove or disprove that for any partition P of a rectangle $[a,b] \subset \mathbb{R}, a < b$, there is a refinement P' of P which is an equal partition.

5.2 The Riemann-Lebesgue theorem

Definition 5.7 (Measure zero set)

A set $A \subset \mathbb{R}^n$ is said to have measure zero in $\mathbb{R}^n$ if for every $\varepsilon > 0$, there exist countably many rectangles $R_1, R_2, \ldots$ in $\mathbb{R}^n$ such that

$$A \subset \bigcup_{i=1}^{\infty} R_i$$

and

$$\sum_{i=1}^{\infty} vol(R_i) < \varepsilon$$

♣

Example 5.7 $\mathbb{Z}$ has measure zero in $\mathbb{R}$.

Solution *Given $\varepsilon > 0$. Define $\phi : \mathbb{N} \to \mathbb{Z}$ by*

$$\phi(n) := \begin{cases} \frac{1}{2}(n-1), & \text{if } n \text{ is odd;} \\ -\frac{1}{2}n, & \text{if } n \text{ is even.} \end{cases}$$

Then ϕ is a bijection. Let

$$R_n := [\phi(n), \phi(n)] = \{\phi(n)\}$$

Then

$$\bigcup_{n=1}^{\infty} R_n = \bigcup_{n=1}^{\infty} \{\phi(n)\} = \mathbb{Z}$$

Since $vol(R_n) = 0$ for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} vol(R_n) = 0 < \varepsilon$$

This shows that $\mathbb{Z}$ has measure zero in $\mathbb{R}$.

Example 5.8 $\mathbb{R} \times \{0\}$ has measure zero in $\mathbb{R}^2$ but $\mathbb{R}$ is not a set of measure zero in $\mathbb{R}$.

Solution *For $n \in \mathbb{N}$, let*

$$R_n := [\phi(n), \phi(n) + 1] \times [0, 0]$$

where $\phi : N \to \mathbb{Z}$ is a bijection. Then

$$\mathbb{R} \times \{0\} = \bigcup_{n=1}^{\infty} R_n$$

and

$$\sum_{n=1}^{\infty} vol(R_n) = \sum_{n=1}^{\infty} 0 = 0$$

Therefore $\mathbb{R} \times \{0\}$ has measure zero in $\mathbb{R}^2$.

To show that $\mathbb{R}$ does not have measure zero in $\mathbb{R}$, we prove by contradiction. Set $\varepsilon = 1$. Assume that there are rectangles $\{S_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ that cover $\mathbb{R}$ and $\sum_{n=1}^{\infty} vol(S_n) < 1$. Since

$$[0,1] \subset \mathbb{R} \subset \bigcup_{n=1}^{\infty} S_n$$

we have

$$1 = vol([0,1]) \leq \sum_{n=1}^{\infty} vol(S_n) < 1$$

which is a contradiction. Therefore $\mathbb{R}$ does not have measure zero in $\mathbb{R}$.

Example 5.9 $\mathbb{S}^1 \subset \mathbb{R}^2$ has measure zero.

Solution *We consider*

$$X = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\} \cap \mathbb{S}^1$$

the portion of $\mathbb{S}^1$ in the first quadrant. Given $\varepsilon > 0$. We first show that we can cover X by rectangles with total volume less that $\frac{\varepsilon}{4}$. By symmetry reason, there are similar covers by rectangles of $\mathbb{S}^1$ in other 3 quadrants. Take $n \in \mathbb{N}$ such that $\frac{1}{\sqrt{n}} < \frac{\varepsilon}{4}$. Let

$$x_i = \frac{i-1}{n}, \quad y_i = \sqrt{1 - x_i^2} = \sqrt{1 - (\frac{i-1}{n})^2}$$

and

$$R_i := [x_i, x_{i+1}] \times [y_{i+1}, y_i]$$

for $i = 1, ..., n$. *Then*

$$\begin{aligned}
vol(R_i) = (x_{i+1} - x_i)(y_i - y_{i+1}) &= \frac{1}{n}\left(\sqrt{1 - (\frac{i-1}{n})^2} - \sqrt{1 - (\frac{i}{n})^2}\right) \\
&= \frac{(1 - (\frac{i-1}{n})^2) - (1 - (\frac{i}{n})^2)}{n\left(\sqrt{1 - (\frac{i-1}{n})^2} + \sqrt{1 - (\frac{i}{n})^2}\right)} \\
&< \frac{(\frac{i}{n} - \frac{i-1}{n})(\frac{i}{n} + \frac{i-1}{n})}{\sqrt{n^2 - (n-1)^2}} \\
&= \frac{2i - 1}{n^2\sqrt{2n - 1}} \\
&< \frac{2i - 1}{n^2\sqrt{n}}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{n} vol(R_i) < \sum_{i=1}^{n} \frac{2i-1}{n^2\sqrt{n}} &= \frac{1}{n^2\sqrt{n}}(\sum_{i=1}^{n}(2i - 1)) \\
&= \frac{1}{n^2\sqrt{n}}(2(\frac{n(n+1)}{2}) - n) \\
&= \frac{n^2}{n^2\sqrt{n}} = \frac{1}{\sqrt{n}} < \frac{\varepsilon}{4}
\end{aligned}$$

This shows that we may cover $\mathbb{S}^1$ *by rectangles with total volume less than* ε *and this means that* $\mathbb{S}^1$ *has measure zero in* $\mathbb{R}^2$.

Proposition 5.2

A countable union of sets of measure zero in $\mathbb{R}^m$ *is a set of measure zero in* $\mathbb{R}^m$. ♠

Proof Let $\{A_n\}_{n=1}^{\infty}$ be a collection of sets of measure zero in $\mathbb{R}^m$. Given $\varepsilon > 0$. Since A_n has measure zero, there are rectangles $\{R_{n,k}\}_{k=1}^{\infty}$ in $\mathbb{R}^m$ such that

$$A_n \subset \bigcup_{k=1}^{\infty} R_{n,k} \text{ and } \sum_{k=1}^{\infty} \text{vol}(R_{n,k}) < \frac{\varepsilon}{2^n}$$

Recall that a countable union of countable sets is countable, the collection $\{R_{n,k}\}_{n,k=1}^{\infty}$ is countable.

Since

$$\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} R_{n,k}$$

and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \text{vol}(R_{n,k}) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon$$

Therefore $\bigcup_{n=1}^{\infty} A_n$ is of measure zero.

Corollary 5.1

A countable set $A \subset \mathbb{R}^n$ has measure zero in $\mathbb{R}^n$. ♡

Proof Write $A = \{a_i\}_{i=1}^{\infty}$ with the convention that if A is a finite set, we set $a_i = a_N$ for $i \geq N$ where $N = |A|$. Since a point is of measure zero, A is a countable union of measure zero sets, by the result above, A has measure zero.

Corollary 5.2

$\mathbb{Q}$ has measure zero in $\mathbb{R}$. ♡

Note that $\mathbb{Q}$ is dense in $\mathbb{R}$, but it has measure zero. The Cantor set C is uncountable, but it is not difficult to show that it also has measure zero.

Definition 5.8 (Set of discontinuities)

Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \to \mathbb{R}$ be a function. The set

$$Disc(f) := \{x \in R | f \text{ is discontinuous at } x\}$$

is called the set of discontinuities of f. ♣

Example 5.10 For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the smallest integer greater or equal to x. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \lfloor x \rfloor$$

Then

$$Disc(f) = \mathbb{Z}$$

Example 5.11 Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x, y) = \begin{cases} \dfrac{x}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Note the along the line $y = 0$,

$$\lim_{x\to 0} g(x, 0) = \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}$$

This limit does not exist. Therefore g is not continuous at $(0, 0)$. A theorem from calculus tells us that if two functions f_1, f_2 are continuous at a point p and $f_2(p) \neq 0$, then $\frac{f_1}{f_2}$ is continuous at p. This result implies that g is continuous on $\mathbb{R} - \{(0, 0)\}$. Therefore $Disc(g) = \{(0, 0)\}$.

Recall that the diameter of a set $A \subset \mathbb{R}^n$ is

$$diam(A) := \sup_{x,y\in A} \{||x - y||\}$$

Definition 5.9 (Oscillation)

Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \to \mathbb{R}$ be a bounded function. The oscillation of f at x is

$$osc_x(f) := \lim_{r\to 0^+} diam f(B_r(x) \cap R)$$

♣

Remark

(i) f is continuous at x if and only if $osc_x(f) = 0$.

(ii) If $x \in R$, $M = \sup_{x\in R}\{f(x)\}$, $m = \inf_{x\in R}\{f(x)\}$, then

$$M - m \geq osc_x(f)$$

The following result gives a complete characterization of Riemann integrable functions.

Theorem 5.1 (The Riemann-Lebesgue theorem)

Let $R \subset \mathbb{R}^n$ be a rectangle. Then f is Riemann integrable if and only if $f : R \to \mathbb{R}$ is a bounded function and the set $Disc(f)$ has measure zero.

♡

Proof Let

$$D_k = \{x \in R | osc_x(f) \geq \frac{1}{k}\}$$

Then

$$Disc(f) = \bigcup_{k=1}^{\infty} D_k$$

Suppose that f is Riemann integrable. By Proposition, f is bounded. Fix $k \in \mathbb{N}$. We are going to show that D_k has measure zero. Given $\epsilon > 0$. By Riemann's integrability criterion, there exists a partition

$$P = \{(t_{1,i_1}, t_{2,i_2}, ..., t_{n,i_n}) | 0 \leq i_j \leq k_j \text{ for } j = 1, 2, ..., n\}$$

of R such that

$$U(f,P) - L(f,P) = \sum_{i_n=1}^{k_n} \cdots \sum_{i_1=1}^{k_1} (M_{i_1,..,i_n} - m_{i_1,...,i_n}) vol(R_{i_1,...,i_n}) < \frac{\epsilon}{k}$$

where

$$R_{i_1,...,i_n} = [t_{1,i_1-1}, t_{1,i_1}] \times [t_{2,i_2-1}, t_{2,i_2}] \times \cdots \times [t_{n,i_n-1}, t_{n,i_n}]$$

is a subrectangle of R with respect to P and

$$M_{i_1,....,i_n} = \sup_{x \in R_{i_1,...,i_n}} \{f(x)\}, \quad m_{i_1,...,i_n} = \inf_{x \in R_{i_1,...,i_n}} \{f(x)\}$$

Let

$$\mathscr{A} := \{(i_1, ..., i_n) \in P | R_{i_1,...,i_n} \cap D_k \neq \emptyset\}$$

If $(i_1, ..., i_n) \in \mathscr{A}$, then

$$M_{i_1,...,i_n} - m_{i_1,...,i_n} \geq \frac{1}{k}$$

Therefore

$$\sum_{(i_1,...,i_n) \in \mathscr{A}} \frac{1}{k} vol(R_{i_1,...,i_n}) \leq \sum_{(i_1,...,i_n) \in \mathscr{A}} (M_{i_1,..,i_n} - m_{i_1,...,i_n}) vol(R_{i_1,...,i_n}) < \frac{\epsilon}{k}$$

and

$$\sum_{(i_1,...,i_n) \in \mathscr{A}} vol(R_{i_1,...,i_n}) < \epsilon$$

Since

$$D_k \subset \bigcup_{(i_1,...,i_n) \in \mathscr{A}} R_{i_1,...,i_n}$$