Advanced calculus II-1

Convergence and differentiation

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Contents of Advanced Calculus II-1

Preface

Advanced calculus occupies the most fundamental position in mathematics training. It is an essential path from elementary numerical calculation to higher-level abstract thinking. Advanced calculus is a two-semester course, four credits per semester, 200 minutes of lectures a week, plus 2 hours of recitation, weekly homework, and four exams in a semester in the mathematics department of National Tsing Hua University of Taiwan. It can be said that it is the most loaded undergraduate course. Many students are quite afraid of it.

The author is trying to write some books that may help students in understanding the materials. In these books, all proofs are explained in detail, easy to understand and complete, with many graphics and colors. Also, they have to be easy to read on mobile phones.

With these ideas in mind, the author produces a series of textbooks: Advanced Calculus I-1, I-2 and Advanced Calculus II-1, II-2.

These books stemmed from lecture notes for courses of advanced calculus that the author taught in the mathematics department of National Tsing Hua University of Taiwan. A dim hope for these books is that students will be more receptive and more willing to spend time in this course.

Mathematical knowledge and the ability of abstract thinking become more and more important in modern sciences and technology industries. Mathematics is indispensable from automated processes and big data processing. Knowledge of calculus is not enough for applied sciences. This may be a reason that regardless of its heavy loading, it attracts students from electrical engineering, computer science, financial engineering, management and medical school to take.

In order to facilitate the use on mobile phones, the author needs to make the files of the books small. The content of Advanced Calculus II is divided into two books: Advanced calculus II-1 and Advanced Calculus II-2. Each contains two midterm exams. Exams and practice exams are all attached to the books. Each section is accompanied by exercises. These books can be regarded as self-complete and suitable for self-study.

Main references of Advanced Calculus II-1 are

1. Real mathematical analysis by Pugh ([**P**]);

2. Elementary classical analysis by Marsden and Hoffman ([**MH**]);
3. Principles of mathematical analysis by Rudin ([**R**]);
4. Wikipedia.

Those beautiful pictures at the end of each chapter are free pictures from pixabay.com.

The latex documentclass “elegantbook”(https://github.com/ElegantLaTeX/ElegantBook) is used to edit this book.

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Chapter 1 Applications of uniform convergence

1.1 Differentiable functions and uniform convergence

We give a brief recall of the definition of uniform convergence (see Chapter 7 of Advanced Calculus I-2). Suppose that X is a set. A sequence of real-valued functions $\{f_n\}_{n=1}^{\infty}$ on X is said to converge uniformly to a function $f : X \to \mathbb{R}$ if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$ and $x \in X$, we have

$$|f_n(x) - f(x)| < \epsilon$$

In this case, we write

$$f_n \rightrightarrows f$$

and say that f is the uniform limit of the sequence $\{f_n\}_{n=1}^{\infty}$.

Example 1.1 Construct a sequence of functions $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to some differentiable function $f : \mathbb{R} \to \mathbb{R}$, but its derivative f' is not the uniform limit of $\{f_n'\}_{n=1}^{\infty}$.

Solution *For $n \in \mathbb{N}$, let*

$$f_n(x) = \frac{\sin(n^2 x)}{n} \quad \text{and} \quad f(x) = 0$$

for $x \in \mathbb{R}$. Then

$$\|f_n - f\|_{\sup} = \|f\|_{\sup} = \frac{1}{n} \to 0$$

as $n \to \infty$. We have $f_n \rightrightarrows f$, but

$$f_n'(x) = \frac{\cos(n^2 x)n^2}{n} = n\cos(n^2 x)$$

which diverges for all $x \in \mathbb{R}$. Therefore the sequence $\{f_n'\}_{n=1}^{\infty}$ does not even converge pointwise to f'.

Example 1.2 Construct a sequence of differentiable functions $\{f_n\}_{n=1}^{\infty}$ on $(-1, 1)$ which converges uniformly to a function f on $(-1, 1)$, but $\{f_n'\}_{n=1}^{\infty}$ converges pointwise to a function g which is not f'. In particular, the following diagram does not commute:

$$\begin{array}{ccc} f_n & \xrightarrow{\rightrightarrows} & f \\ \downarrow{\scriptstyle \frac{d}{dx}} & & \downarrow{\scriptstyle \frac{d}{dx}} \\ f_n' & \longrightarrow & g \neq f' \end{array}$$

Solution *Let*

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}\,,\ x \in (-1, 1)$$

and

$$f(x) = |x|$$

Then

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}$$

which exists for all $x \in (-1, 1)$ and $f_n \to f$ pointwise. Note that

$$\|f_n - f\|_{\sup} = \|\sqrt{x^2 + \frac{1}{n}} - |x|\|_{\sup}$$

and

$$\begin{aligned} \sqrt{x^2 + \frac{1}{n}} - |x| &= \frac{\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}}(\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}) \\ &= \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}} \leq \sqrt{\frac{1}{n}} \to 0 \end{aligned}$$

as $n \to \infty$. So the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $(-1, 1)$. The sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ converges pointwise to the function

$$g(x) = \begin{cases} 1, & \textit{if } x > 0 \\ 0, & \textit{if } x = 0 \\ -1, & \textit{if } x < 0 \end{cases}$$

Note that since f is not differentiable at 0, f' does not exist on $(-1, 1)$.

The problem of the above example is that the sequence of derivatives does not converge uniformly. With this extra condition, we have the following result.

Theorem 1.1

For $n \in \mathbb{N}$, let

$$f_n : (a,b) \to \mathbb{R}$$

be a differentiable function. If $f, g : (a,b) \to \mathbb{R}$ are functions and

$$f_n \rightrightarrows f \text{ and } f_n' \rightrightarrows g$$

uniformly on (a,b), then f is differentiable on (a,b) and

$$f' = g$$

♡

In other words, we have a commutative diagram:

$$\begin{array}{ccc} f_n & \xrightarrow{\rightrightarrows} & f \\ {\scriptstyle \frac{d}{dx}}\downarrow & & \downarrow{\scriptstyle \frac{d}{dx}} \\ f_n' & \xrightarrow{\rightrightarrows} & g = f' \end{array}$$

Proof Fix $x_0 \in (a,b)$. Let

$$\phi_n(t) := \begin{cases} \frac{f_n(t)-f_n(x_0)}{t-x_0}, & \text{if } t \in (a,b) - \{x_0\} \\ f_n'(x_0), & \text{if } t = x_0 \end{cases}$$

$$\phi(t) := \begin{cases} \frac{f(t)-f(x_0)}{t-x_0}, & \text{if } t \in (a,b) - \{x_0\} \\ g(x_0), & \text{if } t = x_0 \end{cases}$$

Claim 1. Each ϕ_n is continuous.

Since each f_n is differentiable on (a,b), each f_n is continuous on (a,b). The function $\frac{1}{t-x_0}$ is continuous on $(a,b) - \{x_0\}$, therefore ϕ_n is continuous on $(a,b) - \{x_0\}$. Since

$$\lim_{t\to x_0} \phi_n(t) = \lim_{t\to x_0} \frac{f_n(t) - f_n(x_0)}{t - x_0} = f_n'(x_0) = \phi_n(x_0)$$

ϕ_n is continuous at x_0 and hence ϕ_n is continuous on (a,b).

Claim 2. $\{\phi_n\}_{n=1}^{\infty}$ converges pointwise to ϕ.

For $t \neq x_0$,

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \frac{f_n(t) - f_n(x_0)}{t - x_0} = \frac{f(t) - f(x_0)}{t - x_0} = \phi(t)$$

For $t = x_0$

$$\lim_{n\to\infty} \phi_n(x_0) = \lim_{n\to\infty} f_n'(x_0) = g(x_0) = \phi(x_0)$$

Therefore $\phi_n \to \phi$ pointwise on (a, b).

Claim 3. $\{\phi_n\}_{n=1}^{\infty}$ **converges uniformly to** ϕ**.**

Given $\epsilon > 0$. Since $f_n' \rightrightarrows g$, there exists $N \in \mathbb{N}$ such that if $n \geq N$,

$$|f_n'(x) - g(x)| < \frac{\epsilon}{2}$$

for all $x \in (a, b)$.

For $m, n \geq N$,

$$\begin{aligned}
|\phi_m(x_0) - \phi_n(x_0)| &= |f_m'(x_0) - f_n'(x_0)| \\
&\leq |f_m'(x_0) - g(x_0)| + |g(x_0) - f_n'(x_0)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

For $m, n \geq N$, $t \neq x_0$,

$$\begin{aligned}
|\phi_m(t) - \phi_n(t)| &= |\frac{(f_m(t) - f_m(x_0)) - (f_n(t) - f_n(x_0))}{t - x_0}| \\
&= |\frac{(f_m - f_n)(t) - (f_m - f_n)(x_0)}{t - x_0}| \\
&= |(f_m - f_n)'(\theta)| \text{ for some } \theta \text{ lies between } t \text{ and } x_0, \\
&= |f_m'(\theta) - f_n'(\theta)| \\
&\leq |f_m'(\theta) - g(\theta)| + |g(\theta) - f_n'(\theta)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Note that the existence of θ is guaranteed by the mean value theorem. Therefore

$$|\phi_m(t) - \phi_n(t)| < \epsilon$$

for all $t \in (a, b)$ when $n, m \geq N$. By the Cauchy criterion, $\{\phi_n\}_{n=1}^{\infty}$ converges uniformly on (a, b). Since the sequence $\{\phi_n\}_{n=1}^{\infty}$ converges to ϕ pointwise, ϕ is the uniform limit of $\{\phi_n\}_{n=1}^{\infty}$.

Claim 4. $f' = g$**.**

Since each ϕ_n is continuous, by a theorem in uniform convergence, the uniform limit ϕ is continuous. In particular, ϕ is continuous at x_0. We have

$$\lim_{t\to x_0}\phi(t)=\phi(x_0)$$

and this means

$$f'(x_0)=\lim_{t\to x_0}\frac{f(t)-f(x_0)}{t-x_0}=g(x_0)$$

Since $x_0\in(a,b)$ is arbitrary, $f'=g$.

Corollary 1.1 (Term-by-term differentiation theorem)

Given a series $\sum_{k=1}^{\infty} f_k$ of differentiable real-valued functions on (a,b). If $\sum_{k=1}^{\infty} f_k$ and $\sum_{k=1}^{\infty} f_k'$ converge uniformly on (a,b), then

$$\left(\sum_{k=1}^{\infty} f_k\right)'=\sum_{k=1}^{\infty} f_k'$$

♡

Proof Let

$$F_n=\sum_{k=1}^{n} f_k \text{ and } F=\sum_{k=1}^{\infty} f_k$$

Since each f_k is differentiable, F_n is differentiable and

$$F_n'=\left(\sum_{k=1}^{n} f_k\right)'=\sum_{k=1}^{n} f_k'$$

By the hypothesis,

$$F_n\rightrightarrows F \text{ and } F_n'\rightrightarrows\sum_{k=1}^{\infty} f_k'$$

By Theorem 1.1,

$$\left(\sum_{k=1}^{\infty} f_k\right)'=F'=\sum_{k=1}^{\infty} f_k'$$

Example 1.3 Show that

$$\left(\sum_{k=1}^{\infty}\frac{\cos kx}{k^3}\right)'=-\sum_{k=1}^{\infty}\frac{\sin kx}{k^2}$$

for $x \in \mathbb{R}$.

Solution *For* $k \in \mathbb{N}, x \in \mathbb{R}$, *let*

$$f_k(x) = \frac{\cos kx}{k^3}$$

Since

$$||f_k||_{\sup} = \frac{1}{k^3}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^3}$ *is convergent by the* p*-series test. By the Weierstrass M-test,* $\sum_{k=1}^{\infty} f_k$ *converges uniformly on* $\mathbb{R}$. *Note that*

$$f_k'(x) = -\frac{\sin kx}{k^2}$$

and

$$||f_k'||_{\sup} = \frac{1}{k^2}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *is convergent by the* p*-series test, again by the Weierstrass M-test,*

$$\sum_{k=1}^{\infty} f_k'$$

converges uniformly on $\mathbb{R}$. *Then by the theorem above, we have*

$$(\sum_{k=1}^{\infty} f_k(x))' = \sum_{k=1}^{\infty} f_k'(x)$$

This shows that

$$(\sum_{k=1}^{\infty} \frac{\cos kx}{k^3})' = -\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$$

Exercise 1.1

1. Let $f_n, f : \mathbb{R} \to \mathbb{R}$ be functions where $n \in \mathbb{N}$. Suppose that if $x_n \to x$, then
$$f_n(x_n) \to f(x)$$
as $n \to \infty$. Show that f is continuous.
2. Let $f : [0, 1) \to \mathbb{R}$ be a uniformly continuous function. Show that there is a unique continuous function $g : [0, 1] \to \mathbb{R}$ such that
$$g(x) = f(x)$$
for all $x \in [0, 1)$.
3. For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a differentiable function with
$$f_n(0) = 0 \text{ and } |f_n'(x)| \le 2$$
for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Suppose that
$$\lim_{n\to\infty} f_n(x) = g(x)$$
for $x \in \mathbb{R}$. Show that g is continuous.

1.2 Riemann integrable functions and uniform convergence

We give a brief recall of the definition of Riemann integrable functions.

Definition 1.1

A partition P of the closed interval $[a, b]$ is a set of points

$$P = \{t_0, t_1, t_2, ..., t_n | a = t_0 < t_1 < \cdots < t_n = b\}$$

where $n \in \mathbb{N}$. A refinement of a partition P of $[a, b]$ is a partition P' of $[a, b]$ such that

$$P \subseteq P'$$

♣

Definition 1.2

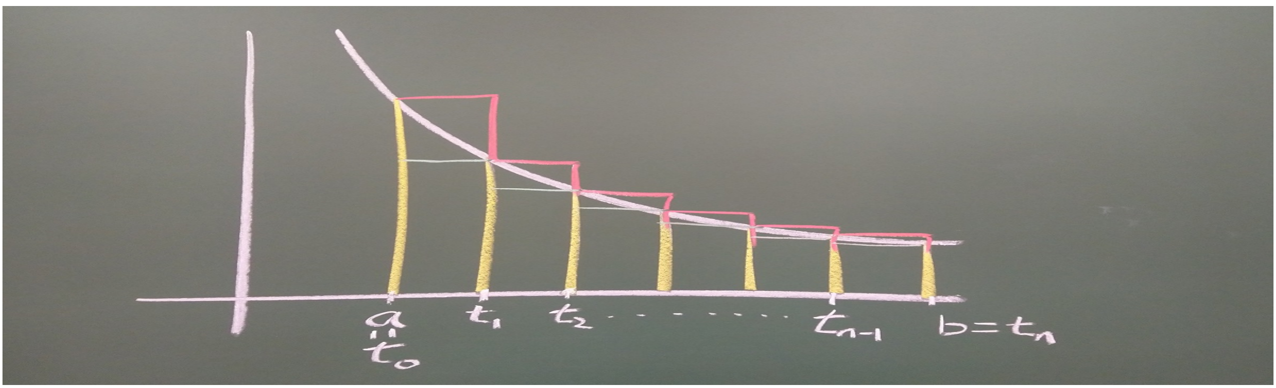
Let $f : [a, b] \to \mathbb{R}$ be a function and $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. The upper sum of f with respect to P is

$$U(f, P) := \sum_{i=0}^{n-1} (\sup_{x \in [t_i, t_{i+1}]} f(x))(t_{i+1} - t_i)$$

The lower sum of f with respect to P is

$$L(f, P) := \sum_{i=0}^{n-1} (\inf_{x \in [t_i, t_{i+1}]} f(x))(t_{i+1} - t_i)$$

♣



Definition 1.3

Let $f : [a,b] \to \mathbb{R}$ be a function. The lower integral of f over $[a,b]$ is

$$\underline{I}(f) := \sup_P L(f,P)$$

and the upper integral of f over $[a,b]$ is

$$\overline{I}(f) := \inf_P U(f,P)$$

where the supreme and infimum are taken over all partitions P of $[a,b]$. If $\underline{I}(f) = \overline{I}(f)$, we say that f is Riemann integrable and denote

$$\int_a^b f(x)dx := \overline{I}(f) = \underline{I}(f)$$

♣

Remark

1. Let P' be a refinement of a partition P of $[a,b]$. From the definition, we have
   (a).
   $$L(f,P) \leq L(f,P')$$
   (b).
   $$U(f,P) \geq U(f,P')$$
2. Observe that if $t_i, t_{i+1} \in P$ and $t_i < s < t_{i+1}$, then
   $$(\inf_{x\in[t_i,t_{i+1}]} f(x))(t_{i+1}-t_i) \leq (\inf_{x\in[t_i,s]} f(x))(s-t_i) + (\inf_{x\in[s,t_{i+1}]} f(x))(t_{i+1}-s)$$
   and
   $$(\sup_{x\in[t_i,t_{i+1}]} f(x))(t_{i+1}-t_i) \geq (\sup_{x\in[t_i,s]} f(x))(s-t_i) + (\sup_{x\in[s,t_{i+1}]} f(x))(t_{i+1}-s)$$
   This gives the following inequalities:
   $$L(f,P) \leq \underline{I}(f) \leq \overline{I}(f) \leq U(f,P)$$

Proposition 1.1

If $f : [a,b] \to \mathbb{R}$ is not a bounded function, then f is not Riemann integrable.

♠

Proof We consider the case

$$\sup_{x\in[a,b]} f(x) = \infty$$

The case

$$\inf_{x\in[a,b]} f(x) = -\infty$$

is proved similarly. Let

$$P = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$$

be a partition of $[a, b]$. Then there exists $t_i \in P$ such that

$$\sup_{x\in[t_i,t_{i+1}]} f(x) = \infty$$

This implies that $U(f, P)$ does not exist and so is $\overline{I}(f)$. Therefore f is not Riemann integrable.

Theorem 1.2 (Riemann's integrability criterion)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if for each $\epsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon$$

♡

Proof If f is Riemann integrable, by the definition, $\overline{I}(f) = \underline{I}(f)$. Since

$$\overline{I}(f) = \inf_P U(f, P) \text{ and } \underline{I}(f) = \sup_P L(f, P)$$

there exist partitions P_1, P_2 of $[a, b]$ such that

$$U(f, P_1) - \overline{I}(f) < \frac{\epsilon}{2}$$

and

$$\underline{I}(f) - L(f, P_2) < \frac{\epsilon}{2}$$

The sum of these two inequalities gives

$$U(f, P_1) - L(f, P_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Let

$$P = P_1 \cup P_2$$

Then

$$L(f, P_2) \leq L(f, P) \leq U(f, P) \leq U(f, P_1)$$

and hence

$$U(f, P) - L(f, P) \leq U(f, P_1) - L(f, P_2) < \epsilon$$

On the other hand, suppose that for each $\epsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon$$

Since

$$L(f, P) \leq \underline{I}(f) \leq \overline{I}(f) \leq U(f, P)$$

we have

$$\overline{I}(f) - \underline{I}(f) \leq U(f, P) - L(f, P) < \epsilon$$

Since ϵ is arbitrary, this means

$$\overline{I}(f) = \underline{I}(f)$$

Proposition 1.2

Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof Given $\epsilon > 0$. Since $[a, b]$ is compact, f is uniformly continuous on $[a, b]$. There exists $\delta > 0$ such that for all $x, y \in [a, b]$, if $|x - y| < \delta$, then

$$|f(x) - f(y)| < \frac{\epsilon}{b - a}$$

Let

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

be a partition of $[a, b]$ such that

$$t_i - t_{i-1} < \delta$$

for $i = 1, ..., n$. Then

$$\sup_{x \in [t_{i-1}, t_i]} f(x) - \inf_{x \in [t_{i-1}, t_i]} f(x) < \frac{\epsilon}{b - a}$$

Hence

$$U(f,P)-L(f,P)=\sum_{i=1}^{n}(\sup_{x\in[x_{i-1},x_i]} f(x)-\inf_{x\in[x_{i-1},x_i]} f(x))(x_i-x_{i-1})$$
$$<\frac{\epsilon}{(b-a)}\sum_{i=1}^{n}(x_i-x_{i-1})$$
$$=\frac{\epsilon}{(b-a)}\cdot(b-a)=\epsilon$$

By the Riemann's integrability criterion, f is Riemann integrable on $[a,b]$.

Definition 1.4

Let $E\subset\mathbb{R}$. The characteristic function of E is the function $\chi_E:\mathbb{R}\to\mathbb{R}$ defined by

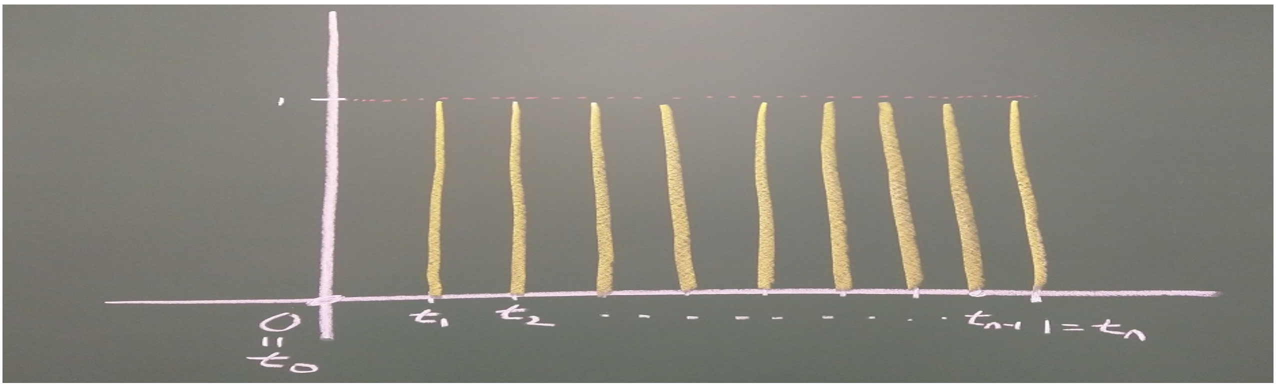
$$\chi_E(x)=\begin{cases}1, & \text{if } x\in E\\ 0, & \text{if } x\notin E\end{cases}$$

♣

Proposition 1.3

The function $\chi_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable over $[0,1]$.

♠



Proof Let $P=\{x_0,x_1,\cdots,x_n\}$ be a partition of $[0,1]$. Note that

$$M_i=\sup\{\chi_{\mathbb{Q}\cap[0,1]}(x)|x\in[x_{i-1},x_i]\}=1$$

and

$$m_i=\inf\{\chi_{\mathbb{Q}\cap[0,1]}(x)|x\in[x_{i-1},x_i]\}=0$$

Thus we have

$$U(\chi_{\mathbb{Q}\cap[0,1]}, P) = \sum_{i=1}^{n} M_i \Delta x_i = 1$$

and

$$L(\chi_{\mathbb{Q}\cap[0,1]}, P) = \sum_{i=1}^{n} m_i \Delta x_i = 0$$

Hence

$$U(\chi_{\mathbb{Q}\cap[0,1]}, P) - L(\chi_{\mathbb{Q}\cap[0,1]}, P) = 1$$

for any partition P of $[0, 1]$. Therefore, by Riemann's integrability criterion, $\chi_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable on $[0, 1]$.

Example 1.4 Since $\mathbb{Q} \cap [0, 1]$ is denumerable, we may list

$$\mathbb{Q} \cap [0, 1] = \{r_1, r_2, r_3, ...\}$$

Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} 1, & \text{if } x \in \{r_1, r_2, ..., r_n\}; \\ 0, & \text{otherwise} \end{cases}$$

Then $f_n \to \chi_{\mathbb{Q}\cap[0,1]}$ pointwise as $n \to \infty$. For $\varepsilon > 0$, take $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \frac{\varepsilon}{n+1}$$

Let

$$P_m = \{\frac{k}{m} | k = 0, 1, ..., m\}$$

Then $L(f_n, P_m) = 0$ and

$$U(f_n, P_m) \leq (n+1)(\frac{1}{m}) < (n+1)\frac{\varepsilon}{n+1} = \varepsilon$$

By the Riemann's integrability criterion, each f_n is Riemann integrable. Since

$$U(f_n, P_m) \to 0$$

as $m \to \infty$, we have

$$\int_0^1 f_n(x)dx = 0$$

for all $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = 0$$

But $\lim_{n\to\infty} f_n = \chi_{\mathbb{Q}\cap[0,1]}$ is not even Riemann integrable, this means that

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

Example 1.5 For $n \geq 2$, define

$$f_n(x) := \begin{cases} n^2 x, & \text{if } x \in [0, \frac{1}{n}]; \\ -n^2(x - \frac{2}{n^2}), & \text{if } x \in [\frac{1}{n}, \frac{2}{n}]; \\ 0, & \text{if } x \in [\frac{2}{n}, 1] \end{cases}$$

Then each f_n is continuous and

$$\int_0^1 f_n(x)dx = 1$$

Note that for $x \in (0,1]$, there exists $N \in \mathbb{N}$ such that

$$\frac{2}{N} < x$$

Then for all $n > N$, $x \in [\frac{2}{n}, 1]$. This means that

$$\lim_{n\to\infty} f_n(x) = 0$$

We have

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

We need the following lemma.

Lemma 1.1

Let $I \subset \mathbb{R}$ be a subset and $f, g : I \to \mathbb{R}$ be bounded functions. Then

$$|\sup_{x\in I} f(x) - \sup_{x\in I} g(x)| \leq \sup_{x\in I} |f(x) - g(x)|$$

and

$$|\inf_{x\in I} f(x) - \inf_{x\in I} g(x)| \leq \sup_{x\in I} |f(x) - g(x)|$$

♡